Formalizing Common Belief with No Underlying Assumption on Individual Beliefs*

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Abstract

This paper formalizes common belief among players with no underlying assumption on their individual beliefs. Especially, players may not be logically omniscient in that they do not believe logical consequences of their beliefs. The key idea is to use a novel concept of common bases: a common basis is an event such that, whenever it is true, every player believes its logical consequences. The common belief in an event obtains when a common basis implies the mutual belief in that event. In the paper, individual beliefs are represented as operators on a general set algebra so that they can be qualitative or probabilistic. If players' beliefs are assumed to be true, then common belief reduces to common knowledge. The formalization nests previous axiomatizations of common belief and common knowledge which have assumed players' logical monotonic reasoning. The paper also studies how common belief inherits properties of individual beliefs.

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1 Introduction

Notions of common belief and common knowledge play indispensable roles whenever multiple players interactively reason with each other. Informally, an event is common belief between Alice and Bob if they mutually believe it, they believe that they believe it, and so on *ad infinitum*. Common belief and common knowledge are scrutinized in

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diverse contexts: computer science (e.g., McCarthy et al. (1978)), game theory and social sciences (e.g., Aumann (1976) and Friedell (1969)), and logic and philosophy (e.g., Lewis (1969)). The foundational nature of these concepts lies in the very fact that a model of knowledge and belief itself is often implicitly assumed to be common belief (or common knowledge) among the players.

The purpose of this paper is to provide a formalization of common belief with no a priori assumption on individual beliefs. Individual beliefs can be contradictory, inconsistent, non-conjunctive, or non-monotonic. That is, a player may believe a contradiction, she may simultaneously believe an event and its negation, she can believe multiple events without believing its conjunction, or she fails to believe some logical consequences of her own beliefs. Especially, logical monotonicity is one of the major ingredients that renders players "rational," which is at the heart of what is referred to as the "logical omniscience" problem. A casual observation suggests that a person can know the rule of a zero-sum game like chess without knowing its optimal strategy. While the players of a game commonly believe the structure of the game and rationality of the (other) players, they may fail to capture logical implications of common belief in rationality such as eliminations of never-best-replies.

Yet, relaxing logical monotonicity while maintaining tractability is no easy task. Consider, for instance, a standard possibility correspondence model of interactive beliefs (e.g., Aumann (1976, 1999), Geanakoplos (1989), and Morris (1996)). Each player has her possibility correspondence, which associates, with each state of the world, the set of states that she considers possible. She believes an event at a state if the event contains the possibility set at that state. Thus, each player's belief at a state is summarized by one possibility set at the state. The notions of mutual and common beliefs are also represented by way of players' possibility correspondences. The tractability of the model hinges on the premise that players are logical reasoners.

Toward a step to analyzing strategic reasoning among players who lack sophisticated logical inferential ability, say, to deduce or compute optimal strategies, this paper provides a framework for capturing interactive beliefs, especially common belief, among them. The formalization of common belief in this paper can also justify the informal sense in which the structure of a game is common belief (or common knowledge) among the players, even if they are not logically omniscient. This paper also clarifies how much logical sophistication individual players need to have in order for common belief to satisfy given properties. To the best of my knowledge, this is the first paper to formalize common belief that does not resort to the premise that players are logical reasoners.[1]

[1]In the literature on interactive beliefs, exceptions are Lismont and Mongin (1994a, 2003). They weaken players' logical monotonicity by proposing an axiom which they call quasi-monotonicity in their syntactic framework. In decision theory, a pioneering and exceptional work is Lipman (1999). Also, Morris (1996) studies a player's belief from her preferences. In his model, roughly, a player's preference relation is a complete ordering if and only if her belief is closed under arbitrary conjunction and logical monotonicity. This is exactly the condition under which her belief is induced by a possibility correspondence. For an overview of logical omniscience problems in computer science

I represent common belief in a set-theoretical (i.e., semantic) framework so as to accommodate various notions of beliefs. If individual players' beliefs are true, then the formalization of common belief reduces to that of common knowledge. Players' beliefs can be probabilistic as well as qualitative. Both knowledge and belief can be analyzed at the same time.[2]

The framework has three components. The first is an underlying set of states of the world. Players reason about some aspects of the underlying state space.

The second component is the collection of events, which are subsets of the state space. Since an event represents a property about the underlying states, the collection of events determines the language available to players when they interactively reason with each other. Especially, logical (set-algebraic) conditions on the collection capture assumptions on depths of players' reasoning. For example, if players engage in any finite depths of interactive reasoning, then the collection of events forms an algebra of subsets of the state space. If players' beliefs are represented by probability measures, then the collection of events forms a σ-algebra. If there is no ordinal restriction on depths of reasoning as in possibility correspondence models, then the collection of events are closed under arbitrary set-algebraic operations. The formalization of common belief in this paper does not hinge on a particular assumption on players' depths of reasoning.

The third component is a player's belief operator, which associates, with each event E, the event that the player believes E.[3] Qualitative and quantitative features of individual beliefs and knowledge are incorporated into the corresponding properties of players' belief operators.[4]

To formalize the common belief operator, I start with defining a new notion that I call common bases. A common basis is a particular type of event known to be publicly evident in the literature (e.g., Milgrom (1981)). An event E is publicly evident if everybody believes E whenever E is true. A common basis is an event E such that everybody believes any logical implication of E whenever E is true. If players' beliefs are (quasi-)monotonic, then a common basis and a publicly-evident event coincide with each other.

Following the terminology by Lewis (1969), I use the notion of common bases literally as a "basis" for common belief in that a common basis is a piece of public information that players can utilize in making their collective inferences.[5] Specifically,

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and logic, see Fagin et al. (2003, Chapter 9) and the references therein.

[2]For example, in an extensive-form game with perfect information, each player forms her belief about the opponents' future plays while she has knowledge about past moves.

[3]The hidden assumption in such a semantic model of players' beliefs is that if two events E and F coincide (i.e., $E = F$) then the beliefs in E and F coincide, even though the denotations of E and F seem to be different. This paper supposes no other restriction on individual players' beliefs.

[4]A player's probabilistic beliefs can be represented by a collection of p-belief operators (Friedell, 1969; Monderer and Samet, 1989). A p-belief operator associates, with each event E, the event that the player assigns a degree of her belief at least p to E (she p-believes E).

[5]Lewis (1969) introduces common knowledge by a notion which he calls a "basis for common

I axiomatize common belief in a way such that the common belief in E is the largest common basis implying the mutual belief in E (i.e., everybody believes E). The idea behind maximality is related to the one that common belief (common knowledge) is formalized as the "infimum" of individual beliefs (knowledge).[6]

Common bases generate any chain of mutual beliefs without any reference to players' logical abilities. If there is a common basis F that implies the mutual belief in an event E, then F generates the mutual belief in the mutual belief in E by itself.[7] Not only does any chain of mutual beliefs hold, but also F implies the very fact that E is common belief. Consequently, if an event E is common belief, then it is common belief that E is common belief, that is, common belief satisfies positive introspection, without assuming logical monotonicity on individual players.

One of the main motivations of the literature behind axiomatizations of common belief (common knowledge) is to provide formalizations that satisfy positive introspection in that the iterative definition of the chain of mutual beliefs (mutual knowledge) may fail it (see, for example, Barwise (1989) and Lismont and Mongin (1994b)).[8] Another is the informal sense in which the structure of a game can be common belief (common knowledge). Gilboa (1988), in his syntactic model, incorporates the statement that the model is common knowledge. Using the fact that the common knowledge of a statement implies the common knowledge of the common knowledge of the statement, he derives the sense in which the model is commonly known.

Positive introspection can in fact be a consequence of the following logical monotonicity of common belief with respect to the implication of common belief. If an event E is common belief and if the common belief in E (not E itself) implies an event F, then the event F is common belief. While common belief may be non-monotone, it is always closed under its logical implication. Positive introspection is a particular case with F being the common belief in E itself. Conceptually, this result implies that if the structure of a game is commonly believed then the players have common belief in any statement derived from the common belief in the structure even if they are not fully logical. Also, if they commonly believe in their rationality, then they have common belief in any logical consequence. The previous formalizations such as the intuitive iterative one may fail this weaker logical monotonicity if the players are not

knowledge." The basis for common knowledge F "indicates to everybody that everybody has reason to believe F," and such F provides players with the chain of mutual knowledge. For Lewis' account of common knowledge, see also Cubitt and Sugden (2003) and Vanderschraaf (1998).

[6]Aumann (1976) defines common knowledge by the infimum of individual players' knowledge partitions. McCarthy et al. (1978) introduce a notion of "any fool" knows.

[7]Contrast this argument with the formalization of common belief by publicly-evident events. If a publicly-evident event F implies the mutual belief in E, then the analysts who assume logical monotonicity on the players can assert that the mutual belief in F (ensured by public evidence) implies the mutual belief in E. The idea of common bases builds this inference into themselves without assuming the players' logical abilities.

[8]As to the chain of mutual knowledge, I also remark that there are somewhat different but related contexts in which the limit of mutual knowledge may not necessarily be common knowledge. Examples include the e-mail game (Rubinstein, 1989) and rationalizability (Lipman, 1994).

logical. This paper shows that if one would like common belief to have this closure under its logical implication, then the formalization of common belief in this paper is the permissive one satisfying this desiderata.

This paper enables one to examine how each logical or introspective property of individual and mutual beliefs leads to the corresponding property of common belief without assuming logical monotonicity. Differently put, this paper can also clarify how much logical sophistication each player needs to have in order for common belief to inherit or satisfy given properties. For example, if mutual belief is monotonic (i.e., if everybody believes an event E and if E implies F, then everybody believes F), then so is common belief.

Common belief inherits truth axiom from mutual belief. Namely, if mutual belief is true (i.e., if everybody believes an event E then the event E is true), then common belief is also true. Especially, if every player's belief is assumed to be true (i.e., players' knowledge instead of their beliefs is analyzed), then common belief reduces to common knowledge. Under this assumption, an event E is common belief at a state ω if and only if (henceforth, sometimes abbreviated as iff) there is a common basis that is true at ω and that implies E.[9] This result implies that if mutual belief is true, then common belief (common knowledge) is monotonic irrespective of whether individual players' beliefs are monotonic. I also demonstrate that if individual players' beliefs are not monotonic or true, then common belief may be non-monotonic.

This paper generalizes the previous formalizations of common belief and common knowledge that have assumed logical monotonicity under weaker monotonicity conditions. The key observation is that an event E is common belief at a state ω if and only if there is a common basis F that is true at ω and that implies the mutual belief in E. When publicly-evident events are common bases, the formalization reduces to Monderer and Samet (1989). I also examine the "fixed point" characterization of the common belief in E as the maximal event F in a way such that F is true if and only if the conjunction of the mutual beliefs in E and F obtains. If every player's belief is (quasi-)monotonic and conjunctive (i.e., a player believes the conjunction of E and F whenever she believes E and F), then the formalization coincides with Friedell (1969), Halpern and Moses (1990), and Lismont and Mongin (1994a,b, 2003). The paper enables one to compare the relations among previous formalizations, for example, in terms of how each formalization presupposes players' logical abilities.

I also relate my formalization to other well-known definitions of common belief. While common belief implies any chain of mutual beliefs, if the iteration of mutual beliefs is a common basis, then the iterative definition reduces to common belief. If mutual belief is countably conjunctive (i.e., if everybody believes a countable number

[9] First, this generalizes such previous formalizations of common knowledge as Aumann (1999), Binmore and Brandenburger (1990), Brandenburger and Dekel (1987), Geanakoplos (1989), Nielsen (1984), and Shin (1993). Second, Appendix B studies common knowledge separately from common belief by calling E to be common knowledge at ω if there is a common basis that is true at ω and that implies E.

of events then everybody believes its conjunction) and if publicly-evident events are common bases, then common belief reduces to the iterative notion. In other words, individual players' full logical monotonicity is not necessary to characterize common belief by the iterative definition. Moreover, if players' beliefs are derived from possibility correspondences, then common belief is also induced from the transitive closure of the possibility correspondence that captures mutual belief.[10]

The paper is structured as follows. Section 2 provides a basic environment. Section 3 provides the main results. Section 3.1 formalizes common belief. Section 3.2 studies how it relates to individual beliefs. Section 3.3 examines how this definition generalizes the previous literature. Section 4 provides concluding remarks. The proofs are relegated to Appendix A. Appendix B separately studies common knowledge.

2 Underlying Framework

In order to define a framework which can accommodate various forms of qualitative or probabilistic beliefs, I begin with technical preliminaries on set algebras. For any infinite cardinal number κ and a set Ω, a subset $\mathcal{D}$ of the power set $\mathcal{P}(\Omega)$ is a *κ-complete algebra* (on Ω) if $\mathcal{D}$ is closed under complementation and under arbitrary union (and consequently intersection) of any sub-collection with cardinality less than κ. Denote the complement of E by E^c or $\neg E$. Also, I follow the conventions that $\emptyset = \bigcup \emptyset$ and that, with Ω being an underlying set, $\Omega = \bigcap \emptyset$. Hence, any κ-complete algebra on Ω contains $\emptyset$ and Ω. An $\aleph_0$-complete algebra is an algebra of sets, and an $\aleph_1$-complete algebra is a σ-algebra (where $\aleph_0$ is the least infinite cardinal and $\aleph_1$ is the least uncountable cardinal).[11] A subset $\mathcal{D}$ of $\mathcal{P}(\Omega)$ is an (∞-)*complete algebra* (on Ω) if $\mathcal{D}$ is closed under complementation and arbitrary union (and intersection).

I move on to providing the underlying framework for representing players' interactive beliefs. Let κ be an infinite cardinal or $\kappa = \infty$, and let I be a non-empty set of *players*. A *κ-belief space* (of I) is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (B_i)_{i \in I} \rangle$. First, the players are reasoning about some aspects of the underlying state space Ω. Second, their objects of reasoning (i.e., the collection of events) are represented by a κ-complete algebra $\mathcal{D}$ on the state space, where κ stands for the limitation (or the maximum lower bound) on the players' depths of reasoning in the sense that the analysts can always examine players' hierarchies of beliefs up to κ in the class of κ-belief spaces as a whole. For example, if $\kappa = \aleph_0$, then the players can engage in finite depths of interactive reasoning. If $\kappa = \aleph_1$, then the players interactively reason about their countable depths

[10]This is implicit in Aumann (1976)'s "reachability" condition. See also Halpern and Moses (1990), Hérves-Beloso and Monteiro (2013), Green (2012), and the references therein. Also, the transitivity of a possibility correspondence is associated with positive introspection (e.g., Binmore and Brandenburger (1990), Geanakoplos (1989), Morris (1996), and Shin (1993)).

[11]Technically, it is without loss to take an infinite regular cardinal κ (Meier, 2006, Remark 1). If an infinite cardinal κ is not regular then any κ-complete algebra is κ^+-complete, where the successor cardinal κ^+ is regular (assuming the axiom of choice). Note that $\aleph_0$ and $\aleph_1$ are regular.

of beliefs (e.g., they reason about the countable chain of mutual beliefs).[12] If $\kappa = \infty$ then there is no limitation on the players' depths of reasoning. Third, $B_i : \mathcal{D} \to \mathcal{D}$ is player i's *belief operator* for each $i \in I$. For each *event* $E \in \mathcal{D}$, the set $B_i(E)$ denotes the event that (i.e., the set of states at which) player i believes E. A player i believes an event E at a state ω if $\omega \in B_i(E)$.

A belief operator can represent qualitative belief and knowledge derived from a possibility correspondence. Players' probabilistic beliefs are also accommodated through p-belief operators on an $\aleph_1$-complete algebra (Friedell, 1969; Monderer and Samet, 1989). A player p-believes an event E at a state ω if her type at ω (which is a mapping from $\mathcal{D}$ into $[0,1]$) assigns a degree of her belief ("probability") at least $p \in [0,1]$ to the event E. Thus, I can accommodate a collection of the players' p-belief operators $(B_i^p)_{(i,p)\in I\times[0,1]}$.[13] While each player's type is usually a probability measure, one can consider p-belief operators induced by a general (especially, non-monotone) set function on an ($\aleph_0$-complete) algebra $\mathcal{D}$.[14] Moreover, one can introduce dynamic or conditional beliefs.[15]

The use of belief operators on a general set algebra makes it possible to analyze such diverse forms of beliefs and knowledge in a unified way. Here, I consider the following nine logical and introspective properties of belief and knowledge. Fix $\langle(\Omega, \mathcal{D}), (B_i)_{i\in I}\rangle$ and $i \in I$.

The first five properties describe logical abilities.

1. *No-Contradiction*: $B_i(\emptyset) = \emptyset$.

2. *Consistency*: $B_i(E) \subseteq (\neg B_i)(E^c)$ for any $E \in \mathcal{D}$.

3. *Monotonicity*: $E \subseteq F$ implies $B_i(E) \subseteq B_i(F)$.

4. *Necessitation*: $B_i(\Omega) = \Omega$.

5. *Non-empty λ-Conjunction* (where λ is a fixed infinite cardinal with $\lambda \leq \kappa$ or $\lambda = \kappa = \infty$): $\bigcap_{E\in\mathcal{E}} B_i(E) \subseteq B_i(\bigcap \mathcal{E})$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$.

[12] In the context of rationalizability, Lipman (1994) studies a game in which players may engage in transfinite levels of reasoning (eliminations of never-best-replies).

[13] Samet (2000, Theorem 2) provides conditions on p-belief operators under which they are derived from a type mapping from the state space to the set of types, where each type is a probability measure. Gaifman (1988) also establishes a related result.

[14] For example, Meier (2006) uses finitely-additive p-belief operators to establish a canonical representation of players' interactive beliefs. Zhou (2010) axiomatizes the properties of finitely-additive p-beliefs and studies its probability logic.

[15] First, Battigalli and Bonanno (1997) study knowledge and qualitative beliefs indexed by time. Second, Di Tillio, Halpern, and Samet (2014) provide conditions on p-belief operators which induce a conditional probability system (CPS) as in Battigalli and Siniscalchi (1999). This suggests that one could also accommodate lexicographic belief systems (LPSs) as in Blume, Brandenburger, and Dekel (1991a,b). See also Brandenburger, Friedenberg, and Keisler (2007), Halpern (2010), Tsakas (2014), and the references therein for relations between CPS and LPS.

First, No-Contradiction says that there is no state at which player i believes a contradiction in the form of $\emptyset$. Second, Consistency states that if player i believes E then she does not believe the negation of E. Third, Monotonicity provides the players with their logical inference ability. If player i believes E and if E implies F, then she believes F. One of the main purposes of this paper is to formalize common belief when the players fail Monotonicity. Fourth, Necessitation means that player i always believes a tautology in the form of Ω. Fifth, Non-empty λ-Conjunction states that if player i believes each of a collection of events (with cardinality less than λ) then she believes its conjunction. Empty Conjunction is identified as Necessitation. I also call Non-empty $\aleph_0$-Conjunction and Non-empty $\aleph_1$-Conjunction, respectively, *Finite Conjunction* and *Countable Conjunction*.

The next three properties are introspective properties.

6. *Truth Axiom*: $B_i(E) \subseteq E$ for any $E \in \mathcal{D}$.

7. *Positive Introspection*: $B_i(\cdot) \subseteq B_i B_i(\cdot)$.

8. *Negative Introspection*: $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$.

Truth Axiom distinguishes between knowledge and belief in that knowledge is assumed to satisfy Truth Axiom in the literature. It states that if player i "knows" E at ω then E is true at ω. Positive Introspection states that if player i believes E then she believes that she believes E. Negative Introspection states that if player i does not believe E then she believes that she does not believe E.

Call an event E to be *self-evident to* player i if $E \subseteq B_i(E)$. That is, i believes E whenever E is true. Denote by $\mathcal{J}_{B_i} := \{E \in \mathcal{D} \mid E \subseteq B_i(E)\}$ the collection of events self-evident to i. Positive Introspection is characterized as $\{B_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\} \subseteq \mathcal{J}_{B_i}$. Negative Introspection is captured by $\{(\neg B_i)(E) \in \mathcal{D} \mid E \in \mathcal{D}\} \subseteq \mathcal{J}_{B_i}$.

The last property, the Kripke property, provides the condition under which a player's belief is derived from a possibility correspondence $b_{B_i} : \Omega \to \mathcal{P}(\Omega)$ defined as

$$b_{B_i}(\omega) := \{\omega' \in \Omega \mid \omega' \in E \text{ for all } E \in \mathcal{D} \text{ with } \omega \in B_i(E)\} = \bigcap \{E \in \mathcal{D} \mid \omega \in B_i(E)\}.$$

The set $b_{B_i}(\omega)$ is regarded as the set of states (not necessarily the event) player i considers possible at ω. Now, the Kripke property states that player i believes E at ω if (and only if) E contains the set of states considered possible at ω.

9. *Kripke property*: for any $(\omega, E) \in \Omega \times \mathcal{D}$, $\omega \in B_i(E)$ if (and only if) $b_{B_i}(\omega) \subseteq E$.

If $\mathcal{D}$ is an ∞-complete algebra, then the Kripke property is equivalent jointly to Monotonicity, Non-empty ∞-Conjunction, and Necessitation (e.g., Morris (1996, Theorem 1) when $\mathcal{D} = \mathcal{P}(\Omega)$).

In the next section, I introduce the notion of common belief (and subsequently the common belief operator) among a group of players I in a κ-belief space, irrespective of assumptions on players' underlying beliefs. Henceforth, I assume that a given κ-belief space $\overrightarrow{\Omega}$ of I satisfies $|I| < \kappa$ so as to introduce the *mutual belief operator* $B_I : \mathcal{D} \to \mathcal{D}$ defined as $B_I(\cdot) := \bigcap_{i \in I} B_i(\cdot)$. The event $B_I(E)$ is the set of states at which every player in I believes E. The above nine logical and introspective properties can be defined analogously for the mutual (and common) belief operators.

To conclude this section, I remark that specifying one's belief operator is equivalent to specifying the collection of events that one believes at each state. Such a collection is referred to as a neighborhood system (or a Montague-Scott structure).[16] The formal equivalence is presented in Remark A.1 in Appendix A.

3 Formalization of Common Belief

3.1 Formalization

Throughout this subsection, fix a κ-belief space $\overrightarrow{\Omega}$ of I. I start with the auxiliary definition of a common basis. Call an event E a *common basis* if everybody believes any logical implication of E whenever E is true. Formally:

Definition 1. An event $E \in \mathcal{D}$ is a *common basis* (among I) if $E \subseteq B_I(F)$ for any $F \in \mathcal{D}$ with $E \subseteq F$. Denote by $\mathcal{J}_I$ the collection of common bases among I.

A common basis is a piece of public information that can be used for inferences even if some players are not logically omniscient. A common basis is a stronger form of a publicly-evident event (Milgrom, 1981), where an event E is publicly evident if it is self-evident to every i: $E \subseteq B_I(E)$, or, $E \in \bigcap_{i \in I} \mathcal{J}_{B_i}$.[17] While a common basis is publicly evident, the converse is not necessarily true (examples will be provided).

In the literature pioneered by Monderer and Samet (1989), an event E is common belief at a state ω if there is a publicly-evident event $F \in \bigcap_{i \in I} \mathcal{J}_{B_i}$ that is true at ω and that implies the mutual belief in E: $\omega \in F \subseteq B_I(E)$. When the players are not logical reasoners, however, how can they deduce chains of mutual beliefs from the event F? If F is a common basis, however, it yields any chain of mutual beliefs by itself regardless of players' reasoning abilities. Thus, call an event E to be *common belief (commonly believed)* at a state ω if there is a common basis F that is true at ω and that implies the mutual belief in E (i.e., $F \subseteq B_I(E)$).

Definition 2. For any $E \in \mathcal{D}$, the set of states at which E is common belief is:

$$\{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_I \text{ such that } \omega \in F \subseteq B_I(E)\}. \tag{1}$$

[16]See, for example, Fagin et al. (2003) and Pacuit (2017). Heifetz (1999) and Lismont and Mongin (1994a,b) use neighborhood systems in formalizing common belief and common knowledge.

[17]Publicly-evident events are also termed as self-evident events (Aumann, 1999), common truisms (Binmore and Brandenburger, 1990), public events (Geanakoplos, 1989), belief closed events (Lismont and Mongin, 1994a,b, 2003; Mertens and Zamir, 1985; Vassilakis and Zamir, 1993), evident knowledge events (Monderer and Samet, 1989), common information (Nielsen, 1984), and so forth.

I introduce two definitions. First, if publicly-evident events are common bases (i.e., $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$), then public evidence itself generates mutual beliefs. To study its implications such as how common belief reflects properties of individual and mutual beliefs, I call it the closure condition on the publicly-evident events.

Definition 3. The collection of publicly-evident events $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is *closed (with respect to common bases)* if $\bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$.

Second, at this level of generality, there is no result asserting that the set of states at which an event E is common belief is itself an event. Especially, if $\mathcal{D}$ is an ($\aleph_0$-complete) algebra of sets but fails to be an $\aleph_1$-complete algebra, then even the intuitive iterative common belief operator may not be well defined because players cannot reason about limits of their mutual beliefs. Suppose, for example, that the analysts would like to study common belief syntactically by finitary languages. Or suppose that players' beliefs are represented by finitely-additive (or non-additive) measures on an ($\aleph_0$-complete) algebra $\mathcal{D}$. In order for the players to reason about common belief, the analysts would need to introduce the common belief operator $C : \mathcal{D} \to \mathcal{D}$ as a primitive of a κ-belief space.

There is a well-defined common belief operator $C : \mathcal{D} \to \mathcal{D}$ that maps each event E to the event that E is common belief consistently with Expression (1) if (and only if) C satisfies, for each $E \in \mathcal{D}$,

$$C(E) := \max\{F \in \mathcal{D} \mid F \subseteq B_I(E) \text{ and } F \in \mathcal{J}_I\}, \tag{2}$$

where "max" is taken with respect to the set inclusion (i.e., on a partially ordered space $\langle \mathcal{D}, \subseteq \rangle$). In other words, $C(E)$ is a well-defined event if and only if $\mathcal{J}_I$ contains a $\subseteq$-maximal element included in $B_I(E)$. Since $\emptyset \in \mathcal{J}_I$, the set in the right-hand side of Expression (2) is always non-empty. Indeed, $\mathcal{J}_I = \{\emptyset\}$ iff $C(\cdot) = \emptyset$. While I almost always restrict attention to the entire player set I, one can analogously define the common belief operator C_G among a (non-empty) subset G of players. By definition, if $G_1 \subseteq G_2 \subseteq I$ then $C_{G_2}(\cdot) \subseteq C_{G_1}(\cdot)$.

If $\mathcal{D}$ is an ∞-complete algebra then $C(E)$ is the union of all $F \in \mathcal{J}_I$ with $F \subseteq B_I(E)$, and thus the common belief operator C is always well defined without incorporating it into a given ∞-belief space as a primitive. I will also show that, under certain logical assumptions on players' beliefs, common belief is well defined in an $\aleph_1$-belief space (i.e., on a measurable space with players' belief operators).

A *κ-belief space (with the common belief operator)* refers to $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, C \rangle$, where $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I} \rangle$ is a κ-belief space with $|I| < \kappa$ and where $\mathcal{D}$ is the common belief operator.

For the rest of this subsection, I provide general properties of the common belief operator C. I start with showing that common belief implies any chain of mutual beliefs. Let $B_I^\alpha(\cdot) := \mathrm{id}_{\mathcal{D}}$ for $\alpha = 0$. For a successor ordinal $\alpha = \beta + 1$, define $B_I^\alpha := B_I \circ B_I^\beta$. For a limit ordinal $\alpha (> 0)$, let $B_I^\alpha(\cdot) := \bigcap_{\beta : 1 \leq \beta < \alpha} B_I^\beta(\cdot)$. In any

κ-belief space, B_I^α is well defined if $|\alpha| < \kappa$. I show that (i) the common belief in an event E implies the mutual belief in E up to any ordinal level α with $1 \leq |\alpha| < \kappa$ and that (ii) if the chains of mutual beliefs themselves form a common basis then they induce common belief.

Proposition 1. *Let α be any ordinal with $1 \leq |\alpha| < \kappa$. Then $C(\cdot) \subseteq B_I^\alpha(\cdot)$. If $E \in \mathcal{D}$ satisfies $\bigcap_{\beta:1\leq\beta\leq\alpha} B_I^\beta(E) \in \mathcal{J}_I$, then $C(E) = \bigcap_{\beta:1\leq\beta\leq\alpha} B_I^\beta(E)$.*

Consider a κ-belief space with $\kappa \geq \aleph_1$ so that countable conjunctions of events are well defined. The first part of Proposition 1 implies that the common belief in E implies the countable chain of mutual beliefs in E, which is taken as the intuitive iterative notion of common belief B_I^ω where the superscript "ω" is (not a state but) the least infinite ordinal. For a given κ-belief space with $\kappa \geq \aleph_1$, an event E is common belief, in the iterative sense, among I at ω if, everybody (in I) believes E at ω, everybody believes that everybody believes E at ω, *ad infinitum.* The second part of Proposition 1 states, for example, that the iterative common belief $B_I^\omega(E)$ reduces to the common belief $C(E)$ when the iterative common belief $B_I^\omega(E)$ itself is a common basis.

As the example provided shortly in Remark 1 (or the one in Remark A.5 in Appendix A) shows, however, the assumption that the iterative common belief $B_I^\omega(E)$ itself is a common basis cannot be dropped even if it is publicly evident. In other words, if $B_I^\omega(E)$ is not a common basis then it may be a proper super-set of $C(E)$.

Two remarks are in order. First, I briefly remark on other iterative definitions of common belief in Remark A.2 in Appendix A. Second, Section 3.3 examines when this iterative definition B_I^ω characterizes common belief.

Next, the common belief operator C always satisfies Positive Introspection irrespective of individual players' beliefs. If an event E is common belief at a state ω, then there is a common basis F satisfying $\omega \in F \subseteq B_I(E)$. Since $C(E)$ is maximal, the common basis F implies the very fact that E is common belief. Since F is a common basis, however, $C(E)$ itself is common belief. The previous literature has shown that the iterative common belief may fail Positive Introspection (see, for example, Barwise (1989) and Lismont and Mongin (1994b)). Thus, the literature has provided alternative axiomatizations of common belief that satisfy Positive Introspection.

Remark 1. The following example shows, however, common belief can fail every property discussed in Section 2 except for Positive Introspection. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. For each $i \in I = \{1, 2\}$, let B_i be defined as in Table 1.

Note that, while B_I and C fail No-Contradiction in the example, consider B_i' with $B_i'(\emptyset) = \emptyset$ and $B_i'(E) = B_i(E)$ for any $E \neq \emptyset$. Since $\emptyset$ is always a common basis and since $B_I'(E) = B_I(E)$ for any $E \neq \emptyset$, the resulting common belief operator C' is given by $C'(\emptyset) = \emptyset$ and $C'(E) = C(E)$ for any $E \neq \emptyset$.

Next, while C may be non-monotone, common belief is logically monotone with respect to the implications of common belief. Positive Introspection can be under-

E	$B_1(E)$	$B_2(E)$	$B_I(E)$	$C(E)$	$(\neg C)(E)$	$C(\neg C)(E)$
$\emptyset$	Ω	Ω	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$
$\{\omega_1\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$
$\{\omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$
$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$
Ω	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$

Table 1: C fails all but for Positive Introspection

stood as a particular case. The common belief operator is also characterized as the maximum operator implying the mutual belief and having this property.

Proposition 2. *1. Let $E, F \in \mathcal{D}$ be such that $C(E) \subseteq F$. Then, $C(E) \subseteq C(F)$. In particular, C always satisfies Positive Introspection.*

2. Let $\tilde{C} : \mathcal{D} \to \mathcal{D}$ be such that (i) $\tilde{C}(E) \subseteq \tilde{C}(F)$ for any $E, F \in \mathcal{D}$ with $\tilde{C}(E) \subseteq F$ and that (ii) $\tilde{C}(\cdot) \subseteq B_I(\cdot)$. Then, $\tilde{C}(\cdot) \subseteq C(\cdot)$.

Suppose an event E denote "rationality" of the players, and suppose E is common belief at a state ω. For any implication F of the common belief in rationality, the formalization of this paper ensures that the players have the common belief in F even if they are not perfectly logical. More informally, suppose that the structure of a game is commonly believed. The players have common belief in any implication of the "common belief assumption" even if they are not fully logical.

As in the above discussion of the failure of Positive Introspection, the intuitive iterative definition may fail the logical monotonicity with respect to its implication. Indeed, Remark 1 provides such an example because, while $\{\omega_1\}$ is publicly evident, there is no mutual belief in some of its implications (e.g., Ω) at ω_1 (another example where B_I satisfies No-Contradiction and Necessitation is in Remark A.5 in Appendix A). This is related to the fact that Proposition 2 (2) still holds even when one replaces the second condition with any order of mutual beliefs $\tilde{C}(\cdot) \subseteq \bigcap_{\beta:1 \leq \beta \leq \alpha} B_I^{\beta}(\cdot)$. The key is that generally not all publicly-evident events are common bases.

Finally, a player i's individual belief B_i and her "common" belief among $\{i\}$ may differ because B_i may fail Positive Introspection. If i's belief is her common basis, i.e., $B_i(\cdot) \in \mathcal{J}_{\{i\}}(:= \{E \in \mathcal{D} \mid E \subseteq B_i(F)$ for any $F \in \mathcal{D}$ with $E \subseteq F\})$, then $B_i = C_{\{i\}}$ (see Remark A.3 in Appendix A).

3.2 Individual, Mutual, and Common Beliefs

I examine how common belief inherits logical and introspective properties from individual and mutual beliefs. I also clarify how logically sophisticated the players need to be in order for common belief to inherit logical and introspective properties. To that end, I introduce Quasi-Monotonicity (Lismont and Mongin, 1994a, 2003). The mutual belief operator B_I satisfies *Quasi-Monotonicity* if $E \subseteq B_I(E) \cap F$ implies $B_I(E) \subseteq B_I(F)$ for any $E, F \in \mathcal{D}$. Quasi-monotonicity can be defined for other operators in the same way. Monotonicity implies Quasi-Monotonicity. Also, if each B_i satisfies Quasi-Monotonicity then so does B_I.

If B_I satisfies Quasi-Monotonicity, then $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed. The converse, however, is not necessarily true (see Remark A.4 in Appendix A). This means that the closure condition on publicly-evident events is a weaker monotonicity condition.

I show how the logical properties of individual and mutual beliefs yield the corresponding properties of common belief.

Proposition 3. *1. If B_I satisfies No-Contradiction (resp. Consistency), then so does C.*

2. If B_I satisfies Quasi-Monotonicity (resp. Monotonicity), then so does C.

3. Every B_i satisfies Necessitation iff B_I satisfies it iff C satisfies it.

4. Let $\bigcap_{i \in I} \mathcal{J}_{B_i}$ be closed. If B_I satisfies Non-empty λ-Conjunction, then so does C.

For the first part of Proposition 3, if some B_i satisfies No-Contradiction and Consistency, respectively, then B_I and consequently C satisfy No-Contradiction and Consistency. The second part is closely related to Lismont and Mongin (2003, Proposition 4). The fourth part says that C does not necessarily inherit a conjunction property from B_I unless otherwise $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$. Note that if each B_i satisfies Non-empty λ-Conjunction then so does B_I, because $\bigcap_{E \in \mathcal{E}} B_I(E) = \bigcap_{E \in \mathcal{E}} \bigcap_{i \in I} B_i(E) = \bigcap_{i \in I} \bigcap_{E \in \mathcal{E}} B_i(E) \subseteq B_I(\bigcap \mathcal{E})$. In the example in Remark 1, both B_I and C fail properties referred in Proposition 3.

Moving on to introspective properties, common belief inherits Truth Axiom from mutual belief. It implies that whether the common belief operator C satisfies Monotonicity hinges also on whether individual players' beliefs are true.

Proposition 4. *Let B_I satisfy Truth Axiom. Then*

$$C(E) = \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_I \text{ such that } \omega \in F \subseteq E\} \text{ for each } E \in \mathcal{D}.$$

Especially, C satisfies Truth Axiom and Monotonicity.

Proposition 4 states that if any one player's belief satisfies Truth Axiom then the resulting common belief satisfies Truth Axiom, Monotonicity, and Positive Introspection. Especially, if players' beliefs are assumed to satisfy Truth Axiom (i.e., knowledge instead of belief is analyzed), then common belief turns out to be common knowledge.[18]

Propositions 3 and 4 imply that if B_I is either monotonic or true then C is monotonic. The example in Remark 1 demonstrates that C may be non-monotonic if B_I fails Monotonicity and Truth Axiom.

Next, I turn to Negative Introspection. Generally, as Colombetti (1993) provides a Kripke frame example (which assumes Monotonicity), Negative Introspection of individual beliefs does not necessarily imply that of C. When it comes to knowledge, however, common knowledge inherits Negative Introspection if publicly-evident events are closed.

Proposition 5. *Let each B_i satisfy Truth Axiom, and let $\bigcap_{i \in I} \mathcal{J}_{B_i}$ be closed. If each B_i additionally satisfies Negative Introspection, then C satisfies Negative Introspection.*

In Proposition 5, common belief fails to inherit Negative Introspection when $\mathcal{J}_I \neq \bigcap_{i \in I} \mathcal{J}_{B_i}$ even if individual players' beliefs are true. See Remark A.5 in Appendix A.

So far, I have examined how common belief inherits the logical and introspective properties of the individual and mutual beliefs. Next, I ask the sense in which common belief can be the "infimum" of players' beliefs. I show that, under the closure condition $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$, common belief can be seen as the "infimum" of players' beliefs in the following sense. Consider a hypothetical individual whose belief satisfies Positive Introspection, and suppose that any event she believes is mutually believed. Then, any event believed by such a hypothetical individual is common belief.

Proposition 6. *Let $\bigcap_{i \in I} \mathcal{J}_{B_i}$ be closed. Let $B : \mathcal{D} \to \mathcal{D}$ satisfy (i) Positive Introspection and (ii) $B(\cdot) \subseteq B_I(\cdot)$. Then, $B(\cdot) \subseteq C(\cdot) \subseteq B_I(\cdot)$.*

The closure condition $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$ relates common bases to the infimum (intersection) of self-evident events to the players by: $E \in \bigcap_{i \in I} \mathcal{J}_{B_i}$ if and only if $E \subseteq C(E)$. Generally, if $E \subseteq C(E)$ then $E \subseteq B_I(E)$, i.e., $E \in \bigcap_{i \in I} \mathcal{J}_{B_i}$. Conversely, if $E \in \bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$ then $\omega \in E$ implies $\omega \in E \subseteq B_I(E)$ and thus $\omega \in C(E)$. This idea generalizes the observation that common knowledge is associated with the publicly-evident events in the past literature.

[18] Proposition 4 generalizes the previous literature on common knowledge and common certainty (i.e., common p-belief with probability $p = 1$) such as: Aumann (1999), Binmore and Brandenburger (1990), Brandenburger and Dekel (1987), Geanakoplos (1989), Nielsen (1984), Shin (1993), and Vassilakis and Zamir (1993). Lismont and Mongin (2003) also studies a notion of common knowledge imposing Quasi-Monotonicity on individual players' beliefs.

3.3 Comparison with the Previous Literature

Here, I demonstrate that the definition of common belief in this paper nests the following previous characterizations. Proposition 7 examines the characterization by Monderer and Samet (1989) in terms of publicly-evident events (Expression (3)) and the "fixed-point" characterization by Friedell (1969), Halpern and Moses (1990), and Lismont and Mongin (1994a,b, 2003) (Expression (6)). Proposition 8 provides conditions under which the iterative definition B_I^{ω} characterizes common belief. Proposition 9 shows that if players' beliefs are induced from a possibility correspondence, then common belief is induced from the transitive closure of players' possibility correspondences.

Proposition 7. *Let $\bigcap_{i \in I} \mathcal{J}_{B_i}$ be closed, and let $E \in \mathcal{D}$. First,*

$$C(E) = \{\omega \in \Omega \mid \textit{there is } F \in \bigcap_{i \in I} \mathcal{J}_{B_i} \textit{ with } \omega \in F \subseteq B_I(E)\} \tag{3}$$

$$= \max\{F \in \mathcal{D} \mid F \subseteq B_I(E) \cap B_I(F)\}. \tag{4}$$

Second, if B_I satisfies Quasi-Monotonicity, then

$$C(E) = \max\{F \in \mathcal{D} \mid F = B_I(E) \cap B_I(F)\}. \tag{5}$$

Third, if B_I additionally satisfies Non-empty $\aleph_0$- (i.e, Finite) Conjunction, then

$$C(E) = \max\{F \in \mathcal{D} \mid F = B_I(E \cap F)\} = \max\{F \in \mathcal{D} \mid F \subseteq B_I(E \cap F)\}. \tag{6}$$

Proposition 7 elucidates the implicit assumptions on players' logical abilities in the previous literature. Remark A.6 in Appendix A discusses counterexamples for the proposition when the preconditions are violated. Generally, the left-hand side of Expression (3) is included in the right-hand side, which is the formalization by Monderer and Samet (1989, Proposition). When publicly-evident events are closed, Expression (3) generalizes Monderer and Samet (1989, Proposition) irrespective of (any other) assumptions on players' beliefs.

If B_I satisfies Quasi-Monotonicity, then $C(E)$ is characterized as the largest fixed point of $f_E(\cdot) := B_I(E) \cap B_I(\cdot)$. As the literature (e.g., Friedell (1969), Halpern and Moses (1990), and Lismont and Mongin (1994a,b, 2003)) formulates common belief and common knowledge in terms of a fixed point using a variant of Tarski's fixed point theorem, the largest event F satisfying $F \subseteq f_E(F)$ satisfies $F = f_E(F)$.

If mutual belief additionally satisfies Finite Conjunction, then the common belief in E is characterized as the greatest event F satisfying $F \subseteq B_I(E \cap F)$. Again, F turns out to be the largest fixed point $F = B_I(E \cap F)$, meaning that F holds (i.e., E is common belief) if and only if everybody believes the conjunction of E and F. This is the characterization by Halpern and Moses (1990).

I also relate the third part of Proposition 7 to Lismont and Mongin (1994a,b, 2003). They define the set of states at which an event E is common belief as

$$\{\omega \in \Omega \mid \text{there is } F \in \bigcap_{i \in I} \mathcal{J}_{B_i} \text{ with } \omega \in B_I(F) \text{ and } F \subseteq E\}. \tag{7}$$

Lismont and Mongin (2003, Propositions 2) show that if B_I is quasi-monotonic then their definition of common belief reduces to the largest fixed point F satisfying $F = B_I(E \cap F)$. Thus, the third part of Proposition 7 also shows that if B_I satisfies Quasi-Monotonicity and Finite Conjunction then the formalizations of common belief by Lismont and Mongin (1994a,b, 2003) and Monderer and Samet (1989) coincide.[19]

Next, I characterize weaker conditions under which the iterative definition fully characterizes common belief.

Proposition 8. *Let* $\overrightarrow{\Omega}$ *be a* κ*-belief space with* $\kappa \geq \aleph_1$ *and* $\bigcap_{i \in I} \mathcal{J}_{B_i}$ *closed.*

1. $C = B_I^{\omega}$ *iff* B_I^{ω} *satisfies Positive Introspection iff* $B_I^{\omega}(\cdot) \subseteq B_I B_I^{\omega}(\cdot)$.
2. *If* B_I *satisfies Countable Conjunction, then* $C = B_I^{\omega}$*. If* B_I *additionally satisfies Truth Axiom, then* $C(E) = B_I^{\omega}(E) \cap E$ *for all* $E \in \mathcal{D}$.

Proposition 8 also shows that, in a κ-belief space $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I} \rangle$ satisfying the preconditions, the players can possibly reason about common belief without introducing the common belief operator as a primitive of the κ-belief space. The intuitive definition B_I^{ω} fails to be the common belief operator C when it fails Positive Introspection $B_I^{\omega}(\cdot) \subseteq B_I^{\omega} B_I^{\omega}(\cdot) (= B_I^{\omega+\omega}(\cdot))$ or public-evidence $B_I^{\omega}(\cdot) \subseteq B_I B_I^{\omega}(\cdot) (= B_I^{\omega+1}(\cdot))$ by the very fact that the chain of mutual beliefs stops at the least infinite ordinal level.[20] If $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is not closed, then these two conditions may not lead to $C = B_I^{\omega}$. The example in Remark 1 ($B_I^{\alpha}(\{\omega_1\}) = \{\omega_1, \omega_2\}$) is such an example (Remark A.5 in Appendix A provides another one). Yet, Proposition 8 shows that in order for the iterative definition to characterize common belief, full logical monotonicity is not necessarily needed.

By Propositions 7 and 8, the iterative definition (B_I^{ω}), the fixed-point definition as well as the one by publicly-evident events, and my definition (Expression (2)) all agree as long as mutual belief satisfies Quasi-Monotonicity and Countable Conjunction.

Lastly, consider a κ-belief space $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I} \rangle$ with $\kappa \geq \aleph_1$ such that the mutual belief operator B_I satisfies the Kripke property. I show that the common belief operator is well defined and has the Kripke property.

[19] Lismont and Mongin (2003, Proposition 3) show that, letting B_I be quasi-monotonic, if E is common belief according to their definition (Expression (7)) then E is common belief in the sense of Monderer and Samet (1989) (i.e., the right-hand side of Expression (3)).

[20] While the first infinite ordinal chain of mutual beliefs is already hard to check in reality (Monderer and Samet, 1989), the definition of common belief as a chain of mutual beliefs is in fact not strong enough to capture introspective properties of common belief (see, for example, Barwise (1989) and Lismont and Mongin (1994b)).

Note first that B_I has the Kripke property if each B_i does so because $b_{B_I}(\cdot) = \bigcup_{i \in I} b_{B_i}(\cdot)$. By induction, each B_I^n has the Kripke property: $b_{B_I^n}(\cdot) = b_{B_I}^n(\cdot)$, where $b_{B_I}^1 := b_{B_I}$ and $b_{B_I}^n(\cdot) := \bigcup_{\omega' \in b_{B_I}(\cdot)} b_{B_I}^{n-1}(\omega')$ for $n \geq 2$.

In light of Proposition 8, let $b_C(\cdot) := \bigcup_{n \in \mathbb{N}} b_{B_I}^n(\cdot)$, i.e., $b_C(\cdot)$ is equal to the transitive closure of $b_{B_I}(\cdot)$. If each B_i has the Kripke property, then $b_C(\omega)$ is the set of states reachable from ω (Aumann, 1976). Namely, ω' is *reachable* from ω if there are sequences $(\omega^j)_{j=1}^m$ of Ω and $(i^j)_{j=1}^{m-1}$ of I with $m \in \mathbb{N} \setminus \{1\}$ such that $\omega = \omega^1$, $\omega' = \omega^m$, and $\omega^{j+1} \in b_{B_{i^j}}(\omega^j)$ for all $j \in \{1, \ldots, m-1\}$.

Now, there is an operator $C : \mathcal{D} \to \mathcal{D}$ such that $C(E) = \{\omega \in \Omega \mid b_C(\omega) \subseteq E\} = B_I^\omega(E)$. Moreover, C is the common belief operator with the Kripke property. Note that the transitivity of b_C is associated with Positive Introspection of C.[21]

Proposition 9. *Let $\langle(\Omega, \mathcal{D}), (B_i)_{i \in I}\rangle$ be a κ-belief space with $\kappa \geq \aleph_1$ such that B_I satisfies the Kripke property. Then, each B_I^n inherits the Kripke property and $b_{B_I^n} = b_{B_I}^n$. Moreover, $C : \mathcal{D} \to \mathcal{P}(\Omega)$ defined by $C(E) := \{\omega \in \Omega \mid b_C(\omega) \subseteq E\}$ is a well-defined common belief operator from $\mathcal{D}$ into itself satisfying the Kripke property, where $b_C(\cdot) := \bigcup_{n \in \mathbb{N}} b_{B_I}^n(\cdot)$. If each B_i satisfies the Kripke property, then $b_C(\omega) = \{\omega' \in \Omega \mid \omega'$ is reachable from $\omega\}$ for each $\omega \in \Omega$.*

4 Concluding Remarks

This paper formalized common belief without assuming any underlying properties on individual beliefs such as players' monotonic reasoning. Players' beliefs can be qualitative as well as quantitative, on a general set algebra. If beliefs are assumed to be correct, common belief becomes common knowledge. When individual beliefs are (quasi-)monotonic, the formalization of the paper reduces to the past literature. The key idea is to define common belief from common bases. The paper investigated the relation among individual, mutual, and common beliefs. While common belief may be non-monotonic, it is closed under its logical implication. The paper studied how much logical sophistication the players need to have in order for a given property of common belief to hold.

For avenues for further research, first, this paper would be useful for studying how various results regarding common knowledge and common belief hinge on (or are robust to) players' logical omniscience issues such as their logical reasoning abilities.

[21] On the one hand, the Kripke property of C states that the correspondence $b_C : \Omega \to \mathcal{P}(\Omega)$ is "(upper-hemi-)continuous" in that $C(E) = \{\omega \in \Omega \mid b_C(\omega) \subseteq E\} \in \mathcal{D}$ for any $E \in \mathcal{D}$. On the other hand, since the common belief operator is a primitive with which to analyze common belief of events, I allow each $b_C(\omega) = \bigcap\{E \in \mathcal{D} \mid \omega \in C(E)\}$ not to be an event. In a model in which players' ($\aleph_1$-)measurable and partitional possibility correspondences $b_i : \Omega \to \mathcal{D}$ are a primitive, Green (2012) shows that the common knowledge partition b_C is universally measurable if $(\Omega, \mathcal{D})$ is a measurable space induced by some Polish topology. See also, for example, Hérves-Beloso and Monteiro (2013) and the references therein for the measurability of the common knowledge partition.

This paper provides conditions under which common belief can be taken as the chain of mutual beliefs even if individual beliefs are not monotonic.

Second, it is interesting to implement the idea of this paper in the context of a logical (precisely, syntactical) system to formalize common knowledge and common belief without imposing individual players' monotonic reasoning.[22] A lattice-theoretical structure on the set of propositions would be a key to defining common belief.

The third is to formulate common belief in enriched domains where each player has collections of (p-)belief operators. Examples of enriched domains (where players retain probabilistic sophistication) include ambiguous beliefs (as in Ahn (2007), Bewley (1986), and Gilboa and Schmeidler (1989)) and lexicographic beliefs. Another instance would be non-standard state space models of (un)awareness as in Heifetz, Meier, and Schipper (2006, 2013). Possibility correspondences on their generalized state space induce knowledge operators on events that satisfy logical and introspective properties. Thus, the question is to formalize common belief from players' belief operators on such a generalized state space that do not necessarily satisfy logical or introspective properties. Their generalized state space consists of multiple sub-spaces, ranked by the degree to which each sub-space can describe different aspects of the world, and thus each event has an additional component indicating the subspace in which the event is described. Since events are ordered (by set inclusion and the ranking of subspaces), notions of self-evidence, public-evidence, and common bases could naturally be extended.

A Proofs

Remark A.1. A *neighborhood system* is a mapping $\mathcal{B}_i : \Omega \to \mathcal{P}(\mathcal{D})$ such that $B_{\mathcal{B}_i}(E) := \{\omega \in \Omega \mid E \in \mathcal{B}_i(\omega)\} \in \mathcal{D}$ for all $E \in \mathcal{D}$. A belief operator B_i induces the neighborhood system defined by $\mathcal{B}_{B_i}(\omega) := \{E \in \mathcal{D} \mid \omega \in B_i(E)\}$. Conversely, a neighborhood system $\mathcal{B}_i : \Omega \to \mathcal{P}(\mathcal{D})$ induces the belief operator $B_{\mathcal{B}_i}$. Belief operators and neighborhood systems are equivalent in the sense that $\mathcal{B}_{B_{\mathcal{B}_i}} = \mathcal{B}_i$ and $B_i = B_{\mathcal{B}_{B_i}}$.

Proof of Proposition 1. By definition, $C(\cdot) \subseteq B_I^1(\cdot)$. Suppose $C(\cdot) \subseteq B_I^\beta(\cdot)$. If $\omega \in C(E)$, then $\omega \in F \subseteq C(E) \subseteq B_I^\beta(E)$ for some $F \in \mathcal{J}_I$, and thus $\omega \in F \subseteq B_I^{\beta+1}(E)$. If $C(E) \subseteq B_I^\beta(E)$ for all $\beta < \alpha$, then $C(E) \subseteq \bigcap_{\beta:1\leq\beta<\alpha} B_I^\beta(E) = B_I^\alpha(E)$.

Next, since $\bigcap_{\beta:1\leq\beta\leq\alpha} B_I^\beta(E) \subseteq B_I(E)$, if $\bigcap_{\beta:1\leq\beta\leq\alpha} B_I^\beta(E) \in \mathcal{J}_I$ then $C(E) \subseteq \bigcap_{\beta:1\leq\beta\leq\alpha} B_I^\beta(E) \subseteq C(E)$. □

Remark A.2. Consider the following two iterative definitions of common belief. First, call an event E common belief among I at ω, if, for any finite sequence of

[22] Some syntactic formalizations of common belief and common knowledge requiring Monotonicity are: Bonanno (1996), Halpern and Moses (1992), and Lismont and Mongin (1994b). Heifetz (1999) allows infinitary languages as I allow $\mathcal{D}$ to be a κ-complete algebra. Lismont and Mongin (1994a, 2003) require Quasi-Monotonicity instead of Monotonicity.

E	$B_1(E)$	$B_2(E)$	$B_I(E)$	$C(E)$	$C_{\{1\}}(E)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_2\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_3\}$	$\{\omega_1\}$	Ω	$\{\omega_1\}$	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_1, \omega_2\}$
$\{\omega_1, \omega_3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
Ω	Ω	Ω	Ω	Ω	Ω

Table A.1: $B_i \neq C_{\{i\}}$ (Remark A.3); and $\bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$ while B_I Fails Quasi-Monotonicity (Remark A.4)

players $(i_1, i_2, \dots, i_n)$ in I, player i_1 believes that player i_2 believes that ... player i_n believes E at ω (i.e., $\omega \in (B_{i_n} \circ \cdots \circ B_{i_2} \circ B_{i_1})(E)$). If E is common belief at a state ω in the sense of $\omega \in C(E)$ then E is common belief at the state in this alternative sense. Under Monotonicity and (Non-empty) λ-Conjunction with $\lambda > |I|$, this iterative definition coincides with the iterative one B_I^{ω}.

Second, call an event E common belief at a state ω if, everybody believes E (i.e., $\omega \in B_I E$), everybody believes that E and everybody believes E (i.e., $\omega \in B_I(E \cap B_I E)$), everybody believes that E and everybody believes that E and everybody believes E (i.e., $\omega \in B_I(E \cap B_I(E \cap B_I E))$), and so forth *ad infinitum*. If B_I satisfies Non-empty $\aleph_0$- (i.e., Finite) Conjunction, then this alternative iterative common belief coincides with the iterative one B_I^{ω}. Or, if B_I satisfies Truth Axiom then the common belief $C(E)$ implies the common belief in E in this alternative sense.

Proof of Proposition 2.

1. Suppose $C(E) \subseteq F$. Since $C(E) \in \mathcal{J}_I$, I get $C(E) \subseteq B_I(F)$ and thus $C(E) \subseteq C(F)$.

2. Fix $E \in \mathcal{D}$, and take $\tilde{C}(E) \subseteq F$. Then, $\tilde{C}(E) \subseteq \tilde{C}(F) \subseteq B_I(F)$. Thus, $\tilde{C}(E) \in \mathcal{J}_I$. Since $\tilde{C}(E) \subseteq B_I(E)$, it follows $\tilde{C}(E) \subseteq C(E)$.

$\square$

Remark A.3. I show that $B_i(\cdot) \in \mathcal{J}_{\{i\}}$ ensures $B_i = C_{\{i\}}$. By definition, $C_{\{i\}}(\cdot) \subseteq B_i(\cdot)$. Conversely, if $\omega \in B_i(E)$ then $\omega \in B_i(E) \subseteq B_{\{i\}}(E)$ and $B_i(E) \in \mathcal{J}_{\{i\}}$. Thus, $\omega \in C_{\{i\}}(E)$. For a counterexample, let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$, and let B_i be as in Table A.1 for each $i \in I = \{1, 2\}$. Then, $B_1(\{\omega_3\}) \notin \mathcal{J}_{\{1\}} (= \mathcal{J}_{B_1})$ and $B_1(\{\omega_3\}) \neq C_{\{1\}}(\{\omega_3\})$.

Remark A.4. Consider the example in Table A.1. While $\mathcal{J}_I = \{\emptyset, \{\omega_2\}, \{\omega_2, \omega_3\}, \Omega\} = \bigcap_{i \in I} \mathcal{J}_{B_i}$, the mutual belief operator B_I fails Quasi-Monotonicity (take, for example, $E = \{\omega_2\}$ and $F = \{\omega_1, \omega_2\}$). Also, C violates Monotonicity.

Proof of Proposition 3. 1. First, by No-Contradiction of B_I, $C(\emptyset) \subseteq B_I(\emptyset) = \emptyset$. Second, by Consistency of B_I, $C(E) \cap C(E^c) \subseteq B_I(E) \cap B_I(E^c) = \emptyset$.

2. First, assume Quasi-Monotonicity on B_I. Let $E \subseteq C(E) \cap F$, and let $\omega \in C(E)$. Since $E \subseteq C(E) \cap F \subseteq B_I(E) \cap F$, there is $E' \in \mathcal{J}_I$ with $\omega \in E' \subseteq B_I(E) \subseteq B_I(F)$. Hence, $\omega \in C(F)$. Second, assume Monotonicity on B_I, and let $E \subseteq F$. If $\omega \in C(E)$ then there is $E' \in \mathcal{J}_I$ with $\omega \in E' \subseteq B_I(E) \subseteq B_I(F)$. Then, $\omega \in C(F)$.

3. If each B_i satisfies Necessitation, then B_I satisfies Necessitation. Now, $\Omega \in \mathcal{J}_I$ and $B_I(\Omega) = \Omega$, leading to $C(\Omega) = \Omega$. If C satisfies Necessitation then $\Omega \subseteq C(\Omega) \subseteq B_I(\Omega) \subseteq B_i(\Omega)$ for all $i \in I$.

4. Let $\mathcal{E}$ be a subset of $\mathcal{D}$ with $0 < |\mathcal{E}| < \lambda$, and suppose $\omega \in \bigcap_{E \in \mathcal{E}} C(E)$. For each $E \in \mathcal{E}$, there is $F_E \in \mathcal{J}_I$ such that $\omega \in F_E \subseteq B_I(E)$. Then, $\omega \in \bigcap_{E \in \mathcal{E}} F_E \subseteq B_I(\bigcap \mathcal{E})$. I have $\bigcap_{E \in \mathcal{E}} F_E \in \bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$ because $\bigcap_{E \in \mathcal{E}} F_E \subseteq \bigcap_{E \in \mathcal{E}} B_I(F_E) \subseteq B_I(\bigcap_{E \in \mathcal{E}} F_E)$. Hence, $\omega \in C(\bigcap \mathcal{E})$.

$\square$

Proof of Proposition 4. If there is $F \in \mathcal{J}_I$ such that $\omega \in F \subseteq E$, then $\omega \in F \subseteq B_I(E)$, and thus $\omega \in C(E)$. This follows without assuming Truth Axiom of B_I. Conversely, if $\omega \in C(E)$ then there is $F \in \mathcal{J}_I$ such that $\omega \in F \subseteq B_I(E) \subseteq E$, where the last set inclusion follows from Truth Axiom of B_I. Finally, Truth Axiom and Monotonicity follows from $C(E) = \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_I \text{ such that } \omega \in F \subseteq E\}$. $\square$

Proof of Proposition 5. It suffices to show $(\neg C)(E) \in \bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$, i.e., $(\neg C)(E) \in \mathcal{J}_{B_i}$ for each $i \in I$. Fix $i \in I$. Since B_i satisfies Truth Axiom, $B_i C(E) \subseteq C(E) \subseteq CC(E) \subseteq B_i C(E)$. Since B_i satisfies Negative Introspection, $(\neg C)(E) = (\neg B_i)C(E) \subseteq B_i(\neg B_i)C(E) = B_i(\neg C)(E)$, i.e., $(\neg C)(E) \in \mathcal{J}_{B_i}$. $\square$

Remark A.5. I provide a counterexample for Proposition 5 when $\mathcal{J}_I \neq \bigcap_{i \in I} \mathcal{J}_{B_i}$ even though each B_i satisfies Truth Axiom. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$, and consider B_i depicted in Table A.2 for each $i \in I = \{1, 2\}$. While each B_i satisfies Negative Introspection, C fails it. Notice that $\mathcal{J}_I = \{\emptyset, \{\omega_1, \omega_3\}, \Omega\} \subsetneq \{\emptyset, \{\omega_2\}, \{\omega_1, \omega_3\}, \Omega\} = \bigcap_{i \in I} \mathcal{J}_{B_i}$. Also, C satisfies Monotonicity even though B_I fails (Quasi-)Monotonicity.

Proof of Proposition 6. It suffices to show $B(\cdot) \in \{E \in \mathcal{D} \mid E \subseteq B(E)\} \subseteq \bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$. Indeed, it follows from Positive Introspection of B and $B(\cdot) \subseteq B_I(\cdot)$. $\square$

Proof of Proposition 7. First, Expression (3) follows because $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$. Next, I show Expression (4). Take any $F \in \mathcal{D}$ with $F \subseteq B_I(E) \cap B_I(F)$. Then, $F \in \bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$ and $F \subseteq B_I(E)$. Thus, $F \subseteq C(E)$. Conversely, by definition, $C(E) \subseteq B_I(E)$. Also, if $\omega \in C(E)$, then there is $F \in \mathcal{J}_I$ such that $\omega \in F \subseteq C(E)$. Since $F \in \mathcal{J}_I$, I have $\omega \in F \subseteq B_I(C(E))$, establishing $C(E) \subseteq B_I(C(E))$.

E	$B_1(E)$	$B_2(E)$	$B_I(E)$	$C(E)$	$(\neg C)(E)$	$C(\neg C)(E)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	Ω	Ω
$\{\omega_1\}$	$\{\omega_1\}$	$\emptyset$	$\emptyset$	$\emptyset$	Ω	Ω
$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\emptyset$	Ω	Ω
$\{\omega_3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	Ω	Ω
$\{\omega_1, \omega_2\}$	$\emptyset$	$\{\omega_2\}$	$\emptyset$	$\emptyset$	Ω	Ω
$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2\}$	$\emptyset$
$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\emptyset$	Ω	Ω
Ω	Ω	Ω	Ω	Ω	$\emptyset$	$\emptyset$

Table A.2: A Counterexample for Proposition 5 due to $\mathcal{J}_I \neq \bigcap_{i \in I} \mathcal{J}_{B_i}$

Second, I show that Quasi-Monotonicity of B_I implies Expression (5). This can be seen as a variant of Tarski's fixed point theorem, stating that the greatest event F satisfying $F \subseteq f_E(F) := B_I(E) \cap B_I(F)$, given that it exists, is the greatest fixed point of $f_E(\cdot)$. See also Lismont and Mongin (2003, Proposition 1).

To obtain Expression (5), I first show that f_E satisfies Quasi-Monotonicity. If $F \subseteq f_E(F) \cap F'$, then $F \subseteq B_I(E) \cap B_I(F) \cap F' \subseteq B_I(F) \cap F'$. Since B_I is quasi-monotonic, $B_I(F) \subseteq B_I(F')$. Thus, $f_E(F) = B_I(E) \cap B_I(F) \subseteq B_I(E) \cap B_I(F') = f_E(F')$.

Now, if $F \subseteq f_E(F) = f_E(F) \cap f_E(F)$ then $f_E(F) \subseteq f_E(f_E(F))$. If F is the largest event satisfying $F \subseteq f_E(F)$, then, since $f_E(F) \subseteq f_E(f_E(F))$, I also have $f_E(F) \subseteq F$.

Third, suppose that B_I satisfies Quasi-Monotonicity and Finite Conjunction. By Finite Conjunction, $B_I(E) \cap B_I(F) \subseteq B_I(E \cap F)$. I show that, by Quasi-Monotonicity, if $F \subseteq B_I(E \cap F)$ then $B_I(E \cap F) \subseteq B_I(E) \cap B_I(F)$. Take $F \in \mathcal{D}$ with $F \subseteq B_I(E \cap F)$. Since $E \cap F \subseteq B_I(E \cap F) \cap E$, I get $B_I(E \cap F) \subseteq B_I(E)$. Also, since $E \cap F \subseteq B_I(E \cap F) \cap F$, I have $B_I(E \cap F) \subseteq B_I(F)$. Thus, $B_I(E \cap F) \subseteq B_I(E) \cap B_I(F)$. $\square$

Remark A.6. I remark on counterexamples for Proposition 7. First, the example in Remark A.5 (i.e., Table A.2) is a counterexample for the first part of Proposition 7 due to $\mathcal{J}_I \neq \bigcap_{i \in I} \mathcal{J}_{B_i}$. It can be seen that the right-hand side of Expression (3) and Expression (4) coincide with $B_I(E)$ (thus, not necessarily $C(E)$) for a given $E \in \mathcal{D}$. Next, for the second part, let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2\}, \mathcal{P}(\Omega))$, and consider: $B_I(\emptyset) = \{\omega_1\}$, $B_I(\{\omega_1\}) = B_I(\{\omega_2\}) = \{\omega_2\}$, and $B_I(\Omega) = \Omega$. Then, $C(\emptyset) = \emptyset$, $C(\{\omega_1\}) = C(\{\omega_2\}) = \{\omega_2\}$, and $C(\Omega) = \Omega$. While $C(E) = \max\{F \in \mathcal{D} \mid F \subseteq B_I(E) \cap B_I(F)\}$ for every $E \in \mathcal{D}$, the event $\max\{F \in \mathcal{D} \mid F = B_I(E) \cap B_I(F)\}$ is not well defined when $E = \emptyset$. Third, Remark B.1 in Appendix B (i.e., Tables B.1 and B.2) provides counterexamples for the third part.

Proof of Proposition 8. 1. If $C = B_I^\omega$ then B_I^ω satisfies Positive Introspection. If B_I^ω satisfies Positive Introspection, then $B_I^\omega(\cdot) \subseteq B_I B_I^\omega(\cdot)$. If $B_I^\omega(\cdot) \subseteq B_I B_I^\omega(\cdot)$, then $B_I^\omega(\cdot) \in \bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$. Since $B_I^\omega(\cdot) \subseteq B_I(\cdot)$, $C(\cdot) \subseteq B_I^\omega(\cdot) \subseteq C(\cdot)$.

2. Fix $E \in \mathcal{D}$. By Proposition 1, $C(E) \subseteq B_I^\omega(E)$. Conversely, since B_I satisfies Countable Conjunction, $B_I^\omega(E) \subseteq \bigcap_{n\in\mathbb{N}} B_I(B_I^n(E)) \subseteq B_I(B_I^\omega(E))$. Thus, $B_I^\omega(E) \in \bigcap_{i\in I} \mathcal{J}_{B_i} = \mathcal{J}_I$. Since $B_I^\omega(E) \subseteq B_I(E)$, $B_I^\omega(E) \subseteq C(E)$.

□

Proof of Proposition 9. I only show that C is a well-defined common belief operator. Fix $E \in \mathcal{D}$. I have $B_I^\omega(E) \in \mathcal{J}_I = \bigcap_{i\in I} \mathcal{J}_{B_i}$ and $B_I^\omega(E) \subseteq B_I(E)$. Thus, if $b_C(\omega) \subseteq E$, then $\omega \in B_I^\omega(E) \subseteq B_I(E)$, and thus $\omega \in C(E)$. Conversely, if $\omega \in C(E)$ then $\omega \in B_I^n(E)$, i.e., $b_{B_I}^n(\omega) \subseteq E$, for all $n \in \mathbb{N}$. Thus, $b_C(\omega) \subseteq E$. □

B Formalization of Common Knowledge

In the main text, Proposition 4 shows that common belief inherits Truth Axiom from mutual belief. Also, the resulting notion of common belief reduces to that of common knowledge in the previous literature. To better understand the properties and relations of common belief and common knowledge, this section formalizes common knowledge separately from common belief.

The structure of this section parallels that of the main text. I begin with defining common knowledge. Next, I study the properties of common knowledge. Finally, I relate my formalization to those in the previous literature. The proofs are relegated to Section B.1.

I introduce the notion of common knowledge as knowledge induced by common bases $\mathcal{J}_I$. Call an event E to be *common knowledge* (among I) at a state ω if there is a common basis $F \in \mathcal{J}_I$ that is true at ω and that implies E. Thus, the set of states at which an event E is common knowledge is:

$$\{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_I \text{ such that } \omega \in F \subseteq E\}. \tag{B.1}$$

I make the following two observations on why this definition captures common knowledge. First, Expression (B.1) captures a notion of knowledge derived from information $\mathcal{J}_I$. At state ω, there is some information $F \in \mathcal{J}_I$ such that F is true at ω and that E is implied (or "proved") by F in the sense that $F \subseteq E$. Second, by the definition of common bases, the common knowledge of E entails the common belief in E. Proposition 4 states that if the mutual belief operator B_I satisfies Truth Axiom then Expression (B.1) coincides with $C(E)$.

I justify the definition of common knowledge by Expression (B.1) on several grounds. To do so, I first characterize the common knowledge operator. As in the main text, suppose that a given κ-belief space satisfies that, for each $E \in \mathcal{D}$,

$$C^*(E) := \max\{F \in \mathcal{D} \mid F \in \mathcal{J}_I \text{ and } F \subseteq E\} \in \mathcal{D}. \tag{B.2}$$

Then, $C^*(E)$ coincides with Expression (B.1). Henceforth, I assume that, for any given κ-belief space, C^* defined by Expression (B.2) is a well-defined operator from

$\mathcal{D}$ into itself. The common knowledge operator $C^* : \mathcal{D} \to \mathcal{D}$ can be axiomatized by the maximal event with the property that if E is common knowledge at ω then there is a common basis F that is true at ω and that implies E. Thus, $C^*(E)$ is the maximal common basis included in E. This implies that if C^* is well defined then so is C. If publicly-evident events are closed then this observation generalizes the idea in the past literature that the common knowledge of E is the maximal publicly-evident event included in E.

To better understand C^* as a common knowledge operator, I examine how it inherits properties of individual and mutual beliefs.

Proposition B.1.

1. *C^* satisfies Positive Introspection, Monotonicity, and Truth Axiom.*

2. *C^* satisfies Necessitation iff every B_i satisfies Necessitation.*

3. *Let $\bigcap_{i \in I} \mathcal{J}_{B_i}$ be closed. If B_I satisfies Non-empty λ-Conjunction, then so does C^*.*

4. *If $K : \mathcal{D} \to \mathcal{D}$ satisfies (i) Positive Introspection, (ii) Monotonicity, (iii) Truth Axiom, and (iv) $K(\cdot) \subseteq B_I(\cdot)$, then $K(\cdot) \subseteq C^*(\cdot)$.*

The first part of Proposition B.1 states that common knowledge, by construction, satisfies Monotonicity and Truth Axiom as well as Positive Introspection. None of these properties hinges on the property of common bases itself. Rather, the fact that C^* satisfies these properties comes from the way in which (common) knowledge is defined as a logical deduction from a collection of events $\mathcal{J}_I$ through Expression (B.1). The second and third parts say that, in terms of Necessitation and (Non-empty λ-) Conjunction, common knowledge and common belief have similar properties.

The fourth part formalizes the sense in which common knowledge can be the "infimum" of players' knowledge. Consider a hypothetical individual whose knowledge satisfies Positive Introspection, Monotonicity, and Truth Axiom. Suppose further that every event that the hypothetical individual "knows" is believed by every player. Then, any event that the hypothetical individual "knows" is common knowledge.

This fourth part, therefore, justifies the definition of common knowledge in the sense that C^* is the strongest knowledge operator that entails common belief. Suppose that $K : \mathcal{D} \to \mathcal{D}$ is a knowledge operator in the sense of satisfying Positive Introspection, Monotonicity, and Truth Axiom. If $K(\cdot) \subseteq C(\cdot)$, then $K(\cdot) \subseteq C^*(\cdot)$.

The three additional remarks are in order. First, consider Negative Introspection. If each B_i satisfies Truth Axiom and Negative Introspection, then it follows from Propositions 4 and 5 that $C^* = C$ satisfies Negative Introspection, provided $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed. Second, since C^* satisfies Truth Axiom, C^* satisfies No-Contradiction and Consistency.

Third, the collection of common bases and the notion of common knowledge induced by common bases are related through $\mathcal{J}_I = \{C^*(E) \in \mathcal{D} \mid E \in \mathcal{D}\} = \{E \in$

$\mathcal{D} \mid E \subseteq C^*(E)\}$. Conceptually, while C^* is induced from $\mathcal{J}_I$, the common knowledge operator C^*, in turn, reproduces the given collection $\mathcal{J}_I$. The proof goes as follows. If $E \in \mathcal{J}_I$ then $E = C^*(E) \in \{C^*(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$. Since $C^*(E) \subseteq C^*C^*(E)$, I have $\{C^*(E) \in \mathcal{D} \mid E \in \mathcal{D}\} \subseteq \{E \in \mathcal{D} \mid E \subseteq C^*(E)\}$. If $E \subseteq C^*(E)$, then $E = C^*(E) \in \mathcal{J}_I$.

Next, I compare the common knowledge operator C^* with the mutual belief operators B_I^α. Consider a κ-belief space with $\kappa \geq \aleph_1$. I get $C^*(E) \subseteq C(E) \cap E \subseteq B_I^\alpha(E)$ for any ordinal α with $0 \leq |\alpha| < \kappa$. If B_I satisfies Truth Axiom and Countable Conjunction and if $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed, then $C^*(E) = C(E) = B_I^\omega(E) \cap E$.

Next, I re-write the common knowledge operator C^* in terms of the largest fixed point of an operator $g_E(F) := B_I(F) \cap E$. For any κ-belief space with $\bigcap_{i \in I} \mathcal{J}_{B_i}$ closed, it follows from Propositions 4 and 7 that

$$C^*(E) = \max\{F \in \mathcal{D} \mid F \subseteq B_I(F) \cap E\}. \tag{B.3}$$

If B_I satisfies Quasi-Monotonicity, then

$$C^*(E) = \max\{F \in \mathcal{D} \mid F = B_I(F) \cap E\}.$$

Since $F \subseteq g_E(F) = B_I(F) \cap E = B_I(F) \cap (B_I(F) \cap E)$, I have $B_I(F) \subseteq B_I(B_I(F) \cap E)$, i.e. $g_E(F) \subseteq g_E(g_E(F))$. Since F is the largest event satisfying $F \subseteq g_E(F)$, $F = g_E(F)$.

When does common knowledge coincide with true common belief in that $C^*(E) = C(E) \cap E$ for all E (without assuming Truth Axiom of C)? On the one hand, $C^*(E) \subseteq C(E) \cap E$ for all $E \in \mathcal{D}$. The next proposition shows that the converse set inclusion also obtains if B_I satisfies Finite Conjunction and $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed. Also, Remark B.1 provides counterexamples when B_I fails Countable Conjunction or $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$.

Proposition B.2. *If $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed and B_I satisfies Finite Conjunction, then $C^*(E) = C(E) \cap E$ for all $E \in \mathcal{D}$.*

Finally, suppose that, for a given κ-belief space with $\kappa \geq \aleph_1$, the mutual belief operator B_I satisfies the Kripke property. I show that the reflexive transitive closure of b_{B_I} is the possibility correspondence that induces C^*.

Proposition B.3. *Let $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I} \rangle$ be a κ-belief space with $\kappa \geq \aleph_1$ and B_I satisfying the Kripke property. Then, $C^* : \mathcal{D} \to \mathcal{P}(\Omega)$ defined by $C^*(E) := \{\omega \in \Omega \mid b_{C^*}(\omega) \subseteq E\}$ is a well-defined common knowledge operator from $\mathcal{D}$ into itself satisfying the Kripke property, where $b_{C^*}(\omega) := b_C(\omega) \cup \{\omega\}$ and $b_C(\omega) := \bigcup_{n \in \mathbb{N}} b_{B_I}^n(\omega)$ for each $\omega \in \Omega$.*

B.1 Proofs

Proof of Proposition B.1.

1. Truth Axiom and Monotonicity hold by construction. Consider Positive Introspection. If $\omega \in C^*(E)$ then $\omega \in F \subseteq E$ for some $F \in \mathcal{J}_I$. Since $F \in \mathcal{J}_I$, $\omega \in F \subseteq C^*(F) \subseteq C^*(E)$. Thus, $\omega \in C^*C^*(E)$.

E	$B_1(E)$	$B_2(E)$	$B_I(E)$	$C(E)$	$C^*(E)$	$C(E) \cap E$	$C^{\mathrm{LM}}(E)$
$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\emptyset$	$\emptyset$	$\{\omega_1\}$
$\{\omega_1\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$
$\{\omega_2\}$	$\{\omega_1, \omega_3\}$	Ω	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\emptyset$	$\emptyset$	$\{\omega_1\}$
$\{\omega_3\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\emptyset$	$\emptyset$	$\{\omega_1\}$
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$
$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\emptyset$	$\{\omega_3\}$	$\{\omega_1\}$
Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω

Table B.1: Common Belief and Common Knowledge Operators C and C^*

2. If C^* satisfies Necessitation then $\Omega \subseteq C^*(\Omega) \subseteq B_i(\Omega)$ for all $i \in I$. If every B_i satisfies Necessitation, then $\Omega \in \mathcal{J}_I$. Thus, $C^*(\Omega) = \Omega$.

3. Let $\mathcal{E}$ be a subset of $\mathcal{D}$ with $0 < |\mathcal{E}| < \lambda$, and suppose $\omega \in \bigcap_{E \in \mathcal{E}} C^*(E)$. For each $E \in \mathcal{E}$, there is $F_E \in \mathcal{J}_I$ such that $\omega \in F_E \subseteq E$. Then, $\omega \in \bigcap_{E \in \mathcal{E}} F_E \subseteq \bigcap \mathcal{E}$. The statement follows because $\bigcap_{E \in \mathcal{E}} F_E \in \bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$.

4. First, $K(E) \in \mathcal{J}_I$ because, if $K(E) \subseteq F$ then $K(E) \subseteq KK(E) \subseteq K(F) \subseteq B_I(F)$. Next, if $\omega \in K(E)$ then $\omega \in K(E) \subseteq E$. Thus $\omega \in C^*(E)$.

$\square$

Proof of Proposition B.2. First, $C^*(E) \subseteq C(E) \cap E$ for all $E \in \mathcal{D}$. In order to establish the converse set inclusion, observe that

$$C(E) \cap E \subseteq B_I(C(E)) \cap B_I(E) \cap E \subseteq B_I(C(E) \cap E) \cap E,$$

where the second set inclusion follows from Finite Conjunction. Now, it follows from Expression (B.3) that $C(E) \cap E \subseteq C^*(E)$. $\square$

Remark B.1. I provide two examples where $C^*(E) \subsetneq C(E) \cap E$ for some $E \in \mathcal{D}$ (while B_I and C violate No-Contradiction in the two examples, $C^*(E) \subsetneq C(E) \cap E$ do not hinge on No-Contradiction as argued in the main text).

In the first example, B_I fails Finite Conjunction. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$, and let B_i be defined as in Table B.1 for each $i \in I = \{1, 2\}$. The mutual belief operator B_I in this example is from the proof of Lismont and Mongin (2003, Proposition 3). The mutual belief operator B_I satisfies Monotonicity and $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i} = \{\emptyset, \{\omega_1\}, \{\omega_1, \omega_3\}, \Omega\}$. The operator B_I fails Finite Conjunction. For example, $B_I(\{\omega_1, \omega_2\}) \cap B_I(\{\omega_1, \omega_3\}) = \{\omega_1, \omega_3\} \not\subseteq \{\omega_1\} = B_I(\{\omega_1\})$.

For $E = \{\omega_2, \omega_3\}$, $C^*(E) = \emptyset \subsetneq \{\omega_3\} = C(E) \cap E$. I also remark that $C \neq C^{\mathrm{LM}}$, where C^{LM} is defined as

$$C^{\mathrm{LM}}(E) := \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_I \text{ such that } \omega \in B_I(F) \text{ and } F \subseteq E\}. \quad \text{(B.4)}$$

E	$B_1(E)$	$B_2(E)$	$B_I(E)$	$C(E)$	$C^*(E)$	$C(E) \cap E$	$C^{\text{LM}}(E)$
$\emptyset$	Ω	Ω	Ω	Ω	$\emptyset$	$\emptyset$	Ω
$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\emptyset$	$\emptyset$	Ω
$\{\omega_2\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_2\}$	Ω
$\{\omega_3\}$	Ω	Ω	Ω	Ω	$\emptyset$	$\{\omega_3\}$	Ω
$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_2\}$	Ω
$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\emptyset$	$\emptyset$	Ω
$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	Ω
Ω	Ω	Ω	Ω	Ω	Ω	Ω	Ω

Table B.2: Common Belief and Common Knowledge Operators C and C^*.

Note that, since $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$, the event $C^{\text{LM}}(E)$ coincides with Expression (7).

This is also a counterexample for the third part of Proposition 7 due to the failure of Finite Conjunction. For $E = \{\omega_2\}$, $C(E) \neq \max\{F \in \mathcal{D} \mid F = B_I(E \cap F)\} = C^{\text{LM}}(E)$.

In the second example, let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$, and let B_i be as in Table B.2 for each $i \in I = \{1, 2\}$. While B_I satisfies Finite Conjunction, it fails $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$. I also remark that Expressions (7) and (B.4) satisfy $\Omega = B_I(\emptyset) \subseteq C^{\text{LM}}(\cdot)$.

Incidentally, this is also a counterexample for the third part of Proposition 7. For example, $\max\{F \in \mathcal{D} \mid F \subseteq B_I(\{\omega_2\} \cap F)\}$ and $\max\{F \in \mathcal{D} \mid F = B_I(\{\omega_1, \omega_3\} \cap F)\}$ are not well defined.

Proof of Proposition B.3. For each $E \in \mathcal{D}$, $C^*(E) = \{\omega \in \Omega \mid b_{C^*}(\omega) \subseteq E\} = \{\omega \in \Omega \mid b_C(\omega) \subseteq E\} \cap E \in \mathcal{D}$. Thus, $C^* : \mathcal{D} \to \mathcal{D}$ is well defined. Since the operator defined by $\{\omega \in \Omega \mid b_{C^*}(\omega) \subseteq E\} (\subseteq B_I(E))$ satisfies Truth Axiom, Positive Introspection, and Monotonicity, it follows that $\{\omega \in \Omega \mid b_{C^*}(\omega) \subseteq E\} \subseteq C^*(E)$. Conversely, if $\omega \in C^*(E)$ then $\omega \in C(E) \cap E$, and thus $b_{C^*}(\omega) \subseteq E$. $\square$

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