An Information Correspondence Approach to Bridging Knowledge-Belief Representations in Economics and Mathematical Psychology*

Satoshi Fukuda†

September 10, 2018

Abstract

This paper develops a model of interactive beliefs and knowledge which I call an information correspondence. The information correspondence assigns multiple information sets at each state. It reduces to a standard possibility correspondence when it assigns a unique information set at each state. This generalization allows one to analyze an agent who fails to believe the conjunction of her own beliefs or a tautology. While a possibility correspondence may not be able to represent probabilistic beliefs, this generalization enables one to study qualitative and probabilistic beliefs in a unified manner. The model also generalizes, in a mathematical sense, a knowledge representation in mathematical psychology known as a surmise function. The paper bridges seemingly different knowledge and belief representations in economics and mathematical psychology. The paper also connects the information correspondence model to knowledge and belief representations in computer science, logic, and philosophy.

Journal of Economic Literature Classification Numbers: C70, D83.
Keywords: Information Correspondences, Possibility Correspondences, Surmise Systems, Knowledge, Qualitative Beliefs, Probabilistic Beliefs

1 Introduction

Representations of beliefs and knowledge have been objects of study in such diverse disciplines as computer science and artificial intelligence, economics and game theory,

*This paper is based on part of the first chapter of my Ph.D. thesis at the University of California at Berkeley. I would like to thank David Ahn, William Fuchs, and Chris Shannon for their encouragement, support, and guidance.

†Department of Decision Sciences and IGIER, Bocconi University, Milan 20136, Italy.

logic and philosophy, and psychology, to name a few. This paper provides a model of interactive beliefs and knowledge, which I call an information correspondence model, with the following two objectives. First, the information correspondence model generalizes a possibility correspondence model in economics and game theory with the following two features: (i) agents' logical reasoning ability is less demanding in a still tractable manner; and (ii) agents' qualitative and probabilistic beliefs are examined in a unified way. Second, this paper bridges, in a mathematical sense, seemingly different knowledge-belief representations between economics and mathematical psychology.

An information correspondence model, as in a possibility correspondence model (e.g., Aumann (1976, 1999), Dekel and Gul (1997), Geanakoplos (1989), and Morris (1996)), has the following three ingredients. The first is a set Ω of states of the world. Each state $\omega \in \Omega$ is supposed to be a complete description of the world in question. Second, each agent reasons about some aspects of the underlying states. Each property of the state space is represented as an event, which is a subset E of the state space Ω. Thus, the second ingredient is the collection of events, which determines the objects of agents' beliefs. Third, each agent has her information correspondence $\mathcal{I}$. It associates, with each state, possibly multiple information sets available at that state. The agent believes an event at a state if the event is implied by some information set at that state. If the information correspondence assigns a single information set at each state, then it reduces to the possibility correspondence.

An agent whose beliefs are represented by an information correspondence, unlike a possibility correspondence, may fail to believe the conjunction of events that she believes or a tautology. The information correspondence model is a tractable generalization of a possibility correspondence model that dispenses with such logical sophistication. To obtain tractability, the only requirement of the model is logical monotonicity: an agent believes logical consequences of her own beliefs. The information correspondence approach, by assuming logical monotonicity, enables the analysts to represent agents' beliefs without specifying the entire collection of events that an agent believes at each state.

The relaxation of agents' conjunction and necessitation properties, which are conceptually at odd with real human reasoning, technically allows the analysts to study both qualitative and probabilistic beliefs under the umbrella of the information correspondence approach. For example, in a dynamic game, while agents have knowledge about past observations, they form probabilistic beliefs about their opponents' future behaviors (see, for example, Dekel and Gul (1997) for the importance of capturing both knowledge and probabilistic beliefs). If an agent exhibits the arbitrary conjunction property (i.e., she believes an arbitrary conjunction of her beliefs) rendered by a possibility correspondence, such belief may not be an event (e.g., in a probabilistic framework where the collection of events forms a σ-algebra, such belief may not be measurable). As I will provide an example, such arbitrary conjunction property may indeed be at odd with probabilistic reasoning because, for example, the agent may not assign probability one to the arbitrary conjunction of events that she believes with probability one. Thus, a possibility correspondence model cannot necessarily represent probabilistic beliefs. In contrast, the information correspondence approach can accommodate various forms of monotonic probabilistic beliefs such as standard countably-additive beliefs, finitely-additive beliefs, and even non-additive beliefs.

The paper then characterizes logical and introspective properties of beliefs. I complete the discussion of the information correspondence approach by demonstrating the generality of the model. For given logical and introspective properties of agents' monotonic beliefs, I demonstrate the equivalence between the information correspondence approach and the belief operator approach where an agent's belief operator maps each event E to the event that she believes E.

The second objective of this paper is to connect an information correspondence model to the knowledge representation in mathematical psychology known as a "surmise system" in the "knowledge space theory" developed by Doignon and Falmagne (1985, 1999, 2016); Falmagne and Doignon (2011).[1] This paper aims to link seemingly different (for their respective aims) knowledge representations in economics and mathematical psychology from a unified mathematical point of view.[2]

To make the connection, I introduce a "surmise system" while keeping the same notations. Since I give two different interpretations to each mathematical object, I attach the quotation mark in referring to the mathematical-psychology literature. The literature studies the knowledge of an agent (e.g., a high school student) regarding subsets of Ω, which consists of "questions" or "items." Thus, the ambient set Ω represents the entire body of knowledge in question (e.g., high-school algebra). While I make the formal connection in the sequel, a "surmise function" is a mapping $\mathcal{I}$ which associates, with each "question" ω, the collection of possibly multiple sets of "items" with the following interpretation: each set of such "items" serves as a possible set of prerequisites for the "question" ω. Thus, if the agent has mastered the "question" ω, then she must have mastered all the "questions" in at least one of the members of $\mathcal{I}(\omega)$. Multiplicity of members of $\mathcal{I}(\omega)$ means multiple ways to master the "question" ω. Thus, the "surmise function" $\mathcal{I}$ encodes all possible (thus not necessarily unique) ways to making inferences at each state.[3] This paper shows that an agent's belief described by a "surmise function" is knowledge that exhibits positive introspection

[1]Doignon and Falmagne (1985) is the pioneering paper on the subject, and Doignon and Falmagne (2016) is a recent survey article. Doignon and Falmagne (1999) is the first survey book, while Falmagne and Doignon (2011) is an enriched edition of it. See also the references therein.

[2]Fukuda (2018) represents an agent's truthful knowledge by a set algebra (a collection of events) such as a σ-algebra or a topology in terms of the agent's logical and introspective reasoning ability. The set algebra, in a particular setting, turns out to correspond to the "knowledge states" in Doignon and Falmagne (1985, 1999, 2016); Falmagne and Doignon (2011).

[3]In light of inferences, Shin (1993) identifies the notion of knowledge with logical provability. An agent knows an event when she can prove it from her "basic knowledge" through use of propositional logic. Within the framework of this paper, an agent believes an event E at a state ω if she can "prove" (in terms of set inclusion) E from one of her information set $F \in \mathcal{I}(\omega)$ (which could be incorrect in that $\omega \notin F$). In Shin (1993), such provable knowledge is characterized by a (reflexive and transitive) possibility correspondence.

(if she knows a set of "questions" then she knows that she knows it) and necessitation (the agent knows a tautology).

The motivation behind bridging these two different knowledge and belief representations comes from the observation that the two knowledge models in economics and mathematical psychology, which I demonstrate are firmly related, have evolved in quite different ways. On the one hand, one feature that is not seen in interactive epistemology in economics and game theory is the development of empirical assessments of an agent's beliefs and knowledge based on the formal model. The mathematical psychology literature (see, Doignon and Falmagne (2016); Falmagne and Doignon (2011) for surveys) has been attempting at constructing and testing a formal knowledge representation in practical contexts. A particular case of "knowledge space theory" referred to as "learning space theory" has developed the assessments of students' knowledge about their academic subjects. For example, the web-based system called **ALEKS** ("**A**ssessment of **LE**arning in **K**nowledge **S**paces") has been used by "millions of students in schools and colleges, and by home schooled students" (Doignon and Falmagne, 2016).

On the other hand, the economics literature has provided features that have not been developed in mathematical psychology. One is consideration of interactive higher-order beliefs, i.e., an agent's belief about other agents' beliefs. Another is unawareness: an agent is unaware of an event in that she does not know it and she does not know that she does not know it; or the agent is unaware of an event in that she lacks the conception that determines the event.[4]

This paper is also closely related to knowledge and belief representations referred to as monotone neighborhood systems in computer science, logic, and philosophy (e.g., Chellas (1980), Fagin et al. (2003), and Pacuit (2017)) and related models of limited reasoning in computer science. A neighborhood system (also called a Montague-Scott structure) is a mapping that associates, with each state of the world, the entire collection of events that an agent believes. A monotone neighborhood system is a neighborhood system such that the agent's belief is logically monotonic. An information correspondence is a "generator" of a monotone neighborhood system, and thus it can describe an agent's beliefs without specifying the entire collection of events that she believes at each state. A monotone neighborhood system is considered to be an information correspondence.

In economics and game theory, such papers as Heifetz (1996, 1999) and Lismont and Mongin (1994a,b) use monotone neighborhood systems to represent notions of common belief and common knowledge (e.g., Aumann (1976) and Friedell (1969)). This paper instead formalizes logical and introspective properties of individual agents' beliefs and knowledge. I also briefly discuss how to introduce notions of common belief and common knowledge into the framework of this paper. Salonen (2009) studies a

[4]Pioneering papers on unawareness include Fagin and Halpern (1987) and Modica and Rustichini (1994, 1999). Some recent models include Heifetz, Meier, and Schipper (2006, 2013). See Schipper (2015) for a survey.

canonical syntactical interactive belief representation by capturing each agent's beliefs as a collection of propositions that she believes.

Monotone neighborhood systems and models of limited reasoning have been studied in computer science, logic, and philosophy. The closest paper in this literature is the logic of local reasoning (or the "society-of-minds") approach by Fagin and Halpern (1987).[5] They study a "boundedly rational" agent who fails to believe (or know) the conjunctions of her own belief (or knowledge). The agent is endowed with a collection of multiple information sets at each state, and she focuses on one information set possible at each time. While an information correspondence is regarded as a purely semantic counterpart of their model, I demonstrate that it can capture probabilistic beliefs by defining it on an appropriate set algebraic structure. This paper takes one step further to characterize various logical and introspective properties. This paper also connects it to the mathematical psychology literature.

The paper is organized as follows. Section 2.1 formally defines information correspondences. It also demonstrates that information correspondences generalize possibility correspondences. Section 2.2 provides examples of information correspondences that cannot be captured by possibility correspondences. Section 2.3 analyzes logical and introspective properties of information correspondences. Section 2.4 studies the equivalence among an information correspondence and alternative knowledge and belief representations in economics and mathematical psychology. Section 3 provides concluding remarks. Proofs are relegated to Appendix A.

2 An Information Correspondence

I represent agents' beliefs (knowledge if it is truthful) on a state space $(\Omega, \mathcal{D})$, where Ω is a set of states of the world and where $\mathcal{D}$ is a collection of events, i.e., subsets of states. To formally define the state space $(\Omega, \mathcal{D})$, I provide the following technical preliminaries on set algebras. Fix a set Ω. Denote by $\mathcal{P}(\Omega)$ the power set of Ω.

Letting κ be an infinite cardinal, a collection $\mathcal{D}$ of subsets of Ω (i.e., $\mathcal{D} \subseteq \mathcal{P}(\Omega)$) is a *$\kappa$-complete algebra* (on Ω) if $\mathcal{D}$ is closed under complementation and under arbitrary union (and intersection) of any sub-collection with cardinality less than κ. Following the conventions, $\bigcup \emptyset := \emptyset \in \mathcal{D}$ and $\bigcap \emptyset := \Omega \in \mathcal{D}$. For instance, an $\aleph_0$-complete algebra is an algebra of sets where $\aleph_0$ is the least infinite cardinal. An $\aleph_1$-complete algebra is a σ-algebra, where $\aleph_1$ is the least uncountable cardinal.[6] A subset $\mathcal{D}$ of $\mathcal{P}(\Omega)$ is an (∞-)*complete algebra* (on Ω) if $\mathcal{D}$ is closed under complementation and is closed under arbitrary union (and intersection). Denote the complement of $E \in \mathcal{P}(\Omega)$ by E^c or $\neg E$.

[5]Thijsse (1993) (see also Thijsse (1992, Chapter 6.6)) calls their model a cluster model. Also, Fagin et al. (2003, Chapter 9.6) and Meyer and Hoek (1995, Chapter 2.9) study this approach.

[6]Technically, as mentioned by Meier (2006, Remark 1), it is with no loss of generality to consider κ-complete algebras for infinite regular cardinals. Note that $\aleph_0$ and $\aleph_1$ are regular.

This general definition of a state space allows one to simultaneously introduce both qualitative and probabilistic beliefs on a state space. This also enables one to study various kinds of probabilistic beliefs.[7] Also, the specification of κ determines the "depths" of agents' reasoning. For example, if the analysts suppose an $\aleph_0$-complete algebra (i.e., an algebra), then agents can reason about their finite depths of reasoning of such a form as, Alice believes that Bob believes that an event E obtains. On an $\aleph_1$-complete algebra (i.e., a σ-algebra), agents can reason about their countable depths of reasoning of such a form as, Alice and Bob believe E, they believe that they believe E, and so forth *ad infinitum*.

Now, I provide a formal definition of a state space, in which I represent agents' beliefs (or knowledge). Henceforth, κ refers to either an infinite cardinal or $\kappa = \infty$. A (κ-)*state space* is a pair $(\Omega, \mathcal{D})$ where Ω is a set of *states* of the world and where $\mathcal{D}$ is a κ-complete algebra on Ω. Each element E of $\mathcal{D}$ is an *event*.

2.1 Definition of an Information Correspondence

I represent each agent's belief (or knowledge) on a state space by an information correspondence. For ease of exposition, unless otherwise stated, I restrict attention to a single agent. The information correspondence retains the spirit of a possibility correspondence in the sense that the agent's belief is logically entailed from her information. The agent is a logical reasoner in that she believes any logical consequence of her own belief. Thus, the only requirement is logical monotonicity, and I dispense with the conjunction and necessitation properties endemic in possibility correspondences. The agent may believe events E and F without believing the conjunction $E \cap F$. She may fail to believe a tautology Ω.

The information correspondence $\mathcal{I}$ associates, with each state ω, a collection of events $\mathcal{I}(\omega) \in \mathcal{P}(\mathcal{D})$ that can be a source of beliefs at the state ω in the following sense: the agent believes an event E at the state ω if there is an event $F \in \mathcal{I}(\omega)$ which is included in E. Each element of $\mathcal{I}(\omega)$ can be understood as a piece of information available to the agent at state ω. Call E an *information set* at ω if $E \in \mathcal{I}(\omega)$.

The information correspondence is a mapping $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ satisfying a certain regularity condition. To define the regularity condition, for any $\Gamma \in \mathcal{P}(\mathcal{D})$, define

$$\uparrow \Gamma := \{E \in \mathcal{D} \mid \text{there is } F \in \Gamma \text{ with } F \subseteq E\}.$$

If Γ is the agent's information (at a particular state) then $E \in \uparrow \Gamma$ means that E is entailed from some information $F \in \Gamma$. I often call Γ an *information collection* in the sense that it is a collection of information sets. Note that $\uparrow \Gamma$ is closed under monotonicity (precisely, set inclusion) in that $\uparrow\uparrow \Gamma \subseteq \uparrow \Gamma$.

[7] For example, Meier (2008) studies knowledge and σ-additive probabilistic beliefs on a σ-algebra. Meier (2006) studies a canonical representation of agent's finitely-additive probabilistic beliefs on a κ-complete algebra.

Formally, an *information correspondence* on a state space $(\Omega, \mathcal{D})$ is a mapping $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ which satisfies the regularity condition that, for each $E \in \mathcal{D}$,

$$B_{\mathcal{I}}(E) := \{\omega \in \Omega \mid E \in \uparrow \mathcal{I}(\omega)\} \in \mathcal{D}. \tag{1}$$

The event $B_{\mathcal{I}}(E)$ is interpreted as the set of states at which the agent has information to support E. Thus, $B_{\mathcal{I}}(E)$ is the event that (i.e., the set of states at which) the agent believes E. Define the *belief operator* $B_{\mathcal{I}} : \mathcal{D} \to \mathcal{D}$ derived from $\mathcal{I}$ through Equation (1). The regularity condition of $\mathcal{I}$ grantees that $B_{\mathcal{I}}$ is a well-defined operator. Higher-order beliefs are generated through iterating the belief operator.

I make four remarks on the belief operator $B_{\mathcal{I}}$. First, observe that $\uparrow \mathcal{I}(\omega)$ is exactly the collection of events that the agent believes at ω. Put differently, $\uparrow \mathcal{I}(\omega) = \{E \in \mathcal{D} \mid \omega \in B_{\mathcal{I}}(E)\}$, that is, $E \in \uparrow \mathcal{I}(\omega)$ if and only if (hereafter, often abbreviated as iff) $\omega \in B_{\mathcal{I}}(E)$. Also, $\uparrow \mathcal{I}$ itself is an information correspondence.

Second, on a related point, the mapping $\uparrow \mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ coincides with a monotone neighborhood system. A neighborhood system assigns, with each state, the collection of events that the agent believes. A neighborhood system is monotone when the agent's belief is monotonic (i.e., if the agent believes E and if E implies F then she believes F). Thus, the information correspondence $\mathcal{I}$ is identified as a monotone neighborhood system iff $\mathcal{I} = \uparrow \mathcal{I}$.

Third, the agent considers an event E *possible* at a state ω if she does not believe E^c at ω. Denote by $L_{\mathcal{I}}$ the agent's possibility operator (i.e., $L_{\mathcal{I}}(E) = (\neg B_{\mathcal{I}})(E^c) \in \mathcal{D}$ for all $E \in \mathcal{D}$).[8] The event that the agent considers E possible is

$$L_{\mathcal{I}}(E) = \{\omega \in \Omega \mid F \cap E \neq \emptyset \text{ for all } F \in \mathcal{I}(\omega)\}. \tag{2}$$

The agent considers E possible at ω when her information set at ω is always not inconsistent with E. Section 2.4.2 connects the possibility operator to a closure operator in Doignon and Falmagne (1985, 1999); Falmagne and Doignon (2011).

Fourth, if $\mathcal{I}$ is singleton-valued (i.e., if $\mathcal{I}(\cdot) = \{P(\cdot)\}$) then it reduces to the possibility correspondence $P : \Omega \to \mathcal{D}$ such (i) that each information/possibility set $P(\omega)$ is an event and (ii) that $\mathcal{I}$ satisfies the regularity condition (i.e., $\{\omega \in \Omega \mid P(\omega) \subseteq E\} \in \mathcal{D}$ for each $E \in \mathcal{D}$).[9]

More generally, I introduce the condition under which $\mathcal{I}$ is identified with a possibility correspondence. Namely, $\mathcal{I}$ satisfies the *Kripke property* if each $\mathcal{I}(\omega)$ contains a minimum element, i.e., there is $P(\omega) \in \mathcal{I}(\omega)$ such that $P(\omega) \subseteq E$ for all

[8] First, possibility (or compatibility) is often considered to be the dual of knowledge or belief (e.g., Hintikka (1962) and Fagin et al. (2003)). Second, in the literature on unawareness, the interpretation of possibility could be problematic if the lack of knowledge of E^c (thus the possibility of E) comes from the unawareness of (the negation of) E (Modica and Rustichini, 1999).

[9] The regularity condition requires the belief operator induced by the possibility correspondence P to be well defined. In other words, the belief operator is not well defined if a given possibility correspondence fails the regularity condition. See Samet (2010) for examples of an $\aleph_0$-state space $(\Omega, \mathcal{D})$ in which a partitional possibility correspondence $P : \Omega \to \mathcal{P}(\Omega)$ (i.e., a partition cell $P(\omega)$ may not be an event) does not induce a well-defined knowledge operator from $\mathcal{D}$ into itself.

$E \in \mathcal{I}(\omega)$. If $\mathcal{I}$ satisfies the Kripke property, then the agent's beliefs are represented by $\mathcal{I}(\cdot) = \{P(\cdot)\}$, because the collection $\uparrow \mathcal{I}(\omega)$ of events that the agent believes at a state ω satisfies $\uparrow \mathcal{I}(\omega) = \{E \in \mathcal{D} \mid P(\omega) \subseteq E\} = \uparrow \{P(\omega)\}$. The following proposition formalizes this argument in terms of the belief operator, and demonstrates that an information correspondence is identified as a possibility correspondence under the Kripke property.

Proposition 1. *An information correspondence $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ satisfies the Kripke property iff for each $\omega \in \Omega$, there is $P(\omega) \in \mathcal{I}(\omega)$ such that $\omega \in B_{\mathcal{I}}(E)$ iff $P(\omega) \subseteq E$ for all $E \in \mathcal{D}$.*

To conclude this subsection, I make a connection to a "surmise function" in Doignon and Falmagne (1985, 1999, 2016); Falmagne and Doignon (2011). They study the knowledge of an agent regarding subsets of Ω, which consists of "questions" or "items." Again, note that I keep the same notations in order to make it easier to see the connections and that I append the quotation mark to the terminologies in the mathematical-psychology literature. Table A.1 in Appendix A lists the correspondence of terminologies.

Doignon and Falmagne (1985, 1999, 2016); Falmagne and Doignon (2011) introduce a "surmise system" $(\Omega, \mathcal{I})$ as a way to model an agent's knowledge. A "surmise function" is a mapping $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{P}(\Omega))$ (i.e., $\mathcal{D} = \mathcal{P}(\Omega)$) which satisfies certain logical and introspective properties of knowledge to be discussed in Section 2.3. Each $\mathcal{I}(\omega)$ is interpreted as encoding all possible (not necessarily unique) ways of inferring a correct response to the "question" ω. Put differently, if the agent is capable of solving the "question" ω, then there exists $E \in \mathcal{I}(\omega)$ such that she is capable of solving all the "questions" in E. Such E (each member of $\mathcal{I}(\omega)$) is referred to as a "clause" (a "background" or a "foundation") for the "question" ω.

2.2 Examples of Information Correspondences

I provide two examples of an information correspondence that cannot be reduced to a possibility correspondence. The first example demonstrates that one can dispense with the agent's conjunctive ability (i.e., the agent believes the conjunction of what she believes) by having multiple information sets while one can dispense with the necessitation property (i.e., the agent believes a tautology) by allowing the information correspondence to be empty-valued. The second demonstrates that an information correspondence can capture both qualitative and quantitative beliefs in a unified manner.

In the first example, let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Let $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ be such that $\mathcal{I}(\omega_1) = \{\{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}\}$ and $\mathcal{I}(\omega_2) = \mathcal{I}(\omega_3) = \emptyset$. The agent believes $\{\omega_1, \omega_2\}$ and $\{\omega_1, \omega_3\}$ at state ω_1 but shes does not believe $\{\omega_1, \omega_2\} \cap \{\omega_1, \omega_3\}$ at that state. The failure of the conjunction property comes from multiple information sets which are not closed under intersection. At state ω_2 or ω_3, she does not believe anything at all, and thus $B_{\mathcal{I}}(\Omega) = \{\omega_1\}$. Necessitation fails because the information correspondence is empty-valued at some states.[10] Generally,

$$B_{\mathcal{I}}(E) = \begin{cases} \{\omega_1\} & \text{if } E \in \{\{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \Omega\} \\ \emptyset & \text{otherwise} \end{cases}. \tag{3}$$

Section 2.3 characterizes logical and introspective properties of beliefs.

The second example demonstrates that an information correspondence, unlike a possibility correspondence, can capture probabilistic beliefs. Consider a measurable space $(\Omega, \mathcal{D}) = ([0,1], \mathcal{B}_{[0,1]})$, where $\mathcal{B}_{[0,1]}$ is the Borel σ-algebra on $[0,1]$. Suppose that the agent's beliefs are dictated by the Lebesgue measure μ on $([0,1], \mathcal{B}_{[0,1]})$ at every state. I first show that her probability-one belief is not induced by a possibility correspondence model. Suppose to the contrary that her probability-one belief is induced by a possibility correspondence $P : \Omega \to \mathcal{P}(\Omega)$: she believes an event $E \in \mathcal{D}$ with probability one at a state ω iff $P(\omega) \subseteq E$. Here, I simply do not impose the assumption that each $P(\omega)$ is measurable (i.e., an event). For each event $E_r := \Omega \setminus \{r\} \in \mathcal{D}$ with $r \in \Omega$, the agent assigns probability-one belief to E_r at each ω. Thus, $P(\omega) \subseteq E_r$ for all $r \in \Omega$. Since it implies $P(\omega) = \emptyset$, her probability-one belief operator $B : \mathcal{D} \to \mathcal{D}$ satisfies $B(\cdot) = \Omega$, that is, she assigns probability one to any event. This is impossible. Since the agent assigns probability one to each event E_r, one can generally assert that she assigns probability one to any *countable* intersection. Thus, this contradiction comes from the arbitrary conjunction rendered by the possibility correspondence.

Next, I construct an information correspondence that can capture the agent's probability-one belief. Let $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ be such that $\mathcal{I}(\omega) := \{E \in \mathcal{D} \mid \mu(E) = 1\}$ for each $\omega \in \Omega$. This is an information correspondence because, for any $E \in \mathcal{D}$,

$$B_{\mathcal{I}}(E) = \begin{cases} \emptyset & \text{if } \mu(E) \in [0,1) \\ \Omega & \text{if } \mu(E) = 1 \end{cases}.$$

If the agent believes an event E with probability one at a state ω, then, since $\mu(E) = 1$, I have $E \in \mathcal{I}(\omega)$ and thus $\omega \in B_{\mathcal{I}}(E)$. Conversely, if $\omega \in B_{\mathcal{I}}(E)$ then there is $F \in \mathcal{I}(\omega)$ (i.e., $\mu(F) = 1$) such that $F \subseteq E$. Thus, the agent believes E with probability one at state ω.

More generally, let $(\Omega, \mathcal{D})$ be a κ-state space, and let $t : \Omega \times \mathcal{D} \to [0,1]$ be a function with the following two properties: (i) for each $\omega \in \Omega$, the mapping $t(\omega, \cdot) : \mathcal{D} \to [0,1]$ is a monotone set function that dictates the agent's probabilistic beliefs at ω (e.g., a non-additive, finitely-additive, or countably-additive probability measure);[11] and

[10] Modica and Rustichini (1994, Section 4) provide an example of a two-states model of unawareness in which the agent's knowledge fails Necessitation. Their knowledge operator can be induced from a simplified version of this example in which $\omega_2 = \omega_3$.

[11] A pioneering work on non-additive beliefs is Schmeidler (1989). I have already mentioned that Meier (2006) constructs a canonical interactive finitely-additive-belief structure.

(ii) $\{\omega \in \Omega \mid t(\omega, E) \geq p\} \in \mathcal{D}$ for all $(p, E) \in [0,1] \times \mathcal{D}$. For each state ω, the set function $t(\omega, \cdot)$ is referred to as the agent's *type* at ω, and the mapping $t : \Omega \times \mathcal{D} \to [0,1]$ is referred to as the agent's *type mapping*. For each $p \in [0,1]$, the agent's *p-belief operator* (Friedell, 1969; Monderer and Samet, 1989) $B^p : \mathcal{D} \to \mathcal{D}$ is defined as $B^p(E) := \{\omega \in \Omega \mid t(\omega, E) \geq p\}$ for each $E \in \mathcal{D}$. For each $p \in [0,1]$, define the mapping $\mathcal{I}_p : \Omega \to \mathcal{P}(\mathcal{D})$ by $\mathcal{I}_p(\omega) := \{E \in \mathcal{D} \mid t(\omega, E) \geq p\}$ for each $\omega \in \Omega$. The mapping $\mathcal{I}_p$ is an information correspondence because $B_{\mathcal{I}_p}(E) := \{\omega \in \Omega \mid E \in\uparrow \mathcal{I}_p(\omega)\} = B^p(E) \in \mathcal{D}$ for each $E \in \mathcal{D}$. Thus, the information correspondence approach can accommodate both probabilistic and non-probabilistic beliefs in a unified manner. The information correspondence approach thus enables one to introduce both qualitative and probabilistic beliefs in, for example, a dynamic game where agents retain knowledge of past observations and beliefs in future actions. The information correspondence approach also enables one to study similarities and differences between qualitative and probabilistic beliefs.

2.3 Properties of an Information Correspondence

This subsection represents logical and introspective properties of beliefs for an information correspondence.

2.3.1 Logical Properties

I introduce four logical properties of beliefs. Recall that, in representing probabilistic beliefs by a set function, the analysts assign, with each state, a set function (usually, a countably-additive probability measure) μ on an $\aleph_1$-state space $(\Omega, \mathcal{D})$ which satisfies certain logical properties such as $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$.[12] With this in mind, I introduce logical properties of an information collection. An information correspondence $\mathcal{I}$ is said to satisfy a given logical property when every $\mathcal{I}(\omega)$ satisfies the given logical property.

Hereafter in this subsection, fix a κ-state space $(\Omega, \mathcal{D})$. First, an information collection $\Gamma \in \mathcal{P}(\mathcal{D})$ satisfies *No-Contradiction* if $\emptyset \notin \Gamma$. In words, Γ does not contain a contradiction in the form of the empty set. Second, Γ is satisfies *Consistency* (or it is *serial*) if $E \cap F \neq \emptyset$ for any $E, F \in \Gamma$. In words, any pair of information $(E, F) \in \Gamma^2$ is not contradictory with each other. Consistency implies No-Contradiction. Third, Γ satisfies *Necessitation* if $\Gamma \neq \emptyset$. That is, Γ contains some information, and thus a tautology is inferred from it. Fourth, Γ satisfies *Non-empty λ-Conjunction* (where λ is an infinite cardinal with $\lambda \leq \kappa$) if, for any $\mathcal{F} \subseteq \Gamma$ with $0 < |\mathcal{F}| < \lambda$, there is $F \in \Gamma$ with $F \subseteq \bigcap \mathcal{F}$. Intuitively, for any given family of information, the information

[12] Thus, a type mapping $t : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \Sigma_{\mathcal{D}})$ is also considered to be a measurable mapping that associates, with each state ω, a countably-additive probability measure $t(\omega)$ on Ω, where $\Sigma_{\mathcal{D}}$ is the smallest σ-algebra on $\Delta(\Omega)$ containing $\{\mu \in \Delta(\Omega) \mid \mu(E) \geq p\}$ for each $(p, E) \in [0,1] \times \mathcal{D}$ as in Heifetz and Samet (1998). Remark A.1 in Appendix A formulates an information correspondence as a (κ-)measurable mapping from a state space into the space of information collections.

collection Γ is rich enough to have another information implying the conjunction of the given family. Now, an information correspondence $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ satisfies a given logical property if every $\mathcal{I}(\omega)$ satisfies it.

Recalling that $\uparrow \Gamma$ corresponds to the collection of events that the agent believes by making inferences from Γ, I formalize the logical properties in terms of $\uparrow \Gamma$ instead of the primitive Γ. As a consequence, the proposition below demonstrates that each logical property of an information correspondence $\mathcal{I}$ embodies the intended definition of the logical property of the belief operator $B_{\mathcal{I}}$.

Proposition 2. *Let Γ be an information collection, and let $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ be an information correspondence.*

1. *(a) Γ satisfies No-Contradiction iff $\uparrow \Gamma$ satisfies it.*

   (b) $\mathcal{I}$ satisfies No-Contradiction iff $\uparrow \mathcal{I}$ satisfies it iff $B_{\mathcal{I}}(\emptyset) = \emptyset$.

2. *(a) Γ satisfies Consistency iff $\uparrow \Gamma$ satisfies it iff $E^c \not\in \uparrow \Gamma$ for any $E \in \uparrow \Gamma$.*

   (b) $\mathcal{I}$ satisfies Consistency iff $\uparrow \mathcal{I}$ satisfies it iff $B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(E^c)$ for all $E \in \mathcal{D}$.

3. *(a) Γ satisfies Necessitation iff $\uparrow \Gamma$ satisfies it iff $\Omega \in \uparrow \Gamma$.*

   (b) $\mathcal{I}$ satisfies Necessitation iff $\uparrow \mathcal{I}$ satisfies it iff $B_{\mathcal{I}}(\Omega) = \Omega$.

4. *(a) Γ satisfies Non-empty λ-Conjunction iff $\uparrow \Gamma$ satisfies it iff $\uparrow \Gamma$ is closed under non-empty λ-intersection: $\bigcap \mathcal{F} \in \uparrow \Gamma$ for any $\mathcal{F} \in \mathcal{P}(\uparrow \Gamma) \setminus \{\emptyset\}$ with $|\mathcal{F}| < \lambda$.*

   (b) $\mathcal{I}$ satisfies Non-empty λ-Conjunction iff $\uparrow \mathcal{I}$ satisfies it iff $\bigcap_{F \in \mathcal{F}} B_{\mathcal{I}}(F) \subseteq B_{\mathcal{I}}\left(\bigcap \mathcal{F}\right)$ for any $\mathcal{F} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{F}| < \lambda$.

Proposition 2 establishes the following three points. First, Proposition 2 restates the properties of the information collection Γ in terms of $\uparrow \Gamma$. For (1a), a contradiction is logically entailed from the information collection if and only if it contains a contradiction. For (2a), Γ satisfies Consistency if and only if an event E and its negation E^c are not logically entailed at the same time. For (3a), Γ satisfies Necessitation if and only if a tautology of the form Ω is logically entailed from Γ. For (4a), Γ satisfies Non-empty λ-Conjunction if and only if $\uparrow \Gamma$ is closed under non-empty λ-intersection as stated in the proposition.

Second, on a related point, Proposition 2 shows that the information collection Γ satisfies a given logical property if and only if the information collection $\uparrow \Gamma$ satisfies it. Thus, the information correspondence $\mathcal{I}$ satisfies a given logical property if and only if the information correspondence $\uparrow \mathcal{I}$ satisfies it. In other words, the logical properties are preserved under the operation of taking "$\uparrow$." This implies that the four logical properties of an information correspondence $\mathcal{I}$ are defined in a way such that the properties only depend on $\uparrow \mathcal{I}$. That is, if two information correspondences $\mathcal{I}$ and $\mathcal{I}'$

induce the same beliefs in that $\uparrow \mathcal{I} = \uparrow \mathcal{I}'$, then the information correspondences $\mathcal{I}$ and $\mathcal{I}'$ share the same logical properties.

Third, Proposition 2 demonstrates that each logical property of the information correspondence $\mathcal{I}$ captures the intended logical property of the belief operator $B_{\mathcal{I}}$. Henceforth, $B_{\mathcal{I}}$ is said to satisfy a given logical property (e.g., No-Contradiction) if $\mathcal{I}$ satisfies it. As an example, the belief operator defined in Equation (3) satisfies No-Contradiction and Consistency but fails Necessitation and Non-empty λ-Conjunction (for any λ). For (1b), No-Contradiction means that there is no state at which the agent believes a contradiction in the form of $\emptyset$. For (2b), Consistency means that if the agent believes E, then she does not believe its negation E^c. Consistency implies No-Contradiction because $B_{\mathcal{I}}(\emptyset) \subseteq B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)$ by monotonicity of $B_{\mathcal{I}}$. For (3b), Necessitation means that the agent always believes a tautology in the form of Ω. Since $B_{\mathcal{I}}$ is monotonic, $B_{\mathcal{I}}$ satisfies Necessitation iff $B_{\mathcal{I}}(B_{\mathcal{I}}(\Omega)) = \Omega$. That is, the agent always believes a tautology if and only if she always believes that she believes a tautology. For (4b), Non-empty λ-Conjunction means that if the agent believes each of a non-empty collection of events with cardinality less than λ, then she believes its conjunction. Under Non-empty $\aleph_0$- (i.e., Finite) Conjunction, Consistency and No-Contradiction are equivalent.

2.3.2 Introspective Properties

Next, I introduce the following eight introspective properties of an information correspondence $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$. I study these eight properties after I define all of them at once. First, $\mathcal{I}$ is *reflexive* (or satisfies *Truth Axiom*) if, for any $(\omega, E) \in \Omega \times \mathcal{D}$, $E \in \mathcal{I}(\omega)$ implies $\omega \in E$. That is, $\mathcal{I}(\omega) \subseteq \{E \in \mathcal{D} \mid \omega \in E\}$ for any $\omega \in \Omega$. In words, the agent's information is always correct at each state.

Second, $\mathcal{I}$ is *secondary reflexive* (or satisfies *Belief in Correct Belief*) if, for any $(\omega, E) \in \Omega \times \mathcal{D}$, there is $F \in \mathcal{I}(\omega)$ such that if $\omega' \in F$ and there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $\omega' \in E$. Roughly, there is always information indicating that if the agent believes E then E is true.

Third, $\mathcal{I}$ is *secondary serial* (or satisfies *Belief in Consistency*) if, for any $(\omega, E) \in \Omega \times \mathcal{D}$, there is $F \in \mathcal{I}(\omega)$ such that if $\omega' \in F$ and there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $H \cap E \neq \emptyset$ for all $H \in \mathcal{I}(\omega')$. Roughly, there is always information implying that if the agent believes E then she does not believe the negation E^c.

Fourth, $\mathcal{I}$ satisfies *Belief in Perfect Reasoning* if, for any $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$, there is $G \in \mathcal{I}(\omega)$ with the following property: if $\omega' \in G$, $E' \subseteq E$ for some $E' \in \mathcal{I}(\omega')$, and if $F' \subseteq (\neg E) \cup F$ for some $F' \in \mathcal{I}(\omega')$, then there is $G' \in \mathcal{I}(\omega')$ such that $G' \subseteq F$. Roughly, there is always information implying that if the agent believes E and $(\neg E) \cup F$ then she believes F. The third and fourth properties are studied in Fagin and Halpern (1987) in their logical framework.

Fifth, $\mathcal{I}$ is *transitive* (or satisfies *Positive Introspection*) if, for any $(\omega, E) \in \Omega \times \mathcal{D}$ with $E \in \mathcal{I}(\omega)$, there is $F \in \mathcal{I}(\omega)$ such that if $\omega' \in F$ then there is $E' \in \mathcal{I}(\omega')$ with

$E' \subseteq E$. Put differently, for any information E at a state, there is another information F at the same state (which can possibly be E itself) such that E is always supported as long as F is true. Transitivity turns out to characterize positive introspection stating that if the agent believes E then she believes that she believes it.

Sixth, $\mathcal{I}$ is *Euclidean* (or satisfies *Negative Introspection*) if the following holds. If $(\omega, E) \in \Omega \times \mathcal{D}$ satisfies $E^c \cap F \neq \emptyset$ for all $F \in \mathcal{I}(\omega)$, then there is $F' \in \mathcal{I}(\omega)$ such that if $\omega' \in F'$ then $E^c \cap F \neq \emptyset$ for any $F \in \mathcal{I}(\omega')$. The Euclidean property turns out to characterize negative introspection stating that if the agent does not believe E then she believes that she does not believe it. Note that if $\mathcal{I}$ is Euclidean then it satisfies Necessitation as $\mathcal{I}(\omega) = \emptyset$ leads to a contradiction.

Taking contrapositive, the Euclidean property of $\mathcal{I}$ is characterized as follows. Let $(\omega, E) \in \Omega \times \mathcal{D}$. If, for any $F \in \mathcal{I}(\omega)$, there are $\omega' \in F$ and $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then there is $F \in \mathcal{I}(\omega)$ with $F \subseteq E$. This means that if the agent does not believe that she does not believe an event E, then she believes E.

Seventh, I introduce a property that turns out to be equivalent to the Euclidean property for reflexive and transitive information correspondences. Namely, $\mathcal{I}$ is *symmetric* if the following obtains. Let $(\omega, E) \in \Omega \times \mathcal{D}$. If, for any $F \in \mathcal{I}(\omega)$, there are $\omega' \in F$ and $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $\omega \in E$. This condition states that if the agent does not believe that she does not believe E then E is true. Equivalently, $\mathcal{I}$ is symmetric if and only if, for any $(\omega, E) \in \Omega \times \mathcal{D}$ with $\omega \in E$, there is $F \in \mathcal{I}(\omega)$ such that $\omega' \in F$ implies $F' \cap E \neq \emptyset$ for all $F' \in \mathcal{I}(\omega')$. That is, for any state ω and event E true at ω, the agent believes that she considers E possible. Note that if $\mathcal{I}$ is symmetric then it satisfies Necessitation as $\mathcal{I}(\omega) = \emptyset$ leads to a contradiction.

Eighth, in order to examine Belief in Perfect Reasoning, I introduce the property that characterizes the agent's belief in her conjunction property. Namely, $\mathcal{I}$ satisfies *Belief in Non-empty λ-Conjunction* if, for any $\omega \in \Omega$ and $\mathcal{F} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{F}| < \lambda$, there is $G \in \mathcal{I}(\omega)$ such that if $\omega' \in G$ and $\mathcal{F} \subseteq \mathcal{I}(\omega')$, then there is $G' \in \mathcal{I}(\omega')$ with $G' \subseteq \bigcap \mathcal{F}$. Roughly, there is always information that implies that her belief satisfies Non-empty λ-Conjunction.

It can be seen that if $\mathcal{I}$ satisfies any of Belief in Correct Belief, Belief in Consistency, Belief in Perfect Reasoning, or Belief in Non-empty λ-Conjunction, then it satisfies Necessitation. Also, if $\mathcal{I}$ is Euclidean or symmetric then it satisfies Necessitation. In other words, the failure of Necessitation implies that of each of these properties. In fact, the information correspondence of the first example in Section 2.2 fails Necessitation. While it satisfies Truth Axiom, it violates all the other seven introspective properties. Now, I restate the introspective properties.

Proposition 3. *Let $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ be an information correspondence. For each introspective property, the following are all equivalent.*

1. *(a) $\mathcal{I}$ is reflexive.*
   (b) $\uparrow \mathcal{I}$ is reflexive.
   (c) $B_{\mathcal{I}}(E) \subseteq E$ for all $E \in \mathcal{D}$.

2. (a) $\mathcal{I}$ *is secondary reflexive.*

   (b) $\uparrow \mathcal{I}$ *is secondary reflexive. That is, for any* $(\omega, E) \in \Omega \times \mathcal{D}$*, there is* $F \in\uparrow \mathcal{I}(\omega)$ *such that if* $\omega' \in F$*, then* $E \in\uparrow \mathcal{I}(\omega')$ *implies* $\omega' \in E$*.*

   (c) $\Omega = B_{\mathcal{I}}((\neg B_{\mathcal{I}})(E) \cup E)$ *for any* $E \in \mathcal{D}$*.*

3. (a) $\mathcal{I}$ *is secondary serial.*

   (b) $\uparrow \mathcal{I}$ *is secondary serial. That is, for any* $(\omega, E) \in \Omega \times \mathcal{D}$*, there is* $F \in\uparrow \mathcal{I}(\omega)$ *such that if* $\omega' \in F$ *and* $E \in\uparrow \mathcal{I}(\omega')$*, then* $E^c \notin\uparrow \mathcal{I}(\omega')$*.*

   (c) $\Omega = B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)))$ *for any* $E \in \mathcal{D}$*.*

4. (a) $\mathcal{I}$ *satisfies Belief in Perfect Reasoning.*

   (b) $\uparrow \mathcal{I}$ *satisfies Belief in Perfect Reasoning. That is, for any* $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$*, there is* $G \in\uparrow \mathcal{I}(\omega)$ *such that if* $\omega' \in G$*,* $E \in\uparrow \mathcal{I}(\omega')$*, and if* $(\neg E) \cup F \in\uparrow \mathcal{I}(\omega')$*, then* $F \in\uparrow \mathcal{I}(\omega')$*.*

   (c) $B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F)) = \Omega$ *for any* $E, F \in \mathcal{D}$*.*

5. (a) $\mathcal{I}$ *is transitive.*

   (b) $\uparrow \mathcal{I}$ *is transitive.*

   (c) *For any* $\omega \in \Omega$*, if* $E \in\uparrow \mathcal{I}(\omega)$ *then* $\{\omega' \in \Omega \mid E \in\uparrow \mathcal{I}(\omega')\} \in\uparrow \mathcal{I}(\omega)$*.*

   (d) $B_{\mathcal{I}}(\cdot) \subseteq B_{\mathcal{I}} B_{\mathcal{I}}(\cdot)$*.*

6. (a) $\mathcal{I}$ *is Euclidean.*

   (b) $\uparrow \mathcal{I}$ *is Euclidean.*

   (c) *If* $E \notin\uparrow \mathcal{I}(\omega)$ *for some* $(\omega, E) \in \Omega \times \mathcal{D}$*, then* $\{\omega' \in \Omega \mid E \notin\uparrow \mathcal{I}(\omega')\} \in\uparrow \mathcal{I}(\omega)$*.*

   (d) $(\neg B_{\mathcal{I}})(\cdot) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(\cdot)$*.*

7. (a) $\mathcal{I}$ *is symmetric.*

   (b) $\uparrow \mathcal{I}$ *is symmetric. That is, let* $(\omega, E) \in \Omega \times \mathcal{D}$*, and suppose that, for any* $F \in\uparrow \mathcal{I}(\omega)$*, there is* $\omega' \in F$ *with* $E \in\uparrow \mathcal{I}(\omega')$*. Then* $\omega \in E$*.*

   (c) $(\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E) \subseteq E$ *for all* $E \in \mathcal{D}$*.*

8. *Belief in Perfect Reasoning is equivalent to Belief in Non-empty* $\aleph_0$*- (i.e., Finite) Conjunction. Generally, the following are equivalent.*

   (a) $\mathcal{I}$ *satisfies Belief in Non-empty* λ*-Conjunction.*

   (b) $\uparrow \mathcal{I}$ *satisfies Belief in Non-empty* λ*-Conjunction. That is, for any* $\omega \in \Omega$ *and* $\mathcal{F} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ *with* $|\mathcal{F}| < \lambda$*, there is* $G \in\uparrow \mathcal{I}(\omega)$ *such that if* $\omega' \in G$ *and* $\mathcal{F} \subseteq\uparrow \mathcal{I}(\omega')$*, then* $\bigcap \mathcal{F} \in\uparrow \mathcal{I}(\omega')$*.*

(c) $B_{\mathcal{I}}(\neg(\bigcap_{F\in\mathcal{F}} B_{\mathcal{I}}(F)) \cup B_{\mathcal{I}}(\bigcap \mathcal{F})) = \Omega$ *for any* $\mathcal{F} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ *with* $|\mathcal{F}| < \lambda$.

Similarly to Proposition 2, Proposition 3 establishes the following two points. First, the information correspondence $\mathcal{I}$ satisfies a given introspective property if and only if the information correspondence $\uparrow \mathcal{I}$ satisfies the given property. Put differently, the introspective properties are also preserved under the operation of taking "$\uparrow$." Thus, if information correspondences $\mathcal{I}$ and $\mathcal{I}'$ induce the same beliefs in that $\uparrow \mathcal{I} = \uparrow \mathcal{I}'$, then $\mathcal{I}$ and $\mathcal{I}'$ share the introspective properties.

Second, each introspective property of the information correspondence $\mathcal{I}$ captures the intended introspective property of the belief operator $B_{\mathcal{I}}$. Henceforth, $B_{\mathcal{I}}$ is said to satisfy a given introspective property (e.g., Truth Axiom) if $\mathcal{I}$ satisfies it. Truth Axiom means that if the agent believes (knows) an event at a state then the event is true at that state. Truth Axiom distinguishes belief and knowledge in that belief can be false while knowledge has to be true. Belief in Correct Belief means that the agent always believes that either she does not believe an event E or otherwise E entails. Likewise, Belief in Consistency means that the agent always believes that her belief is consistent. Moreover, Belief in Perfect Reasoning states that the agent always believes that if she believes E and E implies F then she believes F.[13]

Positive Introspection of $B_{\mathcal{I}}$ states that if the agent believes an event then she believes that she believes it. Negative Introspection states that if the agent does not believe an event then she believes that she does not believe it. Thijsse (1993) proves the equivalence of (5c) and (5d) and that of (6c) and (6d). Proposition 3 (5) and (6) provide the conditions on the primitive $\mathcal{I}$ under which the resulting belief satisfies Positive Introspection and Negative Introspection. I remark that Negative Introspection and Monotonicity imply Belief in Correct Belief. For any $(\omega, E) \in \Omega \times \mathcal{D}$, Negative Introspection implies either $E \in \uparrow \mathcal{I}(\omega)$ or $(\neg B_{\mathcal{I}})(E) \in \uparrow \mathcal{I}(\omega)$. In either case, $(\neg B_{\mathcal{I}})(E) \cup E \in \uparrow \mathcal{I}(\omega)$.

If $\mathcal{I}$ is symmetric, then the resulting property on $B_{\mathcal{I}}$ is often referred to as the axiom B in logic (e.g., Chellas (1980)). If $\mathcal{I}$ is reflexive and transitive, then $\mathcal{I}$ is symmetric if and only if it is Euclidean. This argument generalizes the equivalence of the Euclidean property and symmetry for a reflexive and transitive possibility correspondence to a reflexive and transitive information correspondence. See Remark A.2 in Appendix for additional details.

Next, observe that under Necessitation, Consistency implies Belief in Consistency. Likewise, under Necessitation, Truth Axiom and Non-empty λ-Conjunction imply Belief in Correct Belief and Belief in Non-empty λ-Conjunction, respectively. I provide an example in Remark A.3 in Appendix A that, while $B_{\mathcal{I}}$ violates Consistency, Truth Axiom, and Non-empty λ-Conjunction, the agent can still believe in these properties. Thus, not only can the information correspondence capture the agent whose belief

[13] Fagin and Halpern (1987) study the notion of a narrow-minded agent. Call the agent narrow-minded if, for any $\omega \in \Omega$, there is $E \in \mathcal{I}(\omega)$ such that $\omega' \in E$ implies $\mathcal{I}(\omega') = \{E\}$. As in Fagin and Halpern (1987), this axiom implies Belief in Consistency and Belief in Perfect Reasoning. In contrast, parts (3) and (4) of Proposition 2 fully characterize these two properties.

fails Non-empty λ-Conjunction, but also it can capture the very same agent who believes that her own belief satisfies it.

Next, I remark on the introspective properties of $\mathcal{I}$ when $\mathcal{I}$ is singleton-valued. Since the agent's belief satisfies Monotonicity, Necessitation, and Non-empty κ-Conjunction, $\mathcal{I}(\cdot) = \{P(\cdot)\}$ satisfies Belief in Perfect Reasoning and Belief in Non-empty λ-Conjunction. Now, the introspective properties reduce to the standard definition on the possibility correspondence P. First, $\mathcal{I}$ is reflexive iff $\omega \in P(\omega)$ (for all $\omega \in \Omega$). Second, $\mathcal{I}$ is secondary reflexive iff $\omega' \in P(\omega)$ implies $\omega' \in P(\omega')$. Third, $\mathcal{I}$ is secondary serial iff $\omega' \in P(\omega)$ implies $P(\omega') \neq \emptyset$. Fourth, $\mathcal{I}$ is transitive iff $P(\omega') \subseteq P(\omega)$ for any $\omega' \in P(\omega)$. Fifth, $\mathcal{I}$ is Euclidean iff $\omega' \in P(\omega)$ implies $E^c \cap P(\omega') \neq \emptyset$ for any $E \in \mathcal{D}$ with $E^c \cap P(\omega) \neq \emptyset$. It can be seen that $\mathcal{I}$ is Euclidean iff $\omega' \in P(\omega)$ implies $P(\omega) \subseteq P(\omega')$. Sixth, $\mathcal{I}$ is symmetric if $\omega \in P(\omega')$ implies $\omega' \in P(\omega)$. On a related point, as to logical properties, $\mathcal{I}$ is serial iff $P(\cdot) \neq \emptyset$. Also, $\mathcal{I}$ satisfies No-Contradiction iff $P(\cdot) \neq \emptyset$. No-Contradiction and Consistency are equivalent with each other because $\mathcal{I}(\cdot) = \{P(\cdot)\}$ satisfies Non-empty $\aleph_0$-Conjunction.

To conclude, I discuss the assumptions on a surmise function in Doignon and Falmagne (1985, 1999, 2016); Falmagne and Doignon (2011). To that end, I introduce two additional properties of an information correspondence $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$. First, $\mathcal{I}$ is *strongly transitive* if, for any $(\omega, E) \in \Omega \times \mathcal{D}$ with $E \in \mathcal{I}(\omega)$, if $\omega' \in E$ then there is $E' \in \mathcal{I}(\omega')$ with $E' \subseteq E$. This is the transitivity condition in Doignon and Falmagne (1985, Definition 3.5). Second, $\mathcal{I}$ satisfies the minimality condition if $E = F$ for any $E, F \in \mathcal{I}(\omega)$ with $E \subseteq F$. The idea behind the minimality condition is that, if $E \in \mathcal{I}(\omega)$ is not minimal in that there is $F \in \mathcal{I}(\omega)$ with $F \subsetneq E$, then E is redundant in $\mathcal{I}(\omega)$ in that $\uparrow \mathcal{I}(\omega) = \uparrow (\mathcal{I}(\omega) \setminus \{E\})$.

Now, a "surmise function" is a mapping $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{P}(\Omega))$ (i.e., $\mathcal{D} = \mathcal{P}(\Omega)$) satisfying (i) reflexivity, (ii) strong transitivity, (iii) Necessitation, and (iv) minimality. The agent whose knowledge is represented by a "surmise function" satisfies Truth Axiom, Positive Introspection, and Necessitation.

Three remarks on strong transitivity are in order. First, for a singleton-valued information correspondence, transitivity and strong transitivity are equivalent. Second, while strong transitivity implies transitivity, the converse may not be true. Remark A.4 in Appendix A provides an example of $\mathcal{I}$ which is transitive but not strongly transitive. Third, Remark A.5 in Appendix A shows that $\uparrow \mathcal{I}$ may not be strongly transitive even if $\mathcal{I}$ is.[14]

[14] Fagin and Halpern (1987) consider the following form of transitivity that implies strong transitivity: if $\omega' \in E \in \mathcal{I}(\omega)$ then $E \in \mathcal{I}(\omega')$. Thijsse (1993, Example 4) provides an example where this stronger form of transitivity is not necessary for characterizing Positive Introspection. While it can be seen that the monotone information correspondence $\mathcal{I} = \uparrow \mathcal{I}$ in his example satisfies transitivity but violates strong transitivity, for future use, I provide examples of reflexive information correspondences in Remarks A.4 and A.5 in Appendix A. On a related point, Fagin and Halpern (1987) also consider the following stronger Euclidean property: $\omega' \in E \in \mathcal{I}(\omega)$ implies $\mathcal{I}(\omega') \subseteq \mathcal{I}(\omega)$. Thijsse (1993, Example 5) demonstrates that this stronger Euclidean property does not necessarily characterize Negative Introspection.

Finally, I remark on Negative Introspection. While a standard partitional (i.e., reflexive, transitive, and Euclidean) possibility correspondence in economics and game theory, by construction, presupposes Negative Introspection, a "surmise function" in the "knowledge space theory" (Doignon and Falmagne, 1985, 1999, 2016; Falmagne and Doignon, 2011) does not presuppose Negative Introspection.[15] As Fagin et al. (2003, Chapter 3) argue that "there is no one "true" notion of knowledge" and that "the appropriate notion depends on the application," I believe that the aforementioned difference between economics and mathematical psychology comes from the different contexts in which knowledge is analyzed in these distinct fields.

2.4 Equivalence among Knowledge-Belief Representations

The previous subsection showed that $B_{\mathcal{I}}$ inherits the logical and introspective properties of beliefs imposed on a given $\mathcal{I}$. Here, Section 2.4.1 completes the equivalence between an information correspondence and a monotonic belief operator. This implies that the previous results involving (monotonic) belief operators can be replicated under the framework of information correspondences. Section 2.4.2 formally studies a "surmise function" as a reflexive and transitive information correspondence.

I begin with introducing a particular type of events known as self-evident events in the literature. Letting B be a given belief operator, an event $E \in \mathcal{D}$ is *self-evident* if $E \subseteq B(E)$, i.e., the agent believes E whenever E is true. Denote the collection of self-evident events by $\mathcal{J}_B := \{E \in \mathcal{D} \mid E \subseteq B(E)\}$. If an information correspondence $\mathcal{I}$ is given, then an event E is self-evident (i.e., $E \subseteq B_{\mathcal{I}}(E)$) if and only if for any $\omega \in E$, there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq E$.[16] In Doignon and Falmagne (1985, 1999, 2016); Falmagne and Doignon (2011), a self-evident event turns out to be what they call a "knowledge state." The "knowledge state" is interpreted as a set of "questions" that an agent is capable of solving. The collection of knowledge states is referred to as the "knowledge structure."[17]

[15]First, while partitional knowledge models are prevalent in economics and game theory, non-partitional (reflexive and transitive) possibility correspondence models have also been studied. See, for example, Dekel and Gul (1997), Geanakoplos (1989), Morris (1996), and Shin (1993) for foundations for such non-partitional information processing and characterizations of solution concepts in games. Also, an agent whose knowledge satisfies Negative Introspection cannot be unaware of any event in the sense that if she does not know an event then she knows that she does not know it. See Footnote 4 for unawareness. Second, Negative Introspection has also been investigated in other fields such as logic and philosophy. For example, Hintikka (1962, Chapter 3.8) rejects it.

[16]Using self-evidence, one can introduce common belief among a set I of agents. For each agent i, let $\mathcal{I}_i$ be her information correspondence. Following Monderer and Samet (1989), an event E is *common belief* among agents I at a state ω if there is an event F self-evident to every $i \in I$ such that $\omega \in F \subseteq \bigcap_{i \in I} B_{\mathcal{I}_i}(E)$. If E is common belief, then everybody believes E, everybody believes that everybody believes E, and so on *ad infinitum*. As discussed in Section 1, Heifetz (1996, 1999) and Lismont and Mongin (1994a,b) study common belief using monotone neighborhood systems.

[17]Fukuda (2018) shows the one-to-one correspondence between a belief (knowledge) operator satisfying Truth Axiom, Positive Introspection, and Monotonicity and its self-evident collection.

2.4.1 Information Correspondences and Belief Operators

I define an information correspondence $\mathcal{I}$ from a given monotonic belief operator B in such a way that the induced belief operator $B_{\mathcal{I}}$ coincides with the original operator B. Formally, for a given monotonic belief operator $B : \mathcal{D} \to \mathcal{D}$, an information correspondence $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ is a *generator* of B (or $\mathcal{I}$ *induces* B) if $B = B_{\mathcal{I}}$.

Generally, a given monotonic belief operator B has multiple generators. Information correspondences $\mathcal{I}$ and $\mathcal{I}'$ satisfying $\uparrow \mathcal{I} = \uparrow \mathcal{I}'$ induce the same belief operator $B_{\mathcal{I}} = B_{\mathcal{I}'}$. As the next proposition shows, the simplest way to find a generator of B is to consider the information correspondence $\mathcal{I}_B : \Omega \to \mathcal{P}(\mathcal{D})$ defined by

$$\mathcal{I}_B(\omega) := \{E \in \mathcal{D} \mid \omega \in B(E)\} \text{ for each } \omega \in \Omega. \tag{4}$$

Henceforth, I define the information correspondence $\mathcal{I}_B$ induced from a (monotonic) belief operator B through Equation (4). Since B satisfies Monotonicity, $\mathcal{I}_B = \uparrow \mathcal{I}_B$ and consequently $B = B_{\mathcal{I}_B}$. Moreover, if B has multiple generators, then any generator $\mathcal{I}$ is included in $\mathcal{I}_B$ in the sense that $\mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}(\cdot) = \uparrow \mathcal{I}_B(\cdot) = \mathcal{I}_B(\cdot)$. If the given monotonic belief operator B satisfies Truth Axiom and Positive Introspection, then the proposition demonstrates that one can restrict attention to the self-evident events.

Proposition 4. *Let $(\Omega, \mathcal{D})$ be a κ-state space.*

1. *If $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ is an information correspondence, then $B_{\mathcal{I}}$ inherits the properties of beliefs imposed on $\mathcal{I}$ and $\uparrow \mathcal{I}(\cdot) = \uparrow \mathcal{I}_{B_{\mathcal{I}}}(\cdot)$. Conversely, if B is a monotonic belief operator, then $\mathcal{I}_B$ is a generator of B, i.e., $B = B_{\mathcal{I}_B}$. Any generator $\mathcal{I}$ of B satisfies the properties of beliefs imposed on B and $\mathcal{I}(\cdot) \subseteq \mathcal{I}_B(\cdot)$.*

2. *Let $B : \mathcal{D} \to \mathcal{D}$ satisfy Monotonicity, Truth Axiom, and Positive Introspection. Define $\mathcal{I}_{\mathcal{J}_B} : \Omega \to \mathcal{P}(\mathcal{D})$ by $\mathcal{I}_{\mathcal{J}_B}(\omega) := \{E \in \mathcal{J}_B \mid \omega \in B(E)\}$ for each $\omega \in \Omega$. Then, $\mathcal{I}_{\mathcal{J}_B}$ is a reflexive and (strongly) transitive information correspondence that generates B.*

Multiplicity of generators can be used to compare agents' beliefs. For agents i and j, let $\mathcal{I}_i$ and $\mathcal{I}_j$ be generators of B_i and B_j, respectively. Then, $B_i(\cdot) \subseteq B_j(\cdot)$ iff $\uparrow \mathcal{I}_i(\cdot) \subseteq \uparrow \mathcal{I}_j(\cdot)$. This is also equivalent to: for any $\omega \in \Omega$ and $E \in \mathcal{I}_i(\omega)$, there is $F \in \mathcal{I}_j(\omega)$ such that $F \subseteq E$. In mathematical psychology, Doignon and Falmagne (1999, Definition 3.16) call it an "attribution order" (see also Doignon and Falmagne (1985, Definition 3.3)).

Moreover, $B_i(\cdot) \subseteq B_j(\cdot)$ implies $\uparrow \mathcal{I}_{B_i}(\cdot) \subseteq \uparrow \mathcal{I}_{B_j}(\cdot)$. Also, $\uparrow \mathcal{I}_i(\cdot) \subseteq \uparrow \mathcal{I}_j(\cdot)$ implies $B_{\mathcal{I}_i}(\cdot) \subseteq B_{\mathcal{I}_j}(\cdot)$ through Equation (1). Hence, $\uparrow \mathcal{I}_i(\cdot) \subseteq \uparrow \mathcal{I}_{B_j}(\cdot)$ iff $B_{\mathcal{I}_i}(\cdot) \subseteq B_j(\cdot)$. Following Doignon and Falmagne (1985, 1999); Falmagne and Doignon (2011), which compare different knowledge-belief representations by a Galois connection in order theory, Remark A.6 in Appendix A formalizes this argument as a Galois connection.

Proposition 4 (1) could also hold for probabilistic beliefs. Samet (2000) provides the conditions on p-belief operators to induce a type mapping. Meier (2006) and Zhou (2010) extend the conditions to finitely-additive beliefs.

2.4.2 Reflexive and Transitive Information Correspondences

I study a "surmise function" as a reflexive and transitive information correspondence because the "surmise function" $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{P}(\Omega))$ is regarded as an information correspondence on $(\Omega, \mathcal{P}(\Omega))$ satisfying (i) reflexivity, (ii) strong transitivity, (iii) Necessitation, and (iv) minimality. Doignon and Falmagne (1999, Theorems 3.10 and 6.25) show a one-to-one correspondence between a "surmise function" and a "granular" "knowledge structure."[18] Here, I show a general one-to-one correspondence between a reflexive and (strongly) transitive information correspondence and the collection of self-evident events on a κ-state space by dropping the minimality condition. Observe also that, by Proposition 3, there is a one-to-one correspondence between belief (knowledge) operators satisfying Monotonicity, Truth Axiom, and Positive Introspection and reflexive and transitive information correspondences.

Proposition 5. *Let $(\Omega, \mathcal{D})$ be a κ-state space. Let $\mathcal{J} \in \mathcal{P}(\mathcal{D})$ satisfy*

$$\{\omega \in \Omega \mid \text{there is } F \in \mathcal{J} \text{ with } \omega \in F \subseteq E\} \in \mathcal{J} \text{ for each } E \in \mathcal{D}. \tag{5}$$

The following mapping $\mathcal{I}_{\mathcal{J}} : \Omega \to \mathcal{P}(\mathcal{D})$ is a reflexive and (strongly) transitive information correspondence:

$$\mathcal{I}_{\mathcal{J}}(\omega) := \{E \in \mathcal{J} \mid \omega \in E\} \text{ for each } \omega \in \Omega. \tag{6}$$

If $\Omega \in \mathcal{J}$, then $\mathcal{I}_{\mathcal{J}}(\cdot) \neq \emptyset$. Conversely, if $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ is a reflexive and transitive information correspondence, then $\mathcal{J}_{\mathcal{I}} \in \mathcal{P}(\mathcal{D})$ defined below satisfies Condition (5):

$$\mathcal{J}_{\mathcal{I}} := \{E \in \mathcal{D} \mid \text{if } \omega \in E \text{ then there is } F \in \mathcal{I}(\omega) \text{ with } F \subseteq E\}. \tag{7}$$

If $\mathcal{I}(\cdot) \neq \emptyset$, then $\Omega \in \mathcal{J}_{\mathcal{I}}$. Moreover, starting from $\mathcal{J}$, $\mathcal{J} = \mathcal{J}_{\mathcal{I}_{\mathcal{J}}}$. Starting from $\mathcal{I}$, $\uparrow \mathcal{I} = \uparrow \mathcal{I}_{\mathcal{J}_{\mathcal{I}}}$.

Doignon and Falmagne (1985, 1999, 2016); Falmagne and Doignon (2011) characterize the collection $\mathcal{J}$ of "knowledge states" as a collection of events which are closed under arbitrary union. In contrast, Proposition 5 requires $\mathcal{J}$ to satisfy Condition (5). Now, the closure under arbitrary union turns out to be equivalent to Condition (5) if $\mathcal{D}$ is an (∞-)complete algebra for the following two observations: (i) on the (∞-)complete algebra $\mathcal{D}$, the set of states in Condition (5) reduces to $\bigcup\{F \in \mathcal{J} \mid F \subseteq E\}$; and (ii) $E = \bigcup\{F \in \mathcal{J} \mid F \subseteq E\}$ for each $E \in \mathcal{J}$. Generally, Condition (5) is equivalent to the existence of a maximal event in $\mathcal{J}$ that is included in a given event $E \in \mathcal{D}$. Fukuda (2018) and Samet (2010) study this maximality property to obtain set-algebraic representations of knowledge.

[18] A "knowledge structure" $\mathcal{J}$ is *granular* (Doignon and Falmagne, 1999, Definition 1.35) if, for any $(\omega, E) \in \Omega \times \mathcal{J}$ with $\omega \in E$, there is a minimal $F \in \mathcal{J}$ with $\omega \in F \subseteq E$. Doignon and Falmagne (1985, Theorem 3.7) establish the equivalence between a "surmise function" and a "knowledge structure" when Ω is finite.

In Proposition 5, since $\mathcal{I}_{\mathcal{J}}$ defined by Equation (6) is strongly transitive and since strong transitivity implies transitivity, the proposition also establishes the equivalence between a collection of self-evident events and a reflexive and strongly transitive information correspondence. An example in Remark A.7 in Appendix A, however, shows that $\uparrow \mathcal{I}_{\mathcal{J}}$ may not necessarily be strongly transitive even if $\mathcal{I}_{\mathcal{J}}$ is. Thus, strong transitivity is not necessarily preserved under the operation of taking "$\uparrow$." This also means that the reflexive and transitive information correspondence $\mathcal{I} :=\uparrow \mathcal{I}_{\mathcal{J}}$ (i.e., $\mathcal{I}(\omega) := \{E \in \mathcal{D} \mid \text{there is } F \in \mathcal{J} \text{ with } \omega \in F \subseteq E\}$) can also establish the part of Proposition 5 in place of Equation (6).

Doignon and Falmagne (1999, Theorem 6.25) (and Doignon and Falmagne (1985, Theorem 3.7)) establish the correspondence between a "surmise function" and a "knowledge structure" (i.e., a reflexive and transitive information correspondence and a collection of self-evident events in my context) in terms of a Galois connection. This amounts to proving: $\uparrow \mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}_{\mathcal{J}}(\cdot)$ iff $\mathcal{J}_{\mathcal{I}} \subseteq \mathcal{J}$ for each $\mathcal{I}$ and $\mathcal{J}$. I demonstrate in Remark A.8 in Appendix A that the pair of mappings defined through Equations (6) and (7) forms a Galois connection.

Next, consider a connection between Propositions 4 (2) and 5. If a given monotonic belief operator B satisfies Truth Axiom and Positive Introspection, then the information correspondence $\mathcal{I}_{\mathcal{J}_B} = \{E \in \mathcal{J}_B \mid \omega \in B(E)\}$ in Proposition 4 (2) is indeed equal to $\mathcal{I}_{\mathcal{J}_B} = \{E \in \mathcal{J}_B \mid \omega \in E\}$ in Proposition 5. Note that it can be seen that $\mathcal{J}_B$ satisfies Condition (5).

I remark on the further connections with the "knowledge space theory" of Doignon and Falmagne (1985, 1999, 2016); Falmagne and Doignon (2011). First, let an information correspondence $\mathcal{I}$ be $\mathcal{I}(\cdot) = \{P(\cdot)\}$ (singleton-valued). If $\omega' \in P(\omega)$, then the agent considers ω' *possible* at state ω. Thus, P induces a binary relation also known as an accessibility (or possibility) relation in computer science, logic, and philosophy (e.g., Chellas (1980) and Fagin et al. (2003)). Suppose further that P is reflexive and transitive. In mathematical psychology, if P is reflexive and transitive, then the reflexive and transitive binary relation induced by P turns out to be a "surmise (or precedence) relation" (Doignon and Falmagne, 1985, 1999, 2016; Falmagne and Doignon, 2011). In their context, if $\omega' \in P(\omega)$, then it can be surmised from a correct response to "question" ω that a correct response to "question" ω' is given.

Second, suppose that $\mathcal{I}$ is reflexive and transitive. It turns out that the possibility operator $L_{\mathcal{I}} : \mathcal{D} \to \mathcal{D}$ defined in Equation (2) satisfies the following three properties: (i) $E \subseteq F$ implies $L_{\mathcal{I}}(E) \subseteq L_{\mathcal{I}}(F)$; (ii) $E \subseteq L_{\mathcal{I}}(E)$; and (iii) $L_{\mathcal{I}} L_{\mathcal{I}}(\cdot) \subseteq L_{\mathcal{I}}(\cdot)$. The operator $L_{\mathcal{I}}$ is related to the notion of a closure operator, and a tuple $(\Omega, \{E \in \mathcal{D} \mid L_{\mathcal{I}}(E) \subseteq E\}) = (\Omega, \{E \in \mathcal{D} \mid E^c \in \mathcal{J}_{B_{\mathcal{I}}}\})$ is related to the notion of a closure space (Doignon and Falmagne, 1985, 1999; Falmagne and Doignon, 2011).

3 Conclusion

This paper developed an information correspondence that represents an agent's beliefs about underlying states of the world. It associates, with each state, a set of possibly multiple information sets at that state. Conceptually, it can capture beliefs that may fail the conjunction or necessitation properties. It can also capture both qualitative and quantitative beliefs (e.g., knowledge and probability-one belief) in a unified manner. If it has a unique information set at each state, then it reduces to a possibility correspondence. The paper characterized the logical and introspective properties of beliefs.

This paper connected seemingly different knowledge-belief representations by demonstrating that a "surmise function" in mathematical psychology (Doignon and Falmagne, 1985, 1999, 2016; Falmagne and Doignon, 2011) can be seen as a particular information correspondence. This paper thus provided various logical and introspective properties of a "surmise function." I hope that this paper spurs further ideas in both economics and mathematical psychology as discussed in the introduction.

One interesting direction of future study is to explore interactions of knowledge and probabilistic beliefs, especially belief update on available information. In a standard possibility correspondence model, an agent's type at a state is usually the posterior probability measure conditional on the information set at that state. Another direction is to develop an information correspondence on a generalized state space of the unawareness structure developed by Heifetz, Meier, and Schipper (2006, 2013). In their generalized state space model, a state space consists of multiple subspaces, and a possibility correspondence on such generalized state space can represent an agent's unawareness satisfying certain desirable properties.

A Appendix

Proof of Proposition 1. Let $\mathcal{I}$ be an information correspondence satisfying the Kripke property. Fix $\omega \in \Omega$, and let $P(\omega)$ be the minimum element of $\mathcal{I}(\omega)$. For each $E \in \mathcal{D}$, $\omega \in B_{\mathcal{I}}(E)$ iff $P(\omega) \subseteq E$. Conversely, suppose that, for each $\omega \in \Omega$, there is $P(\omega) \in \mathcal{I}(\omega)$ such that $\omega \in B_{\mathcal{I}}(E)$ iff $P(\omega) \subseteq E$. Fix $\omega \in \Omega$. If $E \in \mathcal{I}(\omega)$, then $\omega \in B_{\mathcal{I}}(E)$ and thus $P(\omega) \subseteq E$. Hence, $P(\omega)$ is the minimum element of $\mathcal{I}(\omega)$. $\square$

Remark A.1. Let $C(\Omega, \mathcal{D}) (\subseteq \mathcal{P}(\mathcal{D}))$ be the set of information collections $\Gamma \in \mathcal{P}(\mathcal{D})$ which respect a certain set of logical properties to be defined in Section 2.3.1. Let $\mathcal{C}(\Omega, \mathcal{D}) (\subseteq \mathcal{P}(C(\Omega, \mathcal{D})))$ be the smallest κ-complete algebra (i.e., the intersection of all κ-complete algebras) including $\{\{\Gamma \in C(\Omega, \mathcal{D}) \mid E \in \uparrow \Gamma\} \in \mathcal{P}(C(\Omega, \mathcal{D})) \mid E \in \mathcal{D}\}$. Now, any measurable mapping $\mathcal{I} : (\Omega, \mathcal{D}) \to (C(\Omega, \mathcal{D}), \mathcal{C}(\Omega, \mathcal{D}))$ is an information correspondence because $B_{\mathcal{I}}(E) = \mathcal{I}^{-1}(\{\Gamma \in C(\Omega, \mathcal{D}) \mid E \in \uparrow \Gamma\}) \in \mathcal{D}$ for each $E \in \mathcal{D}$.

Proof of Proposition 2. By (a) of each logical property, it follows that, in (b), $\mathcal{I}$ satisfies a given logical property if and only if $\uparrow \mathcal{I}$ satisfies the given property.

Notation	Terminology in the Paper	Corresponding Terminology in the Knowledge Space Theory
Ω	State space (Set of states of the world)	Domain (of the body of knowledge)
$\omega \in \Omega$	State	Item or question
$E \in \mathcal{D}$	Event	Set of items (or questions)
$\mathcal{J}$	Collection of self-evident events	Knowledge structure
$E \in \mathcal{J}$	Self-evident event	Knowledge state
$\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$	Information correspondence	Surmise function $(\Omega, \mathcal{I})$: Surmise system
$E \in \mathcal{I}(\omega)$	Information set at ω	Clause (background or foundation) for ω
$\omega' \in P(\omega)$ $(\mathcal{I}(\cdot) = \{P(\cdot)\})$	Possibility (or accessibility) relation	Surmise relation

Table A.1: A list of notations and terminologies: The first column lists key notations of this paper. The second does their terminologies. The third does the corresponding terminologies in Doignon and Falmagne (1985, 1999, 2016); Falmagne and Doignon (2011).

1. (a) I show that if Γ satisfies No-Contradiction then so does $\uparrow \Gamma$ by contraposition. If $\emptyset \in \uparrow \Gamma$, then there is $E \in \Gamma$ with $E \subseteq \emptyset$, i.e., $\emptyset \in \Gamma$. Conversely, if $\emptyset \notin \uparrow \Gamma$, then $\emptyset \notin \Gamma$ because $\Gamma \subseteq \uparrow \Gamma$.

   (b) For each $\omega \in \Omega$, $\emptyset \in \uparrow \mathcal{I}(\omega)$ iff $\omega \in B_{\mathcal{I}}(\emptyset)$. Thus, $\emptyset \notin \uparrow \mathcal{I}(\omega)$ for all $\omega \in \Omega$ iff $B_{\mathcal{I}}(\emptyset) = \emptyset$.

2. (a) Let Γ satisfy Consistency. Suppose to the contrary that there are $E, F \in \uparrow \Gamma$ such that $E \cap F = \emptyset$. Then, there are $E', F' \in \Gamma$ such that $E' \subseteq E$ and $F' \subseteq F$. Thus, $E' \cap F' \subseteq E \cap F = \emptyset$, a contradiction. If $\uparrow \Gamma$ satisfies Consistency then $E^c \notin \uparrow \Gamma$ for any $E \in \uparrow \Gamma$. Finally, suppose that $E^c \notin \uparrow \Gamma$ for any $E \in \uparrow \Gamma$. Suppose to the contrary that there are $E, F \in \Gamma$ with $E \cap F = \emptyset$. Since $F \subseteq E^c$, it follows that $E, E^c \in \uparrow \Gamma$, a contradiction.

   (b) I show that if $\mathcal{I}$ satisfies Consistency then $B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(E^c)$. If $\omega \in B_{\mathcal{I}}(E)$ then $E \in \uparrow \mathcal{I}(\omega)$. Since $E^c \notin \uparrow \mathcal{I}(\omega)$, I have $\omega \in (\neg B_{\mathcal{I}})(E^c)$. Conversely, assume $B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(E^c)$. Take any $\omega \in \Omega$ and $E \in \uparrow \mathcal{I}(\omega)$. Since $\omega \in B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(E^c)$, I have $E^c \notin \uparrow \mathcal{I}(\omega)$.

3. (a) If $\Gamma \neq \emptyset$ then $\uparrow \Gamma \neq \emptyset$. If $\uparrow \Gamma \neq \emptyset$, then there is $E \in \uparrow \Gamma$. Since $E \subseteq \Omega$, I have $\Omega \in \uparrow \Gamma$. If $\Omega \in \uparrow \Gamma$, then there is $E \in \Gamma$ and thus $\Gamma \neq \emptyset$.

   (b) The statement follows because $\Omega \in \uparrow \mathcal{I}(\omega)$ for all $\omega \in \Omega$ iff $B_{\mathcal{I}}(\Omega) = \Omega$.

4. (a) Since $\uparrow \Gamma$ is closed under set inclusion, $\uparrow \Gamma$ satisfies Non-empty λ-Conjunction if and only if $\uparrow \Gamma$ is closed under non-empty λ-intersection.

(b) Take any $\mathcal{F} \subseteq \mathcal{D}$ with $0 < |\mathcal{F}| < \lambda$. Observe that $\bigcap_{F \in \mathcal{F}} B_{\mathcal{I}}(F) \subseteq B_{\mathcal{I}}(\bigcap \mathcal{F})$ iff, for any $\omega \in \Omega$, if $F \in\uparrow \mathcal{I}(\omega)$ for all $F \in \mathcal{F}$ then $\bigcap \mathcal{F} \in\uparrow \mathcal{I}(\omega)$.

$\square$

Proof of Proposition 3. 1. First, I show that (1a) implies (1b). If $E \in\uparrow \mathcal{I}(\omega)$ then there is $F \in \mathcal{I}(\omega)$ with $F \subseteq E$. Then, $\omega \in F \subseteq E$. Second, I show that (1b) implies (1c). If $\omega \in B_{\mathcal{I}}(E)$, then $E \in\uparrow \mathcal{I}(\omega)$, and thus $\omega \in E$. Finally, I show that (1c) implies (1a). If $E \in \mathcal{I}(\omega) \subseteq\uparrow \mathcal{I}(\omega)$, then $\omega \in B_{\mathcal{I}}(E) \subseteq E$.

2. First, I show that (2a) implies (2b). Take $(\omega, E) \in \Omega \times \mathcal{D}$. There is $F \in \mathcal{I}(\omega) \subseteq\uparrow \mathcal{I}(\omega)$ with the following property: if $\omega' \in F$ and $E \in\uparrow \mathcal{I}(\omega')$ then $F' \in \mathcal{I}(\omega')$ for some $F' \subseteq E$. Then, $\omega' \in E$. Second, I show that (2b) implies (2c). Take $(\omega, E) \in \Omega \times \mathcal{D}$. There is $F \in \mathcal{I}(\omega)$ such that $F \subseteq (\neg B_{\mathcal{I}})(E) \cup E$. Thus, $\omega \in B_{\mathcal{I}}((\neg B_{\mathcal{I}})(E) \cup E)$, and hence $\Omega = B_{\mathcal{I}}((\neg B_{\mathcal{I}})(E) \cup E)$. Finally, I show that (2c) implies (2a). Take any $(\omega, E) \in \Omega \times \mathcal{D}$. Since $\omega \in \Omega = B_{\mathcal{I}}((\neg B_{\mathcal{I}})(E) \cup E)$, there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq (\neg B_{\mathcal{I}})(E) \cup E$. If $\omega' \in F$ and if there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then, since $\omega' \in B_{\mathcal{I}}(E)$, I have $\omega' \in E$.

3. First, I show that (3a) implies (3b). Fix $(\omega, E) \in \Omega \times \mathcal{D}$. There is $F \in \mathcal{I}(\omega) \subseteq\uparrow \mathcal{I}(\omega)$ with the following property: if $\omega' \in F$ and if $E \in\uparrow \mathcal{I}(\omega')$ and thus $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $H \cap E \neq \emptyset$ for all $H \in\uparrow \mathcal{I}(\omega')$. Thus, $E^c \notin\uparrow \mathcal{I}(\omega')$. Second, I show that (3b) implies (3c). Fix $E \in \mathcal{D}$, and take $\omega \in \Omega$. I show that $\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)) = (\neg B_{\mathcal{I}})(E) \cup (\neg B_{\mathcal{I}})(E^c) \in\uparrow \mathcal{I}(\omega)$. By supposition, there is $F \in\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F$ and if $\omega' \in B_{\mathcal{I}}(E)$ then $\omega' \in (\neg B_{\mathcal{I}})(E^c)$. Thus, $F \subseteq \neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c))$, and hence $\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)) \in\uparrow \mathcal{I}(\omega)$. Finally, I show that (3c) implies (3a). Fix $(\omega, E) \in \Omega \times \mathcal{D}$. Since $\omega \in \Omega = B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)))$, there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq \neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c))$. Thus, if $\omega' \in F$ and if there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $\omega' \in (\neg B_{\mathcal{I}})(E^c)$. Thus, $H \cap E \neq \emptyset$ for all $H \in \mathcal{I}(\omega')$.

4. First, I show that (4a) implies (4b). For any $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$, there is $G \in \mathcal{I}(\omega) \subseteq\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$, $E \in\uparrow \mathcal{I}(\omega')$, and if $(\neg E) \cup F \in\uparrow \mathcal{I}(\omega')$, then $F \in\uparrow \mathcal{I}(\omega')$. Second, I show that (4b) implies (4c). Fix $E, F \in \mathcal{D}$ and $\omega \in \Omega$. There is $G \in\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$, $\omega' \in B_{\mathcal{I}}(E)$, and if $\omega' \in B_{\mathcal{I}}((\neg E) \cup F)$, then $\omega' \in B_{\mathcal{I}}(F)$. Thus, $G \subseteq \neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F)$. Hence, $\omega \in B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F))$. Finally, I show that (4c) implies (4a). Take $(E, F) \in \mathcal{D}^2$ and $\omega \in \Omega = B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F))$. There is $G \in \mathcal{I}(\omega)$ such that if $\omega' \in G$ then $\omega' \in \neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F)$. If $\omega' \in G$, $E' \in \mathcal{I}(\omega')$ for some $E' \subseteq E$, and if $F' \in \mathcal{I}(\omega')$ for some $F' \subseteq (\neg E) \cup F$, then $\omega' \in B_{\mathcal{I}}(F)$, i.e., there is $G' \in \mathcal{I}(\omega')$ with $G' \subseteq F$.

5. First, I show that (5a) implies (5b). If $E \in\uparrow \mathcal{I}(\omega)$, then there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq E$. Then, there is $G \in \mathcal{I}(\omega)$ such that if $\omega' \in G$ then there is

$F' \in \mathcal{I}(\omega') \subseteq\uparrow \mathcal{I}(\omega')$ such that $F' \subseteq F \subseteq E$. Second, I show that (5b) implies (5c). If $E \in\uparrow \mathcal{I}(\omega)$, then there is $F \in\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F$ then there is $E' \in\uparrow \mathcal{I}(\omega')$ such that $E' \subseteq E$. Then, since $F \subseteq \{\omega' \in \Omega \mid E \in\uparrow \mathcal{I}(\omega')\}$, it follows $\{\omega' \in \Omega \mid E \in\uparrow \mathcal{I}(\omega')\} \in\uparrow \mathcal{I}(\omega)$. Third, I show that (5c) implies (5d). If $\omega \in B_{\mathcal{I}}(E)$, then $E \in\uparrow \mathcal{I}(\omega)$. Then, $\{\omega' \in \Omega \mid E \in\uparrow \mathcal{I}(\omega')\} \in\uparrow \mathcal{I}(\omega)$, i.e., $\omega \in B_{\mathcal{I}}B_{\mathcal{I}}(E)$. Finally, I show that (5d) implies (5a). Fix $\omega \in \Omega$ and $E \in \mathcal{I}(\omega) \subseteq\uparrow \mathcal{I}(\omega)$. Then, $\omega \in B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}B_{\mathcal{I}}(E)$, and thus $\{\omega' \in \Omega \mid E \in\uparrow \mathcal{I}(\omega')\} \in\uparrow \mathcal{I}(\omega)$. Now, there is $F \in \mathcal{I}(\omega)$ such that if $\omega' \in F$ then $E \in\uparrow \mathcal{I}(\omega')$, i.e., there is $E' \in \mathcal{I}(\omega')$ with $E' \subseteq E$.

6. First, I show that (6a) implies (6b). Let $(\omega, E) \in \Omega \times \mathcal{D}$ be such that $E^c \cap F \neq \emptyset$ for all $F \in\uparrow \mathcal{I}(\omega)$. Since $\mathcal{I}(\omega) \subseteq\uparrow \mathcal{I}(\omega)$ and since $\mathcal{I}(\omega)$ is Euclidean, there is $F' \in \mathcal{I}(\omega) \subseteq\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F'$ then $\emptyset \neq E^c \cap F$ for all $F \in\uparrow \mathcal{I}(\omega')$. Second, I show that (6b) implies (6c). If $E \notin\uparrow \mathcal{I}(\omega)$, then $E^c \cap F \neq \emptyset$ for all $F \in\uparrow \mathcal{I}(\omega)$. Then, there is $F' \in\uparrow \mathcal{I}(\omega)$ such that $F' \subseteq \{\omega' \in \Omega \mid E \notin\uparrow \mathcal{I}(\omega')\}$, i.e., $\{\omega' \in \Omega \mid E \notin\uparrow \mathcal{I}(\omega')\} \in\uparrow \mathcal{I}(\omega)$. Third, I show that (6c) implies (6d). If $\omega \in (\neg B_{\mathcal{I}})(E)$, then $E \notin\uparrow \mathcal{I}(\omega)$. Then, $\{\omega' \in \Omega \mid E \notin\uparrow \mathcal{I}(\omega')\} \in\uparrow \mathcal{I}(\omega)$, i.e., $\omega \in B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E)$. Finally, I show that (6d) implies (6a). Let $(\omega, E) \in \Omega \times \mathcal{D}$ be such that $E^c \cap F \neq \emptyset$ for all $F \in \mathcal{I}(\omega)$. Since $E \notin\uparrow \mathcal{I}(\omega)$, I get $\omega \in (\neg B_{\mathcal{I}})(E) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E)$, i.e., $\{\omega' \in \Omega \mid E \notin\uparrow \mathcal{I}(\omega')\} \in\uparrow \mathcal{I}(\omega)$. Thus, there is $F' \in \mathcal{I}(\omega)$ such that if $\omega' \in F$ then $E \notin\uparrow \mathcal{I}(\omega')$, i.e., $E^c \cap F \neq \emptyset$ for all $F \in \mathcal{I}(\omega')$.

7. First, I show that (7a) implies (7b). Fix $(\omega, E) \in \Omega \times \mathcal{D}$. Suppose that, for any $F \in\uparrow \mathcal{I}(\omega)$, there is $\omega' \in F$ with $E \in\uparrow \mathcal{I}(\omega')$. Then, for any $F \in \mathcal{I}(\omega)$, there are $\omega' \in F$ and $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$. Thus, $\omega \in E$. Second, I show that (7b) implies (7c). Fix $E \in \mathcal{D}$. If $\omega \in (\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E)$, then for any $F \in\uparrow \mathcal{I}(\omega)$, $F \not\subseteq (\neg B_{\mathcal{I}})(E)$, i.e., there are $\omega' \in F$ and $E \in\uparrow \mathcal{I}(\omega')$. Then, $\omega \in E$. Finally, I show that (7c) implies (7a). Fix $(\omega, E) \in \Omega \times \mathcal{D}$. Suppose that, for any $F \in \mathcal{I}(\omega)$, there are $\omega' \in F$ and $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$. Thus, for any $F \in \mathcal{I}(\omega)$, $F \not\subseteq (\neg B_{\mathcal{I}})(E)$. Then, $\omega \in (\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E) \subseteq E$.

8. First, I characterize Belief in Non-empty λ-Conjunction. Throughout the proof, let $\mathcal{F} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ satisfy $|\mathcal{F}| < \lambda$. I show that (8a) implies (8b). Fix $\omega \in \Omega$. There is $G \in \mathcal{I}(\omega) \subseteq\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$ and if $\mathcal{F} \subseteq\uparrow \mathcal{I}(\omega')$, then there is $E_F \in \mathcal{I}(\omega')$ with $E_F \subseteq F$ for each $F \in \mathcal{F}$. Thus, there is $G' \in \mathcal{I}(\omega') \subseteq\uparrow \mathcal{I}(\omega')$ such that $G' \subseteq \bigcap \mathcal{F}$. Next, I show that (8b) implies (8c). For each $\omega \in \Omega$, there is $G \in\uparrow \mathcal{I}(\omega)$ such that $G \subseteq \neg(\bigcap_{F \in \mathcal{F}} B_{\mathcal{I}}(F)) \cup B_{\mathcal{I}}(\bigcap \mathcal{F})$. Hence, $\omega \in B_{\mathcal{I}}(\neg(\bigcap_{F \in \mathcal{F}} B_{\mathcal{I}}(F)) \cup B_{\mathcal{I}}(\bigcap \mathcal{F}))$. Next, I show that (8c) implies (8a). Take $\omega \in \Omega = B_{\mathcal{I}}(\neg(\bigcap_{F \in \mathcal{F}} B_{\mathcal{I}}(F)) \cup B_{\mathcal{I}}(\bigcap \mathcal{F}))$. There is $G \in \mathcal{I}(\omega)$ such that if $\omega' \in G$ and if, for each $F \in \mathcal{F}$, there is $E_F \in \mathcal{I}(\omega')$ with $E_F \subseteq F$, then $\omega' \in B_{\mathcal{I}}(\bigcap \mathcal{F})$. Thus, there is $G' \in \mathcal{I}(\omega')$ with $G' \subseteq \bigcap \mathcal{F}$.

   Second, I show that Belief in Perfect Reasoning is equivalent to Belief in Non-empty $\aleph_0$-Conjunction. Without loss, one can restrict attention to the following form. For any $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$, there is $G \in\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$ and $E, F \in\uparrow \mathcal{I}(\omega')$ then $E \cap F \in\uparrow \mathcal{I}(\omega')$. Suppose that $\mathcal{I}$ satisfies Belief in Perfect Reasoning, and take $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$. There is $G \in\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$, $E \in\uparrow \mathcal{I}(\omega)$, and $F \in\uparrow \mathcal{I}(\omega')$, then, since $F \subseteq (\neg E) \cup (E \cap F)$, I have $(\neg E) \cup (E \cap F) \in\uparrow \mathcal{I}(\omega')$ and thus $E \cap F \in\uparrow \mathcal{I}(\omega')$. Conversely, suppose that $\mathcal{I}$ satisfies Belief in Non-empty $\aleph_0$-Conjunction. Take $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$. There is $G \in\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$, $E \in\uparrow \mathcal{I}(\omega)$, and $(\neg E) \cup F \in\uparrow \mathcal{I}(\omega')$, then $E \cap F = E \cap ((\neg E) \cup F) \in\uparrow \mathcal{I}(\omega')$. Since $E \cap F \subseteq F$, I get $F \in\uparrow \mathcal{I}(\omega')$. □

Remark A.2. The statement in the main text can be recast in terms of the belief operator $B_{\mathcal{I}}$ as follows: if $B_{\mathcal{I}}$ satisfies Monotonicity, Truth Axiom, and Positive Introspection, then the Axiom B and Negative Introspection are equivalent. The axiom B implies $(\neg B_{\mathcal{I}})(E) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})B_{\mathcal{I}}(E)$. Since $B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}B_{\mathcal{I}}(E)$ by Positive Introspection and since $B_{\mathcal{I}}$ is monotonic, it follows that $B_{\mathcal{I}}(\neg B_{\mathcal{I}})B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E)$. Thus, $(\neg B_{\mathcal{I}})(E) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E)$, as desired. Conversely, Truth Axiom and Negative Introspection yield $E^c \subseteq (\neg B_{\mathcal{I}})(E^{cc}) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E^{cc})$.

Remark A.3. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Define $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ as follows:

$$\mathcal{I}(\omega) = \begin{cases} \{\{\omega_1, \omega_2\}, \{\omega_2, \omega_3\}\} & \text{if } \omega = \omega_1 \\ \{\{\omega_2\}\} & \text{if } \omega = \omega_2 \\ \{\emptyset\} & \text{if } \omega = \omega_3 \end{cases}.$$

The belief operator $B_{\mathcal{I}}$ is given as follows:

$$B_{\mathcal{I}}(E) = \begin{cases} \{\omega_3\} & \text{if } E \in \{\emptyset, \{\omega_1\}, \{\omega_3\}, \{\omega_1, \omega_3\}\} \\ \{\omega_2, \omega_3\} & \text{if } E = \{\omega_2\} \\ \Omega & \text{if } E \in \{\{\omega_1, \omega_2\}, \{\omega_2, \omega_3\}, \Omega\} \end{cases}.$$

The operator $B_{\mathcal{I}}$ violates No-Contradiction because $\emptyset \in \mathcal{I}(\omega_3)$. In fact, $B_{\mathcal{I}}(\emptyset) = \{\omega_3\}$. Consequently, it violates Truth Axiom and Consistency (indeed, $B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c) = \{\omega_3\}$ for any $E \in \mathcal{D}$). The operator violates Non-empty λ-Conjunction (especially, $\lambda = \aleph_0$): $B_{\mathcal{I}}(\{\omega_1, \omega_2\}) \cap B_{\mathcal{I}}(\{\omega_2, \omega_3\}) = \Omega \not\subseteq \{\omega_2, \omega_3\} = B_{\mathcal{I}}(\{\omega_2\})$. In contrast, it can be seen that $\mathcal{I}$ is secondary reflexive and secondary serial. It also satisfies Belief in Non-empty λ-Conjunction and consequently Belief in Perfect Reasoning.

Remark A.4. I provide an example where $\mathcal{I}$ is not strongly transitive but transitive. While one can give such an example by modifying the example in Remark A.3, for future use I define a reflexive and transitive information correspondence. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Define $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ as $\mathcal{I}(\omega_1) = \mathcal{I}(\omega_3) = \{\{\omega_1, \omega_3\}\}$ and $\mathcal{I}(\omega_2) = \{\{\omega_2\}, \{\omega_1, \omega_2\}\}$. By construction, $\mathcal{I}$ is reflexive.

To see that $\mathcal{I}$ is not strongly transitive, take $E = \{\omega_1, \omega_2\} \in \mathcal{I}(\omega_2)$ and $\omega_1 \in E$. Then, $\{\omega_1, \omega_3\} \in \mathcal{I}(\omega_1)$ and $\{\omega_1, \omega_3\} \not\subseteq E = \{\omega_1, \omega_2\}$. It can be seen, however, that $\mathcal{I}$ is transitive. This can also be verified by the fact that $B_{\mathcal{I}}$ satisfies Positive Introspection:

$$B_{\mathcal{I}}(E) = \begin{cases} \emptyset & \text{if } E \in \{\emptyset, \{\omega_1\}, \{\omega_3\}\} \\ \{\omega_2\} & \text{if } E \in \{\{\omega_2\}, \{\omega_1, \omega_2\}, \{\omega_2, \omega_3\}\} \\ \{\omega_1, \omega_3\} & \text{if } E = \{\omega_1, \omega_3\} \\ \Omega & \text{if } E = \Omega \end{cases}. \tag{A.1}$$

Two additional remarks are in order. First, observe that $\mathcal{I}$ satisfies the Kripke property. Thus, consider $\mathcal{I}'(\cdot) = \{P(\cdot)\}$, where $P(\omega_1) = \{\omega_1, \omega_3\}$, $P(\omega_2) = \{\omega_2\}$, and $P(\omega_3) = \{\omega_1, \omega_3\}$. Now, $\mathcal{I}'$ is strongly transitive.

Second, $B_{\mathcal{I}}$ satisfies all the four logical properties defined in Section 2.3.1. Also, $B_{\mathcal{I}}$ satisfies all the eight introspective properties defined in Section 2.3.2.

Remark A.5. I provide an example where $\mathcal{I}$ is strongly transitive but $\uparrow \mathcal{I}$ is not as in Remark A.4. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Define $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ as follows:

$$\mathcal{I}(\omega) = \begin{cases} \{\{\omega_1, \omega_3\}, \Omega\} & \text{if } \omega = \omega_1 \text{ or } \omega = \omega_3 \\ \{\{\omega_2\}, \Omega\} & \text{if } \omega = \omega_2 \end{cases}. \tag{A.2}$$

First, I show that $\mathcal{I}$ is strongly transitive. Let $\omega \in \Omega$, $E \in \mathcal{I}(\omega)$, and $\omega' \in E$. There is $F = E \in \mathcal{I}(\omega')$ such that $F \subseteq E$. Second, $\uparrow \mathcal{I}$ is written as follows:

$$\uparrow \mathcal{I}(\omega) = \begin{cases} \{\{\omega_1, \omega_3\}, \Omega\} & \text{if } \omega = \omega_1 \text{ or } \omega = \omega_3 \\ \{\{\omega_2\}, \{\omega_2, \omega_3\}, \Omega\} & \text{if } \omega = \omega_2 \end{cases}. \tag{A.3}$$

Third, I show that $\uparrow \mathcal{I}$ is not strongly transitive. Take $E = \{\omega_2, \omega_3\} \in \mathcal{I}(\omega_2)$ and $\omega_3 \in E$. Then, $\{\omega_1, \omega_3\} \not\subseteq \{\omega_2, \omega_3\} = E$ and $\Omega \not\subseteq \{\omega_2, \omega_3\} = E$. I remark that, since $\mathcal{I}$ is transitive, it follows from Proposition 3 that $\uparrow \mathcal{I}$ is transitive. I also remark that the belief operator $B_{\mathcal{I}}$ in this example coincides with that defined by Equation (A.1) in Remark A.4.

Proof of Proposition 4.

1. By Propositions 2 and 3, $B_{\mathcal{I}}$ satisfies the logical and introspective properties of beliefs imposed on $\mathcal{I}$. Next, $\mathcal{I}_{B_{\mathcal{I}}}(\omega) = \{E \in \mathcal{D} \mid \omega \in B_{\mathcal{I}}(E)\} = \{E \in \mathcal{D} \mid E \in \uparrow \mathcal{I}(\omega)\} = \uparrow \mathcal{I}(\omega)$ for all $\omega \in \Omega$. Since $B_{\mathcal{I}}$ is monotonic, $\uparrow \mathcal{I}_{B_{\mathcal{I}}}(\cdot) = \mathcal{I}_{B_{\mathcal{I}}}(\cdot) = \uparrow \mathcal{I}(\cdot)$. Conversely, $B_{\mathcal{I}_B}(E) = \{\omega \in \Omega \mid E \in \uparrow \mathcal{I}_B(\omega)\} = B(E)$ for each $E \in \mathcal{D}$. By Propositions 2 and 3, any generator $\mathcal{I}$ of B satisfies the logical and introspective properties of beliefs imposed on B. As argued in the main text, $\mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}(\cdot) = \uparrow \mathcal{I}_B(\cdot) = \mathcal{I}_B(\cdot)$.

2. First, I show that $\mathcal{I}_{\mathcal{J}_B}$ is an information correspondence that generates B, i.e., $B = B_{\mathcal{I}_{\mathcal{J}_B}}$. Take $E \in \mathcal{D}$ and $\omega \in B(E)$. Since $\omega \in B(E) \subseteq BB(E)$, it follows that $B(E) \in \mathcal{I}_{\mathcal{J}_B}(\omega)$ and $E \in \uparrow \mathcal{I}_{\mathcal{J}_B}(\omega)$. Thus, $\omega \in B_{\mathcal{I}_{\mathcal{J}_B}}(E)$. Conversely, if

$\omega \in B_{\mathcal{I}_{\mathcal{J}_B}}(E)$ then there is $F \in \mathcal{I}_{\mathcal{J}_B}(\omega)$ with $\omega \in F \subseteq E$. Then, $\omega \in F \subseteq B(F) \subseteq B(E)$. Second, $\mathcal{I}_{\mathcal{J}_B}$ is by construction reflexive. Third, I show that $\mathcal{I}_{\mathcal{J}_B}$ is strongly transitive. Let $E \in \mathcal{I}_{\mathcal{J}_B}(\omega)$, i.e., $\omega \in E = B(E)$. If $\omega' \in E$ then there is $E' = E \in \mathcal{I}_{\mathcal{J}_B}(\omega')$ such that $E' \subseteq E$.

$\square$

Remark A.6. I formulate the equivalence between information correspondences and belief operators in Proposition 4 as a Galois connection. To that end, let $(\mathbb{I}, \leq_{\mathbb{I}})$ be the collection of information correspondences on a state space $(\Omega, \mathcal{D})$ with the following pre-order (i.e., reflexive and transitive order): $\mathcal{I} \leq_{\mathbb{I}} \mathcal{I}'$ if and only if, for each $\omega \in \Omega$ and $E \in \mathcal{I}(\omega)$, there is $F \in \mathcal{I}'(\omega)$ such that $F \subseteq E$. In other words, $\uparrow \mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}'(\cdot)$. Let $(\mathbb{B}, \leq_{\mathbb{B}})$ be the collection of monotonic belief operators from $\mathcal{D}$ into itself. Let $\alpha : (\mathbb{I}, \leq_{\mathbb{I}}) \to (\mathbb{B}, \leq_{\mathbb{B}})$ be such that $\alpha(\mathcal{I}) = B_{\mathcal{I}}$ defined as in Equation (1). I show that α is order-preserving. If $\mathcal{I} \leq_{\mathbb{I}} \mathcal{I}'$ then $\uparrow \mathcal{I} \leq_{\mathbb{I}} \uparrow \mathcal{I}'$ and thus $B_{\mathcal{I}} \leq_{\mathbb{B}} B_{\mathcal{I}'}$ by Equation (1). Next, let $\beta : (\mathbb{B}, \leq_{\mathbb{B}}) \to (\mathbb{I}, \leq_{\mathbb{I}})$ be $\beta(B) = \mathcal{I}_B$ as in Equation (4). By construction, β is order-preserving, i.e., if $B \leq_{\mathbb{B}} B'$ then $\mathcal{I}_B \leq_{\mathbb{I}} \mathcal{I}_{B'}$.

Now, I show that (α, β) is a Galois connection, that is, the order-preserving maps α and β on pre-ordered spaces satisfy $\mathcal{I} \leq_{\mathbb{I}} \beta(B)$ if and only if $\alpha(\mathcal{I}) \leq_{\mathbb{B}} B$ for any $(\mathcal{I}, B) \in \mathbb{I} \times \mathbb{B}$. Indeed, if $\alpha(\mathcal{I}) \leq_{\mathbb{B}} B$ then $\mathcal{I} = \mathcal{I}_{B_{\mathcal{I}}} \leq_{\mathbb{I}} \mathcal{I}_B = \beta(B)$. Conversely, if $\mathcal{I} \leq_{\mathbb{I}} \beta(B)$ then $\alpha(\mathcal{I}) = B_{\mathcal{I}} \leq_{\mathbb{B}} B_{\mathcal{I}_B} = B$.

Proof of Proposition 5. Let $\mathcal{J}$ satisfy Condition (5). First, I show that $\mathcal{I}_{\mathcal{J}}$ is an information correspondence. For each $E \in \mathcal{D}$,

$$
\begin{aligned}
&\{\omega \in \Omega \mid \text{there is } F \in \mathcal{I}_{\mathcal{J}}(\omega) \text{ such that } F \subseteq E\} \\
&= \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J} \text{ such that } \omega \in F \subseteq E\} \in \mathcal{J} \subseteq \mathcal{D}.
\end{aligned}
$$

Second, $\mathcal{I}_{\mathcal{J}}$ is reflexive by construction. Third, $\mathcal{I}_{\mathcal{J}}$ is strongly transitive. For any $E \in \mathcal{I}_{\mathcal{J}}(\omega)$ and $\omega' \in E$, $E' = E \in \mathcal{I}_{\mathcal{J}}(\omega')$ satisfies $\omega' \in E' \subseteq E$. Fourth, if $\Omega \in \mathcal{J}$, then $\Omega \in \mathcal{I}_{\mathcal{J}}(\omega)$ for any $\omega \in \Omega$.

Conversely, I show that $\mathcal{J}_{\mathcal{I}}$ satisfies Condition (5). For each $E \in \mathcal{D}$, let $B_{\mathcal{J}_{\mathcal{I}}}(E) := \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_{\mathcal{I}} \text{ such that } \omega \in F \subseteq E\}$. I show that $B_{\mathcal{J}_{\mathcal{I}}}(E) \in \mathcal{J}_{\mathcal{I}}$, i.e., if $\omega \in B_{\mathcal{J}_{\mathcal{I}}}(E)$, then there is $F' \in \mathcal{I}(\omega)$ such that $F' \subseteq B_{\mathcal{J}_{\mathcal{I}}}(E)$. Let $\omega \in B_{\mathcal{J}_{\mathcal{I}}}(E)$. There is $F \in \mathcal{J}_{\mathcal{I}}$ such that $\omega \in F \subseteq E$. Since $\omega \in F$, there is $E' \in \mathcal{I}(\omega)$ such that $E' \subseteq F$. Since $\mathcal{I}$ is transitive, there is $F' \in \mathcal{I}(\omega)$ such that if $\omega' \in F'$ then there is $G \in \mathcal{I}(\omega')$ such that $\omega' \in G \subseteq E' \subseteq F \subseteq E$. Thus, $\omega' \in B_{\mathcal{J}_{\mathcal{I}}}(E)$. Hence, I get $F' \subseteq B_{\mathcal{J}_{\mathcal{I}}}(E)$, as desired.

If $\mathcal{I}(\cdot) \neq \emptyset$, then for any $\omega \in \Omega$, there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq \Omega$. Thus, $\Omega \in \mathcal{J}_{\mathcal{I}}$.

Next, let $E \in \mathcal{J}$. For any $\omega \in E$, I have $E \in \mathcal{I}_{\mathcal{J}}(\omega)$ and $E \subseteq E$. Thus, $E \in \mathcal{J}_{\mathcal{I}_{\mathcal{J}}}$. Conversely, if $E \in \mathcal{J}_{\mathcal{I}_{\mathcal{J}}}$, then

$$
\begin{aligned}
E &= \{\omega \in \Omega \mid \text{there is } F \in \mathcal{I}_{\mathcal{J}}(\omega) \text{ such that } F \subseteq E\} \\
&= \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J} \text{ such that } \omega \in F \subseteq E\} \in \mathcal{J}.
\end{aligned}
$$

Finally, let $\mathcal{I}$ be given. First, I show $\uparrow \mathcal{I}_{\mathcal{J}_\mathcal{I}}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$. If $E \in \mathcal{I}_{\mathcal{J}_\mathcal{I}}(\omega)$, then $\omega \in E \in \mathcal{J}_\mathcal{I}$. Thus, $E \in \uparrow \mathcal{I}(\omega)$. This implies $\mathcal{I}_{\mathcal{J}_\mathcal{I}}(\omega) \subseteq \uparrow \mathcal{I}_{\mathcal{J}_\mathcal{I}}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$. Conversely, suppose that $E \in \uparrow \mathcal{I}(\omega)$. Since $\mathcal{I}$ is transitive, $E' := \{\omega' \in \Omega \mid E \in \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega)$. Thus, there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq E'$. Since $\mathcal{I}$ is reflexive, $\omega \in F \subseteq E'$. If $\omega' \in E'$ then $\omega' \in \{\omega'' \in \Omega \mid E' \in \uparrow \mathcal{I}(\omega'')\}$. That is, if $\omega' \in E'$ then there is $G \in \mathcal{I}(\omega')$ such that $G \subseteq E'$. Thus, $E' \in \mathcal{J}_\mathcal{I}$. Since $\omega \in E'$, I have $E' \in \mathcal{I}_{\mathcal{J}_\mathcal{I}}(\omega)$. By reflexivity, $E' \subseteq E$. Thus, $E \in \uparrow \mathcal{I}_{\mathcal{J}_\mathcal{I}}(\omega)$. $\square$

Remark A.7. Consider the belief operator B defined as in Equation (A.1) in Remark A.4. Let $\mathcal{J} := \mathcal{J}_B = \{\emptyset, \{\omega_2\}, \{\omega_1, \omega_3\}, \Omega\}$. I get the information correspondence $\mathcal{I}_\mathcal{J}$ as in Equation (A.2) in Remark A.5. Then, $\uparrow \mathcal{I}_\mathcal{J}$ is given as in Equation (A.3) in Remark A.5. The arguments in Remark A.5 show that, while $\mathcal{I}_\mathcal{J}$ is strongly transitive, $\uparrow \mathcal{I}_\mathcal{J}$ is not.

Remark A.8. Let $(\mathbb{I}^{rt}, \leq^{rt})$ be the collection of reflexive and transitive information correspondences with the following pre-order as in Remark A.6: $\mathcal{I} \leq^{rt} \mathcal{I}'$ if and only if, for each $\omega \in \Omega$ and $E \in \mathcal{I}(\omega)$, there is $F \in \mathcal{I}'(\omega)$ such that $F \subseteq E$. In other words, $\uparrow \mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}'(\cdot)$. Let $(\mathbb{J}, \subseteq)$ be the space consisting of collections of events $\mathcal{J}(\in \mathcal{P}(\mathcal{D}))$ satisfying Condition (5). Define $\alpha : (\mathbb{I}^{rt}, \leq^{rt}) \to (\mathbb{J}, \subseteq)$ by $\alpha(\mathcal{I}) = \mathcal{J}_\mathcal{I}$ as in Equation (7). I show that the mapping α is order-preserving. Let $\mathcal{I} \leq^{rt} \mathcal{I}'$, and take $E \in \mathcal{J}_\mathcal{I}$. If $\omega \in E$ then there is $F' \in \mathcal{I}'(\omega)$ such that $F' \subseteq E$. Thus, $E \in \mathcal{J}_{\mathcal{I}'}$. Next, define $\beta : (\mathbb{J}, \subseteq) \to (\mathbb{I}, \leq^{rt})$ by $\beta(\mathcal{J}) = \mathcal{I}_\mathcal{J}$ as in Equation (6). I show that β is order-preserving. If $\mathcal{J} \subseteq \mathcal{J}'$, then $\mathcal{I}_\mathcal{J}(\cdot) \subseteq \mathcal{I}_{\mathcal{J}'}(\cdot)$. This implies $\mathcal{I}_\mathcal{J} \leq^{rt} \mathcal{I}_{\mathcal{J}'}$.

Now, I establish (α, β) is a Galois connection, i.e., $\mathcal{I} \leq^{rt} \beta(\mathcal{J})$ if and only if $\alpha(\mathcal{I}) \subseteq \mathcal{J}$ for any $(\mathcal{I}, \mathcal{J}) \in \mathbb{I}^{rt} \times \mathbb{J}$. If $\mathcal{I} \leq^{rt} \beta(\mathcal{J}) = \mathcal{I}_\mathcal{J}$ then $\alpha(\mathcal{I}) \subseteq \alpha \circ \beta(\mathcal{J}) = \mathcal{J}_{\mathcal{I}_\mathcal{J}} = \mathcal{J}$. Conversely, if $\alpha(\mathcal{I}) \subseteq \mathcal{J}$ then $\mathcal{I} \leq^{rt} \beta(\mathcal{J})$ because $\uparrow \mathcal{I} = \uparrow \mathcal{I}_{\mathcal{J}_\mathcal{I}} \leq^{rt} \uparrow \mathcal{I}_\mathcal{J}$.

References

[1] R. J. Aumann. "Agreeing to Disagree". Ann. Statist. 4 (1976), 1236–1239.

[2] R. J. Aumann. "Interactive Epistemology I, II". Int. J. Game Theory 28 (1999), 263–300, 301–314.

[3] B. F. Chellas. Modal Logic: An Introduction. Cambridge University Press, 1980.

[4] E. Dekel and F. Gul. "Rationality and Knowledge in Game Theory". Advances in Economics and Econometrics: Theory and Applications, Seventh World Congress. Ed. by D. M. Kreps and K. F. Wallis. Vol. 1. Econometric Society Monograph. Cambridge University Press, 1997, 87–172.

[5] J.-P. Doignon and J.-C. Falmagne. "Spaces for the Assessment of Knowledge". Int. J. Man Mach. Stud. 23 (1985), 175–196.

[6] J.-P. Doignon and J.-C. Falmagne. Knowledge Spaces. Springer, 1999.

[7] J.-P. Doignon and J.-C. Falmagne. “Knowledge Spaces and Learning Spaces”. New Handbook of Mathematical Psychology. Cambridge University Press, 2016, 274–321.

[8] R. Fagin and J. Y. Halpern. “Belief, Awareness, and Limited Reasoning”. Art. Intell. 34 (1987), 39–76.

[9] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. Reasoning About Knowledge. MIT Press, 2003.

[10] J.-C. Falmagne and J.-P. Doignon. Learning Spaces: Interdisciplinary Applied Mathematics. Springer, 2011.

[11] M. F. Friedell. “On the Structure of Shared Awareness”. Behav. Sci. 14 (1969), 28–39.

[12] S. Fukuda. “Epistemic Foundations for Set-algebraic Representations of Knowledge”. 2018.

[13] J. Geanakoplos. Games Theory without Partitions, and Applications to Speculation and Consensus. Cowles Foundation Discussion Paper No. 914, Yale University. 1989.

[14] A. Heifetz. “Common Belief in Monotonic Epistemic Logic”. Math. Soc. Sci. 32 (1996), 109–123.

[15] A. Heifetz. “Iterative and Fixed Point Common Belief”. J. Philos. Log. 28 (1999), 61–79.

[16] A. Heifetz, M. Meier, and B. C. Schipper. “Interactive Unawareness”. J. Econ. Theory 130 (2006), 78–94.

[17] A. Heifetz, M. Meier, and B. C. Schipper. “Unawareness, Beliefs, and Speculative Trade”. Games Econ. Behav. 77 (2013), 100–121.

[18] A. Heifetz and D. Samet. “Topology-Free Typology of Beliefs”. J. Econ. Theory 82 (1998), 324–341.

[19] J. Hintikka. Knowledge and Belief: An Introduction to the Logic of the Two Notions. Cornell University Press, 1962.

[20] L. Lismont and P. Mongin. “A Non-minimal but Very Weak Axiomatization of Common Belief”. Art. Intell. 70 (1994), 363–374.

[21] L. Lismont and P. Mongin. “On the Logic of Common Belief and Common Knowledge”. Theory Decis. 37 (1994), 75–106.

[22] M. Meier. “Finitely Additive Beliefs and Universal Type Spaces”. Ann. Probab. 34 (2006), 386–422.

[23] M. Meier. “Universal Knowledge-Belief Structures”. Games Econ. Behav. 62 (2008), 53–66.

[24] J.-J. C. Meyer and W. van der Hoek. Epistemic Logic for AI and Computer Science. Cambridge University Press, 1995.

[25] S. Modica and A. Rustichini. "Awareness and Partitional Information Structures". Theory Decis. 37 (1994), 107–124.

[26] S. Modica and A. Rustichini. "Unawareness and Partitional Information Structures". Games Econ. Behav. 27 (1999), 265–298.

[27] D. Monderer and D. Samet. "Approximating Common Knowledge with Common Beliefs". Games Econ. Behav. 1 (1989), 170–190.

[28] S. Morris. "The Logic of Belief and Belief Change: A Decision Theoretic Approach". J. Econ. Theory 69 (1996), 1–23.

[29] E. Pacuit. Neighborhood Semantics for Modal Logic. Springer, 2017.

[30] H. Salonen. "Common Theories". Math. Soc. Sci. 58 (2009), 279–289.

[31] D. Samet. "Quantified Beliefs and Believed Quantities". J. Econ. Theory 95 (2000), 169–185.

[32] D. Samet. "S5 Knowledge without Partitions". Synthese 172 (2010), 145–155.

[33] B. C. Schipper. "Awareness". Handbook of Epistemic Logic. Ed. by H. van Ditmarsch, J. Y. Halpern, W. van der Hoek, and B. P. Kooi. College Publications, 2015, 77–146.

[34] D. Schmeidler. "Subjective Probability and Expected Utility without Additivity". Econometrica 57 (1989), 571–587.

[35] H. S. Shin. "Logical Structure of Common Knowledge". J. Econ. Theory 60 (1993), 1–13.

[36] E. G. C. Thijsse. Partial Logic and Knowledge Representation. Ph.D. Dissertation, Tilburg University. 1992.

[37] E. G. C. Thijsse. "On Total Awareness Logics: With Special Attention to Monotonicity Constraints and Flexibility". Diamonds and Defaults: Studies in Pure and Applied Intensional Logic. Ed. by M. D. Rijke. Kluwer, 1993, 309–347.

[38] C. Zhou. "Probability Logic of Finitely Additive Beliefs". J. Log. Lang. Inf. 19 (2010), 247–282.