On the Consistency among Prior, Posteriors, and Information Sets

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Abstract

This paper studies implications of the consistency conditions among prior, posteriors, and information sets on introspective properties of qualitative belief induced from information sets. The main result reformulates the consistency conditions as: (i) the information sets, without any assumption, almost surely form a partition; and (ii) the posterior at a state is equal to the Bayes conditional probability given the corresponding information set. Implications are as follows. First, each posterior is uniquely determined. Second, qualitative belief reduces to fully introspective knowledge in a "standard" environment. Thus, a care must be taken when one studies non-veridical belief or non-introspective knowledge. Third, an information partition compatible with the consistency conditions is uniquely determined by the posteriors. Fourth, qualitative and probability-one beliefs satisfy truth axiom almost surely. The paper also sheds light on how the additivity of the posteriors yields negative introspective properties of beliefs.

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1 Introduction

Agents in a strategic situation have two forms of beliefs. One is probabilistic beliefs, represented by a notion of types (Harsanyi, 1967-1968). The other is qualitative belief (or knowledge, if it is truthful), represented by information sets (Aumann, 1976). Consider, for instance, a dynamic game where each agent has knowledge about her past observations on the play, while she has probabilistic beliefs about her opponents' future plays. Qualitative belief plays a role when it comes to, say,

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studying consequences of common belief in rationality instead of common knowledge of rationality.[1] These two kinds of beliefs are well studied in a rather separate manner, and somewhat surprisingly, little has been known about how reasoning based on one form of beliefs influences the other.

This paper examines introspective properties of qualitative belief induced by information sets from its relation with prior and posterior beliefs. First, I link prior and posteriors in a way that the prior probability coincides with the expectation of the posterior probabilities with respect to the prior. Second, I relate information sets and posteriors. An agent qualitatively believes (or knows) her own probabilistic beliefs. Also, if the agent qualitatively believes (or knows) something, then she believes it with probability one. I study how these linkages themselves yield introspective properties on qualitative beliefs.

Consider an agent, Alameda, who faces uncertainty about underlying states of the world. On the one hand, Alameda has a prior countably-additive probability measure. She also has a posterior probability measure at each realized state. These dictate her quantitative beliefs. While I analyze how the additivity of each posterior affects her reasoning, for now I assume each posterior to be countably additive. On the other hand, Alameda has a mapping, called a possibility correspondence. It associates, with each state of the world, the set of states that she considers possible (the information set) at that state. I, as an analyst, derive properties of information sets, instead of directly assuming them. The framework is fairly parsimonious.

The main result (Theorem 1) restates the consistency conditions as: (i) Alameda's information sets form a partition almost surely; and (ii) her posterior at each state coincides with the Bayes conditional probability given her corresponding information set. While information sets are usually exogenously assumed to form a partition, the main result demonstrates that the consistency conditions on the agent's qualitative and probabilistic beliefs alone determine Bayes updating.

While the main result has its own interest, I rather use this result to derive a variety of implications. The first implication (Corollary 1) is that the consistency conditions uniquely determine the posterior at each state as the Bayes conditional probability given the associated information set.

The second implication (Corollary 2) is on the introspective properties of qualitative belief. To see this, Alameda's introspective abilities in her qualitative belief are reflected in properties of her information sets. When her information sets form a partition, her qualitative belief becomes knowledge (true belief) with full introspection. Truth axiom obtains: she can only know what is true. Her knowledge satisfies positive introspection: if she knows something then she knows that she knows it. Her knowledge also satisfies negative introspection: if she does not know something then she knows that she does not know it.

[1]See, for instance, Dekel and Gul (1997) for the importance of capturing both knowledge and probabilistic beliefs. See, for example, Stalnaker (1994) for using qualitative and probabilistic beliefs for studying solution concepts of games.

Now, the second implication is that, in a standard countable model where the prior puts positive probability to every state, qualitative belief reduces to fully introspective knowledge. The consistency conditions alone determine this property, and thus qualitative belief reduces to fully introspective knowledge with no a priori assumption on qualitative belief. Also, the notions of qualitative and probability-one beliefs coincide, and each form of beliefs inherits the properties of the other form.

One further implication of Corollary 2 is on the evaluation of a solution concept of a game. If the analysts assume the consistency conditions among prior, posteriors, and information sets, qualitative belief reduces to knowledge even though the analysts would like to study, say, the effect of common belief in rationality instead of common knowledge of rationality.

On a related point, the same issue occurs when the analysts aim to introduce non-introspective knowledge that violates negative introspection because non-introspective knowledge reduces to fully introspective knowledge. Such non-introspective knowledge is associated with unawareness: not only Alameda does not know an event E, but also she does not know that she does not know E (Modica and Rustichini, 1994, 1999). Previous negative results on describing a non-trivial form of unawareness in a standard possibility correspondence model are based on some direct link between qualitative belief (knowledge) and unawareness (Dekel, Lipman, and Rustichini, 1998; Modica and Rustichini, 1994). In contrast, Corollary 2 sheds light on a different channel (i.e., a link between qualitative and probabilistic beliefs) through which a standard state-space model cannot describe a richer form of unawareness.

Another implication of Corollary 2 is the violation of consistency between ex ante and ex post analyses in a non-partitional model. Suppose that the analysts aim to study non-introspective knowledge of a "boundedly rational" agent who is ignorant of her own ignorance. Since the analysts deal with knowledge, assume that knowledge implies probability-one belief. Also, assume that the agent is positively introspective on her own probabilistic beliefs with respect to her knowledge (i.e., if she believes an event with probability at least p (i.e., she p-believes the event), then she knows that she p-believes the event) in the same spirit as she has positive introspection on knowledge. Then, her prior and posteriors must violate the consistency condition, as the previous studies of non-partitional knowledge have demonstrated the inconsistency between ex ante and ex post evaluations of a decision problem.

The third implication (Corollary 3) is on the uniqueness of an information partition. Suppose that the analysts have a model featuring only probabilistic beliefs (i.e., prior and posteriors). The main result implies that if the analysts would like to introduce an information partition (information sets that form a partition) then it is uniquely determined by the posteriors. This result justifies the use of the partition generated by the posteriors in the previous literature (e.g., Battigalli and Bonanno (1999), Halpern (1991), and Vassilakis and Zamir (1993)): if an agent is certain of her own posterior (Harsanyi, 1967-1968; Mertens and Zamir, 1985), then, at each state, she has to be able to infer the set of states that generate the realized posterior.

The resulting sets form a partition. Corollary 3 says that this partition is the unique information partition compatible with the consistency conditions.

The final and fourth implication (Corollary 4) is on the comparison between qualitative and probability-one beliefs. As the previous literature demonstrates, qualitative belief (or knowledge) and probability-one belief may theoretically differ.[2] For example, an agent believes with probability one (i.e., she is certain) that a random draw from $[0, 1]$ is an irrational number but she does not know it (Monderer and Samet, 1989). Corollary 4 demonstrates that, while qualitative and probability-one beliefs may differ, both satisfy truth axiom almost surely.

The paper also sheds light on the role of additivity in introspection (Propositions 1 and 2). Now, suppose that the agent's posteriors are non-additive. First, if she does not p-believe an event E, then she may not be certain that she does not p-believe E. An agent with additive posteriors, however, would be certain of her own probabilistic ignorance. The second related implication is on negative introspection on probabilistic beliefs with respect to qualitative belief. If the agent does not p-believe an event then she may not qualitatively believe that she does not p-believe it. Again, an agent with additive posteriors would qualitatively believe her own probabilistic ignorance.

Related Literature

This paper is related to the following four strands of literature: (i) derivation of Bayes updating from consistency between prior and posterior beliefs, (ii) interaction of knowledge and beliefs, (iii) non-partitional knowledge models, and (iv) the role of additivity in probabilistic reasoning.

In the first strand of literature on Bayes updating, the main result (Theorem 1) is closely related to Gaifman (1988), Mertens and Zamir (1985), and Samet (1999) in a purely probabilistic setting. In a single-agent perspective, these papers study how the consistency conditions between prior and posterior probabilities lead to Bayes updating. I will discuss how these papers relate to the main result (Theorem 1) in Section 3.1. As in Mertens and Zamir (1985), the studies of existence of a common prior (e.g., Feinberg (2000), Golub and Morris (2017), Heifetz (2006), Hellman (2011), and Samet (1998)) also impose the consistency condition that the prior coincides with the expectation of the posteriors with respect to the prior.

The second is an extensive literature on the interaction between knowledge and beliefs in artificial intelligence, computer science, economics, game theory, logic, and philosophy. The consistency conditions between qualitative and probabilistic beliefs imposed in this paper are fairly common in economics and game theory in such contexts as epistemic characterizations of solution concepts for games, existence of a common prior, and canonical structures of agents' knowledge and beliefs (e.g., Aumann (1999), Battigalli and Bonanno (1999), Dekel and Gul (1997), and Meier

[2]Recall, however, that Corollary 2 shows that, in a "standard" countable full-support environment, qualitative and probability-one beliefs coincide.

(2008)). The validity of individual consistency conditions between knowledge and beliefs (i.e, "knowledge entails beliefs" and the knowledge of own beliefs) has been well studied since Hintikka (1962) and Lenzen (1978).

The third is on non-partitional knowledge that fails negative introspection. The studies of non-partitional structures include implications of common knowledge such as generalizations of Agreement theorem of Aumann (1976), studies of solution concepts (Brandenburger, Dekel, and Geanakoplos, 1992; Geanakoplos, 1989), foundations for information processing (errors) (Bacharach, 1985; Morris, 1996; Samet, 1990; Shin, 1993), and unawareness (Dekel, Lipman, and Rustichini, 1998; Modica and Rustichini, 1994, 1999).[3] See also Dekel and Gul (1997).

Fourth, I turn to the role of additivity in probabilistic introspection. In the decision theory literature, such papers as Ghirardato (2001) and Mukerji (1997) study foundations for non-additive beliefs in terms of an agent's imperfect information processing.[4] Although the framework of this paper is quite different from these decision-theoretic papers, these and this papers have the following similar intuition behind why the non-additivity is associated with the lack of introspection. Namely, an agent with non-additive beliefs cannot imagine what the possible states of the world are in her fullest extent.

The paper is organized as follows. Section 2 provides the framework. Section 3 demonstrates the main results. Section 4 provides the concluding remarks. Proofs are in Appendix A.

More specifically, Sections 2 and 3 are structured as follows. Section 2.1 defines the framework. Section 2.2 introduces the probabilistic and non-probabilistic belief operators. Section 2.3 defines the consistency conditions among prior, posteriors, and information sets. Moving on to the results, Section 3.1 provides the main result and the uniqueness of each posterior. Section 3.2 studies introspective properties of qualitative belief. Section 3.3 demonstrates the uniqueness of an information partition. Section 3.4 studies almost sure truth axiom of probabilistic and non-probabilistic beliefs.

2 An Epistemic Model

2.1 An Epistemic Model

This subsection formally defines a framework, which I call an epistemic model, for capturing probabilistic and non-probabilistic beliefs of an agent. For ease of exposition, throughout the paper, I restrict attention to a single agent model.

[3] Propositions 1 and 2 suggest that non-additivity may yield the violation of probabilistic negative introspection: if the agent is not certain of an event, she may not be certain that she is not certain of it. The implication of such violation of probabilistic negative introspection would also be interesting.

[4] A pioneering work on the behavioral foundations for the maximization of subjective expected utility based on non-additive beliefs is Schmeidler (1989).

An *epistemic model* (a *model*, for short) is a tuple $\overrightarrow{\Omega} := \langle \Omega, \Sigma, \mu, P, t \rangle$. First, Ω is a non-empty set of *states* of the world endowed with a σ-algebra Σ. Each element E of Σ is an *event*. I denote the complement of an event E by E^c or $\neg E$. Second, $\mu : \Sigma \to [0,1]$ is a *prior* countably-additive probability measure. Thus, $\langle \Omega, \Sigma, \mu \rangle$ forms a probability space.

Third, $P : \Omega \to \Sigma$ is a *possibility correspondence* with the measurability condition that $\{\omega \in \Omega \mid P(\omega) \subseteq E\} \in \Sigma$ for each $E \in \Sigma$. It associates, with each state ω, the set of states considered possible at that state. Note that $P(\omega)$ is assumed to be an event about which the agent herself reasons. The measurability condition will be used to introduce the qualitative belief operator. Thus, the possibility correspondence P dictates the agent's qualitative belief on $\langle \Omega, \Sigma \rangle$. I assume that μ and P jointly satisfy $\mu(P(\cdot)) > 0$. This joint assumption is commonly assumed in the literature on standard partitional knowledge models (e.g., Aumann (1976)).

Fourth, $t : \Omega \times \Sigma \to [0,1]$ is a *type mapping* satisfying the following two measurability conditions. The type mapping t dictates the agent's probabilistic beliefs on $\langle \Omega, \Sigma \rangle$. The first assumption is: for each $E \in \Sigma$, the mapping $t(\cdot, E) : \Omega \to [0,1]$ satisfies $(t(\cdot, E))^{-1}([p,1]) = \{\omega \in \Omega \mid t(\omega, E) \geq p\} \in \Sigma$ for all $p \in [0,1]$. This assumption allows the agent to reason about whether her degree of belief in an event E is at least p. This assumption will be used to define the agent's p-belief operators. Mathematically, this condition also means that each $t(\cdot, E) : \langle \Omega, \Sigma \rangle \to \langle [0,1], \mathcal{B}_{[0,1]} \rangle$ is measurable with respect to the Borel σ-algebra $\mathcal{B}_{[0,1]}$ on $[0,1]$. The second is: (i) $(\uparrow t(\omega)) := \{\omega' \in \Omega \mid t(\omega, \cdot) \leq t(\omega', \cdot)\} \in \Sigma$ and (ii) $(\downarrow t(\omega)) := \{\omega' \in \Omega \mid t(\omega', \cdot) \leq t(\omega, \cdot)\} \in \Sigma$ for all $\omega \in \Omega$. Note that $t(\omega, \cdot) \leq t(\omega', \cdot)$ means $t(\omega, E) \leq t(\omega', E)$ for all $E \in \Sigma$. I use similar abbreviations throughout the paper. If $\tilde{\omega} \in (\uparrow t(\omega))$, then $t(\tilde{\omega}, E)$ is always at least as high as $t(\omega, E)$ for any $E \in \Sigma$.

Remarks on the second assumption are in order. Letting $[t(\cdot)] := (\uparrow t(\cdot)) \cap (\downarrow t(\cdot))$, the set $[t(\omega)]$ consists of states $\tilde{\omega}$ indistinguishable from ω in that $t(\omega, \cdot) = t(\tilde{\omega}, \cdot)$. Intuitively, if the agent is perfectly certain of her probabilistic beliefs, then, at each state ω, she would be able to infer that the realization must be in $[t(\omega)]$. The second assumption ensures that each $[t(\omega)]$ is an object of the agent's beliefs.

While the measurability of $[t(\cdot)]$ is a standard assumption, here instead I assume the measurability of $(\uparrow t(\cdot))$ and $(\downarrow t(\cdot))$ in order to see two different ways in which the agent can be certain of her probabilistic beliefs later in Propositions 1 and 2. To see this point, by letting $[0,1]_{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$,

$$(\uparrow t(\omega)) = \bigcap_{E \in \Sigma} \{\tilde{\omega} \in \Omega \mid t(\tilde{\omega}, E) \geq t(\omega, E)\} = \bigcap_{(p,E) \in [0,1]_{\mathbb{Q}} \times \Sigma : t(\omega, E) \geq p} \{\omega' \in \Omega \mid t(\omega', E) \geq p\} \text{ and}$$

$$(\downarrow t(\omega)) = \bigcap_{E \in \Sigma} \{\tilde{\omega} \in \Omega \mid t(\tilde{\omega}, E) \leq t(\omega, E)\} = \bigcap_{(p,E) \in [0,1]_{\mathbb{Q}} \times \Sigma : t(\omega, E) \leq p} \{\omega' \in \Omega \mid t(\omega', E) \leq p\}.$$

Note that, in the third term of each expression, p can also range over $[0,1]$ instead of $[0,1]_{\mathbb{Q}}$. Thus, when the agent reasons about $(\uparrow t(\cdot))$, she only uses her positive belief of the form, "I believe an event with probability at least p." That is, when she does not believe an event E with probability at least p, she does not take this information into account in inferring the true state of the world. In contrast, when the agent reasons about $(\downarrow t(\cdot))$, she only uses her negative belief of the form, "I do not believe an event with probability at least p."

The distinction between $(\uparrow t(\cdot))$ and $[t(\cdot)]$ (and between $(\downarrow t(\cdot))$ and $[t(\cdot)]$) is somewhat related to Ghirardato (2001) and Mukerji (1997) in the decision theory literature, which characterize non-additivity from the agent's "perception" (see also Bonanno (2002) and Lipman (1995)). To see this point, interpret t as a mapping from Ω into the collection of set functions (with some given properties). On the one hand, $[t(\omega)]$ can be considered to be $t^{-1}(\{t(\omega)\}) := \{\tilde{\omega} \in \Omega \mid t(\omega)(\cdot) = t(\tilde{\omega})(\cdot)\}$. Thus, at state ω, the agent is assumed to be able to observe a singleton $\{t(\omega)\}$ so that she is able to infer that the true state is in $t^{-1}(\{t(\omega)\})$. On the other hand, $(\uparrow t(\omega))$ can be regarded as $t^{-1}(\{\mu \mid \mu(\cdot) \geq t(\omega)(\cdot)\}) := \{\tilde{\omega} \in \Omega \mid t(\tilde{\omega})(\cdot) \geq t(\omega)(\cdot)\}$. At state ω, the agent is assumed to be able to observe a (generally) non-singleton set $\{\mu \mid \mu(\cdot) \geq t(\omega)(\cdot)\}$. In Ghirardato (2001) and Mukerji (1997), the agent has a limited observation on consequences or signals (instead of own beliefs here).

The distinction among $(\uparrow t(\cdot))$ and $(\downarrow t(\cdot))$ matters when the agent's belief is non-additive. That is, if each $t(\omega, \cdot)$ is additive, then $(\uparrow t(\cdot)) = (\downarrow t(\cdot)) = [t(\cdot)]$.[5] Note that, in this case, part (ii) of the second assumption is implied by part (i).

To conclude the remarks on the second assumption, the following standard assumption ensures the second assumption: if Σ is generated by a countable algebra Σ_0 and if each $t(\omega, \cdot)$ is countably additive, then it follows from Samet (1999, Proposition 1) that each $[t(\omega)]$ is measurable:

$$[t(\omega)] = \bigcap_{(p,E) \in [0,1]_{\mathbb{Q}} \times \Sigma_0 : t(\omega,E) \geq p} \{\omega \in \Omega \mid t(\omega, E) \geq p\} \in \Sigma.$$

For each $\omega \in \Omega$, I call $t(\omega, \cdot)$ the *type* at ω. If a realized state is ω, the type $t(\omega, \cdot)$ at ω assigns, with each event E, the posterior probability of E. The idea behind the type mapping $t : \Omega \times \Sigma \to [0, 1]$ is a Markov kernel when each $t(\omega, \cdot)$ is a countably-additive probability measure (Gaifman, 1988; Samet, 1998, 2000). Here, each type $t(\omega, \cdot)$ is assumed to be a general set function. I do not assume any property of a set function on each type $t(\omega, \cdot)$ at this point, as I study how each type inherits the properties of the prior μ by imposing the link between the prior μ and the type mapping t.

An epistemic model is a general framework for capturing the agent's probabilistic and non-probabilistic beliefs. In Section 2.2, the agent's probabilistic beliefs are represented as p-belief operators induced by the type mapping t, while her qualitative belief is represented by her qualitative belief operator induced by her possibility

[5] Suppose $t(\omega, \cdot) \leq t(\omega', \cdot)$. If $t(\omega, E) < t(\omega', E)$ for some $E \in \Sigma$, then $1 = t(\omega, E) + t(\omega, E^c) < t(\omega', E) + t(\omega', E^c) = 1$, a contradiction.

correspondence P. In Section 2.3, I link (i) the prior μ and the type mapping t and (ii) the possibility correspondence P and the type mapping t.

2.2 Probabilistic and Non-Probabilistic Belief Operators

With the framework of an epistemic model in mind, I introduce probabilistic and non-probabilistic belief operators and their introspective properties. The agent's probabilistic beliefs are captured by *p-belief operators* (Friedell, 1969; Lenzen, 1978; Monderer and Samet, 1989). For each $E \in \Sigma$ and $p \in [0,1]$, define $B^p(E) := \{\omega \in \Omega \mid t(\omega, E) \geq p\} \in \Sigma$. The event $B^p(E)$ is the set of states at which the agent *p-believes* E, i.e., she assigns probability at least p to E.

I consider the following two introspective properties of probabilistic beliefs. A given model $\overrightarrow{\Omega}$ satisfies *Certainty of (p-)Beliefs* if $t(\cdot, [t(\cdot)]) = 1$ (Gaifman, 1988; Mertens and Zamir, 1985; Samet, 1999, 2000). Certainty of Beliefs states that the type at state ω puts probability one to the set of states indistinguishable from ω according to the type mapping t. To restate, if the state of the world is ω and if the agent has a perfect understanding of her own type mapping, she would be able to infer that the state is in $[t(\omega)]$ by unpacking the possible types.[6] If $\overrightarrow{\Omega}$ satisfies Certainty of Beliefs and if each $t(\omega, \cdot)$ is monotonic (i.e, $E \subseteq F$ implies $t(\omega, E) \leq t(\omega, F)$), then (i) $B^p(\cdot) \subseteq B^1 B^p(\cdot)$ and (ii) $(\neg B^p)(\cdot) \subseteq B^1(\neg B^p)(\cdot)$. Part (i) states that if the agent p-believes an event E then she 1-believes that she p-believes E. Part (ii), on the other hand, states that if the agent does not p-believe an event E then she 1-believes that she does not p-believe E. Thus, Certainty of Beliefs implies full introspection in the above sense. Moreover, the idea of Certainty of Beliefs plays an important role in the construction of a universal Harsanyi type space (Mertens and Zamir, 1985).

If Σ in a given model is generated by a countable algebra and if each $t(\omega, \cdot)$ is a countably-additive probability measure, then it follows from Samet (2000, Theorem 3) that the model satisfies Certainty of Beliefs if and only if $B^p(\cdot) \subseteq B^1 B^p(\cdot)$.

Full introspection associated with Certainty of Beliefs comes from the fact that, at each state ω, the agent always puts probability one to the set of states that are indistinguishable from ω. I now disentangle the negative introspective property $((\neg B^p)(\cdot) \subseteq B^1(\neg B^p)(\cdot))$ and Certainty of Beliefs.

I say that a model $\overrightarrow{\Omega}$ satisfies *Positive Certainty of (p-)Beliefs* if $t(\cdot, (\uparrow t(\cdot))) = 1$. At each state ω, the agent puts probability one to the set of states $\tilde{\omega}$ such that $t(\omega, \cdot) \leq t(\tilde{\omega}, \cdot)$. If the model satisfies Positive Certainty of Beliefs and if each $t(\omega, \cdot)$ is monotonic, then $B^p(\cdot) \subseteq B^1 B^p(\cdot)$. Indeed, I show the sense in which Positive Certainty of Beliefs captures the positive introspective property $B^p(\cdot) \subseteq B^1 B^p(\cdot)$.

Proposition 1. *Suppose that a given model $\overrightarrow{\Omega}$ satisfies the following: (i) Σ is finite; (ii) each $t(\omega, \cdot)$ is monotonic; and (iii) each $t(\omega, \cdot)$ satisfies that if $t(\omega, E) = t(\omega, F) =$*

[6] Technically, Gaifman (1988) and Samet (1999) require $t(\omega, [t(\omega)]) = 1$ μ-almost surely.

1 *then* $t(\omega, E \cap F) = 1$. *Then, the model satisfies Positive Certainty of Beliefs if and only if* $B^p(\cdot) \subseteq B^1 B^p(\cdot)$.

Proposition 1 provides the sense in which Positive Certainty of Beliefs characterizes the positive introspective property of the form $B^p(\cdot) \subseteq B^1 B^p(\cdot)$, while (standard) Certainty of Beliefs implies both $B^p(\cdot) \subseteq B^1 B^p(\cdot)$ and $(\neg B^p)(\cdot) \subseteq B^1(\neg B^p)(\cdot)$. Again, Positive Certainty of Beliefs and the assumption (ii) yield $B^p(\cdot) \subseteq B^1 B^p(\cdot)$.

The main implication of Proposition 1 is the role of additivity in introspection of probabilistic beliefs. That is, if each $t(\omega, \cdot)$ is additive then Certainty of Beliefs and Positive Certainty of Beliefs coincide as $(\uparrow t(\cdot)) = [t(\cdot)]$. Thus, under the setting of Proposition 1, if each $t(\omega, \cdot)$ is additive, then the positive introspective property (in the sense of $B^p(\cdot) \subseteq B^1 B^p(\cdot)$) implies the negative introspective property ($(\neg B^p)(\cdot) \subseteq B^1(\neg B^p)(\cdot)$).[7]

I make two additional remarks on Proposition 1. First, I assume Σ to be finite (and consequently the condition (iii)) because it is well known in measure theory that an infinite Σ is uncountable. Together with (i), the positive introspective property of the form $B^p(\cdot) \subseteq B^1 B^p(\cdot)$ implies Positive Certainty of Beliefs. Second, if, for instance, a monotonic type $t(\omega, \cdot)$ is convex (i.e., $t(\omega, E) + t(\omega, F) \leq t(\omega, E \cap F) + t(\omega, E \cup F)$), then the condition (iii) is met. That is, if Ω is finite and if each $t(\omega, \cdot)$ is a convex capacity then the assumptions in Proposition 1 are met.[8]

Next, I turn to the agent's qualitative belief. The possibility correspondence P induces the *qualitative belief operator* $K : \Sigma \to \Sigma$ defined by $K(E) := \{\omega \in \Omega \mid P(\omega) \subseteq E\} \in \Sigma$ for each $E \in \Sigma$. The event $K(E)$ is the set of states at which the agent qualitatively believes E. It can be seen that K satisfies: (i) Monotonicity: $E \subseteq F$ implies $K(E) \subseteq K(F)$; (ii) (Countable) Conjunction: $\bigcap_{n \in \mathbb{N}} K(E_n) \subseteq K(\bigcap_{n \in \mathbb{N}} E_n)$; and (iii) Necessitation: $K(\Omega) = \Omega$.

The following are well known (e.g., Dekel and Gul (1997), Geanakoplos (1989), and Morris (1996)). First, K satisfies Truth Axiom ($K(E) \subseteq E$) iff P is reflexive (i.e., $\omega \in P(\omega)$). Second, K satisfies Positive Introspection ($K(\cdot) \subseteq KK(\cdot)$) iff P is transitive (i.e., $\omega' \in P(\omega)$ implies $P(\omega') \subseteq P(\omega)$). Third, K satisfies Negative Introspection ($(\neg K)(\cdot) \subseteq K(\neg K)(\cdot)$) iff P is Euclidean (i.e., $\omega' \in P(\omega)$ implies $P(\omega) \subseteq P(\omega')$). Also, Truth Axiom and Negative Introspection imply Positive Introspection (e.g., Aumann (1999)).

Note that no such introspective assumption on P is imposed a priori. Thus, K need not be the "knowledge" operator that satisfies Truth Axiom, although I denote the qualitative belief operator by K especially to distinguish it from the p-belief operator B^p. Instead of assuming axioms on P, I derive properties of P from how qualitative and probabilistic beliefs interact with each other.

[7] This remark applies to the aforementioned setting where Σ is generated by a countable algebra and where each $t(\omega, \cdot)$ is a countably-additive probability measure.

[8] See, for example, Ghirardato (2001), Mukerji (1997), and Schmeidler (1989) for convex capacities (or a stronger notion of belief functions) in decision theory.

Note also that the domain of the qualitative belief operator K is the σ-algebra Σ. Technically, for any subset A of Ω, the set of states $\{\omega \in \Omega \mid P(\omega) \subseteq A\}$ is a subset of Ω. Thus, in principle, it is possible to define the qualitative belief operator from the entire power set of Ω into itself.[9] I, however, restrict attention to Σ as the domain of the qualitative belief operator K for the following two reasons.

First, in an epistemic model $\overrightarrow{\Omega}$, the state space Ω is paired with the σ-algebra Σ. The collection of events Σ describes objects of the agent's qualitative and quantitative beliefs. If her beliefs are introduced from some logical system (which allows countable operations), the collection of events would form a σ-algebra, not the power set of a state space. In dealing with agent's (agents') reasoning, it would be important to specify the language that agent(s) can utilize. Since the domain Σ in an epistemic model represents objects of the agent's probabilistic and non-probabilistic beliefs, the natural domain of the agent's qualitative belief operator is Σ.

Second, on a related point, specifying the domain Σ determines the depth of reasoning. Suppose that Σ is not a σ-algebra but an algebra. For any finite length, the following form of event is well defined: the event that she qualitatively believes that she qualitatively believes that ... an event E holds. Formally, $K^n(E) \in \Sigma$ for any $n \in \mathbb{N}$. The intersection of all these events, however, may not be well defined, that is, $K^\infty(E) = \bigcap_{n \in \mathbb{N}} K^n(E)$ may not belong to Σ. In this sense, assigning the domain Σ as an algebra provides the agent with finite depths of reasoning. Likewise, if Σ is a σ-algebra, the agent can reason about her countable depths of reasoning. But the conjunction of an uncountable number of events may not be an event. As I argued the importance of the specification of the language that agent(s) can employ, the existence and non-existence of a canonical ("universal" or "terminal") interactive type/belief/knowledge space hinges on such specification of domain (see, for example, Fukuda (2017), Heifetz and Samet (1998), and Meier (2005, 2008)). Thus, in an epistemic model, the domain of the qualitative belief operator K is taken to be the σ-algebra Σ.

2.3 Relations among Prior, Posteriors, and Information Sets

The previous section introduced the agent's probabilistic and non-probabilistic beliefs from her type mapping and possibility correspondence, respectively. Now, I relate (i) the prior and the type mapping and (ii) the type mapping and the possibility correspondence (i.e., probabilistic and non-probabilistic beliefs). Throughout this subsection, I fix a model $\overrightarrow{\Omega}$.

First, I assume that the prior and the type mapping jointly satisfy the invariance condition that the prior probability of an event E coincides with the expectation of the posteriors of E with respect to the prior. Formally, the model satisfies *Invariance*

[9] Probabilistic assessments are also extended to all subsets of the state space Ω. For any subset A of Ω, the inner measure induced by a measure ν is defined as $\sup\{\nu(E) \in [0,1] \mid E \in \Sigma \text{ and } E \subseteq A\}$.

if

$$\mu(\cdot) = \int_\Omega t(\omega, \cdot)\mu(d\omega).$$

This consistency condition is especially used to characterize (existence of) a common prior as discussed in the introduction. Note also that, in accordance with this literature, one can define the prior μ as a (countably-additive) probability measure that satisfies the Invariance condition. Heifetz, Meier, and Schipper (2013) also impose this condition in their probabilistic model of unawareness.

Second, I introduce two introspective properties that relate non-probabilistic and probabilistic beliefs. These conditions are commonly imposed in economics and game theory when knowledge and probabilistic beliefs are present. The model $\overrightarrow{\Omega}$ satisfies *Entailment* if $t(\cdot, P(\cdot)) = 1$. Entailment states that, at each state, the agent assigns probability one to the set of states that she considers possible at that state. Since $\omega \in K(P(\omega))$, if each type $t(\omega, \cdot)$ is monotonic then Entailment is expressed, in terms of operators, as the stronger condition $K(\cdot) \subseteq B^1(\cdot)$. If qualitative belief reduces to knowledge, then Entailment states that knowledge entails probability-one belief. See also Battigalli and Bonanno (1999), Dekel and Gul (1997), and Hintikka (1962) for assessments of Entailment.

Next, $\overrightarrow{\Omega}$ satisfies *Self-Evidence of (p-)Beliefs* if $\omega' \in P(\omega)$ implies $t(\omega, \cdot) \leq t(\omega', \cdot)$, i.e., $P(\omega) \subseteq (\uparrow t(\omega))$. It provides a consistency requirement between the possibility correspondence P and the type mapping t: if the agent considers ω' possible at ω, then, as long as she assigns probability at least p to an event E at ω, she also assigns probability at least p to E at ω'. If each $t(\omega, \cdot)$ is additive, then Self-Evidence of Beliefs is re-written as $P(\omega) \subseteq [t(\omega)]$ (i.e., if $\omega' \in P(\omega)$ then $t(\omega, \cdot) = t(\omega', \cdot)$). If qualitative belief reduces to knowledge, then the latter condition ($P(\omega) \subseteq [t(\omega)]$) is commonly imposed in economics and game theory.[10] I, however, formulate Self-Evidence of beliefs so as to examine the effect of the additivity of types. The following proposition shows that Self-Evidence of Beliefs captures positive introspection.

Proposition 2. *1. The model satisfies Self-Evidence of Beliefs if and only if $B^p(\cdot) \subseteq K(B^p)(\cdot)$ for all $p \in [0, 1]$.*

2. $P(\cdot) \subseteq (\downarrow t(\cdot))$ if and only if $(\neg B^p)(\cdot) \subseteq K(\neg B^p)(\cdot)$ for all $p \in [0, 1]$.

The first part of Proposition 2 states that Self-Evidence of Beliefs means: whenever the agent p-believes an event E, she knows that she p-believes E.[11] Proposition 2 also states that $P(\cdot) \subseteq [t(\cdot)]$ if and only if $B^p(\cdot) \subseteq K(B^p)(\cdot)$ and $(\neg B^p)(\cdot) \subseteq K(\neg B^p)(\cdot)$.

[10] In philosophy, Hintikka (1962, Section 3.7) rejects Self-Evidence of Beliefs. See also Lenzen (1978, Chapter 4) for critical assessments of the rejection of this axiom.

[11] Note that I do not impose the following form of introspection: $B^p(E) \subseteq B^p K(E)$ (i.e., whenever the agent p-believes an event E, she p-believes that she qualitatively believes E). This form of interaction between "knowledge and belief (certainty)" is considered in computer science, logic, and philosophy (see, Halpern (1996) and Lenzen (1978) and the references therein).

Proposition 2 again disentangles the role of the additivity of types by considering $(\uparrow t(\cdot))$ and $(\downarrow t(\cdot))$.

I make the following two remarks on implications of Invariance, Entailment and Self-Evidence of Beliefs. First, Entailment and Self-Evidence of Beliefs imply Positive Certainty of Beliefs, provided that each $t(\omega, \cdot)$ is monotonic.

Second, suppose Invariance and Self-Evidence of Beliefs. Then, $\mu(E) = 0$ if and only if $t(\omega, E) = 0$ for all $\omega \in \Omega$. The "if" part follows from Invariance. For the "only if" part, $\mu(E) = 0$ and $t(\omega, E) > 0$ imply

$$0 = \int_{\Omega} t(\tilde{\omega}, E)\mu(d\tilde{\omega}) \geq \int_{(\uparrow t(\omega))} t(\tilde{\omega}, E)\mu(d\tilde{\omega}) \geq t(\omega, E)\mu(\uparrow t(\omega)) \geq t(\omega, E)\mu(P(\omega)) > 0,$$

which is impossible (recall the assumption made in Section 2.1 that $\mu(P(\cdot)) > 0$ as in the standard possibility correspondence model). I say that $E \subseteq F$ μ-almost surely (μ-a.s.) if $\mu(E \setminus F) = 0$. The above argument implies the consistency condition that $E \subseteq F$ μ-a.s. if and only if $E \subseteq F$ $t(\omega, \cdot)$-a.s. for all $\omega \in \Omega$, provided Invariance and Self-Evidence of Beliefs.

To conclude, call the model $\overrightarrow{\Omega}$ *regular* if each $t(\omega, \cdot)$ is a finitely-additive probability measure (precisely, charge) and if the model satisfies Invariance (the consistency condition between the prior and the type mapping), Entailment and Self-Evidence of Beliefs ($t(\cdot, P(\cdot)) = 1$ and $P(\cdot) \subseteq (\uparrow t(\cdot))$, which are the consistency conditions between qualitative and probabilistic beliefs). While I require each type $t(\omega, \cdot)$ to be finitely additive, Theorem 1 in Section 3 demonstrates that each $t(\omega, \cdot)$ turns out to be countably additive in any regular model. Given that the agent's types are assumed to be finitely additive, if a given model satisfies Positive Certainty of Beliefs then it also satisfies Certainty of Beliefs.

3 Results

3.1 Main Result and Uniqueness of the Type Mapping

I present the main result that fully characterizes a regular model. It states that a model is regular if and only if (i) each type $t(\omega, \cdot)$ is derived, through Bayes updating, from the prior μ conditional on the information set $P(\omega)$; and (ii) the information sets form a partition almost surely.

Theorem 1. *A model $\overrightarrow{\Omega}$ is regular if and only if (i) $t(\cdot, \cdot) = \mu(\cdot \mid P(\cdot))$, (ii) $P(\cdot) \subseteq [t(\cdot)]$, and (iii) $P(\cdot) \supseteq [t(\cdot)]$ μ-almost surely.*

Three remarks on Theorem 1 are in order. First, if a model is regular, then each finitely-additive type $t(\omega, \cdot)$ turns out to be countably additive as it inherits countable additivity from the prior μ by part (i). Second, suppose that μ is only finitely additive while keeping all the other assumptions (note that the integrals with respect to μ are well defined). The "only if" part still holds. That is, the consistency conditions still imply that each type $t(\omega, \cdot)$ is the Bayes conditional probability $\mu(\cdot \mid P(\cdot))$ and that the collection of information sets $\{P(\omega)\}_{\omega\in\Omega}$ almost surely forms a partition. The "if" part holds when $\{[t(\cdot)]\}$ forms a finite partition.

Third, since part (iii) only requires $P(\cdot) \supseteq [t(\cdot)]$ μ-almost surely, $\{P(\omega)\}_{\omega\in\Omega}$ may fail to be a partition. Thus, the qualitative belief operator K may violate Truth Axiom ($K(E) \subseteq E$), Positive Introspection ($K(\cdot) \subseteq KK(\cdot)$), and Negative Introspection ($(\neg K)(\cdot) \subseteq K(\neg K)(\cdot)$). The model, however, satisfies introspection of the form $B^p(\cdot) \subseteq KB^p(\cdot)$ and $(\neg B^p)(\cdot) \subseteq K(\neg B^p)(\cdot)$. Section 3.2 examines a special case where Ω is countable, $\Sigma = 2^\Omega$, and where $\mu(\{\cdot\}) > 0$. There, $P(\cdot) = [t(\cdot)]$ forms a partition, that is, the agent's qualitative belief satisfies Truth Axiom, Positive Introspection, and Negative Introspection. Moreover, the qualitative belief operator K and 1-belief operator B^1 coincide. Section 3.4 compares the qualitative belief and probability-one belief operators in detail.

Fourth, in part (iii), if one considers the set of ω such that $P(\cdot) \supseteq [t(\cdot)]$, it may be the case that the μ-measure of such a set may not be equal to 1, provided that such a set is measurable (Example 2 in Appendix A is such an example). Since $P(\omega)$ dictates the agent's qualitative belief at state ω, I consider, at each state ω, whether $P(\cdot) \supseteq [t(\cdot)]$ holds in a certain sense. Part (iii) says that $P(\omega) \supseteq [t(\omega)]$ holds μ-almost surely at every state $\omega \in \Omega$.

Theorem 1 relates to the following previous results on the consistency conditions between prior and posteriors where probabilistic beliefs are sole primitives. First, Mertens and Zamir (1985) ask when an agent's posterior beliefs are derived from her (or common) prior conditional on her information. Mertens and Zamir (1985, Proposition 4.2) show that if a given probabilistic belief model $\langle \Omega, \Sigma, \mu, t\rangle$ satisfies Invariance and Certainty of Beliefs then the agent's type $t(\omega, E)$ turns out to be the Bayes conditional probability $\mu(E \mid [t(\omega)])$ whenever it is well defined.[12]

Second, following Samet (1999), call a probabilistic belief model $\langle \Omega, \Sigma, \mu, t\rangle$ to be Bayesian if it satisfies Certainty of Beliefs (μ-almost surely) and Invariance. Gaifman (1988) and Samet (1999) characterize a Bayesian model by the consistency requirement that the prior conditioned on some specification of the posterior beliefs must agree with the specification.[13]

These results and Theorem 1 of this paper *derive* Bayes updating from epistemic properties within a model. Theorem 1 states that, in an environment in which an agent's probabilistic and non-probabilistic beliefs are both present, the interaction between prior and posteriors (i.e., Invariance) and the ones between posteriors and possibility correspondence (i.e., Entailment and Self-Evidence of Beliefs) give rise to

[12]In the similar way to the proof of Theorem 1, the following can be established. Suppose that, for a given model $\overrightarrow{\Omega}$, each $t(\omega, \cdot)$ is countably additive. Then, the model satisfies Invariance and Certainty of Beliefs if and only if $t(\cdot,\cdot) = \mu(\cdot \mid [t(\cdot)])$ whenever the right-hand side is well defined.

[13]More specifically, a model is Bayesian if and only if $\mu\left(E_i \mid \bigcap_{k=1}^n B^{p_k}(E_k)\right) \geq p_i$ for any $(p_i, E_i)_{i=1}^n$. See also the references therein.

Bayes updating within the model, provided each type is at least finitely additive.

From now on, I examine the implications of Theorem 1. The first immediate corollary is the uniqueness of a type mapping. Not only does Theorem 1 justify Bayes updating conditional on information sets, but also it implies that if a model is regular then the type mapping is uniquely determined by the other two ingredients of the model (namely, the prior μ and the possibility correspondence P) through Bayes conditional probabilities.

Corollary 1. *Let $\langle \Omega, \Sigma, \mu, P, t \rangle$ and $\langle \Omega, \Sigma, \mu, P, t' \rangle$ be regular models (where Ω, Σ, μ, and P are common). Then, $t = t'$.*

3.2 Partitional Properties of Qualitative Belief

Call a model $\overrightarrow{\Omega}$ *discrete* if Ω is countable, $\Sigma = 2^\Omega$, and if $\mu(\{\cdot\}) > 0$.[14] Under this "standard" setting, the following corollary demonstrates that qualitative belief necessarily becomes knowledge and that probability-one belief and knowledge coincide.

Corollary 2. *A discrete model $\overrightarrow{\Omega}$ is regular if and only if (i) $P(\cdot) = [t(\cdot)]$ and (ii) $t(\cdot, \cdot) = \mu(\cdot \mid P(\cdot))$. In this case, $K = B^1$ (satisfies Truth Axiom, Positive Introspection, and Negative Introspection).*

In a regular discrete model, the possibility correspondence $P(\cdot)$ exactly coincides with $[t(\cdot)]$, and knowledge and certainty (probability-one belief) coincide with each other, irrespective of assumptions on information sets. Thus, $K = B^1$ satisfies Truth Axiom, Positive Introspection, Negative Introspection, (Countable) Conjunction, Monotonicity, and Necessitation.

Moreover, Corollary 2 implies that possibility coincides with assigning positive probability in the sense that $P(\omega) = \{\omega' \in \Omega \mid t(\omega, \{\omega'\}) > 0\}$ for all $\omega \in \Omega$.[15] In a probabilistic belief model, an information set at a state is often defined as the support of the type at that state. Section 3.3 demonstrates that if $\{P(\omega)\}_{\omega \in \Omega}$ forms a partition then it is uniquely determined by $P(\cdot) = [t(\cdot)]$.

Corollary 2 suggests that a care must be taken of the agent's qualitative belief if the analysts study an epistemic characterization of a solution concept for a game. One of the most standard ways to study an epistemic characterization of a solution concept for a game is to characterize agents' beliefs by a prior, posteriors, and a possibility correspondence on a state space for each agent. If the possibility correspondence is partitional, then the model captures fully introspective knowledge and probabilistic beliefs. As discussed in the introduction, for example, such a model can capture

[14] In a discrete model, $\Sigma = 2^\Omega$ is generated by a countable algebra Σ_0. For example, let $\Sigma_0 = \{E \in 2^\Omega \mid E \text{ is finite or } E^c \text{ is finite}\}$.

[15] Halpern (1991) studies certainty (probability-one belief) by defining the "support relation" between two states by $t(\omega, \{\omega'\}) > 0$. Samet (1998) studies the (Markov transition) matrix generated by $(t(\omega, \{\omega'\}))_{\omega, \omega' \in \Omega}$. Morris (1996) derives qualitative belief from preferences, and under certain condition, the notion of possibility reduces to assigning positive probability.

an agent's fully introspective knowledge about her past observations and her beliefs about her opponents' future plays in a dynamic game.[16]

I examine the implications of Corollary 2 when the information sets do not form a partition. Suppose that the analysts introduce qualitative belief instead of knowledge when they study, for example, implications of common belief in rationality instead of common knowledge of rationality (see, for example, Stalnaker (1994)). Qualitative belief violates Truth Axiom if the information sets fail reflexivity. Corollary 2 points to the importance of figuring out the relations between prior, posteriors, and information sets in a discrete model because qualitative belief reduces to (fully introspective) knowledge despite the analysts' purpose.

Another case is where the information sets (only) fail to be Euclidean so that the resulting model captures non-introspective knowledge that violates Negative Introspection. Suppose $P(\cdot) \neq [t(\cdot)]$ due to the failure of Negative Introspection (of K) in a discrete model. I consider non-partitional models in broader contexts, as the same conclusion as above can already be drawn with respect to solution concepts of games and the implications of common knowledge such as Agreement theorem (Aumann, 1976) in non-partitional models. Now, the model has to violate either Invariance, Entailment, Self-Evidence of Beliefs, or the assumption that each type $t(\omega, \cdot)$ is finitely additive. Note, however, that μ is assumed to be a countably-additive probability measure in my framework.

Since the information sets represent the agent's knowledge, assume Entailment: if the agent knows an event then she 1-beliefs the event. Also, assume Self-Evidence of Beliefs. Just as the agent's knowledge is positively introspective, if she p-believes an event E then she knows that she p-believes E. Suppose further that each type $t(\omega, \cdot)$ is finitely additive. Then, Corollary 2 implies that the model has to violate Invariance. This observation sheds light on the comparison between ex ante and ex post analyses or the value of information in non-partitional knowledge models as discussed by, for example, Dekel and Gul (1997) and Geanakoplos (1989). Namely, this observation provides an intuition behind why "dynamic inconsistency" occurs in a decision problem with a non-partitional (i.e., reflexive and transitive) environment.[17] For another example, if the model is assumed to satisfy Invariance, Entailment, and Self-Evidence of Beliefs, then some type $t(\omega, \cdot)$ may be non-additive.

A natural question arises as to probabilistic belief updating in a non-partitional environment. In the literature featuring non-partitional information sets, it has been assumed that an agent updates her probabilistic assessment according to the Bayes rule given a non-partitional information set. As Dekel and Gul (1997, p.147) put it, however, "there are essentially no results that justify stapling traditional frameworks

[16] Another approach is a (product) type space approach. See, for example, Battigalli and Bonanno (1999), Dekel and Gul (1997), Dekel and Siniscalchi (2015), and Stalnaker (1994) and the references therein for epistemic characterizations of solution concepts.

[17] Gaifman (1988) also discusses the violation of a Bayesian belief model $\langle \Omega, \Sigma, \mu, t \rangle$ in terms of "dynamic inconsistency" between ex ante and ex post analyses.

together with non-partitions."[18] There is room for investigating probabilistic beliefs of an agent compatible with the information processing errors due to the failure of Negative Introspection.

3.3 Uniqueness of a Partition Consistent with Probabilistic Beliefs

In a probabilistic belief model $\langle \Omega, \Sigma, \mu, t \rangle$, an information set at a state is often externally introduced as the support of the type at that state in the literature (e.g., Battigalli and Bonanno (1999), Halpern (1991), and Vassilakis and Zamir (1993) in various settings). In a discrete model, it means that the agent considers ω' possible (according to her type mapping t) at state ω if $t(\omega, \{\omega'\}) > 0$.

The next corollary shows that if there is a partition $\{P(\omega)\}_{\omega \in \Omega}$ such that the resulting model $\langle \Omega, \Sigma, \mu, P, t \rangle$ is regular, then $P(\cdot)$ has to coincide with $[t(\cdot)]$. Roughly, if the analysts would like to introduce an agent's fully introspective knowledge (i.e., knowledge introduced by a partition) in a probabilistic belief model $\langle \Omega, \Sigma, \mu, t \rangle$, then the unique possibility correspondence $P(\cdot)$ which makes the resulting model regular is $[t(\cdot)]$. The uniqueness part provides a justification for introducing an information set by the support of a type. In a discrete model, one obtains: $P(\omega) = [t(\omega)] = \{\omega' \in \Omega \mid t(\omega, \{\omega'\}) > 0\}$.

Corollary 3. *1. Let $\overrightarrow{\Omega}$ be a model. The following are equivalent.*

(a) (i) $\{P(\omega)\}_{\omega \in \Omega}$ forms a partition, (ii) Invariance, (iii) Entailment, (iv) Self-Evidence of Beliefs, and (v) each $t(\omega, \cdot)$ is a finitely-additive probability measure.

(b) $P(\cdot) = [t(\cdot)]$ and $t(\cdot, \cdot) = \mu(\cdot \mid P(\cdot))$.

2. Let $\overrightarrow{\Omega}$ be a model satisfying Invariance, Entailment, and Self-Evidence of Beliefs. Suppose further that each $t(\omega, \cdot)$ is a finitely-additive probability measure. Then, $\{P(\omega)\}_{\omega \in \Omega}$ forms a partition (i.e., K satisfies Truth Axiom, Positive Introspection, and Negative Introspection) if and only if $P(\cdot) = [t(\cdot)]$.

3. Let $\overrightarrow{\Omega}$ be a model such that $P(\cdot) = [t(\cdot)]$ and that each $t(\omega, \cdot)$ is a finitely-additive probability measure. The model satisfies Entailment and Invariance if and only if $t(\cdot, \cdot) = \mu(\cdot \mid P(\cdot))$.

[18] First, to the best of my knowledge, no result has been put forward until now in a single standard state-space framework. Second, the probabilistic approach to unawareness by Heifetz, Meier, and Schipper (2013) considers an extended structure consisting of multiple sub-state-spaces. While an agent's beliefs satisfy Invariance within each sub-space, she exhibits unawareness within the entire enriched model.

The second and third parts of Corollary 3 follow from the first. The second part establishes the uniqueness of the possibility correspondence P compatible with the consistency conditions. Thus, if the analysts introduce knowledge together with probabilistic beliefs in a consistent way, the possibility correspondence is uniquely determined by $P(\cdot) = [t(\cdot)]$.

The third part states that, under the assumption that the possibility correspondence satisfies $P(\cdot) = [t(\cdot)]$, the "Bayes conditional property" (i.e., $t(\cdot, \cdot) = \mu(\cdot \mid P(\cdot))$) characterizes Entailment and Invariance.

3.4 Almost-Sure Truth Axiom of Probability-One and Qualitative Beliefs

Lastly, I compare the qualitative belief and probability-one belief operators in a regular model. It is theoretically important to understand the differences between qualitative and probability-one beliefs. Here, I examine these possible differences in terms of belief operators. First, as is known in the literature (e.g., Monderer and Samet, 1989; Vassilakis and Zamir, 1993), it is not necessarily the case that $B^1 = K$ (even) μ-almost surely. Second, while the probability-one belief satisfies Positive Introspection and Negative Introspection ($B^1(\cdot) \subseteq B^1 B^1(\cdot)$ and $(\neg B^1) \subseteq B^1(\neg B^1)(\cdot)$) by Certainty of Beliefs, it is not necessarily the case that the qualitative belief operator satisfies both properties μ-almost surely. Examples 1 and 2 in Appendix A demonstrate these two facts. Yet, I show below that the probability-one belief and qualitative belief operators satisfy Truth Axiom μ-almost surely.

Corollary 4. *In any regular model, B^1 satisfies Truth Axiom μ-almost surely in that $\mu(B^1(E) \setminus E) = 0$ for all $E \in \Sigma$. Consequently, K also satisfies Truth Axiom μ-almost surely. Indeed, B^1 and K satisfy Truth Axiom $t(\omega, \cdot)$-almost surely for all $\omega \in \Omega$.*

I make the following two technical remarks on Corollary 4. First, in comparison with Corollary 2, in a regular discrete model, $B^1 = K$ satisfies Truth Axiom, Positive Introspection, and Negative Introspection. Corollary 4 states that, generally, the 1-belief and qualitative belief operators satisfy Truth Axiom μ-almost surely in any regular model. Second, to obtain almost-sure Truth Axiom of B^1, it is enough for a given model to satisfy Invariance, Certainty of Beliefs, and $\mu([t(\cdot)]) > 0$. Thus, this part of Corollary 4 also holds in a probabilistic belief model $\langle \Omega, \Sigma, \mu, t \rangle$.

Brandenburger and Dekel (1987) obtain the closely related result in their setting where an agent's (countably-additive) type mapping is introduced as a posterior conditional on a sub-σ-algebra that dictates her information and where "knowledge" is defined as probability-one belief. Then, the "knowledge" (or probability-one belief) operator satisfies Truth Axiom $t(\omega, \cdot)$-almost surely for all $\omega \in \Omega$ (Brandenburger and Dekel, 1987, Property P.4.). Note that, in their model, B^1 also satisfies Truth Axiom μ-almost surely. Also, Halpern (1991, Proposition 4.3) establishes that B^1 satisfies

Truth Axiom μ-almost surely when the type mapping does not depend on states (i.e., $t(\omega, \cdot) = t(\omega', \cdot)$ for all $\omega, \omega' \in \Omega$).[19]

4 Conclusion

This paper studied implications of the consistency conditions among prior, posteriors, and information sets on introspective properties of qualitative belief induced from information sets. The consistency conditions are: (i) the prior probability is equal to the expectation of the posterior probabilities (Invariance); (ii) qualitative belief entails probability-one belief (Entailment); and (iii) qualitative belief in one's own probabilistic beliefs (Self-Evidence of Beliefs). The main benchmark result (Theorem 1) is: a model satisfies the consistency conditions if and only if the agent's information sets form a partition almost surely and her posteriors coincide with the Bayes conditional probabilities given information sets.

I studied four implications. First, the posterior at each state is uniquely determined (Corollary 1). Second, in a standard countable state space such that the prior puts positive probability to each state, the information sets necessarily form a partition (Corollary 2). I discussed the implication of this result on situations when the information sets do not form a partition (i.e., qualitative belief that violates Truth Axiom or non-introspective knowledge that violates Negative Introspection).

Third, in a model of probabilistic beliefs, if one would like to introduce fully introspective knowledge, then Corollary 3 states that a unique partition compatible with the consistency conditions is the partition generated by the type mapping. This result justifies the definition of an information partition by the support of each type in the previous literature. Forth, Corollary 4 demonstrates that, while qualitative and probability-one beliefs may differ, both satisfy Truth Axiom almost surely.

I also studied, in Propositions 1 and 2, how the additivity of types plays a role in negative introspection of beliefs. As avenues for future research, it is interesting to scrutinize a link between prior and posteriors (or an "updating rule") that is consistent with non-partitional information processing. In so doing, it is also interesting to explore the role of additivity.

A Appendix

Proof of Proposition 1. Assume Positive Certainty of Beliefs. If $\omega \in B^p(E)$ then $(\uparrow t(\omega)) \subseteq B^p(E)$. Thus, (ii) implies $1 = t(\omega, (\uparrow t(\omega))) \leq t(\omega, B^p(E))$, i.e., $\omega \in B^1 B^p(E)$.[20] Conversely, $\omega \in B^{t(\omega,E)}(E) \subseteq B^1 B^{t(\omega,E)}(E)$ for any $(\omega, E) \in \Omega \times \Sigma$. By

[19] Invariance implies that if the type mapping is independent of states then $\mu(\cdot) = t(\omega, \cdot)$ for all $\omega \in \Omega$.

[20] Since $[t(\omega)] \subseteq (\neg B^p)(E)$ for all $\omega \in (\neg B^p)(E)$, it can be shown that Certainty of Beliefs and monotonicity (ii) imply $(\neg B^p)(\cdot) \subseteq B^1(\neg B^p)(\cdot)$ (as well as $B^p(\cdot) \subseteq B^1 B^p(\cdot)$).

(i) and (iii),

$$\omega \in \bigcap_{E\in\Sigma} B^1 B^{t(\omega,E)}(E) \subseteq B^1\left(\bigcap_{E\in\Sigma}\{\tilde{\omega}\in\Omega \mid t(\tilde{\omega},E)\geq t(\omega,E)\}\right) = B^1(\uparrow t(\omega)).$$

Thus, $t(\omega, (\uparrow t(\omega))) = 1$. □

Proof of Proposition 2.

1. Assume Self-Evidence of Beliefs. If $\omega \in B^p(E)$, then $t(\omega', E) \geq t(\omega, E) \geq p$ for any $\omega' \in P(\omega)$. Thus, $P(\omega) \subseteq B^p(E)$, i.e., $\omega \in K(B^p)(E)$. Conversely, fix $(\omega, E) \in \Omega \times \Sigma$. Since $\omega \in B^{t(\omega,E)}(E) \subseteq K(B^{t(\omega,E)})(E)$, if $\omega' \in P(\omega)$ then $\omega' \in B^{t(\omega,E)}(E)$, i.e., $t(\omega', E) \geq t(\omega, E)$.

2. Assume $P(\cdot) \subseteq (\downarrow t(\cdot))$. If $\omega \in (\neg B^p)(E)$ and if $\omega' \in P(\omega)$, then $t(\omega', E) \leq t(\omega, E) < p$. Thus, $P(\omega) \subseteq (\neg B^p)(E)$, i.e., $\omega \in K(\neg B^p)(E)$. Conversely, let $\omega' \in P(\omega)$ and $E \in \Sigma$. If $t(\omega, E) = 1$ then $t(\omega', E) \leq t(\omega, E)$. If $t(\omega, E) < 1$, let $\varepsilon > 0$ be such that $q_\varepsilon := t(\omega, E) + \varepsilon < 1$. Since $\omega \in (\neg B^{q_\varepsilon})(E) \subseteq K(\neg B^{q_\varepsilon})(E)$, I have $\omega' \in P(\omega) \subseteq (\neg B^{q_\varepsilon})(E)$. That is, $t(\omega', E) < t(\omega, E) + \varepsilon$. Letting $\varepsilon \to 0$ yields $t(\omega', E) \leq t(\omega, E)$.

□

Proof of Theorem 1. The "Only If" Part. Suppose that $\overrightarrow{\Omega}$ is regular. First, Self-Evidence of Beliefs and the additivity of each $t(\omega, \cdot)$ imply that $P(\cdot) \subseteq [t(\cdot)]$. Especially, $\mu([t(\cdot)]) \geq \mu(P(\cdot)) > 0$. Second, since each $t(\omega, \cdot)$ is additive, the model satisfies Certainty of Beliefs. I now show that, by Invariance and Certainty of Beliefs, $t(\cdot, \cdot) = \mu(\cdot \mid [t(\cdot)])$. While this part is essentially Mertens and Zamir (1985, Proposition 4.2), I provide a proof for completeness. By Invariance and Certainty of Beliefs, I have

$$\begin{aligned}\mu([t(\omega)]) &= \left(\int_{[t(\omega)]} t(\omega', [t(\omega)])\mu(d\omega')\right) + \left(\int_{[t(\omega)]^c} t(\omega', [t(\omega)])\mu(d\omega')\right)\\ &\geq \int_{[t(\omega)]} t(\omega, [t(\omega)])\mu(d\omega') = \mu([t(\omega)]).\end{aligned}$$

Thus, I obtain $\int_{[t(\omega)]^c} t(\omega', [t(\omega)])\mu(d\omega') = 0$. Now, it follows from Invariance, Certainty of Beliefs and the additivity of each $t(\omega, \cdot)$ that

$$\begin{aligned}\mu(E \cap [t(\omega)]) &= \int_{\Omega} t(\omega', E \cap [t(\omega)])\mu(d\omega') = \int_{[t(\omega)]} t(\omega', E \cap [t(\omega)])\mu(d\omega')\\ &= t(\omega, E \cap [t(\omega)])\mu([t(\omega)]) = t(\omega, E)\mu([t(\omega)]).\end{aligned}$$

Thus, $\mu(E \mid [t(\omega)]) = t(\omega, E)$.

Third, Entailment implies $1 = t(\cdot, P(\cdot)) = \mu(P(\cdot) \mid [t(\cdot)])$. I have $\mu([t(\omega)]) = \mu(P(\omega) \cap [t(\omega)]) = \mu(P(\omega))$. Thus, $[t(\cdot)] \subseteq P(\cdot)$ μ-a.s. Fourth, it can now be seen that $t(\omega, E) = \mu(E \mid [t(\omega)]) = \mu(E \mid P(\omega))$.

The "If" Part. Each $t(\omega, \cdot) = \mu(\cdot \mid P(\omega))$ is a countably-additive probability measure. Entailment follows because $t(\omega, P(\omega)) = \mu(P(\omega) \mid P(\omega)) = 1$. Self-Evidence of Beliefs holds by supposition. I show Invariance. Since $\mu([t(\omega)]) = \mu(P(\omega)) > 0$, let $([t(\omega_n)])_n$ be a countable partition of Ω. Since it follows from the assumptions that $t(\omega_n, E) = \mu(E \mid P(\omega_n)) = \mu(E \mid [t(\omega_n)])$, I obtain

$$\mu(E \cap [t(\omega_n)]) = \mu([t(\omega_n)])t(\omega_n, E) = \int_{[t(\omega_n)]} t(\omega', E)\mu(d\omega').$$

By summing over all n, I obtain Invariance:

$$\mu(E) = \sum_n \mu(E \cap [t(\omega_n)]) = \sum_n \int_{[t(\omega_n)]} t(\omega', E)\mu(d\omega') = \int_\Omega t(\omega, E)\mu(d\omega).$$

□

Proof of Corollary 2. First, the equivalence follows from Theorem 1 and the supposition that $\mu(\{\cdot\}) > 0$. Second, it suffices to show $B^1(\cdot) \subseteq K(\cdot)$ as Entailment implies the converse set inclusion $K(\cdot) \subseteq B^1(\cdot)$. If $\omega \in B^1(E)$ then $t(\omega, E) = 1$. Then, $\mu(E \cap P(\omega)) = \mu(P(\omega))$, and thus $\mu(P(\omega) \cap E^c) = 0$. Since $\mu(\{\cdot\}) > 0$, I have $P(\omega) \cap E^c = \emptyset$, i.e., $P(\omega) \subseteq E$. Thus, $\omega \in K(E)$. □

Proof of Corollary 3. It is sufficient to show the "only if" (i.e, "(1a) ⇒ (1b)") part of the first statement. Self-Evidence of Beliefs and the additivity of each $t(\omega, \cdot)$ imply $P(\cdot) \subseteq [t(\cdot)]$. Thus, the partition $\{P(\omega)\}_{\omega\in\Omega}$ is a refinement of $\{[t(\omega)]\}_{\omega\in\Omega}$. Suppose to the contrary that $\omega' \in [t(\omega)] \setminus P(\omega)$. Since $P(\omega) \cap P(\omega') = \emptyset$, it follows that $\mu(P(\omega)) < \mu(P(\omega)) + \mu(P(\omega')) = \mu(P(\omega') \sqcup P(\omega)) \leq \mu([t(\omega)])$, a contradiction. Thus, $[t(\cdot)] = P(\cdot)$. □

Example 1. As in Monderer and Samet (1989, p.176), suppose that the agent is reasoning about the realization of a random draw from $[0, 1]$. Let $\langle \Omega, \Sigma, \mu \rangle = \langle [0, 1], \mathcal{B}_{[0,1]}, \mu \rangle$, where μ is the Lebesgue measure. Let $P(\cdot) = [0, 1]$, i.e., the agent considers every number possible at each realization. Her qualitative belief reduces to (degenerate) knowledge in that she only knows that the draw is from $[0, 1]$ at each state. Her type at each ω is $t(\omega, \cdot) = \mu(\cdot)$. By construction, the model is regular. At any realization, the agent does not know that the draw is an irrational number, as she does not observe the realization. She, however, believes with probability one that the draw is an irrational number. In fact, she 1-believes any event E at any state ω as long as $\mu(E) = 1$. Thus, for any $E \in \Sigma \setminus \{\Omega\}$ with $\mu(E) = 1$, it follows that $B^1(E) \setminus K(E) = \Omega$ and $\mu(B^1(E) \setminus K(E)) = 1$. In this example, while the agent's qualitative belief is fully introspective knowledge, probability-one belief and knowledge differ.

Example 2. As in Example 1, suppose that the agent is reasoning about the realization of a random draw from $\Omega = [0,1]$. The agent's prior μ is the Lebesgue measure on $\langle \Omega, \Sigma \rangle = \langle [0,1], \mathcal{B}_{[0,1]} \rangle$. At each realization $\omega \in [\frac{1}{2}, 1] \cup [0,1]_{\mathbb{Q}}$, the agent considers $P(\omega) = [0,1] \setminus \mathbb{Q}$ possible. At each $\omega \in [0, \frac{1}{2}) \setminus \mathbb{Q}$, the agent considers $P(\omega) = [0,1]$ possible. The agent's type at each ω remains unchanged: $t(\omega, \cdot) = \mu(\cdot)$. The model is regular. The agent's qualitative belief violates all of Truth Axiom, Positive Introspection, and Negative Introspection. While Truth Axiom holds μ-almost surely (as a consequence of Corollary 4), the introspection properties do not even in this sense. For example, if $E = [0,1] \setminus \mathbb{Q}$ then $\mu(K(E) \setminus KK(E)) = \frac{1}{2}$ and $\mu((\neg K)(E) \setminus K(\neg K)(E)) = \frac{1}{2}$.

Proof of Corollary 4. Take $E \in \Sigma$. If $B^1(E) \cap E^c = \emptyset$ then $\mu(B^1(E) \cap E^c) = 0$. Suppose not. If $\omega \in B^1(E) \cap E^c$, then $1 = t(\omega, E) = \mu(E \mid [t(\omega)])$ and $\omega \in [t(\omega)] \cap E^c$. Then, $\mu(E \cap [t(\omega)]) = \mu([t(\omega)])$ and $0 = \mu([t(\omega)] \cap E^c)$. There is a countable $(\omega_n)_n$ such that $\mu(B^1(E) \cap E^c) \leq \mu(\bigcup_n ([t(\omega_n)] \cap E^c)) = 0$. I also get $\mu(K(E) \setminus E) \leq \mu(B^1(E) \setminus E) = 0$. Finally, the last statement follows from the remark in the main text. □

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