Can the Crowd be Introspective? Modeling Distributed Knowledge from Collective Information through Inference*

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January 19, 2019

Abstract

This paper studies distributed knowledge among agents who possibly have contradictory beliefs with each other. The paper formalizes distributed knowledge as knowledge logically deduced from agents' collective information, consisting of events that some agent believes whenever they are true. As a result, distributed knowledge is true, monotonic, and positively introspective, even though agents' beliefs are not. Agents' false beliefs do not lead to distributed knowledge. Distributed knowledge can fail negative introspection even if agents' beliefs satisfy it. Agents cannot necessarily have distributed knowledge of the lack of distributed knowledge of an event. Thus, agents can be collectively unaware of events. If agents' beliefs are true, monotonic, positively introspective, and conjunctive, then distributed knowledge coincides with knowledge possessed by the least knowledgeable "wise man" who knows everything each agent knows.

Keywords: Group Knowledge; Distributed Knowledge; Collective Information; Beliefs; Knowledge; Negative Introspection

1 Introduction

Suppose that a group of agents are reasoning about underlying states of the world, say, the weight of an ox in a poultry exhibition (Galton, 1907a,b, 1908). They possess qualitative beliefs about states of the world—possible weights and relevant aspects of the ox. Their beliefs may not be true and may even be contradictory with each other. While the observation by Galton (1907a,b, 1908) is that the average of agents'

*I would like to thank Frederik Van De Putte for his insightful comments.

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estimations more or less correctly points to the actual weight of the ox, this paper studies qualitative features of group knowledge about states of the world—the weight of the ox—and about group knowledge itself. If the agents share relevant information, what can the group "know" about the weight of the ox and group knowledge itself? To understand qualitative features of knowledge of the group, as opposed to belief of each individual, I study group knowledge in terms of (i) what information the agents share and (ii) how they collectively make inferences about the world.

One cannot overemphasize the importance of group knowledge dispersed among agents in an inconsistent form in such fields as computer science and artificial intelligence, economics and the social sciences, logic, and philosophy. In economics, as Hayek (1945) points out, it is one of the most fundamental facts in economic and social activities that the knowledge possessed by the agents in a society is dispersed among them in possibly contradictory or incomplete forms.[1] Moving on to computer science and multi-agent systems (e.g., Fagin et al. (2003) and Meyer and Hoek (1995)), interpret agents as computer-based systems such as databases with contradictory pieces of information. A natural question is how to aggregate inconsistent sources of information into one contradiction-free system (see, for example, Grégoire and Konieczny (2006)).[2]

Specifically, this paper studies a notion of group knowledge termed as *distributed knowledge*, which is first independently studied in computer science, logic, and philosophy by Halpern and Moses (1990) and Hilpinen (1977).[3] In the original context, a group of agents have distributed knowledge of a statement p if someone who were informed of everything that each agent knows would know p. For example, suppose that two agents, Ann and Bob, respectively know statements q and $(q \to p)$. Even though none of them knows p, combining their knowledge together, Ann and Bob have distributed knowledge of p.

While the previous studies have assumed that individual agents have fully logical, fully introspective, and truthful knowledge (and thus so does group knowledge), this paper formalizes distributed knowledge among agents whose beliefs are not necessarily truthful, consistent, or logically monotonic. Indeed, the paper presents a formalization of distributed knowledge that is, in a certain minimal sense, logically monotonic, introspective, and truthful, independently of any particular assumptions

[1]This paper does not take into account the incentives of agents to share or misreport information (see Arrow (1962) for the public-good nature of information) or the costs or computational complexity associated with sharing information. However, the analyses of this paper as to the formalization of distributed knowledge and as to how distributed knowledge inherits properties of individual beliefs would be independent of the incentive problems in themselves.

[2]The question of this paper is also related to belief revision pioneered by Alchourrón, Gärdenfors, and Makinson (1985) (e.g., adding a piece of information that is inconsistent with existing sources of information so that the revised sources are consistent) and judgment aggregation that studies aggregation of agents' opinions (see, for example, Grossi and Pigozzi (2014)).

[3]First, Hilpinen (1977) calls it "impersonal knowledge." Second, see Halpern and Moses (1990, Footnote 1) and the references therein for their original terminology "implicit knowledge."

on agents' beliefs.

In order for an event E to be distributed knowledge among agents whose beliefs may be inconsistent with each other, I define distributed knowledge in a way such that (i) the agents pool their information and that (ii) they make a collective inference to deduce E from their collective information. I define agents' collective information as those events that some agent believes whenever they are true. This paper systematically studies how the content of agents' collective information affects qualitative aspects of distributed knowledge. The paper also studies how distributed knowledge inherits properties of individual beliefs.

This paper sheds light on the way in which distributed knowledge is generated from individual beliefs and on the extent to which properties of distributed knowledge are attributable to those of individual beliefs. Distributed knowledge formalized in this paper is truthful, monotonic, and positively introspective. That is, not only is distributed knowledge consistent (i.e., if an event E is distributed knowledge then the negation $\neg E$ is not distributed knowledge), but also can agents only have truthful distributed knowledge (i.e., if some event is distributed knowledge then it is true). Thus, when agents' beliefs are not true, their false beliefs may not lead to distributed knowledge. Even if agents are not logical reasoners, any logical consequence of distributed knowledge is distributed knowledge. If agents have distributed knowledge of E, then they have distributed knowledge of the fact that E is distributed knowledge. These three properties embody the idea that a group of agents posses meaningful group knowledge.

Distributed knowledge is not necessarily negatively introspective when individual beliefs are not truthful or conjunctive (i.e., agents may not know a conjunction of what they know). That is, if agents do not possess distributed knowledge of an event E, then it is not necessarily the case that they have distributed knowledge of the fact that E is not distributed knowledge. In other words, after aggregating collective information and making collective inferences about the world, agents may be collectively unaware of E.[4] The result sheds new light on a limitation of group knowledge in that after pooling their information, agents may still not be able to resolve their own unawareness. Indeed, if agents' beliefs are truthful and conjunctive as in the previous studies, then distributed knowledge inherits negatively introspection.

If agents are positively-introspective and logical reasoners as in the previous studies, distributed knowledge turns out to be knowledge of the least knowledgeable hypothetical individual ("wise man") who knows everything that each agent knows (e.g., Halpern and Moses (1990)). The formalization of distributed knowledge in this paper also accommodates Hilpinen (1977) if agents' knowledge is positively introspective.

I move on to the related literature. Within the literature on distributed knowledge pioneered by Halpern and Moses (1990) and Hilpinen (1977), first, this paper

[4]An agent is unaware of an event E if she does not know it and she does not know that she does not know it. Pioneering papers on unawareness include Fagin and Halpern (1987) and Modica and Rustichini (1994, 1999). See Schipper (2015) for a recent overview.

generalizes distributed knowledge among agents whose beliefs may violate logical or introspective properties. Pacuit (2017) and Van De Putte and Klein (2018) study notions of distributed belief without assuming logical or introspective properties on individual beliefs in a semantic (i.e., set-theoretical) framework.[5] Unlike this paper, distributed belief may be inconsistent, and thus agents may have false group beliefs. While the notion of distributed belief is of interest, this paper asks how one can model distributed knowledge, which, by definition, is truthful and thus consistent. In a special case in which agents' beliefs are truthful, logical, and positively introspective, distributed belief studied in Pacuit (2017) and Van De Putte and Klein (2018) and distributed knowledge studied in this paper coincide.

Second, while this paper formalizes distributed knowledge in a semantic framework, there is a strand of literature in computer science, logic, and philosophy that provides a syntactic characterization of distributed knowledge and that establishes the equivalence (i.e., soundness and completeness) between semantics and syntax. See Fagin, Halpern, and Vardi (1992), Fagin et al. (2003), Halpern and Moses (1992), Hoek and Meyer (1992, 1997), and Meyer and Hoek (1995).[6] Unlike the previous literature, however, this paper does not presuppose any property on each agent's belief such as logical monotonicity.[7] While I do not study a syntax system or any associated computational (e.g., model-checking or satisfiability) issue, the paper provides hints on how axioms of distributed knowledge would look like when each individual agent's belief or knowledge is modeled in a weaker (i.e., non-normal) system.

On a related point, the previous literature has shown some subtle dissonances between semantic and syntactic interpretations of distributed knowledge (e.g., Ågotnes and Wáng (2017), Gerbrandy (1999, Chapter 3), Hoek, Linder, and Meyer (1999), Hoek and Meyer (1992), and Roelofsen (2007)). For example, if each agent's fully logical and introspective knowledge is represented by a partition on an underlying state space, then distributed knowledge is introduced by the coarsest common refinement (i.e., the intersection of partitions). Each agent's partition cell containing a state is written as the arbitrary intersection of events believed by the agent at that state, which may not be modally definable (e.g., Samet (2010)). In the complete axiomatization of Hoek and Meyer (1992), however, the partition dictating distributed knowledge is finer than the coarsest common refinement, i.e., the equality is not modally defin-

[5]Van De Putte and Klein (2018) provide a logical framework that has, for a set of individual modal (e.g., belief) operators, the group modal (e.g., distributed belief) operator. In the context of epistemic logic, this is closely related to distributed belief studied in Pacuit (2017).

[6]First, another notion of group knowledge referred to as common knowledge (e.g., Aumann (1976), Friedell (1969), and McCarthy et al. (1978)) is often present. Fukuda (2018c) analyzes common knowledge and common belief without assuming properties on individual agents' beliefs in a semantic framework. While this paper focuses (only) on distributed knowledge, one can introduce notions of common belief and common knowledge into the framework of this paper.

[7]The only implicit assumption is the "rule of equivalence," as in neighborhood semantics (e.g., Chellas (1980), Pacuit (2017), and Fagin et al. (2003)): if any two events E and F are set-theoretically equivalent then so are beliefs/knowledge of E and F.

able. Another instance is the "principle of full communication" by Hoek, Linder, and Meyer (1999). They show that the idea that distributed knowledge of a proposition is obtained through communication may not necessarily be well defined in an arbitrary state space. See also Gerbrandy (1999, Chapter 3) and Roelofsen (2007).

Since the purpose of this paper is to explore properties of distributed knowledge, I provide a semantic framework in which one can always define distributed knowledge by allowing arbitrary union and intersection of events. Thus, for example, if individual agents' beliefs are represented by partitions then distributed knowledge in this paper is indeed characterized by the coarsest common refinement. However, in the framework of this paper, I discuss how distributed knowledge can be defined even when languages are restricted to be finite (i.e., when the collection of events only forms an algebra of sets so that arbitrary union or intersection may not be well defined). On a related point, I also characterize distributed knowledge when agents' beliefs are represented by non-partitional information sets (i.e., information correspondences (e.g., Aumann (1976, 1999), Geanakoplos (1989), and Morris (1996)) in economics or Kripke frames (e.g., Chellas (1980) and Fagin et al. (2003)) in computer science, logic, and philosophy). Namely, distributed knowledge is characterized by the intersection of the "reflexive transitive" (i.e., "truthful-and-positively-introspective") closures of agents' information sets.

Third, distributed knowledge has to do with what agents can come to know after implicitly or explicitly communicating their information.[8] In economics and game theory, such pioneering papers as Bacharach (1985), Cave (1983), and Geanakoplos and Polemarchakis (1982) study how communication among agents with different information leads to agreement of information. Tallon, Vergnaud, and Zamir (2004) consider agents who may have erroneous beliefs. The literature looks at how communication protocols influence information sharing, which may or may not lead to distributed knowledge. As in the literature on modeling distributed knowledge, this paper does not model an exact communication process but studies how the contents of agents' beliefs qualitatively affect group knowledge.

The paper is structured as follows. Section 2 presents an underlying framework. Section 3 formulates distributed knowledge. Section 4 studies properties of distributed knowledge. Section 5 provides concluding remarks. Proofs are relegated to Appendix A.

2 Framework

My technical framework builds on a semantic representation of agents' beliefs by their belief operators. Throughout the paper, let I be a non-empty set of *agents*. A *belief*

[8] Ågotnes and Wáng (2017) point out the temporal inconsistencies between how agents' knowledge is updated and how a proposition to be distributed knowledge is evaluated as a result of communication.

space (of I) is a tuple $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (B_i)_{i \in I}\rangle$.

First, Ω is a non-empty set of *states* of the world. The state space is a sample space on which agents' beliefs are represented. Second, $\mathcal{D}$ is a subset of the power set $\mathcal{P}(\Omega)$ which I call the *domain.* The domain $\mathcal{D}$ consists of *events* about which agents reason. Each event $E \in \mathcal{D}$ is a subset of Ω which describes a certain aspect of the state space. I assume that $\mathcal{D}$ is a complete algebra on Ω. That is, if E is an event, then its complement $\neg E$ (also denoted by E^c) is also an event. If $\mathcal{E}$ is an arbitrary subset of $\mathcal{D}$, then its intersection and its union are events, i.e., $\bigcap \mathcal{E} := \bigcap_{E \in \mathcal{E}} E \in \mathcal{D}$ and $\bigcup \mathcal{E} := \bigcup_{E \in \mathcal{E}} E \in \mathcal{D}$, with the conventions that $\emptyset = \bigcup \emptyset \in \mathcal{D}$ and $\Omega = \bigcap \emptyset \in \mathcal{D}$. Third, each agent i's *belief operator* $B_i : \mathcal{D} \to \mathcal{D}$ associates, with each event $E \in \mathcal{D}$, the event that (i.e., the set of states at which) agent i believes E.

On the one hand, the domain $\mathcal{D}$ is not necessarily the power set $\mathcal{P}(\Omega)$. If the following separation condition holds, then $\mathcal{D} = \mathcal{P}(\Omega)$: for any two distinct states $\omega, \omega' \in \Omega$ with $\omega \neq \omega'$, there is $E \in \mathcal{D}$ with $\omega \in E$ and $\omega' \notin E$. This follows because the separation condition implies $\{\omega\} = \bigcap\{E \in \mathcal{D} \mid \omega \in E\} \in \mathcal{D}$. A finite belief space satisfying this separation condition roughly corresponds with a finite distinguishing model studied by Hoek, Linder, and Meyer (1999).

On the other hand, the domain $\mathcal{D}$ is not only closed under finite intersection (and union) but also under arbitrary intersection (and union). If one introduces a set of syntactical formulas $\mathcal{L}$ closed under finite conjunction and disjunction, then the domain consequently forms an algebra of sets $[\![\mathcal{L}]\!]$ through an interpretation function $[\![\cdot]\!]$, which is closed under finite intersection and union instead of arbitrary intersection and union. The assumption that the domain $\mathcal{D}$ forms a complete algebra is for technical convenience. As I will discuss, by this assumption, distributed knowledge will always be well defined for any sub-group of agents in any belief space.

The generality of the framework comes from the facts that the analysts can assign logical and introspective properties of agents' beliefs one by one and that logical and introspective properties can be expressed by properties of the agents' belief operators in a translucent way. As I will discuss shortly, the framework can accommodate a standard possibility correspondence model in economics and game theory, i.e., a Kripke frame in computer science, logic, and philosophy. Indeed, as long as a semantic representation of agents' beliefs is concerned, this approach is the most general (or minimal) in the following two senses: (i) the framework does not impose properties of agents' beliefs a priori; and (ii) the only restriction is the closure of one's beliefs in logical equivalence: if two events E and F are set-theoretically equivalent (i.e., $E = F$) then the corresponding beliefs are equivalent (i.e., $B_i(E) = B_i(F)$).

I remark that another equivalent way to express agents' beliefs is to use neighborhood systems (see, for example, Chellas (1980), Fagin et al. (2003), and Pacuit (2017) for neighborhood systems; see Fukuda (2018c) for the formal equivalence). An agent i's neighborhood system assigns, with each state ω, the collection of events that she believes at ω. The belief operator of i induces her neighborhood system by collecting the events that i believes at each state. Conversely, i's neighborhood system induces, for each event E, the set of states at which i believes E.

Now, I introduce the following four logical properties of beliefs. First, B_i satisfies *Monotonicity* if $B_i(E) \subseteq B_i(F)$ for any $E, F \in \mathcal{D}$ with $E \subseteq F$. That is, if agent i believes E and if E implies F, then she believes F. Second, B_i satisfies *Necessitation* if $B_i(\Omega) = \Omega$. Agent i believes any tautology in the form of Ω at any state. Third, B_i satisfies *Non-empty Conjunction* if $\bigcap_{E \in \mathcal{E}} B_i(E) \subseteq B_i(\bigcap \mathcal{E})$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$. If agent i believes each of a collection of events then she believes its conjunction. Combining Non-empty Conjunction and Necessitation, B_i satisfies *Conjunction* if $\bigcap_{E \in \mathcal{E}} B_i(E) \subseteq B_i(\bigcap \mathcal{E})$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{D})$.

Fourth, B_i satisfies the *Kripke property* (or B_i is *normal*) if, for each $(\omega, E) \in \Omega \times \mathcal{D}$, $\omega \in B_i(E)$ if and only if (hereafter, often abbreviated as iff) $E \supseteq b_{B_i}(\omega) := \bigcap \{F \in \mathcal{D} \mid \omega \in B_i(F)\}$. The mapping $b_{B_i} : \Omega \to \mathcal{D}$ is referred to as i's possibility correspondence in economics and game theory (see, for example, Aumann (1976, 1999), Geanakoplos (1989), and Morris (1996)). Agent i considers ω' possible at ω if $\omega' \in b_{B_i}(\omega)$. Assigning i's possibility correspondence is equivalent to assigning the accessibility/possibility relation in modal logic (see, for example, Chellas (1980) and Fagin et al. (2003)). Agent i believes E at ω if (and only if) E contains the set of states considered possible at ω. The Kripke property can be characterized as Monotonicity and Conjunction as in Morris (1996, Theorem 1).[9]

Next, I introduce the following three introspective properties of beliefs. First, B_i satisfies *Truth Axiom* if $B_i(E) \subseteq E$ for any $E \in \mathcal{D}$. Truth Axiom distinguishes between knowledge and belief. It means that if agent i "knows" E at ω then E is true at ω. Second, B_i satisfies *Positive Introspection* if $B_i(\cdot) \subseteq B_i B_i(\cdot)$. That is, if agent i believes E then she believes that she believes E. Third, B_i satisfies *Negative Introspection* if $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$. It states that if agent i does not believe E then she believes that she does not believe E. It can be seen that Truth Axiom and Negative Introspection yield Positive Introspection and Necessitation.

To conclude this section, I introduce additional definitions. Call an event E to be *self-evident to* agent i if $E \subseteq B_i(E)$, i.e., i believes E whenever E is true. Denote by $\mathcal{J}_{B_i}$ the collection of events self-evident to i.

Call a mapping $b : \Omega \to \mathcal{D}$ a *possibility correspondence* if $B_b(E) := \{\omega \in \Omega \mid b(\omega) \subseteq E\} \in \mathcal{D}$ for any $E \in \mathcal{D}$. If B_i satisfies the Kripke property then b_{B_i} is a possibility correspondence because $B_i = B_{b_{B_i}}$. Fix a possibility correspondence $b : \Omega \to \mathcal{D}$. First, $b : \Omega \to \mathcal{D}$ is *reflexive* if $\omega \in b(\omega)$ for all $\omega \in \Omega$. It is well known that b is reflexive iff B_b satisfies Truth Axiom. Second, $b : \Omega \to \mathcal{D}$ is *transitive* if $\omega' \in b(\omega)$ implies $b(\omega') \subseteq b(\omega)$. It is well known that b is transitive iff B_b satisfies Positive Introspection. Third, $b : \Omega \to \mathcal{D}$ is *Euclidean* if $\omega' \in b(\omega)$ implies $b(\omega) \subseteq b(\omega')$. It is well known that b is Euclidean iff B_b satisfies Negative Introspection. Fourth, the *reflexive transitive closure* of b is the smallest possibility correspondence $\overline{b} : \Omega \to \mathcal{D}$

[9]The characterization of the Kripke property using Conjunction hinges on arbitrary intersection of events. See Fukuda (2018b) and Samet (2010) (in the case of "$S5$") for a general characterization of the Kripke property when a collection of events may not necessarily form a complete algebra.

which is reflexive and transitive. That is, if $b' : \Omega \to \mathcal{D}$ is a reflexive and transitive possibility correspondence with $b(\cdot) \subseteq b'(\cdot)$, then $\bar{b}(\cdot) \subseteq b'(\cdot)$.

3 Formalization of Distributed Knowledge

I define distributed knowledge by knowledge logically induced from a pool of agents' collective information in the following two steps. The first step presents a general formalization of knowledge logically deduced from a collection of information. The second step formalizes what it means by a pool of agents' collective information.

In the first step, letting $\mathcal{I}$ be the collection of events (i.e., $\mathcal{I} \subseteq \mathcal{D}$), define an operator $K_{\mathcal{I}} : \mathcal{D} \to \mathcal{D}$ as follows. For each $E \in \mathcal{D}$, let

$$K_{\mathcal{I}}(E) := \{\omega \in \Omega \mid \text{there is } F \in \mathcal{I} \text{ with } \omega \in F \subseteq E\} = \bigcup \{F \in \mathcal{I} \mid F \subseteq E\}. \quad (1)$$

The operator $K_{\mathcal{I}}$ is the knowledge operator in the sense that an event E is known at ω (i.e., $\omega \in K_{\mathcal{I}}(E)$) if and only if E is derived from an event $F \in \mathcal{I}$ that is true at ω (i.e., $\omega \in F \subseteq E$). Also, since $K_{\mathcal{I}}(E)$ is itself an event, one can nest knowledge.[10] In fact, as it will turn out, $K_{\mathcal{I}}$ is positively introspective: $K_{\mathcal{I}}(\cdot) \subseteq K_{\mathcal{I}} K_{\mathcal{I}}(\cdot)$. That is, knowledge of an event E implies knowledge of knowledge of E.

The idea of defining knowledge from a collection of information is related to knowledge-belief representations in various strands of literature. First, the logic-of-local-reasoning model by Fagin and Halpern (1987) studies an agent's belief inferred from multiple pieces of information (see also Fagin et al. (2003), Meyer and Hoek (1995), and Thijsse (1993)). In the context of this paper, let $\mathcal{N}_{\mathcal{I}} : \Omega \to \mathcal{P}(\mathcal{D})$ be such that $\mathcal{N}_{\mathcal{I}}(\omega) := \{F \in \mathcal{D} \mid \omega \in F \in \mathcal{I}\}$. Thus, each $\mathcal{N}_{\mathcal{I}}(\omega)$ is the collection of correct information available at ω. An agent knows an event E at ω if she can infer E from some information in $\mathcal{N}_{\mathcal{I}}(\omega)$. Letting $\mathcal{N}_{\mathcal{I}}^{\uparrow}(\omega) := \{F \in \mathcal{D} \mid E \subseteq F \text{ for some } E \in \mathcal{N}_{\mathcal{I}}(\omega)\}$, it turns out that $K_{\mathcal{I}}(E) = \{\omega \in \Omega \mid E \in \mathcal{N}_{\mathcal{I}}^{\uparrow}(\omega)\}$.

Three remarks are in order. First, while the collection of events $\mathcal{I}$ that determines knowledge is independent of any particular states, the resulting knowledge defined though the operator $K_{\mathcal{I}}$ or the neighborhood function $\mathcal{N}_{\mathcal{I}}^{\uparrow}$ is dependent on states. Second, $\mathcal{N}_{\mathcal{I}}^{\uparrow}$ can be seen as a monotone neighborhood system. Third, the knowledge representation by Doignon and Falmagne (1985, 2016) and Falmagne and Doignon (2011) in the mathematical-psychology literature is also related to the logic-of-local-reasoning model (see Fukuda (2018a)).

Second, a collection of information that represents knowledge often takes a form of a set algebra in such diverse literature as computer science, economics, logic, mathematics, philosophy, and psychology. While I do not provide a detailed overview,

[10] The assumption that $\mathcal{D}$ is a complete algebra plays a role in guaranteeing that $K_{\mathcal{I}}(E)$ is itself an event. If, for example, $\mathcal{D}$ is restricted to be an algebra of sets, then $\mathcal{I}$ and $\mathcal{D}$ would be assumed to have the property that, for any $E \in \mathcal{D}$, the $\subseteq$-maximal event included in E exists in $\mathcal{I}$. See Fukuda (2018b) and Samet (2010).

the idea behind the formalization of the knowledge operator $K_{\mathcal{I}}$ is related to Fukuda (2018b), which shows the following: if a given knowledge operator K satisfies Truth Axiom, Positive Introspection, and Monotonicity, then the knowledge operator is recovered from the collection of self-evident events $\mathcal{J}_K$ in the sense that $K = K_{\mathcal{J}_K}$. Since this paper does not impose any property on an individual's beliefs, the equivalence $B_i = K_{\mathcal{J}_{B_i}}$ may not hold.

Since the domain $\mathcal{D}$ is a complete algebra, when a given knowledge operator K satisfies Truth Axiom, Positive Introspection, and Monotonicity, it can be seen that $\mathcal{J}_K$ is closed under arbitrary union. In contrast, $\mathcal{J}_{B_i}$ may fail to have such a property. Doignon and Falmagne (1985, 2016) and Falmagne and Doignon (2011) identify a collection of events that is closed under arbitrary union with a collection of information that represents one's knowledge. If K additionally satisfies Necessitation and Finite Conjunction ($K(E) \cap K(F) \subseteq K(E \cap F)$), then K can be identified with the interior operator associated with a topological space $(\Omega, \mathcal{J}_K)$.[11] Also, when knowledge additionally satisfies Negative Introspection, then knowledge is represented by a sub-algebra. See, for example, Aumann (1999), Bacharach (1985), Fukuda (2018b), Samet (2010), and the references therein.

Third, Shin (1993) studies notions of individual and common knowledge in terms of provability. An agent knows a proposition e at a state if e is proved in a logical system from a set of propositions that are available at that state. Intuitively, the set of propositions that are available at a state ω corresponds to the set of events $F \in \mathcal{I}$ such that $\omega \in F$. Provability roughly corresponds to the set inclusion. In Shin (1993), each agent's knowledge operator can be identified with the interior operator of the topology that dictates the agent's knowledge as well.

The second step is to define agents' collective information in terms of a collection of events. That is, I define $\mathcal{I}$ that has been used in the first step. Fix a group of agents $G \in \mathcal{P}(I)$. I define the distributed knowledge operator $D_G : \mathcal{D} \to \mathcal{D}$ of G by $D_G := K_{\mathcal{I}_G}$, where

$$\mathcal{I}_G := \left\{ \bigcap_{i \in H} E_i \in \mathcal{D} \;\middle|\; \text{there is } H \in \mathcal{P}(G) \setminus \{\emptyset\} \text{ with } E_i \in \mathcal{J}_{B_i} \text{ for each } i \in H \right\}. \quad (2)$$

Expression (2) states that the collective information $\mathcal{I}_G$ among G consists of the conjunction of self-evident events among a non-empty sub-group H of agents. Technically, I allow $G = \emptyset$ because $\mathcal{I}_G = \emptyset$ and $D_\emptyset(\cdot) = K_\emptyset(\cdot) = \emptyset$.

This definition has the following two salient features. First, $\mathcal{I}_G$ includes the collection $\bigcup_{i \in G} \mathcal{J}_{B_i}$ of events that are self-evident to some agent. That is, $E \in \mathcal{I}_G$ whenever $E \subseteq B_i(E)$ for some $i \in G$ (i.e., agent i believes E whenever E is true). Thus, an event E is distributed knowledge if E is derived from another event F which some agent i believes whenever it obtains.

[11] See, for example, Barwise and Etchemendy (1990), Chagrov and Zakharyaschev (1997), Pacuit (2017), Parikh, Moss, and Steinsvold (2007), Vickers (1989), and the references therein for the strands of literature that study knowledge and information from a topological standpoint.

The second feature takes care of the way in which agents' information is combined. Suppose that two events E and F are available to a hypothetical individual whose knowledge is associated with distributed knowledge among G in the sense that $E \in \mathcal{J}_{B_i}$ and $F \in \mathcal{J}_{B_j}$. The second condition states that the hypothetical individual can combine E and F into one piece of information $E \cap F$ to make inferences.

The third feature is that each element of the collective information $\mathcal{I}_G$ is an event $\bigcap_{i \in H} E_i$ formed from agents' self-evident events. While each E_i could be i's belief $B_i(F_i)$ for some $F_i \in \mathcal{D}$ with $B_i(F_i) \in \mathcal{J}_{B_i}$, I define distributed knowledge in a way that distributed knowledge of an event is deduced from a conjunction of agents' collective information itself.[12] I will characterize (in Proposition 4 in Section 4) when distributed knowledge can simply be deduced from events of the form $\bigcap_{i \in H} B_i(E_i)$ where there is $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ such that $E_i \in \mathcal{D}$ for each $i \in H$.

Section 4 examines how the definition of distributed knowledge in this paper nests those in the previous literature when agents' beliefs are truthful, monotonic, conjunctive, and/or introspective. Section 4 also provides (simple) examples.

I remark that, in principle, this section provides many potential ways to formalize distributed knowledge by varying the content of $\mathcal{I}_G$. First, one can impose different assumptions on conjunctions of the collective information. As I have already argued that the hypothetical individual whose knowledge is associated with distributed knowledge would be able to combine pieces of information that are self-evident to individual agents, I do not assume $\mathcal{I}_G = \bigcup_{i \in G} \mathcal{J}_{B_i}$. In contrast, one may let $\mathcal{I}_G$ be the smallest collection that (i) includes $\bigcup_{i \in G} \mathcal{J}_{B_i}$ and that (ii) is closed under non-empty intersection. I show that if each B_i satisfies Conjunction then distributed knowledge can indeed be characterized as knowledge deduced from this definition of $\mathcal{I}_G$.

Instead of analyzing and comparing different collections, I will concentrate on one fixed collection $\mathcal{I}_G$ defined in Expression (2). I also study properties of distributed knowledge that are common irrespective of the particular choice of $\mathcal{I}_G$ by looking at properties of the knowledge operator $K_{\mathcal{I}_G}$.

4 Properties of Distributed Knowledge

This section examines properties of distributed knowledge. Henceforth, I often suppress G and denote $D = D_G$ when I deal with a generic group of agents $G \in \mathcal{P}(I)$. I start with properties of distributed knowledge that hold irrespective of agents' beliefs and a particular choice of $\mathcal{I}_G$.

Proposition 1. *1. If $G_1, G_2 \in \mathcal{P}(I)$ satisfy $G_1 \subseteq G_2$, then $D_{G_1}(\cdot) \subseteq D_{G_2}(\cdot)$.*

[12] Three remarks are in order. First, under Positive Introspection, agents' beliefs are self-evident in that $B_i(\cdot) \in \mathcal{J}_{B_i}$. Thus, E_i may indeed be $B_i(F_i)$ for some $F_i \in \mathcal{D}$. Second, under Truth Axiom, $E_i = B_i(E_i)$ for any $E_i \in \mathcal{J}_{B_i}$ so that one could replace $\bigcap_{i \in H} E_i$ with $\bigcap_{i \in H} B_i(E_i)$ in Expression (2). Third, generally $\bigcap_{i \in H} E_i \subseteq \bigcap_{i \in H} B_i(E_i)$ for $(E_i)_{i \in H}$ in Expression (2). Thus, "distributed knowledge" deduced from $\bigcap_{i \in H} B_i(E_i)$ is distributed knowledge deduced from $\bigcap_{i \in H} E_i$ in Expression (2).

2. *Let $G \in \mathcal{P}(I)$, and take $G_j \in \mathcal{P}(G)$ for each $j \in J$ with $G = \bigcup_{j \in J} G_j$. Denote by D_G^J the distributed knowledge operator among the group of hypothetical agents whose knowledge operators are $(D_{G_j})_{j \in J}$. Then, $D_G = D_G^J$.*

3. *The distributed knowledge operator D satisfies Truth Axiom, Monotonicity, and Positive Introspection.*

The first part of Proposition 1 states that distributed knowledge is monotonic in group, as desired. Intuitively, this is because the amount of collective information is monotonic in group and because distributed knowledge is logically derived from collective information.

The second part states that distributed knowledge is independent of the way in which knowledge is aggregated within a group of agents. In order to find distributed knowledge among a group G of agents, one can first find distributed knowledge among each subgroup G_j and then find distributed knowledge among hypothetical individuals who represent the knowledge possessed by subgroups $(G_j)_{j \in J}$. Note that the subgroups may have overlapping agents. The intuition follows from the facts that the amounts of collective information are identical $\mathcal{I}_G = \bigcup_{j \in J} \mathcal{I}_{G_j}$ and that distributed knowledge is logically deduced from collective information.

This second part shows that distributed knowledge formalized here is free from "informational cascade" to the extent that the amount of collective information is unaffected by the way in which information is shared. Letting a group of agents be $G = \{1, \dots, n\}$, compare two ways in which distributed knowledge is formed. One is to extract agents' collective information simultaneously (i.e., in one step) to form directly $\mathcal{I}_G$. In the other, one could find distributed knowledge among G to sequentially find the distributed knowledge operator among two agents whose belief operators are respectively $D_{\{1,\dots,k-1\}}$ and B_k.

The third part says that, by the way in which events become distributed knowledge, distributed knowledge is true, monotonic, and positively introspective irrespective of agents' individual beliefs. This part hinges on the way in which knowledge is defined through Expression (1) irrespective of a particular choice of $\mathcal{I}_G$.[13]

As I formalize distributed knowledge as opposed to distributed "belief," distributed knowledge is truthful and monotonic. Especially, even if individual agents' beliefs are contradictory or non-monotonic, resulting distributed knowledge is consistent and monotonic. That is, if E is distributed knowledge then $\neg E$ is not distributed knowledge, and a contradiction in the form of $\emptyset$ can never be distributed knowledge. An agent's false belief does not lead to distributed knowledge because distributed knowledge is truthful. At this point, one can assert that individual belief B_i and distributed knowledge among the single individual $D_{\{i\}}$ may not coincide (detailed examinations will be provided in Proposition 3 below).

[13]As is already discussed, Fukuda (2018b) studies the sense in which knowledge can be represented by a set algebra, and the three properties of Truth Axiom, Monotonicity, and Positive Introspection play a key role in the representation.

Not only is distributed knowledge truthful and monotonic, but also it is positively introspective: if an event E is distributed knowledge, then it is distributed knowledge that E is distributed knowledge. Positive Introspection of distributed knowledge could be justified together with that of common belief or common knowledge (Aumann, 1976; Friedell, 1969), which is often considered to be the dual notion of distributed knowledge. The idea behind axiomatizations (especially, the "fixed-point" axiomatization) of common belief is to ensure Positive Introspection of common belief because the intuitive definition of common belief by the iteration of mutual beliefs may not necessarily satisfy Positive Introspection (see, for example, Barwise (1989) and Lismont and Mongin (1994)).[14] In fact, if agents' knowledge is truthful, monotonic, positively introspective, and conjunctive, then distributed knowledge turns out to be the "supremum" of individual agents' knowledge: distributed knowledge can be regarded as knowledge of the least knowledgeable "wise man" who knows everything that each agent knows (Halpern and Moses, 1990). In this case, common knowledge is considered to be the "infimum" of individual agents' knowledge as McCarthy et al. (1978) define the notion of what "any fool" knows.

Now, I study how distributed knowledge reflects logical properties of individual beliefs. The next result shows how distributed knowledge inherits Necessitation, (Non-empty) Conjunction, and the Kripke property from individual beliefs.

Proposition 2. *Let $G \in \mathcal{P}(I)$.*

1. (a) *The distributed knowledge operator D_G satisfies Necessitation if and only if, for each $\omega \in \Omega$, there is $F \in \mathcal{I}_G$ such that $\omega \in F$. Especially, if B_i satisfies Necessitation for some $i \in G$, then D_G satisfies Necessitation.*

   (b) *If B_i satisfy Necessitation for every $i \in G$, then $D_G = K_{\mathcal{I}^*_G}$, where*

$$\mathcal{I}^*_G = \left\{ \bigcap_{i \in G} E_i \in \mathcal{D} \;\middle|\; E_i \in \mathcal{J}_{B_i} \text{ for each } i \in G \right\}.$$

2. *If B_i satisfies (Non-empty) Conjunction for every $i \in G$, then so does D_G. Moreover, $D_G = K_{\mathcal{I}^*_G}$, where*

$$\mathcal{I}^*_G = \left\{ \bigcap \mathcal{E} \in \mathcal{D} \;\middle|\; \mathcal{E} \subseteq \bigcup_{i \in H} \mathcal{J}_{B_i} \text{ for some } H \in \mathcal{P}(G) \setminus \{\emptyset\} \right\}.$$

3. *If B_i satisfies Conjunction (i.e., Necessitation and Non-empty Conjunction) for every $i \in G$, then D_G satisfies the Kripke property. Its possibility correspondence $b_D : \Omega \to \mathcal{D}$ satisfies $b_D(\cdot) = \bigcap_{i \in G} \beta_{B_i}(\cdot)$, where $\beta_i(\omega) := \bigcap\{F \in \mathcal{D} \mid \omega \in F \subseteq B_i(F)\}$. Moreover, b_D is reflexive and transitive.*

[14] The iteration of mutual beliefs means that an event E is common belief at a state if, everybody believes E at ω, everybody believes that everybody believes E at ω, and so forth *ad infinitum*.

4. *Suppose that B_i satisfies the Kripke property for every $i \in G$. Then, each β_{B_i} is a reflexive transitive closure of b_{B_i}. Consequently, the following also hold.*

   (a) *If each B_i satisfies Positive Introspection, then $\beta_{B_i}(\omega) = b_{B_i}(\omega) \cup \{\omega\}$ and $b_D(\omega) = \left(\bigcap_{i \in I} b_{B_i}(\omega)\right) \cup \{\omega\}$.*

   (b) *If each B_i satisfies Truth Axiom and Positive Introspection, then $b_{B_i} = \beta_{B_i}$ and $b_D(\cdot) = \bigcap_{i \in I} b_{B_i}(\cdot)$.*

Remarks on Proposition 2 are in order. First, the first part of Proposition 2 (1a) characterizes Necessitation in terms of $\mathcal{I}_G$. If no agent's beliefs satisfies Necessitation, then distributed knowledge may or may not satisfy Necessitation. On the one hand, suppose that each B_i satisfies Monotonicity but fails Necessitation. If $\omega \notin B_i(\Omega)$ for all $i \in G$, then $\omega \notin D(\cdot)$. In words, no event can be distributed knowledge at state ω. Especially, D fails Necessitation.

On the other hand, consider two agents who have opposing beliefs: while agent $i \in G$ believes E is true whenever E obtains, agent $j \in G$ believes $\neg E$ is true whenever E does not obtain. If E is true at ω, then Ω is distributed knowledge at ω, as $\omega \in E \subseteq \Omega$. If E is false, then Ω is distributed knowledge at ω, as $\omega \in E^c \subseteq \Omega$.

Second, Proposition 2 (2) shows that if each B_i satisfies Conjunction then distributed knowledge is also obtained through taking arbitrary conjunction of agents' information. When agents' beliefs do not satisfy (Non-empty) Conjunction, however, distributed knowledge does not necessarily satisfy it (an example is provided in the proof of Proposition 2 in the Appendix). Also, the example shows that even if agents' beliefs satisfy Non-empty Conjunction, distributed knowledge would be different from knowledge deduced from $\bigcup_{i \in G} \mathcal{J}_{B_i}$ because the hypothetical agent who is associated with distributed knowledge could take conjunctions of events self-evident across different agents.

Third, Proposition 2 (3) and (4) imply that if each B_i satisfies Conjunction, then $D_{\{i\}}$ satisfies the Kripke property. Indeed, $D_{\{i\}}$ is induced by the possibility correspondence β_{B_i}. If B_i satisfies the Kripke property, then $D_{\{i\}}$ is the "maximal" belief operator satisfying Truth Axiom, Positive Introspection, the Kripke property, and $D_{\{i\}}(\cdot) \subseteq B_i(\cdot)$ (i.e., if $K : \mathcal{D} \to \mathcal{D}$ satisfies Truth Axiom, Positive Introspection, the Kripke property, and $K(\cdot) \subseteq B_i(\cdot)$, then $K(\cdot) \subseteq D_{\{i\}}(\cdot)$). Thus, Proposition 2 (3) states that distributed knowledge among a group G is generated from distributed knowledge among $\{i\}$ for all $i \in G$ through $b_D(\cdot) = \bigcap_{i \in G} \beta_{B_i}(\cdot)$.

Fourth, Proposition 2 (3) and (4) imply that $b_D(\cdot)$ is a reflexive and transitive possibility correspondence which includes $\bigcap_{i \in G} b_{B_i}(\cdot)$ (i.e., $\bigcap_{i \in G} b_{B_i}(\cdot) \subseteq b_D(\cdot)$). This sheds light on defining distributed knowledge from agents' possibility correspondences because the previous literature (e.g., Halpern and Moses (1990) and Meyer and Hoek (1995)) introduces the possibility correspondence that dictates D_G from the intersection of agents' possibility correspondences $\bigcap_{i \in G} b_{B_i}$ when agents' beliefs are fully introspective and satisfy the Kripke property. First, distributed knowledge can be derived from a possibility correspondence as long as agents' beliefs are conjunctive.

Second, even if each B_i satisfies the Kripke property, $b_D(\cdot)$ may not necessarily be the intersection of agents' possibility correspondences $\bigcap_{i\in G} b_{B_i}$. Third, it is not necessarily the case that $b_D(\cdot) = \bigcap_{i\in G} \beta_{B_i}(\cdot)$ is the reflexive transitive closure of $\bigcap_{i\in G} b_{B_i}(\cdot)$ (see Example A.1 in Appendix A). That is, in order to obtain the possibility correspondence that dictates D from agents' beliefs which may violate either Truth Axiom or Positive Introspection, one needs to take the reflexive transitive closure of each b_{B_i} (which is β_{B_i}) instead of the reflexive transitive closure of $\bigcap_{i\in G} b_{B_i}(\cdot)$.

Now, I study relations between distributed knowledge and individual beliefs. The next proposition identifies: the conditions under which individual knowledge and distributed knowledge of a single agent coincide; and the sense in which distributed knowledge is the "supremum" of individual knowledge in terms of knowledgeability.

Proposition 3. *Let $G \in \mathcal{P}(I) \setminus \{\emptyset\}$, and let $i \in G$.*

1. *(a) If B_i satisfies Truth Axiom and Positive Introspection, then $B_i(\cdot) \subseteq D_{\{i\}}(\cdot) \subseteq D_G(\cdot)$.*

   (b) If B_i satisfies Monotonicity, then $D_{\{i\}}(\cdot) \subseteq B_i(\cdot)$.

   (c) $B_i = D_{\{i\}}$ iff B_i satisfies Truth Axiom, Monotonicity, and Positive Introspection.

2. *Let $K_G : \mathcal{D} \to \mathcal{D}$ satisfy Monotonicity, Non-empty Conjunction, and $\bigcup_{i\in G} D_{\{i\}}(\cdot) \subseteq K_G(\cdot)$. Then, $D_G(\cdot) \subseteq K_G(\cdot)$.*

Proposition 3 (1a) implies that if an agent i's belief satisfies Truth Axiom and Positive Introspection, then her individual belief (knowledge) leads to distributed knowledge among $\{i\}$. This is because i's belief $B_i(E)$ is self-evident to her. In contrast, if i's beliefs are monotonic as in (1b), then distributed knowledge among $\{i\}$ implies her belief. Indeed, (1c) demonstrates that distributed knowledge among $\{i\}$ coincides with i's belief if and only if her beliefs satisfy these three axioms of Truth Axiom, Positive Introspection, and Monotonicity. As I compare an individual agent's belief and group knowledge, it is not necessarily the case that one's belief coincides with the group knowledge of one agent.

I also remark on how distributed knowledge shares the properties of individual agents' beliefs in an extreme case in which agents' beliefs are identical. Assume $B_i = B_j$ for all $i, j \in G$. In order to focus only on the aggregation of agents' beliefs, assume further that (every) B_i satisfies Non-empty Conjunction. Then, $D_G = D_{\{i\}}$. Proposition 3 implies that whenever an agent's beliefs violate Truth Axiom, Positive Introspection, or Monotonicity, it may not necessarily be the case that $B_i = D_G$ ((counter-)examples are in the proof of Proposition 3).

Proposition 3 (2) is related to Proposition 2 (3) and (4). To see this, assume that each B_i satisfies Conjunction. Then, D_G satisfies Conjunction and Monotonicity (i.e., the Kripke property). By Proposition 3 (2), D_G is the "minimal" belief operator satisfying $\bigcup_{i\in G} D_{\{i\}}(\cdot) \subseteq D_G(\cdot)$ (as well as Truth Axiom and Positive Introspection).

Since $D_{\{i\}}$ is induced by β_{B_i}, $D_G(\cdot)$ is induced by $b_D(\cdot) = \bigcap_{i \in G} \beta_{B_i}(\cdot)$. I will discuss (in Corollary 1) the sense in which distributed knowledge is associated with the least knowledgeable "wise man" as knowledgeable as any individual agent (Halpern and Moses, 1990). Also, I remark that if $K_G : \mathcal{D} \to \mathcal{D}$ satisfies Monotonicity, Non-empty Conjunction, and $\bigcup_{i \in G} B_i(\cdot) \subseteq K_G(\cdot)$, then $D_G(\cdot) \subseteq K_G(\cdot)$. This is because, by Monotonicity, distributed knowledge among $\{i\}$ implies i's belief (i.e., $D_{\{i\}}(\cdot) \subseteq B_i(\cdot)$).

Using Proposition 3 (1a), I demonstrate that if agents' beliefs (knowledge) satisfy Truth Axiom and Positive Introspection then distributed knowledge can be seen as knowledge logically deduced from agents' individual knowledge.

Proposition 4. *Let*

$$\mathcal{I}_G^* = \left\{ \bigcap_{i \in H} B_i(E_i) \in \mathcal{D} \;\middle|\; E_i \in \mathcal{D} \text{ for each } i \in H \in \mathcal{P}(G) \setminus \{\emptyset\} \right\}.$$

If each B_i satisfies Truth Axiom, then $D_G(\cdot) \subseteq K_{\mathcal{I}_G^*}(\cdot)$. If each B_i satisfies Positive Introspection, then $K_{\mathcal{I}_G^*}(\cdot) \subseteq D_G(\cdot)$.

In Proposition 4, if each B_i satisfies Positive Introspection, then any event deduced by putting agents' beliefs together is distributed knowledge. If each B_i satisfies Truth Axiom, then distributed knowledge is deduced from pooling agents' beliefs.

The characterization of distributed knowledge in Proposition 4 is related to the original definition of distributed knowledge (impersonal knowledge) by Hilpinen (1977). His definition of distributed knowledge would reduce to the following: an event E is distributed knowledge among G at state ω if there are $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $E_i \in \mathcal{D}$ for each $i \in H$ such that $\omega \in \bigcap_{i \in H} B_i(E_i)$ and $\bigcap_{i \in H} E_i \subseteq E$.

Generally, if an event E is distributed knowledge at a state ω (in the formulation of this paper) then E is distributed knowledge at ω (in the above sense). For the converse, suppose that agents' beliefs satisfy Truth Axiom and Positive Introspection as in Proposition 4. Then, the definition of distributed knowledge by Hilpinen (1977) implies (coincides with) that of this paper. In other words, the definition of distributed knowledge in this paper can accommodate the original definition of distributed knowledge (impersonal knowledge) by Hilpinen (1977) if each agent has positively-introspective and truthful knowledge.

The definition of Hilpinen (1977), however, would reflect (monotonic) "collective belief" (or "distributed belief") when it comes to agents' beliefs for the following grounds. His notion of distributed knowledge may fail Truth Axiom if some agents' beliefs do not satisfy Truth Axiom (in contrast, if every agent's belief satisfies Truth Axiom, then his notion of distributed knowledge satisfies Truth Axiom). If each agent's belief satisfies Conjunction, then it can be seen that an event E is distributed knowledge (impersonal knowledge) at a state ω iff $\bigcap_{i \in G} b_{B_i}(\omega) \subseteq E$. If each B_i satisfies Truth Axiom and Positive Introspection, this characterization coincides with that in Proposition 2 (3).

Pacuit (2017) and Van De Putte and Klein (2018) study "distributed belief" which does not impose logical monotonicity: an event E is distributed belief among G at state ω if there are $(E_i)_{i\in G} \in \mathcal{D}^G$ such that $\omega \in \bigcap_{i\in G} B_i(E_i)$ and $E = \bigcap_{i\in G} E_i$. If every B_i satisfies Monotonicity and Necessitation, then this definition of distributed belief coincides with that by Hilpinen (1977). Thus, on the one hand, if each B_i satisfies Truth Axiom, Positive Introspection, Monotonicity, and Necessitation, this definition of distributed belief and the definition of distributed knowledge in this paper coincide with each other.

On the other hand, if some agents fail Truth Axiom or Positive Introspection, the definition of distributed belief by Pacuit (2017) and Van De Putte and Klein (2018) and the definition of distributed knowledge in this paper may be different. To see this point, suppose that each B_i satisfies the Kripke property. While their distributed belief (and indeed Hilpinen (1977)'s definition of distributed knowledge/belief) can be captured by $\bigcap_{i\in G} b_{B_i}(\cdot)$, distributed knowledge of this paper can be captured by $\bigcap_{i\in G} \beta_i(\cdot)$, which may properly include $\bigcap_{i\in G} b_{B_i}(\cdot)$. Thus, if an event E is distributed knowledge at a state then E is distributed belief at that state. Again, it may not necessarily be the case that distributed knowledge is characterized by the reflexive transitive closure of distributed belief $\bigcap_{i\in G} b_{B_i}(\cdot)$ (see Example A.1 in Appendix A).

Now, I provide the following corollary of Propositions 2, 3, and 4.

Corollary 1. *Let $G \in \mathcal{P}(I)$. Suppose that B_i satisfies Truth Axiom, Monotonicity, Positive Introspection, and Non-empty Conjunction for each $i \in G$.*

1. *D_G satisfies Truth Axiom, Monotonicity, Positive Introspection, Non-empty Conjunction, and $\bigcup_{i\in G} B_i(\cdot) \subseteq D_G(\cdot)$.*
2. *Let $K_G : \mathcal{D} \to \mathcal{D}$ satisfy Truth Axiom, Monotonicity, Positive Introspection, Non-empty Conjunction, and $\bigcup_{i\in G} B_i(\cdot) \subseteq K_G(\cdot)$. Then, $\bigcup_{i\in G} B_i(\cdot) \subseteq D_G(\cdot) \subseteq K_G(\cdot)$.*

Corollary 1 sheds light on conditions under which distributed knowledge becomes the "supremum" of agents' knowledge (or the least knowledgeable "wise man" (Halpern and Moses, 1990)). Assume that agents' beliefs (knowledge) satisfy Truth Axiom, Positive Introspection, Monotonicity, and Non-empty Conjunction. Then, distributed knowledge also satisfies these axioms. Moreover, distributed knowledge is associated with knowledge of the least knowledgeable hypothetical individual who knows everything that each agent knows. As the examples in the Appendix show, without any of Truth Axiom, Monotonicity, Positive Introspection, or Non-empty Conjunction, it may not necessarily be the case that distributed knowledge is associated with the "supremum" of agents' knowledge.

Conjunction cannot be dropped conceptually because the hypothetical individual who dictates distributed knowledge can take an intersection of agents' self-evident events. If one restricts attention to agents' truthful, monotonic, and positively introspective knowledge, then the knowledge operator of the "wise man" is characterized by $K_{\mathcal{I}_G^*}$ with $\mathcal{I}_G^* = \bigcup_{i\in G} \mathcal{J}_{B_i}$.

If agents' knowledge additionally satisfies Necessitation, while each i's knowledge is characterized by her (reflexive and transitive) possibility correspondence b_{B_i}, distributed knowledge is associated with $\bigcap_{i \in G} b_{B_i}$. This observation generalizes the introduction of distributed knowledge by $\bigcap_{i \in G} b_{B_i}$ in Halpern and Moses (1990), in which each b_{B_i} forms a partition of the state space Ω (i.e., each B_i also satisfies Negative Introspection). Thus, the definition of distributed knowledge coincides with that of Halpern and Moses (1990) if each agent's knowledge is "S5" (i.e., it satisfies the Kripke property, Truth Axiom, Positive Introspection, and Negative Introspection). The previous argument also shows that Negative Introspection could be dropped (i.e., agents' knowledge can be "S4").

When each agent's knowledge satisfies the Kripke property, Truth Axiom, and Positive Introspection, each agent's knowledge is characterized by an Alexandroff topology $\mathcal{J}_{B_i}$, a topology which is closed under arbitrary intersection (see the references in Footnote 11). The corollary states that $\mathcal{J}_{D_G}$ is indeed the supremum Alexandroff topology (the coarsest Alexandroff topology which is as fine as every agent's Alexandroff topology).

Doignon and Falmagne (2016) and Falmagne and Doignon (2011) document empirical procedures to describe agents' knowledge, (partly) based on the idea that an agent's knowledge can be represented by a collection of events that is closed under arbitrary union (see also the references therein). Particularly, the web-based system referred to as ALEKS (Assessment of LEarning in Knowledge Spaces) has been used by "millions of students in schools and colleges, and by home schooled students" to assess students' knowledge on scholarly subjects (Doignon and Falmagne, 2016). While each agent's knowledge would be described by $\mathcal{J}_{B_i}$ (which is indeed closed under arbitrary union), distributed knowledge among a group of students would be described by the supremum $\mathcal{J}_{D_G}$.

Finally, I study how distributed knowledge inherits and violates Negative Introspection. It would imply that, even after aggregating agents' information, agents as a group may still be unable to resolve their unawareness.

Proposition 5. *1. Suppose that each B_i satisfies Non-empty Conjunction, Truth Axiom, and Negative Introspection. Then, D_G satisfies Negative Introspection. In this case, D_G is also re-written as either of the following two ways. First, $D_G(E) = \{\omega \in \Omega \mid \bigcap_{i \in G} b_{B_i}(\omega) \subseteq E\}$ for all $E \in \mathcal{D}$. Second, $D_G = K_{\mathcal{I}^*_G}$, where $\mathcal{I}^*_G$ is the complete algebra generated by $\bigcup_{i \in G} \mathcal{J}_{B_i}$. Indeed, $\mathcal{I}^*_G = \mathcal{J}_{D_G}$.*

2. Suppose that some B_i fails either Non-empty Conjunction or Truth Axiom. Then, D_G may fail Negative Introspection even if every B_i satisfies it.

In Proposition 5 (1), notice that Truth Axiom and Negative Introspection yield Positive Introspection and Necessitation. By Proposition 2, the distributed knowledge operator D_G satisfies the Kripke property with its Euclidean possibility correspondence $\bigcap_{i \in G} b_{B_i}$. Since the Euclidean property is preserved under intersection, D_G

E	$B_1(E)$	$B_2(E)$	$B_3(E)$	$D_{\{1,2\}}(E)$	$D_{\{3\}}(E)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\emptyset$	$\{\omega_1\}$	$\emptyset$
$\{\omega_2\}$	$\{\omega_2, \omega_3\}$	$\emptyset$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_3\}$	$\emptyset$	$\emptyset$	$\{\omega_3\}$	$\emptyset$	$\{\omega_3\}$
$\{\omega_1, \omega_2\}$	Ω	$\{\omega_1\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_1, \omega_3\}$
$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\emptyset$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
Ω	Ω	Ω	Ω	Ω	Ω

Table 1: (Counter-)examples for Proposition 5 (Examples 1 and 2)

satisfies Negative Introspection. Note that the statement holds even if each agent's beliefs (knowledge) are non-monotonic. If agents' beliefs are monotonic then her belief is characterized by the partitional possibility correspondence b_{B_i}, and the supremum partition (coarsest common refinement) $\bigcap_{i \in G} b_{B_i}$ characterizes distributed knowledge.

While I relegate the proof of the last assertion in Proposition 5 (1) to the Appendix, it states that if each agent has fully introspective and conjunctive knowledge, then distributed knowledge is associated with the knowledge deduced from the complete algebra $\mathcal{I}_G^*$ generated by agents' self-evident events $\bigcup_{i \in G} \mathcal{J}_{B_i}$. Distributed knowledge does not have any more than the complete algebra $\mathcal{I}_G^*$ in that $\mathcal{J}_D$ coincides with $\mathcal{I}_G^*$.

Proposition 5 (2) contrasts with Colombetti (1993). In the context of common belief, Colombetti (1993) provides a Kripke-frame example in which common belief (which may violate Truth Axiom) fails to inherit Negative Introspection from agents' beliefs. Fukuda (2018c) shows that common belief (common knowledge) inherits Negative Introspection if agents' beliefs (knowledge) are truthful and satisfy a certain form of monotonicity.

Now, I prove the second statement of Proposition 5 by providing (counter-)examples. The first example exhibits a violation of Truth Axiom while the second does Non-empty Conjunction.

Example 1. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$, and let B_1 and B_2 be as in Table 1. Both B_1 and B_2 satisfy the Kripke property, Positive Introspection, and Negative Introspection. While B_2 satisfies Truth Axiom, B_1 fails it. The distributed knowledge operator $D_{\{1,2\}}$, depicted in Table 1, violates Negative Introspection. Also, $B_2(\cdot) \subseteq D_{\{1,2\}}(\cdot) \subseteq B_1(\cdot)$.

To get an idea, suppose that agents 1 and 2 are reasoning about a company's performance. At state ω_1, the company's performance is bad and the company is not window-dressing. At state ω_2, the performance is good and the company is not window-dressing. At state ω_3, however, while the company is not performing well, it is window-dressing (so that the company's performance may appear good).

Agent 1 believes, at state $\omega \in \{\omega_1, \omega_2\}$, any event containing ω. At state ω_3, she believes any event containing ω_2. Thus, at ω_1, agent 1 considers $\{\omega_1\}$ possible. At $\omega \in \{\omega_2, \omega_3\}$, she (naively) considers $\{\omega_2\}$ possible.

Agent 2 believes, at state ω_1, any event containing ω_1. At state $\omega \in \{\omega_2, \omega_3\}$, she believes $\{\omega_2, \omega_3\}$ and Ω. Thus, at ω_1, agent 2 considers $\{\omega_1\}$ possible. At $\omega \in \{\omega_2, \omega_3\}$, she (cautiously) considers $\{\omega_2, \omega_3\}$ possible.

Consider the agents' distributed knowledge. While D satisfies the Kripke property, b_D does not partition Ω. While $b_D(\omega) = \{\omega\}$ for each $\omega \in \{\omega_2, \omega_3\}$, $b_D(\omega_3) = \{\omega_2, \omega_3\}$. That is, while the hypothetical individual whose knowledge is associated with distributed knowledge would consider $\{\omega_2\}$ possible at ω_2 (at which the company is not window-dressing), the hypothetical individual would consider $\{\omega_2, \omega_3\}$ (the company's real performance is bad) possible at ω_3 (at which the company is window-dressing). Now, let $E = \{\omega_2\}$ (i.e., the company is not window-dressing and is performing bad). When the company is window-dressing (i.e., at ω_3), E is not distributed knowledge. At the same time, at ω_3, it is not distributed knowledge that E is not distributed knowledge. Thus, to the extent that the failure of Negative Introspection of an agent's knowledge is associated with unawareness, the agents are "collectively unaware" of the event E at state ω_3 in the sense that E is not distributed knowledge at ω_3 and the event that E is not distributed knowledge is not distributed knowledge at ω_3.[15] Observe that agent 2's beliefs (knowledge) are fully introspective: if she does not believe (know) any event F at a state then she believes (knows) that she does not believe (know) F at the state. In other words, there is no possibility for agent 2 to be unaware of any event. □

Example 2. I introduce and consider agent 3 in the previous example. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$, and let B_3 be as in Table 1. While B_3 satisfies Truth Axiom, Negative Introspection, Positive Introspection, and Necessitation, it violates Non-empty Conjunction and Monotonicity. Note that any operator satisfying Truth Axiom, Negative Introspection, and Monotonicity also satisfies Conjunction (see Fukuda (2018b)). For simplicity, consider the distributed knowledge operator $D_{\{3\}}$ depicted in Table 1. It violates Negative Introspection.

At state ω_1 (at which the company's performance is "truly" bad), agent 3 believes $\{\omega_1, \omega_2\}$ (the company is not window-dressing), $\{\omega_1, \omega_3\}$ (the company's performance

[15] Following Modica and Rustichini (1994, 1999), I define unawareness as the lack of knowledge: an agent is unaware of an event E if she does not know E and she does not know that she does not know E (see also Fagin and Halpern (1987), Heifetz, Meier, and Schipper (2006), and Schipper (2015) for concepts of unawareness). It would be interesting to examine collective unawareness from the lack of "concept" as in Heifetz, Meier, and Schipper (2006)'s unawareness structure consisting of multiple state spaces (see, for example, Board, Chung, and Schipper (2011), Halpern and Rêgo (2008), and Schipper (2015) for equivalent representations of unawareness). However, I stick to the standard-state-space framework as introducing multiple state spaces of an unawareness structure would require lots of additional definitions and as the extension of the distributed knowledge operator D_G (especially, Expression (1)) to an unawareness structure, while interesting, would seem to be straightforward.

is bad), and a tautology Ω. Thus, agent 3 cannot identify the realization ω_1 at ω_1, but otherwise she has correct beliefs. At state $\omega \in \{\omega_2, \omega_3\}$, agent 3 is able to identify the realization ω, i.e., she is able to discern whether the company is window-dressing. She, however, does not believe $\{\omega_2, \omega_3\}$. □

I remark that even if each agent does not exhibit any unawareness, group knowledge may exhibit unawareness. As a somewhat trivial example, consider agent 4 with $B_4 = B_3$. Then, $D_{\{3,4\}} = D_3$ exhibits unawareness while B_3 and B_4 do not.[16]

One may wonder how the way in which information is aggregated (i.e., the choice of collective information $\mathcal{I}_G$) affects the negative introspection property of distributed knowledge. To see this point, suppose that each B_i satisfies Truth Axiom and Negative Introspection as in Example 2. Then, Necessitation holds, and $\mathcal{J}_{B_i}$ is closed under complementation. Thus, $D = K_{\mathcal{I}^*_G}$, where

$$\mathcal{I}^*_G = \left\{ \bigcap_{i \in G} E_i \in \mathcal{D} \;\middle|\; E_i \in \mathcal{J}_{B_i} \text{ or } E_i^c \in \mathcal{J}_{B_i} \text{ for each } i \in G \right\}.$$

Example 2 suggests that $K_{\mathcal{I}^*_G}$ defined above may fail Negative Introspection. Thus, even if each agent i can utilize either $E_i \in \mathcal{J}_{B_i}$ or $E_i^c \in \mathcal{J}_{B_i}$, distributed knowledge may fail Negative Introspection. Suppose, instead, that each agent pools the complete algebra $\mathcal{J}^*_{B_i}$ generated by $\mathcal{J}_{B_i}$. Letting

$$\mathcal{I}^*_G = \left\{ \bigcap_{i \in H} E_i \in \mathcal{D} \;\middle|\; \text{there is } H \in \mathcal{P}(G) \setminus \{\emptyset\} \text{ with } E_i \in \mathcal{J}^*_{B_i} \text{ for each } i \in H \right\},$$

the distributed knowledge operator $D_G = K_{\mathcal{I}^*_G}$ is induced by the partitional possibility correspondence $\bigcap_{i \in G} \bigcap \{E_i \in \mathcal{J}^*_{B_i} \mid \omega \in E_i\}$ and satisfies Negative Introspection. Indeed, if B_i satisfies Truth Axiom, Monotonicity, and Negative Introspection (which imply Positive Introspection and Conjunction), then $\mathcal{J}_{B_i}$ itself is a complete algebra.

5 Conclusion

This paper provided a formalization of distributed knowledge inferred from agents' collective information consisting of events that some agent always believes whenever they are true. Irrespective of agents' beliefs, distributed knowledge monotonically increases in group (size). Distributed knowledge is true, monotonic, and positively introspective even if agents' beliefs are not. The formalization of distributed knowledge in this paper nests the previous ones under particular assumptions on agents' beliefs (knowledge).

[16] By Proposition 3 (1a), if B_i satisfies Truth Axiom and Positive Introspection then $(\neg D_G)(\cdot) \subseteq (\neg B_i)(\cdot)$. The proposition, however, does not necessarily imply $(\neg D_G)(\neg D_G)(\cdot) \subseteq (\neg B_i)(\neg B_i)(\cdot)$. Indeed, $(\neg D_{\{3,4\}})(\neg D_{\{3,4\}})(\{\omega_2, \omega_3\}) = \Omega \not\subseteq \emptyset = (\neg B_3)(\neg B_3)(\{\omega_2, \omega_3\})$.

Distributed knowledge inherits logical properties from individual agents. When agents' beliefs are induced by their possibility correspondences, the possibility correspondence that dictates distributed knowledge is the intersection of the reflexive transitive closures of agents' possibility correspondences.

Distributed knowledge of one agent coincides with her belief if and only if her belief is true, monotonic, and positively introspective. If agents' beliefs (knowledge) are also conjunctive, distributed knowledge coincides with knowledge of the least knowledgeable "wise man" who is at least as knowledgeable as any individual agent.

Under Truth Axiom, Non-empty Conjunction, and Negative Introspection, distributed knowledge inherits Negative Introspection. If agents' beliefs are not truthful or not conjunctive, however, distributed knowledge may not inherit Negative Introspection. In this case, distributed knowledge exhibits "collective unawareness." It implies that agents, after aggregating their collective information, may not be able to resolve their unawareness.

Finally, an interesting avenue for further research is to scrutinize processes by which distributed knowledge is obtained. Examples are to incorporate particular communication processes or agents' decisions to share information. A syntactic approach would also be interesting.

A Appendix

Proof of Proposition 1. The first statement follows because $\mathcal{I}_{G_1} \subseteq \mathcal{I}_{G_2}$.

For the second statement, assume $G \neq \emptyset$ because the statement vacuously holds under $G = \emptyset$. Suppose $\omega \in D^J_G(E)$. There are $J' \in \mathcal{P}(J) \setminus \{\emptyset\}$ and $E_j \in \mathcal{J}_{D_{G_j}}$ for each $j \in J'$ such that $\omega \in \bigcap_{j \in J'} E_j \subseteq E$. Since $\omega \in E_j \subseteq D_{G_j}(E_j)$, there are $H_j \in \mathcal{P}(G_j) \setminus \{\emptyset\}$ and $F_i \in \mathcal{J}_{B_i}$ for all $i \in H_j$ such that $\omega \in \bigcap_{i \in H_j} F_i \subseteq E_j$. Then, $\omega \in \bigcap_{i \in \bigcup_{j \in J'} H_j} F_i = \bigcap_{j \in J'} \bigcap_{i \in H_j F_i} F_i \subseteq \bigcap_{j \in J'} E_j \subseteq E$. Since $\bigcup_{j \in J'} H_j$ is a non-empty subset of G, it follows that $\omega \in D_G(E)$. Conversely, assume $\omega \in D_G(E)$. Then, there are $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $E_i \in \mathcal{J}_{B_i}$ for all $i \in H$ such that $\omega \in \bigcap_{i \in H} E_i \subseteq E$. Now, let $J_H := \{j \in J \mid G_j \cap H \neq \emptyset\}$. For each $j \in J_H$, $\omega \in E^j := \bigcap_{i \in G_j \cap H} E_i \in \mathcal{J}_{D_{G_j}}$. This is because, for any $\omega' \in E^j$, $\omega' \in \bigcap_{i \in G_j \cap H} E_i \subseteq E^j$ with $E_i \in \mathcal{J}_{B_i}$ for each $i \in G_j \cap H$, and thus $\omega' \in D_{G_j}(E^j)$. Hence, $\omega \in \bigcap_{j \in J_H} E^j = \bigcap_{j \in J_H} \bigcap_{i \in G_j \cap H} E_i = \bigcap_{i \in H} E_i \subseteq E$. Thus, $\omega \in D^J_G(E)$.

For the third statement, Truth Axiom and Monotonicity hold by construction. Thus, I prove Positive Introspection. If $\omega \in D(E)$ then there is $F \in \mathcal{I}_G$ such that $\omega \in F \subseteq E$. Since $\omega \in F \subseteq F$ and $F \in \mathcal{I}_G$, I have $\omega \in F \subseteq D(F) \subseteq D(E)$. Thus, $\omega \in DD(E)$. □

Proof of Proposition 2. 1. (a) Suppose that, for each $\omega \in \Omega$, there is $F \in \mathcal{I}_G$ such that $\omega \in F$. Since $F \subseteq \Omega$, $\omega \in D_G(\Omega)$. Conversely, for any $\omega \in$

$\Omega = D_G(\Omega)$, there is $F \in \mathcal{I}_G$ such that $\omega \in F \subseteq \Omega$. If $B_i(\Omega) = \Omega$ then $\Omega \in \mathcal{J}_{B_i} \subseteq \mathcal{I}_G$, and D_G satisfies Necessitation as well.

(b) It suffices to show $D(\cdot) \subseteq K_{\mathcal{I}^*_G}(\cdot)$. If $\omega \in D(E)$, then there are $H \in \mathcal{P}(G)\setminus\{\emptyset\}$ and $E_i \in \mathcal{J}_{B_i}$ for each $i \in H$ such that $\omega \in \bigcap_{i\in H} E_i \subseteq E$. Now, letting $E_j = \Omega$ for each $j \in G \setminus H$, I have $\omega \in \bigcap_{i\in G} E_i \subseteq E$.

2. First, observe that if B_i satisfies Non-empty Conjunction then $\mathcal{J}_{B_i}$ is closed under non-empty intersection. If $\mathcal{E} \in \mathcal{P}(\mathcal{J}_{B_i}) \setminus \{\emptyset\}$, then $\bigcap \mathcal{E} \subseteq \bigcap_{E\in\mathcal{E}} B_i(E) \subseteq B_i(\bigcap \mathcal{E})$, i.e., $\bigcap \mathcal{E} \in \mathcal{J}_{B_i}$.

   Second, I show that D satisfies Non-empty Conjunction. Take $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$, and let $\omega \in \bigcap_{E\in\mathcal{E}} D(E)$. For each $E \in \mathcal{E}$, there is $F_E \in \mathcal{I}_G$ such that $\omega \in F_E \subseteq E$. Thus, $\omega \in \bigcap_{E\in\mathcal{E}} F_E \subseteq \bigcap \mathcal{E}$. To establish $\omega \in D\left(\bigcap \mathcal{E}\right)$, it is enough to show that $\bigcap_{E\in\mathcal{E}} F_E \in \mathcal{I}_G$. For each $E \in \mathcal{E}$, there are $H_E \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $F^i_E \in \mathcal{J}_{B_i}$ with $i \in H_E$ such that $F_E = \bigcap_{i\in H_E} F^i_E$. Letting $H = \bigcup_{E\in\mathcal{E}} H_E$, I show $\bigcap_{E\in\mathcal{E}} F_E = \bigcap_{i\in H} \bigcap \mathcal{F}_i$, where $\mathcal{F}_i := \{F^i_E \in \mathcal{J}_{B_i} \mid E \in \mathcal{E} \text{ satisfies } i \in H_E\}$. By the previous argument, $\bigcap \mathcal{F}_i \in \mathcal{J}_{B_i}$, and thus $\omega \in \bigcap_{E\in\mathcal{E}} F_E = \bigcap_{i\in H} \bigcap \mathcal{F}_i \in \mathcal{I}_G$.

   Suppose $\omega \in \bigcap_{i\in H} \bigcap \mathcal{F}_i$. For any $E \in \mathcal{E}$, I have $\omega \in \bigcap_{i\in H_E} F^i_E = F_E$. Conversely, suppose $\omega \in F_E = \bigcap_{i\in H_E} F^i_E$ for all $E \in \mathcal{E}$. Fix $i \in H$. For any $E \in \mathcal{E}$ with $i \in H_E$, I have $\omega \in \bigcap_{j\in H_E} F^j_E \subseteq F^i_E$. Thus, $\omega \in \bigcap \mathcal{F}_i$, and consequently $\omega \in \bigcap_{i\in H} \bigcap \mathcal{F}_i$.

   Third, to prove $K_{\mathcal{I}^*_G} = D$, it is enough show that $K_{\mathcal{I}^*_G}(\cdot) \subseteq D(\cdot)$. If $\omega \in K_{\mathcal{I}^*_G}(E)$, then there are $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $\mathcal{E} \subseteq \bigcup_{i\in H} \mathcal{J}_{B_i}$ such that $\omega \in \bigcap \mathcal{E} \subseteq E$. Since $\omega \in D(F)$ for each $F \in \mathcal{E}$ and since D satisfies Non-empty Conjunction, it follows $\omega \in \bigcap_{F\in\mathcal{E}} D(F) \subseteq D(\bigcap \mathcal{E}) \subseteq D(E)$.

   Finally, I provide a counterexample when agents' beliefs violate (Non-empty) Conjunction. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Let B_1 and B_2 be as in Table A.1. Let $G = \{1, 2\}$. Then, $D = D_G$, depicted in Table A.1, fails (Non-empty) Conjunction.

   I remark that this counterexample is also the one for the following variant of second part of Corollary 1 stating that D_G is the supremum of individual knowledge satisfying Truth Axiom, Monotonicity, and Positive Introspection (Formally, suppose that B_i satisfies Truth Axiom, Monotonicity, and Positive Introspection for every $i \in G$. Then, $D_G(\cdot) \subseteq K_G(\cdot)$ for any $K_G : \mathcal{D} \to \mathcal{D}$ satisfying Truth Axiom, Monotonicity, Positive Introspection, and $\bigcup_{i\in G} B_i(\cdot) \subseteq K_G(\cdot)$). In the example depicted in Table A.1, $K_G(\cdot) := K_{\bigcup_{i\in G} \mathcal{J}_{B_i}}(\cdot) = B_1(\cdot) \cup B_2(\cdot)$ satisfies $K_G(\cdot) \subsetneq D_G(\cdot)$.

3. Since B_i satisfies Conjunction (Necessitation and Non-empty Conjunction), similarly to the previous argument, $\mathcal{J}_{B_i}$ is closed under arbitrary intersection. I show that $\omega \in D(E)$ iff $\bigcap_{i\in G} \beta_{B_i}(\omega) \subseteq E$. Suppose that $\omega \in D(E)$. By (1b), there are $F_i \in \mathcal{J}_{B_i}$ for all $i \in G$ such that $\omega \in \bigcap_{i\in G} F_i \subseteq E$. Then,

E	$B_1(E)$	$B_2(E)$	$D(E)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_2\}$	$\emptyset$	$\emptyset$	$\{\omega_2\}$
$\{\omega_3\}$	$\emptyset$	$\emptyset$	$\{\omega_3\}$
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\emptyset$	$\{\omega_1, \omega_2\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\emptyset$	$\{\omega_1, \omega_3\}$
$\{\omega_2, \omega_3\}$	$\emptyset$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
Ω	Ω	Ω	Ω

Table A.1: Distributed knowlegde may fail (Non-empty) Conjunction

$\omega \in \bigcap_{i\in G} \beta_{B_i}(\omega) \subseteq \bigcap_{i\in G} F_i \subseteq E$. Conversely, suppose that $\bigcap_{i\in G} \beta_{B_i}(\omega) \subseteq E$. First, $\omega \in \bigcap_{i\in G} \beta_{B_i}(\omega)$. Second, since $\beta_{B_i}(\omega) \in \mathcal{J}_{B_i}$ (recall that $\mathcal{J}_{B_i}$ is closed under arbitrary intersection), $\bigcap_{i\in G} \beta_{B_i}(\omega) \in \mathcal{I}_G$. Thus, $\omega \in D(E)$.

Finally, I show that $b_D(\cdot) = \bigcap_{i\in G} \beta_{B_i}(\cdot)$ is reflexive and transitive by proving that each β_{B_i} is reflexive and transitive. By construction, β_{B_i} is reflexive. Next, let $\omega' \in \beta_{B_i}(\omega)$ and $\omega'' \in \beta_{B_i}(\omega')$. Take any $F \in \mathcal{J}_{B_i}$ with $\omega \in F$. Then, $\omega' \in F$ and consequently $\omega'' \in F$. Thus, $\omega'' \in \beta_{B_i}(\omega)$.

4. I show that β_{B_i} is the reflexive transitive closure of b_{B_i}. First, I have already shown that β_{B_i} is reflexive and transitive. Second, I show that $b_{B_i}(\cdot) \subseteq \beta_{B_i}(\cdot)$. If $\omega' \in b_{B_i}(\omega) = \bigcap\{E \in \mathcal{D} \mid \omega \in B_i(E)\}$, then for any $F \in \mathcal{D}$ with $\omega \in F \subseteq B_i(F)$, I have $\omega' \in F$. Third, let b' be a reflexive and transitive possibility correspondence satisfying $b_{B_i}(\cdot) \subseteq b'(\cdot)$. Thus, $B_{b'}(\cdot) \subseteq B_i(\cdot)$. Let $\omega' \in \beta_{B_i}(\omega)$ (i.e., $\omega' \in F$ if $\omega \in F \subseteq B_i(F)$). Now, for any $F \in \mathcal{D}$ with $\omega \in B_{b'}(F)$, I have $B_{b'}(F) \subseteq B_{b'}B_{b'}(F) \subseteq B_i B_{b'}(F)$, and $\omega' \in B_{b'}(F) \subseteq F$. Thus, $\omega' \in b'(\omega) = \bigcap\{F \in \mathcal{D} \mid \omega \in B_{b'}(F)\}$.

   (a) If B_i satisfies Positive Introspection, then b_{B_i} is transitive and $b_{B_i}(\omega) \cup \{\omega\}$ is the reflexive transitive closure of b_{B_i}.

   (b) If B_i satisfies Truth Axiom and Positive Introspection, then $b_{B_i} = \beta_{B_i}$.

□

Example A.1. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Table A.2 depicts B_1, B_2, and $D = D_{\{1,2\}}$. I have: $b_{B_1}(\omega_1) = b_{B_1}(\omega_2) = \{\omega_1\}$ and $b_{B_1}(\omega_3) = \emptyset$; and $b_{B_2}(\omega_1) = b_{B_2}(\omega_3) = \{\omega_1\}$ and $b_{B_2}(\omega_2) = \{\omega_3\}$. Since each β_{B_i} is the reflexive transitive closure of b_{B_i}, I obtain: $\beta_{B_1}(\omega_1) = \{\omega_1\}$, $\beta_{B_1}(\omega_2) = \{\omega_1, \omega_2\}$, $\beta_{B_1}(\omega_3) = \{\omega_3\}$; and $\beta_{B_2}(\omega_1) = \{\omega_1\}$, $\beta_{B_2}(\omega_2) = \Omega$, and $b_{B_2}(\omega_3) = \{\omega_1, \omega_3\}$. Then, I get: $\bigcap_{i\in G} b_{B_i}(\omega_1) = \{\omega_1\}$ and $\bigcap_{i\in G} b_{B_i}(\omega_2) = \bigcap_{i\in G} b_{B_i}(\omega_3) = \emptyset$; and $\bigcap_{i\in G} \beta_{B_i}(\omega_1) = \{\omega_1\}$, $\bigcap_{i\in G} \beta_{B_i}(\omega_2) = \{\omega_1, \omega_2\}$,

E	$B_1(E)$	$B_2(E)$	$D(E)$
$\emptyset$	$\{\omega_3\}$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	Ω	$\{\omega_1,\omega_3\}$	$\{\omega_1\}$
$\{\omega_2\}$	$\{\omega_3\}$	$\emptyset$	$\emptyset$
$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_2\}$	$\{\omega_3\}$
$\{\omega_1,\omega_2\}$	Ω	$\{\omega_1,\omega_3\}$	$\{\omega_1,\omega_2\}$
$\{\omega_1,\omega_3\}$	Ω	Ω	$\{\omega_1,\omega_3\}$
$\{\omega_2,\omega_3\}$	$\{\omega_3\}$	$\{\omega_2\}$	$\{\omega_3\}$
Ω	Ω	Ω	Ω

Table A.2: $\bigcap_{i\in G}\beta_{B_i}(\cdot)$ is not necessarily the reflexive transitive closure of $\bigcap_{i\in G} b_{B_i}(\cdot)$

and $\bigcap_{i\in G} b_{B_i}(\omega_3)=\{\omega_3\}$. Thus, $\bigcap_{i\in G}\beta_{B_i}$ is not the reflexive transitive closure $\bar{b}$ of $\bigcap_{i\in G} b_{B_i}$, as $\bar{b}(\omega)=\{\omega\}$ for each $\omega\in\Omega$. □

Proof of Proposition 3. 1. (a) By Positive Introspection, $B_i(E)\in\mathcal{J}_{B_i}$. If $\omega\in B_i(E)$ then $\omega\in B_i(E)\subseteq E$ by Truth Axiom, and thus $\omega\in D_{\{i\}}(E)$. By Proposition 1, $D_{\{i\}}(\cdot)\subseteq D_G(\cdot)$.

Next, I provide counterexamples when B_i violates Truth Axiom or Positive Introspection. First, let $(\Omega,\mathcal{D})=(\{\omega_1\},\{\emptyset,\{\omega_1\}\})$. Let $B_1(\cdot)=\{\omega_1\}$, violating Truth Axiom. Then, $D_{\{i\}}(E)=E$ for each $E\in\mathcal{D}$, and $B_1(\emptyset)=\{\omega_1\}\not\subseteq\emptyset=D_{\{1\}}(\emptyset)$.

Second, let $(\Omega,\mathcal{D})=(\{\omega_1,\omega_2\},\mathcal{P}(\{\omega_1,\omega_2\}))$. Define B_1 as follows: $B_1(\emptyset)=B_1(\{\omega_2\})=\emptyset$, $B_1(\{\omega_1\})=\{\omega_1\}$, and $B_1(\Omega)=\{\omega_2\}$. Since $B_1(\Omega)=\{\omega_2\}\not\subseteq\emptyset=B_1B_1(\Omega)$, B_1 violates Positive Introspection. The distributed knowledge operator $D_{\{1\}}$ satisfies: $D_{\{1\}}(\emptyset)=D_{\{1\}}(\{\omega_2\})=\emptyset$ and $D_{\{1\}}(\{\omega_1\})=D_{\{1\}}(\Omega)=\{\omega_1\}$. Now, $B_1(\Omega)=\{\omega_2\}\not\subseteq\{\omega_1\}=D_{\{1\}}(\Omega)$.

(b) If $\omega\in D_{\{i\}}(E)$, then there is $F\in\mathcal{J}_{B_i}$ such that $\omega\in F\subseteq E$. Since B_i is monotone, $\omega\in F\subseteq B_i(F)\subseteq B_i(E)$. I provide a counterexample. Let $(\Omega,\mathcal{D})=(\{\omega_1,\omega_2\},\mathcal{P}(\Omega)\})$. Let $B_1(E)=E$ if $E\neq\Omega$, and let $B_1(\Omega)=\emptyset$. Then, $D_{\{i\}}(E)=E$ for each $E\in\mathcal{D}$, and $B_1(\Omega)=\emptyset\subsetneq\Omega=D_{\{1\}}(\Omega)$.

(c) This part follows from the previous parts and Proposition 1.

2. If $\omega\in D(E)$, then there are $H\in\mathcal{P}(G)\setminus\{\emptyset\}$ and $F_i\in\mathcal{J}_{B_i}$ for each $i\in H$ such that $\omega\in\bigcap_{i\in H}F_i\subseteq E$. Then,

$$\omega\in\bigcap_{i\in H}F_i\subseteq\bigcap_{i\in H}D_{\{i\}}(F_i)\subseteq\bigcap_{i\in H}K_G(F_i)\subseteq K_G\left(\bigcap_{i\in H}F_i\right)\subseteq K_G(E).$$

□

Proof of Proposition 4. Suppose that each B_i satisfies Positive Introspection. Since $\{B_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\} \subseteq \mathcal{J}_{B_i}$, I have $\mathcal{I}_G^* \subseteq \mathcal{I}_G$, and thus $K_{\mathcal{I}_G^*}(\cdot) \subseteq D_F$.

Next, suppose that each B_i satisfies Truth Axiom. Take $\omega \in D_G(E)$. There are $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $E_i \in \mathcal{J}_{B_i}$ for all $i \in H$ such that $\omega \in \bigcap_{i \in H} E_i \subseteq E$. Since $\bigcap_{i \in H} E_i = \bigcap_{i \in H} B_i(E_i)$, it follows $\omega \in K_{\mathcal{I}_G^*}(E)$. □

Proof of Proposition 5. Here I prove $D_G = K_{\mathcal{I}_G^*}$ and $\mathcal{I}_G^* = \mathcal{J}_{D_G}$, where $\mathcal{I}_G^*$ is the smallest complete algebra including $\bigcup_{i \in G} \mathcal{J}_{B_i}$ (the proofs of the other assertions are in the main text).

First, I show $D_G = K_{\mathcal{I}_G^*}$. Since $\mathcal{I}_G \subseteq \mathcal{I}_G^*$, it follows that $D_G(\cdot) \subseteq K_{\mathcal{I}_G^*}(\cdot)$. Conversely, suppose $\omega \in K_{\mathcal{I}_G^*}(E)$. There is $F \in \mathcal{I}_G^*$ such that $\omega \in F \subseteq E$. Since $\mathcal{I}_G^*$ is the smallest complete algebra including $\bigcup_{i \in G} \mathcal{J}_{B_i}$ and since each $\mathcal{J}_{B_i}$ is closed under complementation, it can be seen that $F = \bigcap_{\lambda \in \Lambda} \bigcup_{\mu_\lambda \in M_\lambda} F_{\lambda,\mu_\lambda}$ with each $F_{\lambda,\mu_\lambda} \in \bigcup_{i \in G} \mathcal{J}_{B_i}$. Since D_G satisfies Necessitation, if $\Lambda = \emptyset$ then $F = E = \Omega$ and $\omega \in D_G(E)$. Thus, assume $\Lambda \neq \emptyset$. For each $\lambda \in \Lambda$, choose μ_λ and $i_\lambda \in G$ with $\omega \in F_{\lambda,\mu_\lambda} \in \mathcal{J}_{B_{i_\lambda}}$. Let $H = \{i_\lambda \in G \mid \lambda \in \Lambda\}$. For each $i \in H$, define $F_i := \bigcap_{\lambda \in \Lambda : i = i_\lambda} F_{\lambda,\mu_\lambda} \in \mathcal{J}_{B_i}$. Then, $\omega \in \bigcap_{i \in H} F_i \subseteq E$ and $\bigcap_{i \in H} F_i \in \mathcal{I}_G$. Thus, $\omega \in D_G(E)$.

Second, I show $\mathcal{I}_G^* = \mathcal{J}_{D_G}$. For any $E \in \mathcal{I}_G^*$, if $\omega \in E$ then $\omega \in E \subseteq E$ and thus $E \subseteq D_G(E)$. Hence, $\mathcal{I}_G^* \subseteq \mathcal{J}_{D_G}$. Conversely, take $E \in \mathcal{J}_{D_G}$. For any $\omega \in E$, $\omega \in D_G(E)$. Since Truth Axiom and Negative Introspection imply Necessitation, there are $E_i^\omega \in \mathcal{J}_{B_i} \subseteq \mathcal{I}_G^*$ for all $i \in G$ such that $\omega \in \bigcap_{i \in G} E_i^\omega \subseteq E$. Then, $E = \bigcup_{\omega \in E} \bigcap_{i \in G} E_i^\omega \in \mathcal{I}_G^*$. □

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