The Existence of Universal Qualitative Belief Spaces*

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Abstract

This paper establishes the existence of a canonical representation of players' interactive beliefs with a number of desirable features. Players' beliefs can be qualitative, truthful (i.e., knowledge), or probabilistic (e.g., countably-additive, finitely-additive, or non-additive). Players' logical and introspective properties can be specified one by one. The canonical model is the "largest" interactive belief model to which any particular model can be mapped in a unique belief-preserving way. The canonical model incorporates all possible ways in which players' interactive beliefs are described. Each state of the canonical model encodes players' interactive beliefs at that state within itself in a coherent manner.

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1 Introduction

Consider a group of players who reason interactively about unknown external values, *states of nature* S, such as the payoffs and strategies in a game. Each player must reason about states of nature and about each other's beliefs about states of nature, and so on. This paper constructs the first formal framework general enough to represent any conceivable form of interactive beliefs irrespective of nature of beliefs. In particular, beliefs can be probabilistic or qualitative including knowledge. An arbitrary structure will capture some possible aspects of players' interactive reasoning but will generally exclude others. I construct a sufficiently rich belief space that includes all possible forms of reasoning about beliefs. The construction enables analysis of formal questions regarding beliefs without ad hoc restrictions on the nature of reasoning. Claims regarding beliefs can be logically disassociated from extraneous structure that is not immediately related to interactive beliefs.

A model of beliefs (a belief space) consists of the following three ingredients. The first ingredient is a set Ω. Each element $\omega \in \Omega$ is a list of possible specifications of the prevailing state of nature $s \in S$ and players' interactive beliefs regarding states of nature S (i.e., their beliefs about states of nature S, their beliefs about their beliefs about S, and so on). Call each specification ω a state (of the world).

The second ingredient is the set of statements about which the players can reason. These statements, specified as subsets of states of the world Ω, are referred to as events. A belief space requires a description of the language available to the players, modeled as a collection of subsets of Ω which I call the domain.

The third ingredient is players' belief operators defined on the domain. For each event E, player i's belief operator assigns the set of states at which she believes E, i.e., the event that i believes E. Iterative applications of this operator generates higher-order interactive reasoning. One can represent various notions of qualitative or probabilistic beliefs by imposing properties of those beliefs on belief operators.

In applications, typically a specific model of beliefs is assumed a priori. This leaves open the possibility that some relevant aspects of reasoning are excluded. To address this, the main result of the paper (Theorem 1 in Section 3) is to demonstrate the existence of a universal belief space into which any belief space is embedded in a unique manner that maintains all the structure of that smaller space. That is, any form of reasoning in the smaller space can be retrieved in the universal space in a unique way. The existence result ensures that players' interactive beliefs in a strategic situation can be modeled by the belief space approach without neglecting any form of reasoning. Moreover, each state of the universal space specifies players' interactive beliefs at that state within itself (Proposition 1). Thus, the universal belief space leaves no relevant aspect of players' interactive beliefs unspecified. For any statement about players' interactive beliefs, the statement is true at some state of some belief space if and only if it is true at some state of the universal space; and the statement is true at all states of the universal space if and only if it is true at all states of any belief space (Proposition 2). I also prove that the space is complete in the sense that it includes all possible forms of reasoning (Proposition 3). It reveals what form of reasoning is indeed lost in the specific smaller space.

I construct a universal belief space under a variety of assumptions on players' logical and introspective abilities. My result is theoretically interesting in that the existence of a universal belief space is unrelated to assumptions on players' beliefs. For example, my paper reconciles the previous existence results on canonical probabilistic belief structures and the previous non-existence results on canonical knowledge (or more general qualitative belief) structures. At the same time, it is substantively interesting because I establish the canonical representation of beliefs even when players are less than "perfectly rational" in terms of their logical or introspective abilities.

My framework nests partitional (Aumann, 1976) and non-partitional possibility correspondence models of knowledge and qualitative belief by identifying the conditions on players' belief operators under which their beliefs are induced from information sets on the underlying states of the world. Each player's information set associated with a state represents the set of states she considers possible at that state. While a player in a partitional model is logically omniscient and is fully introspective about what she knows and what she does not know, a player in a non-partitional model may, for example, fail Negative Introspection—she does not know a certain event, and she does not know that she does not know it.[1] My framework also nests other forms of possibility correspondence models of qualitative beliefs which may

[1]Non-partitional models are motivated in part by notions of unawareness (e.g., Fagin and Halpern (1987), Modica and Rustichini (1994, 1999), and Schipper (2015)). The study of non-partitional models ranges from implications of common knowledge and common belief (e.g., Agreement theorems (Aumann, 1976)) to solution concepts in game theory. See, for example, Bacharach (1985), Binmore and Brandenburger (1990), Brandenburger, Dekel, and Geanakoplos (1992), Dekel and Gul (1997), Geanakoplos (1989), Morris (1996), Samet (1990), and Shin (1993).

not necessarily be truthful.[2] I can further relax players' logical abilities inherent in possibility correspondence models. For example, players may fail to believe logical consequences of their beliefs.

Qualitative beliefs may play an important role in characterizing solution concepts such as common belief in rationality in a game (Stalnaker, 1994) or especially in a game with ordinal payoff structures that do not admit probabilistic beliefs (Bonanno and Tsakas, 2018). One can also introduce qualitative beliefs as in Bjorndahl, Halpern, and Pass (2013) in the context of psychological games (Battigalli and Dufwenberg, 2009; Geanakoplos, Pearce, and Stacchetti, 1989) where players' interactive beliefs themselves enter into their preferences.

I establish the existence of a universal belief space in a way such that beliefs can be probabilistic as in type spaces (Harsanyi, 1967-1968): each player has a type mapping that associates, with each state, a probability distribution on the underlying states.[3] As Samet (2000) demonstrates the correspondence between a type mapping and a collection of p-belief operators (Monderer and Samet, 1989), the main result of this paper also implies the existence of a universal probabilistic belief space. Indeed, my framework admits a wide variety of assumptions on probabilistic beliefs by varying corresponding properties on p-belief operators. Intuitively, probabilistic beliefs can reduce to whether a player believes an event with probability p or not. Technically, my construction of a universal qualitative belief space follows the topology-free construction of a universal probabilistic type space by Heifetz and Samet (1998b).

My framework can endow players with both knowledge and belief on a general domain.[4] Indeed, the consideration of the domain of knowledge has often been neglected. Standard possibility correspondence models of knowledge allow any subset of states Ω to be an object of knowledge. Under such specification, knowledge and

[2]In the literature, knowledge is distinguished from belief in that a player can only know what is true while she can believe something false.

[3]The existence of a universal type/belief space is pioneered by Armbruster and Böge (1979), Böge and Eisele (1979), and Mertens and Zamir (1985) with certain topological assumptions on underlying nature states. Their topological constructions are extended by, for example, Brandenburger and Dekel (1993) and Pintér (2005).

[4]Such consideration would be needed (i) if one analyzes each player's knowledge about her own strategy and her beliefs about her opponents' strategies (e.g., Dekel and Gul (1997)) or (ii) if one analyzes players' knowledge about their past observed moves and their beliefs about past unobserved moves and future moves in an extensive-form game (e.g., Battigalli and Bonanno (1997)). In a continuous model, knowledge and probability-one belief would differ. For example, a player believes with probability one that a random number drawn from the interval $[0, 1]$ is irrational while she does not know it (Monderer and Samet, 1989).

probabilistic beliefs may be incompatible with each other when the knowledge of an event is not in the domain of the probability space. This incompatibility has been one of the issues that have hampered the epistemic analyses of players' knowledge and beliefs. Not only is my framework capable of capturing both knowledge and belief, but also the framework admits a universal space.

In conclusion, I provide a framework capable of capturing interactive beliefs in the following ways: (i) the framework contains a universal belief space; (ii) the framework admits a wide variety of assumptions on players' logical and introspective abilities, and thus various notions of qualitative and probabilistic beliefs and knowledge; and (iii) the framework allows one to specify the language that the players can use through selecting feasible domains. Moreover, the framework is amenable to tractable representations of interactive beliefs in the sense that it nests and generalizes previous interactive belief models such as possibility correspondences and type mappings.

The paper is organized as follows. The rest of this section provides a technical overview of the main result. Section 2 defines a belief space, properties of beliefs, and a universal belief space. Section 3 constructs a universal belief space (Theorem 1). Section 4 characterizes the universal belief space as the "largest" set describing players' interactive beliefs in a coherent manner. Section 5 discusses the main result. Section 5.1 discusses why the framework admits a universal knowledge space in contrast to the previous non-existence results. Section 5.2 discusses further applications such as probabilistic beliefs and dynamics. Proofs are relegated to Appendix A.

1.1 Technical Overview

Heifetz and Samet (1998a) demonstrate that a universal standard partitional knowledge space generically does not exist, where a standard partitional knowledge space allows any subset of underlying states of the world to be an object of knowledge. They show that, unlike σ-additive probabilistic beliefs, a non-trivial sequence of interactive knowledge can develop beyond any given ordinal. The negative results are also obtained by Fagin (1994), Fagin et al. (1999), Fagin, Halpern, and Vardi (1991), and Heifetz and Samet (1999). Moreover, Meier (2005) shows, by invoking Heifetz and Samet (1998a) above, that there is no universal qualitative belief (or knowledge) space even when qualitative belief is represented by a more general non-partitional possibility correspondence (i.e., a general Kripke frame). If there were a universal knowledge space in a class of such general knowledge spaces, then one could construct a universal partitional knowledge space from the given class, which is impossible.

How do my positive results reconcile with the negative results? What plays a crucial role in establishing a universal knowledge space (a universal belief space where beliefs are assumed to be truthful) is to specify a set algebra as objects of players' knowledge, i.e., a specification of the language that the players are allowed to use in their reasoning.

To see this point, let κ be an infinite cardinal number. Call a collection of subsets of underlying states Ω a *κ-algebra* (a shorthand for a κ-complete algebra) if it is closed under complementation and under union (and consequently intersection) of any sub-collection with cardinality less than κ. Thus, if a certain set is an object of knowledge, then so is its complement. If each of a collection of events is an object of knowledge, then so are its union and intersection, provided that the collection has cardinality less than κ. The power set of Ω is always κ-complete. For example, a κ-algebra subsumes an algebra of sets if κ is the least infinite cardinal number $\aleph_0$. A κ-algebra subsumes a σ-algebra if κ is the least uncountable cardinal number $\aleph_1$. Call a knowledge space (a belief space where players' beliefs are assumed to be truthful) a κ-knowledge space if its domain is a κ-algebra.

Specifying the domain of each knowledge space by a κ-algebra amounts to determining the language available to the players in reasoning about their interactive knowledge. Namely, any κ-knowledge space can capture players' interactive knowledge of a form, player i knows that player j knows that ..., up to the level of κ. For example, any $\aleph_0$-knowledge space can capture any finite level of players' interactive knowledge as $\kappa = \aleph_0$ is the least infinite cardinal number. Likewise, any $\aleph_1$-knowledge space can capture any countable level of players' interactive knowledge. Thus, κ-knowledge space can accommodate an infinite knowledge hierarchy of the form, Alice knows that Bob knows that Alice knows..., which would naturally emerge when one considers common knowledge among players (e.g., Aumann (1976, 1999)). Transfinite numbers of reasoning would also be necessary if one considers implications of common knowledge of rationality in a general setup (e.g., Lipman (1994)).

With this property in mind, the main theorem (Theorem 1 in Section 3) establishes that there is a universal κ-belief space by taking care of all the possible transfinite levels of interactive beliefs up to κ in a given class of κ-belief spaces. I do so for each class of belief spaces which respects given assumptions on players' beliefs. In particular, a universal κ-knowledge space exists when players' qualitative beliefs are assumed to be truthful.

My construction has the following three implications. First, I circumvent the above mentioned non-existence results by explicitly specifying a domain of qualitative belief (or knowledge) as a κ-algebra. On the one hand, the previously mentioned negative results imply that a sequence of interactive knowledge can generally develop beyond any depth of reasoning in a discontinuous way if any subset of states of the world is an object of knowledge. On the other hand, once I specify the language available to the players as a κ-algebra, any κ-knowledge space can generally take into consideration players' interactive knowledge up to the ordinality of κ.

Thus, I turn the previously mentioned negative results into the positive result in the following two ways. First, I enlarge a class of knowledge spaces by allowing the domain of a knowledge space to be a κ-algebra. Second, I find a universal κ-knowledge space by keeping track of all possible forms of reasoning up to the depth of κ attained in the given class of κ-knowledge spaces. Thus, unlike a universal (σ-additive) type space, my universal knowledge (or generally, qualitative belief) space usually has transfinite (precisely, κ) hierarchies of interactive knowledge (beliefs) incorporating all possible forms of interactive reasoning up to the depth of κ.[5]

Observe the following analogy with σ-additive beliefs. The domain of each type space is a σ-algebra because a σ-additive probability measure may not necessarily be defined on the power set. That is, the domain specification is implicitly incorporated in type spaces. Put differently, the domain of any σ-additive type space (σ-algebra) is the language available to the players in reasoning up to any countable form of interactive beliefs. The domain of any $\aleph_1$-qualitative-belief space (σ-algebra) is the language available to the players in reasoning up to any countable form of interactive qualitative beliefs. While I keep track of any form of interactive beliefs up to the ordinality of $\aleph_1$ in establishing a universal belief space, the continuity property of σ-additive beliefs (or more precisely, the continuity of the operation "Δ") guarantees that the least infinite depth of interactive beliefs can determine any subsequent countable order of players' interactive beliefs to establish a universal σ-additive type space (see, for example, Fagin et al. (1999) and Heifetz and Samet (1998b)).[6] To elaborate on this

[5]In decision theory, Lipman (1991) studies a canonical set consisting of transfinite sequences of decision procedures to pick decision procedures.

[6]Moss and Viglizzo (2004, 2006) reformulate and generalize σ-additive type spaces as coalgebras for a certain endofunctor F, which is related to the functor Δ. They show that a universal type

point further, suppose that players' beliefs are finitely additive. Meier (2006) shows, in a similar way to Heifetz and Samet (1998a), that a universal finitely-additive belief space does not exist if all subsets are required to be measurable (see also Fagin et al. (1999, Example 4.5)). On the other hand, Meier (2006) also shows that a universal finitely-additive belief space exists once players' beliefs are defined on a κ-algebra.

The second implication of my construction is that existence hinges on the specification of a domain rather than on assumptions on players' beliefs. Third, I assert that there is a universal partitional (non-partitional) κ-knowledge space if I explicitly specify domains of such partitional (non-partitional) κ-knowledge spaces.

More specifically, the existence result of a universal knowledge space is related to the following two previous positive results. First, Meier (2008) demonstrates the existence of a universal knowledge-belief space when players' knowledge operators operate only on given measurable subsets of the space on which players' probabilistic beliefs are defined. My framework nests Meier (2008) as a special class of $\aleph_1$-knowledge(-belief) spaces under his assumptions on players' knowledge which may not necessarily be induced from possibility correspondences. In addition to his result, this paper shows the existence of a universal κ-knowledge space for various notions of knowledge including the one that is induced from partitional possibility correspondences.

Second, Aumann (1999) constructs what he calls a canonical knowledge system (of a finitary epistemic $S5$ logic), where each state of the world is a "complete and coherent" set of formulas describing finite levels of players' interactive knowledge.[7] Theorem 2 in Section 4 provides an alternative construction of a universal qualitative belief (or knowledge) space by generalizing and modifying the idea of Aumann (1999)'s canonical knowledge system for any combination of assumptions on players' beliefs and for any domain (i.e., for any κ).

space (i.e., a terminal coalgebra) is expressed as the set of descriptions of each point (type profile together with a state of nature) in all coalgebras, endowed with measurable and coalgebra structures. Furthermore, their terminal coalgebra T is isomorphic to its image of the endofunctor $F(T)$, which establishes the "(belief-)completeness" of T (see Brandenburger (2003) and Brandenburger and Keisler (2006) for (belief-)completeness).

[7] Meier (2012) axiomatizes classes of belief/type spaces and shows that the space of all maximally consistent sets of formulas of his infinitary probability logic (i.e., the canonical space) is a universal space, which is isomorphic to the universal type space constructed by Heifetz and Samet (1998b). Zhou (2010) studies a canonical infinitary finitely-additive probability logic.

2 Belief Spaces

Throughout the paper, denote by I a non-empty set of players. Let S be a non-empty set of *states of nature*, endowed with a sub-collection $\mathcal{A}_S$ of the power set $\mathcal{P}(S)$. An element of S is regarded as a specification of the exogenous values (e.g., strategies and payoff functions) that are relevant to the strategic interactions among the players. Each element $E \in \mathcal{A}_S$, an *event of nature*, plays a role of a "proposition" regarding states of nature S about which players interactively reason.

In order to specify a language the players are allowed to use in making inferences about nature states S and their interactive beliefs, endow $\mathcal{A}_S$ with a "logical" (precisely, a set-algebraic) structure. To that end, I introduce the following three technical definitions. First, denote by κ an infinite cardinal number or a symbol ∞. Second, for an underlying set Ω, call a subset $\mathcal{D}$ of $\mathcal{P}(\Omega)$ or a pair $(\Omega, \mathcal{D})$ itself a *κ-complete algebra* (*κ-algebra*, for short) if $\mathcal{D}$ is closed under complementation and is closed under arbitrary union (and consequently intersection) of any sub-collection with cardinality less than κ (i.e, closed under *κ-union* and *κ-intersection*). For $\kappa = \infty$, $\mathcal{D}$ is closed under complementation and is closed under arbitrary union and intersection (i.e., an ∞-algebra is a *complete algebra*). I follow the conventions that $\emptyset = \bigcup \emptyset \in \mathcal{D}$ and that, with Ω being an underlying set, $\Omega = \bigcap \emptyset \in \mathcal{D}$. For example, an $\aleph_0$-algebra is an algebra of sets, because $\aleph_0$ is the least infinite cardinal. An $\aleph_1$-algebra is a σ-algebra, because $\aleph_1$ is the least uncountable cardinal. Third, denote by $\mathcal{A}_\kappa(\cdot)$ the smallest κ-algebra (i.e., the intersection of all κ-algebras) including a given collection.

Assume that a given set of nature states $(S, \mathcal{A}_S)$ is a κ-algebra for some fixed κ chosen by the outside analysts. Put differently, identify it with $(S, \mathcal{A}_\kappa(\mathcal{A}_S))$.[8] This assumption means the following: (i) if E is a nature event (i.e., an object of players' beliefs regarding nature states S), then so is its complement E^c (also denote it by $\neg E$); if each $E \in \mathcal{E}$ with $|\mathcal{E}| < \kappa$ is a nature event, then so are its union $\bigcup \mathcal{E} := \bigcup_{E \in \mathcal{E}} E$ and its intersection $\bigcap \mathcal{E} := \bigcap_{E \in \mathcal{E}} E$. As I will discuss later, κ turns out to determine possible depth of players' reasoning.

Two remarks on a κ-algebra $(S, \mathcal{A}_S)$ are in order. First, the outside analysts would assume $\kappa > |I|$ in order for the "language" to be fine enough to refer to mutual beliefs: statements regarding all the (possible subsets of) players. Second, as mentioned in

[8]For example, let $(S_i, \mathcal{A}_{S_i})$ be a κ-algebra for each $i \in I$, and let $S := \prod_{i \in I} S_i$. Then, let $\mathcal{A}_S$ be the product κ-algebra $\mathcal{A}_\kappa(\{\pi_i^{-1}(E_i) \in \mathcal{P}(S) \mid E_i \in \mathcal{A}_{S_i} \text{ for some } i \in I\})$, where $\pi_i : S \to S_i$ is the projection.

Meier (2006, Remark 1), it is without loss to restrict attention to κ-algebras for infinite regular cardinals κ or $\kappa = \infty$. If an infinite cardinal κ is not regular then any κ-algebra is a κ^+-algebra, where the successor cardinal κ^+ is known to be regular (supposing the axiom of choice). Note that $\aleph_0$ and $\aleph_1$ are regular.

2.1 Belief Spaces

I define a model of players' beliefs in which belief operators on some "sample space" induce players' interactive beliefs regarding $(S, \mathcal{A}_S)$.

Definition 1 (Belief Space). *A κ-belief space of I on $(S, \mathcal{A}_S)$ (a belief space, for short) is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta \rangle$ with the following three properties.*

1. *$(\Omega, \mathcal{D})$ is a κ-algebra. Call Ω the set of states of the world (the state space). Call each $E \in \mathcal{D}$ an event (of the world).*
2. *$B_i : \mathcal{D} \to \mathcal{D}$ is player i's belief operator for each $i \in I$. For each $E \in \mathcal{D}$, $B_i(E)$ denotes the event that (i.e., the set of states at which) player i believes E. A player $i \in I$ believes an event $E \in \mathcal{D}$ at a state $\omega \in \Omega$ if $\omega \in B_i(E)$.*
3. *$\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_S)$ is a measurable mapping: $\Theta^{-1}(E) \in \mathcal{D}$ for any $E \in \mathcal{A}_S$.*

By Condition (3), any set-algebraic ("logical") operations in $\mathcal{A}_S$ are preserved in the domain $\mathcal{D}$.[9] The mapping Θ can be regarded as a pair of mappings (Θ, Θ^{-1}) such (i) that Θ maps from Ω into S while Θ^{-1} maps $\mathcal{A}_S$ into $\mathcal{D}$ and (ii) that $\omega \in \Theta^{-1}(E)$ in $(\Omega, \mathcal{D})$ if and only if (iff, for short) $\Theta(\omega) \in E$ in $(S, \mathcal{A}_S)$.

While a standard partitional model assumes any subset of an underlying state space Ω to be an object of beliefs (i.e., $\mathcal{D} = \mathcal{P}(\Omega)$), my framework is more general and explicit about what players can reason. First, it is often desirable to capture players' probabilistic beliefs in addition to their knowledge (or qualitative beliefs). The framework of this paper allows for treating both knowledge and beliefs together on a κ-algebra (primarily, σ-algebra) without imposing the standard assumption (e.g., Aumann (1976)) that players' partitions are at most countable. Second, in the literature on logical foundations of state space models, events are generated by some

[9] In the setting of Footnote 8 in which κ-algebras $((S_i, \mathcal{A}_{S_i}))_{i \in I}$ are given, consider measurable maps $\Theta_i : (\Omega, \mathcal{D}) \to (S_i, \mathcal{A}_{S_i})$. Then, $\Theta : \Omega \ni \omega \mapsto \Theta(\omega) = (\Theta_i(\omega))_{i \in I} \in S$ is measurable by construction.

logical system, and thus the domain may only form a κ-algebra for some κ, depending on the given logical system.[10]

Next, I define properties of qualitative beliefs.

Definition 2 (Propeties of Beliefs). *Let $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta \rangle$ be a κ-belief space. Fix $i \in I$.*

1. *The following are logical properties of B_i.*
   (a) *No-Contradiction: $B_i(\emptyset) = \emptyset$.*
   (b) *Consistency: $B_i(E) \subseteq (\neg B_i)(E^c)$ for any $E \in \mathcal{D}$.*
   (c) *Monotonicity: $B_i(E) \subseteq B_i(F)$ for any $E, F \in \mathcal{D}$ with $E \subseteq F$.*
   (d) *Necessitation: $B_i(\Omega) = \Omega$.*
   (e) *Non-empty λ-Conjunction ($\lambda \leq \kappa$ is fixed): $\bigcap_{E \in \mathcal{E}} B_i(E) \subseteq B_i(\bigcap \mathcal{E})$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$.*[11]
2. *B_i satisfies the Kripke property if, for each $(\omega, E) \in \Omega \times \mathcal{D}$, $\omega \in B_i(E)$ if(f) $E \supseteq b_{B_i}(\omega) := \bigcap \{E \in \mathcal{D} \mid \omega \in B_i(E)\}$.*
3. *The following are introspective properties of B_i.*
   (a) *Truth Axiom: $B_i(E) \subseteq E$ (for any $E \in \mathcal{D}$).*
   (b) *Positive Introspection: $B_i(\cdot) \subseteq B_i B_i(\cdot)$.*
   (c) *Negative Introspection: $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$.*

No-Contradiction means that a player does not believe any contradiction in the form of the empty set. Consistency means that, if a player believes an event E then she does not believe its negation E^c. In other words, she does not believe E and E^c at the same time. Monotonicity says that if a player believes some event then she believes any of its logical consequences. Necessitation means that a player believes any form of tautology such as $E \cup E^c$ expressed as Ω. Non-empty λ-Conjunction implies that a player believes any non-empty conjunction of events (with cardinality less than λ) if

[10] Set-theoretical (semantic) models of knowledge and belief where events are based on propositions include such papers as Aumann (1999), Bacharach (1985), Samet (1990, 2010), and Shin (1993). In fact, it turns out later (in Section 3) that the domain of the universal belief space is generated by events corresponding to an "infinitary language" defined by nature and players' beliefs.

[11] If $\lambda = \kappa = \infty$, then there is no restriction on the cardinality of $\mathcal{E}$.

she believes each event. Necessitation is identified as the empty conjunction property since $\Omega = \bigcap \emptyset$.

The Kripke property provides the condition under which B_i is induced from the possibility correspondence $b_{B_i} : \Omega \to \mathcal{P}(\Omega)$. The information (or possibility) set $b_{B_i}(\omega) = \{\omega' \in \Omega \mid \omega' \in E \text{ for all } E \in \mathcal{D} \text{ with } \omega \in B_i(E)\}$ consists of states at which i considers possible at ω. The Kripke property can be thought of as a logical property of beliefs.

Truth Axiom says that a player can only "know" what is true. Truth Axiom distinguishes belief and knowledge in the sense that belief can be false while knowledge has to be true. Positive Introspection states that if a player believes some event then she believes that she believes it. Negative Introspection states that if a player does not believe some event then she believes that she does not believe it.

So far, this section has defined belief spaces (Definition 1) and the properties of belief operators (Definition 2). A belief space resides in a given class of belief spaces satisfying given assumptions on players' beliefs. To conclude, four remarks are in order. First, the framework can admit different properties of beliefs for different players. Players may also have multiple kinds of "belief" operators.[12] Second, some property of beliefs implies another.[13]

Third, my framework nests possibility correspondence models. Under the Kripke property, each introspective property can be expressed as the corresponding property of the possibility correspondence. First, B_i satisfies Truth Axiom iff b_{B_i} is reflexive (i.e., $\omega \in b_{B_i}(\omega)$ for all $\omega \in \Omega$). Second, B_i satisfies Positive Introspection iff b_{B_i} is transitive (i.e., if $\omega' \in b_{B_i}(\omega)$ then $b_{B_i}(\omega') \subseteq b_{B_i}(\omega)$). Third, B_i satisfies Negative Introspection iff b_{B_i} is Euclidean (i.e., if $\omega' \in b_{B_i}(\omega)$ then $b_{B_i}(\omega) \subseteq b_{B_i}(\omega')$). Thus, b_{B_i} forms a partition iff B_i satisfies Truth Axiom, Positive Introspection, and Negative Introspection. Likewise, b_{B_i} is reflexive and transitive iff B_i satisfies Truth Axiom and Positive Introspection.

Fourth, in order to accommodate Truth Axiom, the state space Ω in a belief space may not necessarily have a product structure. A type space Ω (as opposed to a belief space of Mertens and Zamir (1985)) is typically the product of the nature states

[12] Let $\{0, 1\} \times I$ be the set of players, where player i's knowledge operator (which satisfies Truth Axiom) is given by $K_i := B_{(0,i)}$ while her qualitative belief operator is given by $B_i := B_{(1,i)}$.

[13] For instance, Truth Axiom implies No-Contradiction and Consistency. Consistency and Necessitation jointly imply No-Contradiction. Negative Introspection together with Truth Axiom imply Positive Introspection (e.g., Aumann (1999, p. 270)).

$(S, \mathcal{A}_S)$ and players' type spaces $((T_i, \mathcal{T}_i))_{i \in I}$, all of which form a κ-algebra. Thus, Ω is the Cartesian product $\Omega = S \times \prod_{i \in I} T_i$ and $\mathcal{D}$ is the product κ-algebra. At each state $\omega = (s, (t_i)_{i \in I})$, player i's beliefs are assumed to depend only on t_i. Thus, for any event $E \in \mathcal{D}$, her belief $B_i(E)$ would satisfy $B_i(E) = S \times E^i \times \prod_{j \in I \setminus \{i\}} T_j$ for some $E^i \in \mathcal{T}_i$. Such B_i would often violate Truth Axiom.

2.2 A Universal Belief Space

The main objective of the paper (Theorem 1 in Section 3) is to demonstrate the existence of a universal κ-belief space for any infinite (regular) cardinal κ and for any combination of assumptions on players' beliefs. To that end, I define a universal belief space in a given class of belief spaces. It is a belief space to which every belief space in the given class is uniquely mapped in a belief-preserving manner. I start by formalizing the notion of a belief-preserving mapping, a belief morphism.

Definition 3 (Belief Morphism)**.** *Let* $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta \rangle$ *and* $\overrightarrow{\Omega'} = \langle (\Omega', \mathcal{D}'), (B'_i)_{i \in I}, \Theta' \rangle$ *be belief spaces of a given class. A (belief) morphism* $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ *is a measurable mapping* $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ *satisfying: (i)* $\Theta' \circ \varphi = \Theta$*; and (ii)* $B_i(\varphi^{-1}(E')) = \varphi^{-1}(B'_i(E'))$ *for each* $(i, E') \in I \times \mathcal{D}'$*.*

Condition (i) requires that the same nature state prevail for two associated belief spaces. By Condition (ii), players' beliefs are preserved from one space to another in that player i believes an event E' at $\varphi(\omega)$ iff she believes $\varphi^{-1}(E')$ at ω.

For any belief space $\overrightarrow{\Omega}$, the identity map $\mathrm{id}_\Omega : \Omega \to \Omega$ is a morphism from $\overrightarrow{\Omega}$ into itself. Denote by $\mathrm{id}_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega}$ the identity (belief) morphism. Next, call a belief morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ a *(belief) isomorphism*, if there is a morphism $\psi : \overrightarrow{\Omega'} \to \overrightarrow{\Omega}$ such that $\psi \circ \varphi = \mathrm{id}_{\overrightarrow{\Omega}}$ and $\varphi \circ \psi = \mathrm{id}_{\overrightarrow{\Omega'}}$. In other words, a morphism φ is an isomorphism if φ is bijective and its inverse φ^{-1} is a morphism. If φ is an isomorphism then its inverse φ^{-1} is unique. Belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ are *isomorphic*, if there is an isomorphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$.

Now, I define a universal belief space. It "includes" all belief spaces in that any belief space can be mapped uniquely to the universal space by a morphism.

Definition 4 (Universal Belief Space)**.** *Fix a class of* κ*-belief spaces of* I *on* $(S, \mathcal{A}_S)$*. A belief space* $\overrightarrow{\Omega^*}$ *is universal if, for any belief space* $\overrightarrow{\Omega}$*, there is a unique morphism* $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$*.*

Fix an infinite cardinal κ (or $\kappa = \infty$), a non-empty set of players I, a κ-algebra of nature states $(S, \mathcal{A}_S)$, and assumptions on players' beliefs. Then, any composite of two morphisms is a morphism; composites of morphisms are associative; and an identity morphism satisfies the identity law. Thus, the collection of all κ-belief spaces of I on $(S, \mathcal{A}_S)$ forms a *category*, where each belief space $\overrightarrow{\Omega}$ is an *object* and a belief morphism is a *morphism*. In the language of category theory, a universal belief space is a *terminal (final)* object in the category of belief spaces. Thus, a universal belief space could also be called a terminal belief space (see also Brandenburger and Keisler (2006, Section 11)). However, I stick to the terminology, *universal*, throughout the paper. As it is well known in category theory that a terminal object is unique up to isomorphism, a universal belief space is unique up to belief isomorphism.

3 Construction of a Universal Belief Space

Throughout this section, fix a non-empty set of players I, an infinite regular cardinal κ, a κ-algebra $(S, \mathcal{A}_S)$ of nature states, and assumptions on players' beliefs. A belief space refers to a κ-belief space of I on $(S, \mathcal{A}_S)$ in the given category.

I demonstrate the existence of a universal (κ-)belief space by employing the "expressions-descriptions" approach (Heifetz and Samet (1998b) and Meier (2006, 2008)). I do so without imposing any restriction on a κ-algebra $(S, \mathcal{A}_S)$.[14]

The construction of a universal belief space consists of six steps.[15] The first step is to inductively define *expressions*, syntactic formulas that express events defined solely in terms of nature and players' interactive beliefs. Any event of nature $E \in \mathcal{A}_S$ is an object of beliefs, so that any such E is an expression. Since objects of beliefs are closed under κ-union, κ-intersection, complementation, and the players' beliefs, define the corresponding syntactic operations.[16]

Definition 5 (Expressions)**.** *Let λ be an infinite cardinal with $\lambda \leq \kappa$. The set of all λ-expressions $\mathcal{L}^I_\lambda(\mathcal{A}_S)$ is the smallest set satisfying the following.*

1. *Every $E \in \mathcal{A}_S$ is a λ-expression.*

[14]Meier (2006, 2008) assumes the following "separative" condition on $(S, \mathcal{A}_S)$: for any distinct $s, s' \in S$, there is $E \in \mathcal{A}_S$ with $s \in E$ and $s' \notin E$. Then, $\{s\} = \bigcap\{E \in \mathcal{A}_S \mid s \in E\}$ for each $s \in S$, though it may be the case that $\{s\} \notin \mathcal{A}_S$.

[15]Figure 1 in the Appendix illustrates how the definitions and lemmas in this section relate with each other.

[16]Expressions constitute an infinitary language as in Fagin (1994) and Heifetz (1997).

2. *If $\mathcal{E}$ is a set of λ-expressions with $|\mathcal{E}| < \lambda$, then so is $(\bigwedge \mathcal{E})$, where $S := \bigwedge \emptyset$ and identify $\bigwedge \mathcal{E} := \bigcap \mathcal{E}$ if $\mathcal{E}$ is a subset of $\mathcal{A}_S$ with $|\mathcal{E}| < \lambda$.*

3. *If e is a λ-expression then so is $(\neg e)$, where $(\neg E) := E^c$ for all $E \in \mathcal{A}_S$.*

4. *If e is a λ-expression, then so is $(\beta_i(e))$ for each $i \in I$.*

For $\lambda = \kappa$, call each κ-expression simply an expression, and denote $\mathcal{L} = \mathcal{L}^I_\kappa(\mathcal{A}_S)$.

Remarks are in order. First, for ease of notation, I often add or omit parentheses in denoting expressions (and in other occurrences). Second, if $\mathcal{E}$ is a set of expressions with $|\mathcal{E}| < \kappa$, then let $(\bigvee \mathcal{E}) := \neg(\bigwedge\{(\neg e) \in \mathcal{L} \mid e \in \mathcal{E}\})$ with the convention that $\bigvee \emptyset := \emptyset$. Third, I interchangeably denote, for instance, $e_1 \wedge e_2 = \bigwedge\{e_1, e_2\}$ and $e_1 \vee e_2 = \bigvee\{e_1, e_2\}$.[17] Fourth, I interchangeably denote $\bigwedge_{j \in J} e_j = \bigwedge\{e_j \mid j \in J\}$ and $\bigvee_{j\in J} e_j = \bigvee\{e_j \mid j \in J\}$ when expressions are indexed by some set J. Fifth, denote $(e \to f) := ((\neg e) \vee f)$ and $(e \leftrightarrow f) := ((e \to f) \wedge (f \to e))$.

The set $\mathcal{L}$ incorporates all the hierarchies of interactive beliefs regarding $(S, \mathcal{A}_S)$ up to the ordinality of κ. The following remark shows how the set of expressions $\mathcal{L}$ is inductively generated from the nature states $(S, \mathcal{A}_S)$ in κ steps.

Remark 1 (Restatement of Expressions $\mathcal{L}$)**.** Let $\mathcal{L}_0 := \mathcal{A}_S$, and let λ be an infinite ordinal with $\lambda \le \kappa$. For any ordinal α with $0 < \alpha \le \lambda$, define

$$\mathcal{L}_\alpha := \mathcal{L}'_\alpha \cup \{(\neg e) \mid e \in \mathcal{L}'_\alpha\} \cup \left\{ \bigwedge \mathcal{F} \;\middle|\; \mathcal{F} \subseteq \mathcal{L}'_\alpha \text{ and } 0 < |\mathcal{F}| < \kappa \right\}, \text{ where}$$

$$\mathcal{L}'_\alpha := \left(\bigcup_{\beta<\alpha} \mathcal{L}_\beta \right) \cup \bigcup_{i \in I} \{\beta_i(e) \mid e \in \bigcup_{\beta<\alpha} \mathcal{L}_\beta\}.$$

Then, $\mathcal{L}^I_\lambda(\mathcal{A}_S) = \mathcal{L}_\lambda$. In particular, $\mathcal{L} = \mathcal{L}_\kappa$.

Intuitively, each $e \in \mathcal{L}_\alpha$ is an expression of "depth at most α." Remark 1 states that $\mathcal{L}$ consists exactly of expressions of "depth at most κ," i.e., logical formulas expressing interactive beliefs regarding $(S, \mathcal{A}_S)$ up to the ordinality of κ.

While expressions themselves are defined independently of any particular belief space, for any belief space $\overrightarrow{\Omega}$, I can recursively identify each expression with an event in $\overrightarrow{\Omega}$ (i.e., an element of $\mathcal{D}$), by the measurability condition on Θ in $\overrightarrow{\Omega}$.

[17] For example, I simply do not distinguish $e_1 \vee e_2$ and $e_2 \vee e_1$. Similarly, since $\{e, e\} = \{e\}$, I simply identify $(e \wedge e)$ as e. These could be augmented by defining $(\bigwedge \mathcal{F})$ for an ordinal sequence of expressions $\mathcal{F}$ instead of a set of expressions.

Definition 6 (Expressions Identified as Events)**.** *Fix a κ-belief space $\overrightarrow{\Omega}$. Inductively define the mapping $[\![\cdot]\!]_{\overrightarrow{\Omega}} : \mathcal{L} \to \mathcal{D}$, the semantic interpretation function of $\overrightarrow{\Omega}$, as follows.*

1. $[\![E]\!]_{\overrightarrow{\Omega}} := \Theta^{-1}(E)$ *for every* $E \in \mathcal{A}_S$.
2. $[\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}} := \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}}$ *for any* $\mathcal{E} \in \mathcal{P}(\mathcal{L})$ *with* $|\mathcal{E}| < \kappa$.
3. $[\![\neg e]\!]_{\overrightarrow{\Omega}} := \neg [\![e]\!]_{\overrightarrow{\Omega}} \left(= ([\![e]\!]_{\overrightarrow{\Omega}})^c\right)$ *for each expression* e.
4. $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega}} := B_i([\![e]\!]_{\overrightarrow{\Omega}})$ *for each* $i \in I$ *and expression* e.

Call $[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D}$ *the denotation of* e *in* $\overrightarrow{\Omega}$.

The semantic interpretation function of a given belief space is, by recursion, uniquely extended from Θ^{-1}. It gives the semantic meaning of an expression e in the sense that $[\![e]\!]_{\overrightarrow{\Omega}}$ is the set of states of the world in which the expression e holds.[18] By recursion, a morphism preserves semantics.

Remark 2 (Morphism Preserves Semantics/Meanings of Expressions)**.** If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism, then $[\![\cdot]\!]_{\overrightarrow{\Omega}} = \varphi^{-1}([\![\cdot]\!]_{\overrightarrow{\Omega'}})$.

The proof is similar to that of Heifetz and Samet (1998b, Proposition 4.1) and Meier (2006, Proposition 2). For events of nature, since φ is a morphism, $[\![\cdot]\!]_{\overrightarrow{\Omega}} = \Theta^{-1}(\cdot) = \varphi^{-1}(\Theta')^{-1}(\cdot) = \varphi^{-1}([\![\cdot]\!]_{\overrightarrow{\Omega'}})$. Then, use the property that φ^{-1} commutes with set-algebraic properties and belief operators.

Before moving on to the second step, I define the following semantic notions and see how a morphism preserves these notions. Later on, I will characterize (in Proposition 2) how a universal belief space can capture statements that are "valid" or "satisfiable" in particular belief spaces.

Definition 7 (Semantic Properties)**.**

1. *An expression* $e \in \mathcal{L}$ *is valid in a belief space* $\overrightarrow{\Omega}$ *(written* $\models_{\overrightarrow{\Omega}} e$*) if* $[\![e]\!]_{\overrightarrow{\Omega}} = \Omega$. *If* e *is valid in any belief space (of the given category), then* e *is valid (written* $\models e$*) (in the given category).*
2. *A set of expressions* $\Phi \in \mathcal{P}(\mathcal{L})$ *is satisfiable in* $\overrightarrow{\Omega}$ *if there is a state* $\omega \in \Omega$ *such that* $\omega \in [\![f]\!]_{\overrightarrow{\Omega}}$ *for all* $f \in \Phi$. *If there is a belief space* $\overrightarrow{\Omega}$ *such that* Φ *is satisfiable, then* Φ *is satisfiable.*

[18] While I do not discuss implications of finite-depth reasoning, one could analyze players' finite depth/level-α reasoning in any belief space $\overrightarrow{\Omega}$ by restricting attention to events $[\![e]\!]_{\overrightarrow{\Omega}}$ with $e \in \mathcal{L}_\alpha$.

3. *An expression $e \in \mathcal{L}$ is a semantic consequence of Φ in $\overrightarrow{\Omega}$ (written $\Phi \models_{\overrightarrow{\Omega}} e$) if, $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$ holds whenever $\omega \in [\![f]\!]_{\overrightarrow{\Omega}}$ for all $f \in \Phi$. If $\Phi \models_{\overrightarrow{\Omega}} e$ for any belief space $\overrightarrow{\Omega}$, then $e \in \mathcal{L}$ is a semantic consequence of Φ (written $\Phi \models e$).*

I make four remarks on the notion of validity. First, irrespective of assumptions on players' beliefs, such expressions as S and $(e \vee (\neg e))$ are valid. Second, in a category of belief spaces where, for example, Truth Axiom is always assumed for player i, an expression of the form $(\beta_i(e) \to e)$ is always valid. In contrast, $(\beta_i(e) \to e)$ is not valid in a category of belief spaces where Truth Axiom is not assumed for player i. Third, it is possible that a given expression f is valid in some belief space $\overrightarrow{\Omega}$ of a given category due to a particular representation of states and players' beliefs (i.e., in a particular context) while it may not be valid in other belief spaces of the same category.

Fourth, let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ be a morphism. By Remark 2, any valid expression e in $\overrightarrow{\Omega'}$ is also valid in $\overrightarrow{\Omega}$. If Φ is satisfiable in $\overrightarrow{\Omega}$, then so is it in $\overrightarrow{\Omega'}$. Suppose further that φ is surjective. If e is a semantic consequence of Φ in $\overrightarrow{\Omega}$, then so is it in $\overrightarrow{\Omega'}$. I will examine (in Proposition 2) the implications of these statements for a universal belief space.

The second step is to define *descriptions* by the set of expressions and the nature state that obtain at each state of each belief space. Since nature states and expressions reside in different spaces, define a description to be a subset of the disjoint union $S \sqcup \mathcal{L}(:= \{(0, s) \in \{0\} \times S \mid s \in S\} \cup \{(1, e) \in \{1\} \times \mathcal{L} \mid e \in \mathcal{L}\})$ (throughout the paper, I denote by $(0, s)$ and $(1, e)$ elements of $S \sqcup \mathcal{L}$). While this definition of the description is different from the one in the previous literature, this definition uniquely identifies the corresponding nature state for each description without any condition (e.g., the separative condition in Footnote 14) on $(S, \mathcal{A}_S)$.

Definition 8 (Descriptions)**.** *For any belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$, define $D(\omega)$, the description of ω, by*

$$D(\omega) := \{\Theta(\omega)\} \sqcup \{e \in \mathcal{L} \mid \omega \in [\![e]\!]_{\overrightarrow{\Omega}}\} := \{(0, \Theta(\omega))\} \cup \{(1, e) \in \{1\} \times \mathcal{L} \mid \omega \in [\![e]\!]_{\overrightarrow{\Omega}}\}.$$

Descriptions have two roles in constructing a universal belief space. First, I will construct a universal belief space so that its underlying state space Ω^* is the set of all descriptions of states of the world ranged over all belief spaces (in the given category):

$$\Omega^* := \{\omega^* \in \mathcal{P}(S \sqcup \mathcal{L}) \mid \omega^* = D(\omega) \text{ for some } \overrightarrow{\Omega} \text{ and } \omega \in \Omega\}. \tag{1}$$

Second, regard D as a mapping $D : \Omega \to \Omega^*$ (I denote $D_{\overrightarrow{\Omega}} : \Omega \to \Omega^*$ when I stress its domain) for any belief space $\overrightarrow{\Omega}$, and hence call D the *description map*. The description map D turns out to be a unique morphism.

Two remarks are in order. First, Ω^* is not empty because there is a belief space $\overrightarrow{\Omega}$ with $\Omega \neq \emptyset$ in the given category. For any assumptions on players' beliefs, consider a belief space $\overrightarrow{\{s\}} := \langle(\{s\}, \mathcal{P}(\{s\})), (\mathrm{id}_{\mathcal{P}(\{s\})})_{i \in I}, \Theta\rangle$ where $s \in S$ and $\Theta : \{s\} \ni s \mapsto s \in S$. Each $B_i = \mathrm{id}_{\mathcal{P}(\{s\})}$ satisfies all the properties of beliefs in Definition 2.

Second, Ω^* depends on the choice of a category of belief spaces. Consider any two categories of belief spaces where assumptions on players' beliefs in the first are also imposed in the second. Denoting by Ω^{1*} and Ω^{2*} the spaces constructed according to Equation (1), $\Omega^{2*} \subseteq \Omega^{1*}$ holds by construction.

Before the third step, I remark that a morphism preserves the descriptions.

Remark 3 (Morphism Preserves Descriptions)**.** If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism, then $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \varphi$.

To see this, fix belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ and $(\omega, \omega') \in \Omega \times \Omega'$. Then, $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$ iff (i) $\Theta(\omega) = \Theta'(\omega')$; and (ii) $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega' \in [\![e]\!]_{\overrightarrow{\Omega'}}$ for all $e \in \mathcal{L}$. Thus, $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$ means that the outside analysts would consider states ω and $\varphi(\omega)$ to be equivalent in terms of a prevailing nature state and prevailing expressions, abstracting away from physical representations of $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$.[19] By Remark 2, both conditions are met for any $(\omega, \varphi(\omega)) \in \Omega \times \Omega'$ where $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism.

I discuss two implications of Remark 3. First, a belief space $\overrightarrow{\Omega}$ is *non-redundant* (Mertens and Zamir, 1985, Definition 2.4) (or *non-flabby* (Fagin, 1994)) if its description map D is injective. In other words, for any distinct ω and ω', either $\Theta(\omega) \neq \Theta(\omega')$ or they are separated by (a sub-κ-algebra) $\mathcal{D}_\kappa := \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$.[20]

[19] This notion of equivalence (identicalness or indistinguishability) is closely related to Fagin (1994, Section 4) and Mertens and Zamir (1985). Also, as the description map turns out to be a unique morphism into a universal belief space, this notion of equivalence corresponds to one notion of bisimulations called "behavioral equivalence" (Kurz, 2000) in theoretical computer science. See Remark A.1 in the Appendix. For notions of bisimulations ("observational equivalence"), see, for instance, Jacobs and Rutten (2012), Kurz (2000), Rutten (2000), and the references therein.

[20] The sub-κ-algebra $\mathcal{D}_\kappa$ turns out to be written solely in terms of the primitives of the belief space as "$\mathcal{C}_\kappa$" in Definition 10 of Section 5.1.

Second, Remark 3 implies that if $\overrightarrow{\Omega'}$ is non-redundant then there is at most one morphism from a given space $\overrightarrow{\Omega}$ into $\overrightarrow{\Omega'}$.[21] If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ and $\psi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ are morphisms then $D_{\overrightarrow{\Omega'}} \circ \varphi = D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \psi$. Since $D_{\overrightarrow{\Omega'}}$ is injective, $\varphi = \psi$. I will show that the description map $D_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is a unique morphism by demonstrating that $D_{\overrightarrow{\Omega^*}}$ is the identity map.

The third step is to define the collection of events $\mathcal{D}^*$ on Ω^* (i.e., the domain of the candidate universal belief space). Since each expression e corresponds to an object of beliefs, define the set $[e]$ of descriptions that make e true (i.e., the set of descriptions that contain e) to be objects of players' beliefs on Ω^*. Formally, for each $e \in \mathcal{L}$, define the set of descriptions $[e] := \{\omega^* \in \Omega^* \mid (1, e) \in \omega^*\}$. Let $\mathcal{D}^* := \{[e] \in \mathcal{P}(\Omega^*) \mid e \in \mathcal{L}\}$. I show that $\mathcal{D}^*$ is a legitimate domain.

Lemma 1 (Domain of Candidate Universal Space)**.** *$(\Omega^*, \mathcal{D}^*)$ is a κ-algebra. Moreover, for any belief space $\overrightarrow{\Omega}$, the description map $D : (\Omega, \mathcal{D}) \to (\Omega^*, \mathcal{D}^*)$ is a measurable mapping such that $D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}}$ for any $e \in \mathcal{L}$.*

The property that $D_{\overrightarrow{\Omega}}^{-1}([\cdot]) = [\![\cdot]\!]_{\overrightarrow{\Omega}}$ exhibits duality between the semantic interpretation function $[\![\cdot]\!]_{\overrightarrow{\Omega}} : \mathcal{L} \to \mathcal{D}$ (which, by recursion, is unique) and the description map $D_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ (which turns out to be a unique morphism). Note also that the sub-κ-algebra $\mathcal{D}_\kappa$ is the one induced by $D_{\overrightarrow{\Omega}}$ in the sense that $\mathcal{D}_\kappa = \{D_{\overrightarrow{\Omega}}^{-1}([e]) \in \mathcal{D} \mid e \in \mathcal{L}\}$.

The fourth step is to construct the mapping $\Theta^* : \Omega^* \to S$ that associates with each state $\omega^* \in \Omega^*$ the unique nature state s contained in ω^* (i.e., $(0, s) \in \omega^*$).

Lemma 2 (Mapping of Candidate Universal Space)**.** *There is a measurable mapping $\Theta^* : (\Omega^*, \mathcal{D}^*) \to (S, \mathcal{A}_S)$ with the following two properties: (i) $\Theta^*(D(\omega)) = \Theta(\omega)$ for any belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$; and (ii) $(\Theta^*)^{-1}(E) = [E] \in \mathcal{D}^*$ for all $E \in \mathcal{A}_S$.*

The fifth step is to introduce players' beliefs. Define players' beliefs regarding $\mathcal{D}^*$ in a way that player i believes an event $[e]$ at a state ω^* iff ω^* contains $\beta_i(e)$ (i.e., $(1, \beta_i(e)) \in \omega^*$). I show that this is well defined: if expressions e and f are equivalent in the sense that $(1, e) \in \omega^*$ iff $(1, f) \in \omega^*$, then $\beta_i(e)$ and $\beta_i(f)$ are equivalent in the

[21] An object having this property is called extensional in Kurz (2000). This is also related to the "coinduction proof principle" in theoretical computer science (e.g., Jacobs and Rutten (2012), Kurz (2000), Rutten (2000), and the references therein): if $\overrightarrow{\Omega}$ is non-redundant, then in order for two states ω and ω' in Ω to be the same, it is enough to show that $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega}}(\omega')$ (i.e., they are "behaviorally equivalent").

same sense.[22] This equivalence depends on assumptions imposed on players' beliefs. For example, if Positive Introspection and Truth Axiom are imposed on player i, then $\beta_i(e)$ and $\beta_i\beta_i(e)$ are equivalent. I will examine (in Remark 4 and Proposition 1) how assumptions on players' beliefs are encoded within Ω^* itself.

Lemma 3 (Belief Operators of Candidate Universal Space)**.** *Fix $i \in I$. Define $B_i^* : \mathcal{D}^* \to \mathcal{D}^*$ by $B_i^*([e]) := [\beta_i(e)]$ for each $e \in \mathcal{L}$. Then, B_i^* is a well-defined belief operator which inherits the properties of beliefs imposed in the given category. Moreover, for any belief space $\overrightarrow{\Omega}$, $D^{-1}(B_i^*([e])) = B_i(D^{-1}([e]))$ for all $[e] \in \mathcal{D}^*$.*

Lemma A.1 in the Appendix shows that each B_i^* inherits various properties satisfied in the given category beyond Definition 2. Section 5.2 discusses such properties of beliefs in the contexts of dynamic and probabilistic beliefs.

I remark on two additional results proved in Lemma A.1. First, if there is a belief space $\overrightarrow{\Omega}$ which fails a given property with respect to $[\![e]\!]_{\overrightarrow{\Omega}}$, then the belief operator B_i^* fails that property with respect to $[e]$. Thus, B_i^* satisfies the properties of beliefs for player i that are common among all the belief spaces in the given category. That is, B_i^* satisfies the properties that the outside analysts exactly would like to impose.[23]

Second, for belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ which may reside in different categories, if $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ is a surjective measurable mapping such that $B_i\varphi^{-1}(\cdot) = \varphi^{-1}B_i'(\cdot)$, then the belief operator B_i' inherits the properties of B_i.

So far, $\overrightarrow{\Omega^*} = \langle(\Omega^*, \mathcal{D}^*), (B_i^*)_{i\in I}, \Theta^*\rangle$ is a belief space of the given category such that, for any belief space $\overrightarrow{\Omega}$, the description map $D : \Omega \to \Omega^*$ is a morphism. Before the next (final) step, I examine how each state $\omega^* \in \Omega^*$ is structured in the following two senses. The first is the logical structure of each state $\omega^* \in \Omega^*$, following Aumann (1999). Recall that each state ω^* contains expressions that hold at ω^* (i.e., $(1, e) \in \omega^*$ iff $\omega^* \in [e]$) as well as the corresponding nature state $s = \Theta^*(\omega^*)$. Based on this fact, the following remark examines which expressions a given state ω^* contains. Each ω^* is *coherent*: if ω^* contains an expression e then it does not contain $(\neg e)$. Each ω^* is *complete*: if ω^* does not contain e then it contains $(\neg e)$. Every ω^* is logically closed.

[22] Proposition 2 establishes that if there are a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ such that $[\![e]\!]_{\overrightarrow{\Omega}} \neq [\![f]\!]_{\overrightarrow{\Omega}}$ then e and f are not equivalent (i.e., $[e] \neq [f]$). Thus, such identifications (equivalences) of expressions (or assumptions on specifications of players' interactive beliefs) are minimal in $\overrightarrow{\Omega^*}$.

[23] Roy and Pacuit (2013) define "substantive" and "structural" assumptions in a syntactic interactive epistemic model. In their framework, a universal structure, if it exists, minimizes substantive assumptions and validates only structural assumptions.

It also contains valid expressions such as S (i.e., expressions that hold in any belief space of the given class).

Remark 4 (Logical Properties of Each State)**.** Fix $\omega^* \in \Omega^*$.

1. For each $e \in \mathcal{L}$, $(1, e) \notin \omega^*$ iff $(1, (\neg e)) \in \omega^*$.

2. For any $e, f \in \mathcal{L}$, if $(1, e) \in \omega$ and $(1, (e \to f)) \in \omega$ then $(1, f) \in \omega$.

3. For any $\mathcal{E} \in \mathcal{P}(\mathcal{L})$ with $|\mathcal{E}| < \kappa$, $(1, \bigwedge \mathcal{E}) \in \omega^*$ iff $(1, e) \in \omega^*$ for all $e \in \mathcal{E}$.

The second is how the belief space $\overrightarrow{\Omega^*}$ resolves the following form of self-reference: players' beliefs are defined on states while states are supposed to completely describe the world.[24] Recall that each player's beliefs at each state ω^* are built within the state ω^* itself in the sense that $\omega^* \in B_i^*([e])$ iff $(1, \beta_i(e)) \in \omega^*$. The following proposition shows how each player's beliefs at each state are encoded within the state itself.

Proposition 1 (Beliefs within Each State)**.** *Fix $\omega^* \in \Omega^*$ and $i \in I$.*

1. *For each $e \in \mathcal{L}$, either $(1, \beta_i(e)) \in \omega^*$ or $(1, (\neg\beta_i)(e)) \in \omega^*$.*

2. *For each $e \in \mathcal{L}$, at least one of $(1, \beta_i(e)) \in \omega^*$, $(1, \beta_i(\neg e)) \in \omega^*$, or $(1, (\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e)) \in \omega^*$ holds. Exactly one of them holds iff i's beliefs satisfy Consistency.*

3. *For each $e \in \mathcal{L}$, exactly one of the following conditions holds: $(1, \beta_i(e)) \in \omega^*$, $(1, (\neg\beta_i)(e) \wedge \beta_i(\neg\beta_i)(\neg e)) \in \omega^*$, or $(1, (\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg\beta_i)(e)) \in \omega^*$. The third condition never occurs iff i's beliefs satisfy Negative Introspection.*

The first part of Proposition 1 states that each state ω^* completely describes i's beliefs in the sense that, for any $e \in \mathcal{L}$, the state ω^* contains exactly one of the above two expressions denoting "i believes e" or "i does not believe e." The second and third parts characterize how the space Ω^* encodes such properties of i' beliefs as Consistency and Negative Introspection. Proposition 1 is related to some of consistency conditions of Gilboa (1988) for a state to completely describe the world. Section 4 (Definition 9 and Theorem 2) will characterize how states encode properties

[24] See, for example, Aumann (1976, 1987, 1999), Bacharach (1985), Binmore and Brandenburger (1990), Brandenburger and Dekel (1993), Brandenburger, Dekel, and Geanakoplos (1992), Dekel and Gul (1997), Fagin et al. (1999), Gilboa (1988), Roy and Pacuit (2013), and Werlang and Tan (1992).

of players' beliefs by demonstrating that properties imposed on player i's beliefs by the outside analysts are expressed within Ω^*.

The sixth step finally establishes that the description map D is a unique morphism. To that end, I show that the description map from $\overrightarrow{\Omega^*}$ into itself is the identity map.

Lemma 4 (Description Map $D_{\overrightarrow{\Omega^*}}$)**.** *The description map* $D_{\overrightarrow{\Omega^*}} : \overrightarrow{\Omega^*} \to \overrightarrow{\Omega^*}$ *is the identity morphism.*

I prove Lemma 4 by showing $[\cdot] = [\![\cdot]\!]_{\overrightarrow{\Omega^*}}$. This property means that the semantics of e at ω^* is determined solely by whether $(1, e) \in \omega^*$. The lemma implies that the belief space $\overrightarrow{\Omega^*}$ is non-redundant. Moreover, it implies that $D_{\overrightarrow{\Omega}}$ is a unique morphism: if $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is a morphism then $D_{\overrightarrow{\Omega}}(\cdot) = D_{\overrightarrow{\Omega^*}}(\varphi(\cdot)) = \varphi(\cdot)$. Thus, I establish the main result, the existence of a universal belief space.

Theorem 1 ($\overrightarrow{\Omega^*}$ is Universal)**.** *The space* $\overrightarrow{\Omega^*} = \langle(\Omega^*, \mathcal{D}^*), (B_i^*)_{i\in I}, \Theta^*\rangle$ *is a universal belief space of* I *on* $(S, \mathcal{A}_S)$ *for each given category of belief spaces.*

As discussed in Section 2.2, a universal belief space exists uniquely up to isomorphism. Generally, $\overrightarrow{\Omega}$ is universal iff the description map $D_{\overrightarrow{\Omega}}$ is an isomorphism. For the rest of this section, I study how the universal belief space Ω^* "includes" other belief spaces.

First, for any state ω of any particular belief space $\overrightarrow{\Omega}$, states $\omega \in \Omega$ and $D(\omega) \in \Omega^*$ are equivalent in the sense that the same state of nature $\Theta(\omega) = \Theta^*(D(\omega)) \in S$ prevails and the same set of expressions regarding nature and players' interactive beliefs obtains. This is because $D(\omega) = D_{\overrightarrow{\Omega^*}}(D(\omega))$. To restate, for any representation $\overrightarrow{\Omega}$ of players' interactive beliefs regarding $(S, \mathcal{A}_S)$ and for any realization $\omega \in \Omega$, the prevailing nature state and the prevailing set of expressions at ω are encoded in the state $D(\omega)$ of the universal belief space. Hence, $\overrightarrow{\Omega^*}$ is terminal in the sense of Friedenberg (2010): for any state $\omega \in \Omega$ of any belief space $\overrightarrow{\Omega}$, there is a unique state $\omega^* = D(\omega)$ in the universal belief space $\overrightarrow{\Omega^*}$ such that ω and ω^* induce the same belief hierarchies. Especially, $\{e \in \mathcal{L} \mid \omega \in B_i([\![e]\!]_{\overrightarrow{\Omega}})\} = \{e \in \mathcal{L} \mid \omega^* \in B_i^*([e])\}$ for every $i \in I$.

Second, any non-redundant belief space $\overrightarrow{\Omega}$ is, by definition, embedded into $\overrightarrow{\Omega^*}$: there is a belief (sub-)space $\overrightarrow{D(\Omega)} := \langle(D(\Omega), \mathcal{D}^* \cap D(\Omega)), (B_i')_{i\in I}, \Theta^*|_{D(\Omega)}\rangle$ such that $D : \overrightarrow{\Omega} \to \overrightarrow{D(\Omega)}$ is a bijective morphism, where $\mathcal{D}^* \cap D(\Omega) := \{[e] \cap D(\Omega) \mid [e] \in \mathcal{D}^*\}$ and $B_i'([e] \cap D(\Omega)) := B_i^*([e]) \cap D(\Omega)$. Any property of players' beliefs satisfied in $\overrightarrow{\Omega}$

holds in $\overrightarrow{D(\Omega)}$. Thus, in a category of belief spaces in which Necessitation holds for every player, $\overrightarrow{D(\Omega)}$ is *belief-closed* (Mertens and Zamir, 1985): $B_i'(D(\Omega)) = D(\Omega)$ for all $i \in I$.

Third, the following proposition, an implication of $[\cdot] = [\![\cdot]\!]_{\overrightarrow{\Omega^*}}$, shows that such semantic notions as satisfiability, semantic consequence, and validness in $\overrightarrow{\Omega^*}$ are informationally robust in the sense that they do not depend on particular belief spaces.

Proposition 2 (Informational Robustness)**.** *Let $e \in \mathcal{L}$, and let $\Phi \in \mathcal{P}(\mathcal{L})$.*

1. *Φ is satisfiable iff Φ is satisfiable in $\overrightarrow{\Omega^*}$.*

2. *e is a semantic consequence of Φ in every belief space iff e is a semantic consequence of Φ in $\overrightarrow{\Omega^*}$. Similarly, e is valid in every belief space iff e is valid in $\overrightarrow{\Omega^*}$.*

The first part of Proposition 2 states that the universal space $\overrightarrow{\Omega^*}$ exhausts all possible sets of satisfiable expressions (in some belief space $\overrightarrow{\Omega}$) within $\overrightarrow{\Omega^*}$. Put differently, if expressions Φ hold at some state ω in some belief space $\overrightarrow{\Omega}$, then the expressions Φ hold at $D(\omega)$ in $\overrightarrow{\Omega^*}$. This result reflects the insight by Moss and Viglizzo (2004, 2006) that their terminal object (for a "measure polynomial functor") consists of all satisfied theories (descriptions) of all points in all objects.

The second part of Proposition 2 implies that if expressions e and f satisfy $[\![e]\!]_{\overrightarrow{\Omega}} \neq [\![f]\!]_{\overrightarrow{\Omega}}$ for some belief space $\overrightarrow{\Omega}$, then $([e] =)[\![e]\!]_{\overrightarrow{\Omega^*}} \neq [\![f]\!]_{\overrightarrow{\Omega^*}}(= [f])$. Suppose for instance that expressions e and $\beta_i(f)$ happen to satisfy $[\![e]\!]_{\overrightarrow{\Omega'}} = B_i'[\![f]\!]_{\overrightarrow{\Omega'}}$ in a particular representation (or a particular "context") $\overrightarrow{\Omega'}$. If another belief space $\overrightarrow{\Omega}$ distinguishes these expressions in that $[\![e]\!]_{\overrightarrow{\Omega}} \neq B_i[\![f]\!]_{\overrightarrow{\Omega}}$, it follows in the universal belief space $\overrightarrow{\Omega^*}$ that $[\![e]\!]_{\overrightarrow{\Omega^*}} \neq B_i^*[\![f]\!]_{\overrightarrow{\Omega^*}}$. Put differently, if $[\![e]\!]_{\overrightarrow{\Omega^*}} = B_i^*[\![f]\!]_{\overrightarrow{\Omega^*}}$, then it is always the case in any belief space that $[\![e]\!]_{\overrightarrow{\Omega}} = B_i[\![f]\!]_{\overrightarrow{\Omega}}$.

Fourth, define the following set Ω^{**} consisting of a profile of the nature state and sets of expressions that individual players believe at some state of some belief space player by player. I show that the universal belief space $\overrightarrow{\Omega^*}$ exhausts nature and players' beliefs in that the bijection exists between the universal space Ω^* and Ω^{**}. Formally, define

$$\Omega^{**} := \{(s, \Psi) \in S \times \mathcal{P}(\mathcal{L})^I \mid (s, \Psi) = (\Theta(\omega), (\{e \in \mathcal{L} \mid \omega \in B_i([\![e]\!]_{\overrightarrow{\Omega}})\})_{i \in I}) \text{ for some belief space } \overrightarrow{\Omega} \text{ and } \omega \in \Omega\}.$$

Proposition 3 (Universal Space Exhausts Interactive Beliefs)**.** *The mapping* $\Omega^* \ni \omega^* \mapsto (\Theta^*(\omega^*), (\{e \in \mathcal{L} \mid \omega^* \in B_i^*([e])\})_{i \in I}) \in \Omega^{**}$ *is bijective. In particular,*

$$\Omega^{**} = \{(\Theta^*(\omega^*), (\{e \in \mathcal{L} \mid \omega \in B_i^*([e])\})_{i \in I}) \in S \times \mathcal{P}(\mathcal{L})^I \mid \omega^* \in \Omega^*\}. \tag{2}$$

The universal space Ω^* exhausts all possible forms of interactive beliefs that can realize at some state of some belief space in a way that any two distinct states in Ω^* describe different belief hierarchies. Thus, as Equation (2) shows, Ω^{**} is obtained by restricting attention to the universal space. For each player i, each state ω^* contains all the relevant information about i's beliefs at ω^* within itself because, for any expression e, whether i believes $[e]$ or not at ω^* is well defined according to $\{e \in \mathcal{L} \mid \omega^* \in B_i^*([e])\}$. Put differently, the universal space Ω^* is (belief-)complete in the sense that each state ω^* is in a one-to-one relation with the state of nature $s = \Theta^*(\omega^*)$ and the profile of a set of expressions that each player believes at the state ω^*.[25]

Note that the space $\Omega^{**}(\subseteq S \times \mathcal{P}(\mathcal{L})^I)$ would not necessarily be a product space. In contrast, if one would restrict attention to a product state space Ω (and rule out Truth Axiom), then Ω^{**} would be written as a product space because, for each player i, her belief $B_i(\cdot)$ would depend only on her "type space T_i."

4 The Universal Belief Space as Coherent Sets of Descriptions

The last section has established the existence of a universal belief space by collecting all possible states that can realize in some belief space. Each state in the universal space consists of a set of expressions (together with a state of nature) satisfiable at some state of some belief space. Thus, for a general infinite regular cardinal κ (especially with $\kappa \geq \aleph_1$ where infinitary operations are allowed), states in the universal κ-belief space would generally be different from the collection of "maximally consistent" sets of expressions (together with a state of nature) in some syntax system, which are often used to prove the "completeness" theorem of the syntax system.[26]

[25] Brandenburger and Keisler (2006) define a notion of (belief-)completeness in terms of a language.

[26] The discrepancy between a semantic notion of satisfiability and a syntactic notion of maximal consistency would emerge when infinitary operations are allowed (Karp, 1964). See also Heifetz

The question arises as to how (or whether) one can characterize each state ω^* and the set $\overrightarrow{\Omega^*}$ in an explicit way. Here I characterize the universal (κ-)belief space $\overrightarrow{\Omega^*}$ in terms of the largest set consisting of coherent and complete descriptions. This is the first formal result linking a universal belief space constructed by the expression-description approach of Heifetz and Samet (1998b) and Aumann (1999)'s idea of a canonical knowledge system (of a finitary epistemic $S5$ logic).

I characterize the universal belief space Ω^* as the largest *coherent set of descriptions* in light of the characterization of a universal probabilistic type space as the largest set of "coherent" belief hierarchies by Brandenburger and Dekel (1993) and Mertens and Zamir (1985). To that end, call a subset Ω of $\mathcal{P}(S \sqcup \mathcal{L})$ a coherent set of descriptions if (i) each element $\omega \in \Omega$ is a complete and coherent set of expressions together with a unique nature state; if (ii) each element ω reflects assumptions on players' beliefs; and if (iii) the set Ω as a whole induces players' beliefs in a well-defined manner. Thus, I establish an alternative characterization of the universal belief space obtained in Section 3. This characterization holds irrespective of a given cardinality κ and assumptions on players' beliefs. Formally:

Definition 9 (Coherent Set of Descriptions)**.** *Call a subset Ω of $\mathcal{P}(S \sqcup \mathcal{L})$ a coherent set of descriptions if it satisfies the following conditions.*

1. *Each element $\omega \in \Omega$ satisfies the following properties.*

   (a) *There is a unique $s \in S$ such that $(0, s) \in \omega$. Moreover, $(1, E) \in \omega$ for all $E \in \mathcal{A}_S$ with $s \in E$.*

   (b) *If $(1, e) \in \omega$ and $(1, (e \to f)) \in \omega$ then $(1, f) \in \omega$.*

   (c) *For any $\mathcal{E} \in \mathcal{P}(\mathcal{L})$ with $|\mathcal{E}| < \kappa$, $(1, \bigwedge \mathcal{E}) \in \omega$ iff $(1, e) \in \omega$ for all $e \in \mathcal{E}$.*

   (d) *Coherency: For each $e \in \mathcal{L}$, if $(1, (\neg e)) \in \omega$ then $(1, e) \notin \omega$.*

   (e) *Completeness: For each $e \in \mathcal{L}$, if $(1, e) \notin \omega$ then $(1, (\neg e)) \in \omega$.*

   (f) *Depending on assumptions on players' beliefs, ω contains any instance of the following expressions.*

      i. *No-Contradiction: $(\emptyset \leftrightarrow \beta_i(\emptyset))$.*

      ii. *Consistency: $(\beta_i(e) \to (\neg \beta_i)(\neg e))$.*

(1997), Meier (2012), Moss and Viglizzo (2004, 2006), and Zhou (2010) for this point.

iii. *Non-empty λ-Conjunction:* $((\bigwedge_{e \in \mathcal{E}} \beta_i(e)) \to \beta_i(\bigwedge \mathcal{E}))$ *with* $0 < |\mathcal{E}| < \lambda$.

iv. *Necessitation:* $\beta_i(S)$.

v. *Truth Axiom:* $(\beta_i(e) \to e)$.

vi. *Positive Introspection:* $(\beta_i(e) \to \beta_i \beta_i(e))$.

vii. *Negative Introspection:* $((\neg\beta_i)(e) \to \beta_i(\neg\beta_i)(e))$.

2. *The set* Ω *satisfies the following conditions.*

   (a) *If expressions* e *and* f *satisfy* $(1, (e \leftrightarrow f)) \in \omega$ *for all* $\omega \in \Omega$*, then* $(1, (\beta_i(e) \leftrightarrow \beta_i(f))) \in \omega$ *for all* $\omega \in \Omega$.

   (b) *Let Monotonicity be assumed for player* i*. If expressions* e *and* f *satisfy* $(1, (e \to f)) \in \omega$ *for all* $\omega \in \Omega$*, then* $(1, (\beta_i(e) \to \beta_i(f))) \in \omega$ *for all* $\omega \in \Omega$.

   (c) *Let the Kripke property be assumed for player* i*. Then,* $(1, \beta_i(e)) \in \omega$ *for any* $(e, \omega) \in \mathcal{L} \times \Omega$ *with the following condition: if* $\omega' \in \Omega$ *satisfies* $(1, f) \in \omega'$ *for all* $f \in \mathcal{L}$ *with* $(1, \beta_i(f)) \in \omega$*, then* $(1, e) \in \omega'$.

First, Condition (1a) states that each state of the world ω describes a corresponding nature state s in a well-defined manner. Also, the state ω contains those nature events $E \in \mathcal{A}_S$ that are true at s (i.e., $s \in E$).[27] Conditions (1b) through (1e) are logical requirements on how the world is described (recall Proposition 4 for each $\omega^* \in \Omega^*$).

Second, each condition in (1f) describes how each state of the world describes the corresponding property of players' beliefs. Proposition 1 has provided related characterizations for Consistency and Negative Introspection for Ω^*. Monotonicity and the Kripke property are described in (2b) and (2c), respectively.

Third, Condition (2a) requires that if two expressions e and f are equivalent in the sense that every ω contains $(e \leftrightarrow f)$ then expressions $\beta_i(e)$ and $\beta_i(f)$ are equivalent in the same sense. This condition allows one to define players' belief operators in a way such that if two expressions e and f correspond to the same event then the events associated with the beliefs in e and f are the same. Together with the conditions on beliefs in the last paragraph (i.e., (1f), (2b), and (2c)), assumptions on players'

[27] By Condition (1d), Condition (1a) implies that, for a unique $s \in S$ with $(0, s) \in \omega$, $(1, E) \in \omega$ iff $s \in E$ for any $E \in \mathcal{A}_S$. Especially, $(1, S) \in \omega$ and $(1, \emptyset) \notin \omega$.

beliefs are encoded within Ω itself in the sense that the resulting belief operators satisfy given assumptions.

Now, I characterize the universal belief space Ω^* as the largest set of coherent descriptions.

Theorem 2 (Ω^* as Largest Coherent Set of Descriptions)**.** *The set Ω^* constructed in Section 3 is the largest coherent set of descriptions: for any set Ω of coherent descriptions, there is a belief space $\overrightarrow{\Omega}$ such that its description map $D_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map and thus $\Omega \subseteq \Omega^*$.*

5 Concluding Discussion

This paper has constructed a universal belief space for various assumptions on players' beliefs (Theorem 1). Within a given class of κ-belief spaces, the players are able to reason about their interactive beliefs up to the ordinal depth of κ. The universal space exhausts all possible forms of interactive beliefs up to depth κ that can realize at some state of some belief space. Explicitly, it is the largest set of coherent descriptions that reflects assumptions on players' beliefs (Theorem 2).

Each state of the universal space encodes players' interactive beliefs at that state within itself in a coherent and complete manner (Propositions 1 and 3). In the universal belief space, only the explicit assumptions on properties of players' beliefs made by the outside analysts are imposed. That is, the universal space is free from implicit assumptions imposed by how the model is represented. At the same time, the universal space exhausts any statement regarding players' interactive beliefs that holds at some state of some belief space (Proposition 2).

Now, the main result (Theorem 1) implies the existence of a universal knowledge space where players' knowledge is induced from (partitional or more generally non-partitional) possibility correspondences, contrary to the previous negative results.

Section 5.1 compares the existence of a universal knowledge space with the previous non-existence results. It discusses how domain specifications together with an appropriate notion of morphism (which reflects domain specifications) have an essential role in constructing a universal knowledge space. As discussed above, the basic idea is that any κ-knowledge space can capture players' interactive knowledge of the ordinal depth up to κ. Theorem 1 establishes the existence of a universal κ-knowledge space within such class of κ-knowledge spaces.

The previous existence results of a universal type (probabilistic belief) space hinge on the domain specification. Because players' countably-additive beliefs are represented on a σ-algebra (i.e., an $\aleph_1$-algebra), the outside analysts rather implicitly restrict attention to countable hierarchies of beliefs. Section 5.2 shows that the methodology of this paper can be applied to the existence of a universal belief space for various notions of probabilistic beliefs. That is, this paper establishes the existence of a universal belief space irrespective of nature of players' beliefs.

5.1 Comparison with the Previous Negative Results

To compare my existence result with the previous negative results (e.g., Fagin et al. (1999), Heifetz and Samet (1998a), and Meier (2005)), consider the notion of a rank of a standard partitional knowledge space (where the domain is the power set of underlying states) by Heifetz and Samet (1998a).[28] Heifetz and Samet (1998a) demonstrate that there is no universal standard partitional knowledge space on the following two grounds. First, a morphism preserves the ranks. Second, there is a standard partitional knowledge space with arbitrarily high rank. Hence, for any candidate universal standard partitional knowledge space, there exists a standard partitional knowledge space with a higher rank, and thus the candidate space must not be universal.

I extend the notion of a rank to that of a κ-rank of a κ-belief space while maintaining the idea that the κ-rank represents the maximal ordinality of interactive beliefs in a given κ-belief space. I extend the notion to account for the fact that not all subsets are expressible within the κ-algebra of a given κ-belief space.

Definition 10 (κ-Rank)**.** *The κ-rank of a κ-belief space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{A}_S)$ is the least ordinal α with $\mathcal{C}_\alpha = \mathcal{C}_{\alpha+1}$, where the sequence $(\mathcal{C}_\alpha)_\alpha$ is defined as follows:*

$$\mathcal{C}_\alpha := \begin{cases} \mathcal{A}_\kappa(\{\Theta^{-1}(E) \in \mathcal{D} \mid E \in \mathcal{A}_S\})(= \Theta^{-1}(\mathcal{A}_S)) & \text{if } \alpha = 0 \\ \mathcal{A}_\kappa\left(\left(\bigcup_{\beta<\alpha} \mathcal{C}_\beta\right) \cup \bigcup_{i \in I}\{B_i(E) \in \mathcal{D} \mid E \in \bigcup_{\beta<\alpha} \mathcal{C}_\beta\}\right) & \text{if } \alpha > 0 \end{cases}.$$

I show: (i) a morphism between κ-belief spaces preserves the κ-ranks; and (ii) the κ-rank of any κ-belief space is at most κ.

[28] Fagin (1994) considers a closely related concept, "distinguishing ordinals," in his logical system.

Proposition 4 (κ-Rank of a Universal κ-Belief Space). *1. If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism between κ-belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$, then the κ-rank of $\overrightarrow{\Omega'}$ is at least as high as that of $\overrightarrow{\Omega}$.*

2. The κ-rank of any κ-belief space $\overrightarrow{\Omega}$ is at most κ.

By Proposition 4 (1), any two isomorphic κ-belief spaces have the same κ-rank. Especially, the κ-rank of a universal κ-belief space is unique. The second part hinges on the fact that the set of expressions $\mathcal{L}(= \mathcal{L}_\kappa^I(\mathcal{A}_S))$ consists of expressions that involve player's interactive beliefs of the ordinality up to κ (Remark 1). Specifically, define $\mathcal{D}_\alpha := \mathcal{A}_\kappa(\{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}_\alpha\})$ for each ordinal $\alpha \leq \kappa$, where $\mathcal{L}_\alpha$ is defined as in Remark 1. Note that $\mathcal{D}_\kappa = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$. I show in the proof that $\mathcal{D}_\alpha = \mathcal{C}_\alpha$ for each ordinal $\alpha \leq \kappa$. Then, $\mathcal{D}_\kappa = \mathcal{C}_\kappa = \mathcal{C}_{\kappa+1}$, i.e., the κ-rank of $\overrightarrow{\Omega}$ is at most κ. Also, since $\{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\} = \mathcal{D}_\kappa = \mathcal{C}_\kappa$, one can check whether a given κ-belief space $\overrightarrow{\Omega}$ is non-redundant through its primitives alone (i.e., $\mathcal{C}_\kappa$).

Two remarks on Proposition 4 (2) are in order. First, using the terminologies of Definition 10, Heifetz and Samet (1998a) study ∞-ranks of (particular) ∞-belief spaces. Thus, for an infinite regular cardinal κ, Heifetz and Samet (1998a)'s non-existence argument does not apply to a given class of κ-belief spaces. Second, it is important to take care of all the κ levels of interactive beliefs in order to incorporate possible "discontinuity" of qualitative beliefs (knowledge).[29] The existence of a universal qualitative belief (knowledge) space does not hinge on the continuity of beliefs (knowledge) itself by keeping track of all possible transfinite belief hierarchies (see also Zhou (2010) in the context of finitely-additive beliefs). Thus, one needs to take care of transfinite (generally κ) hierarchies of beliefs when beliefs are not continuous.

In contrast, one only needs to take care of all possible finite belief hierarchies for countably-additive probabilistic beliefs. As it turns out in Section 5.2, the framework of this paper can accommodate probabilistic beliefs by considering players' p-belief operators $(B_i^p)_{(i,p)\in I\times[0,1]}$. Countably-additive beliefs are continuous with respect to a monotone sequence of events (see Definition 11 (2g) and (2h)).[30] Within the class of

[29] Heifetz and Samet (1998a,b) attribute the non-existence of a universal standard partitional knowledge space to the "lack of continuity" of knowledge with respect to an increasing sequence of events. Fagin, Halpern, and Vardi (1991), Fagin (1994), Fagin et al. (1999), and Heifetz and Samet (1999) attribute the non-existence of the space of all coherent hierarchies of knowledge to the lack of "continuity" property of knowledge structures, as opposed to that of σ-additive probability measures. See also Meier (2006, 2008) for the discussion of the use of infinitary expressions.

[30] Unlike qualitative belief or probability-one belief alone, the continuity of beliefs with respect to

countably-additive $\aleph_1$(-probabilistic)-belief spaces, as the set of expressions in Heifetz and Samet (1998b) forms a finitary language, the universal belief space has the $\aleph_1$-rank $\aleph_0$.

By fixing the language that the players are allowed to use in reasoning about their interactive beliefs (within the ordinality of κ), or by making the domain of a belief space explicit, a morphism (a description map) preserves interactive beliefs in a given (κ-)belief space to the universal (κ-)belief space. At the same time, such preservation concerns only to the extent that (κ-)expressions are preserved.

To conclude, I discuss how my results reconcile with the existence of a universal partitional κ-knowledge space. First, fix any infinite regular cardinal κ. Since I can identify the conditions on each player's knowledge operator under which her knowledge is induced from a partition (namely, Truth Axiom, Negative Introspection, and the Kripke property), Theorem 1 demonstrates that there is a universal partitional κ-knowledge space. Theorem 2 generalizes Aumann (1999)'s idea of a canonical space to characterize the universal κ-belief (in particular, partitional κ-knowledge) space within the class of κ-belief (partitional κ-knowledge) spaces where the domain of each space is explicitly formulated as a κ-algebra.

On the contrary, consider the ∞-knowledge spaces of I on $(S, \mathcal{P}(S))$ (with $|S| \geq 2$ and $|I| \geq 2$).[31] In this case, the notion of an ∞-rank is equivalent to that defined by Heifetz and Samet (1998a). Thus, contrary to the case where κ is an infinite regular cardinal, there is no universal ∞-knowledge space on $(S, \mathcal{P}(S))$ satisfying all the logical and introspective properties (provided that $|S| \geq 2$ and $|I| \geq 2$). Moreover, this non-existence result shows, as in Meier (2005), that there is no universal ∞-belief space in a category of belief spaces on $(S, \mathcal{P}(S))$ which includes, as a subclass, the category of belief spaces satisfying all the logical and introspective properties. With respect to the previous discussion, the collection of "∞-expressions" (Definition 5 with $\kappa = \infty$) is too large to be a set in the realm of the standard set theory.

5.2 Probabilistic Beliefs and Further Applications

I show that the existence of a universal belief space itself does not hinge on whether beliefs are qualitative or probabilistic. Section 5.2.1 establishes the existence of a

an increasing sequence of events (Definition 11 (2h)) requires degrees of p-beliefs.

[31] While I assume $\mathcal{A}_S = \mathcal{P}(S)$ for ease of exposition, note that if $\mathcal{A}_S$ satisfies the separative property (see Footnote 14) then $\mathcal{A}_S = \mathcal{P}(S)$.

universal probabilistic belief space for countably-additive, finitely-additive, and non-additive (not-necessarily-additive) beliefs. Conceptually, I demonstrate that the existence of a universal belief space is established under the same condition (i.e., the domain specification) irrespective of whether beliefs are probabilistic or qualitative (or knowledge). Technically, the framework nests, for example, Heifetz and Samet (1998b), Meier (2006), and Pintér (2012), and establishes the existence of a universal non-additive belief space irrespective of any continuity property on beliefs.[32]

Moving on to further applications, Section 5.2.2 briefly discusses a universal belief space for conditional probability systems (CPSs) (Battigalli and Siniscalchi, 1999; Guarino, 2017) by using conditional p-belief operators (Di Tillio, Halpern, and Samet, 2014). Section 5.2.3 studies a situation in which players' knowledge and qualitative belief are indexed by time as in Battigalli and Bonanno (1997). Note that one can also combine knowledge and probabilistic beliefs as in Meier (2008). Also, one can analyze other qualitative notions of players' minds. Section 5.2.4 briefly discusses the existence of a universal knowledge-unawareness space within the category of standard state space models of knowledge and unawareness.

5.2.1 Universal Probabilistic Belief Space

The framework of this paper can accommodate probabilistic beliefs. As Samet (2000) establishes the equivalence between a type mapping and p-belief operators to represent countably-additive beliefs, I formulate a probabilistic belief space using p-belief operators. Let $(S, \mathcal{A}_S)$ be an $\aleph_1$-algebra. Denote by $\Delta(\Omega)$ the set of countably-additive probability measures on an $\aleph_1$-algebra $(\Omega, \mathcal{D})$. Let Σ_Δ be the σ-algebra on $\Delta(\Omega)$ generated by $\{\{\mu \in \Delta(\Omega) \mid \mu(E) \geq p\} \in \mathcal{P}(\Delta(\Omega)) \mid (E, p) \in \mathcal{D} \times [0, 1]\}$ as in Heifetz and Samet (1998b).

Definition 11 (Probabilistic Belief Space)**.** *A probabilistic belief space of I on $(S, \mathcal{A}_S)$ is a tuple $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (B_i^p)_{(i,p)\in I\times[0,1]}, \Theta\rangle$ with the following properties.*

[32] As discussed in Section 5.1, if players' beliefs are countably additive as in Heifetz and Samet (1998b), then a finitary language suffices to generate the universal countably-additive probabilistic belief space. Also, Heifetz and Samet (1998b) represent players' beliefs by a product of players' type spaces while this paper represents players' beliefs by a single non-product state space Ω (if each player is always "certain" of her beliefs, then the non-product universal belief space is isomorphic to the product of nature states and players' type spaces; see, for example, Mertens and Zamir (1985)). In the literature, the latter non-product structure is referred to as a "belief space" (Mertens and Zamir, 1985). These remarks apply to Section 5.2.2, which establishes the existence of a universal belief space for conditional probability systems (CPSs).

1. *$(\Omega, \mathcal{D})$ is an $\aleph_1$-algebra and the mapping $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_S)$ is measurable.*

2. *For each $i \in I$, $(B_i^p)_{p \in [0,1]}$ is player i's p-belief operators $B_i^p : \mathcal{D} \to \mathcal{D}$ satisfying the properties below. For each $E \in \mathcal{D}$, $B_i^p(E)$ is the event that player i believes E with probability at least p (i.e., she p-believes E).*

   (a) *$B_i^0(\cdot) = \Omega$.*

   (b) *If $p_n \uparrow p$ then $B_i^{p_n}(\cdot) \downarrow B_i^p(\cdot)$.*

   (c) *Monotonicity: If $E \subseteq F$ then $B_i^p(E) \subseteq B_i^p(F)$.*

   (d) *Normalization: $B_i^1(\Omega) = \Omega$.*

   (e) *Super-additivity: $B_i^p(E \cap F) \cap B_i^q(E \cap (\neg F)) \subseteq B_i^{p+q}(E)$ for $p + q \leq 1$.*

   (f) *Sub-additivity: $(\neg B_i^p)(E) \cap (\neg B_i^q)(F) \subseteq (\neg B_i^{p+q})(E \cup F)$ for $p + q \leq 1$.*

   (g) *Continuity-from-above: If $E_n \downarrow E$ then $B_i^p(E_n) \downarrow B_i^p(E)$.*

   (h) *Continuity-from-below: If $E_n \uparrow E$ then $B_i^p(E) = \bigcap_{r \in \mathbb{N}: p - \frac{1}{r} \geq 0} \bigcup_{n \in \mathbb{N}} B_i^{p - \frac{1}{r}}(E_n)$.*

   (i) *Certainty-of-Beliefs: For any $(\omega, E) \in \Omega \times \mathcal{D}$, $[t_{B_i}(\omega)] \subseteq E$ implies $\omega \in B_i^1(E)$, where*

$$[t_{B_i}(\omega)] := \left(\bigcap_{(p,E) \in [0,1] \times \mathcal{D} : \omega \in B_i^p(E)} B_i^p(E) \right) \cap \left(\bigcap_{(p,E) \in [0,1] \times \mathcal{D} : \omega \in (\neg B_i^p)(E)} (\neg B_i^p)(E) \right).$$

In a probabilistic belief space, player i's beliefs are represented by her *type mapping* $t_{B_i} : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \Sigma_\Delta)$, a measurable mapping defined by $t_{B_i}(\omega)(E) := \sup\{p \in [0,1] \mid \omega \in B_i^p(E)\}$ for each $(\omega, E) \in \Omega \times \mathcal{D}$. At each state $\omega \in \Omega$, player i's beliefs are captured by a countably-additive probability measure $t_{B_i}(\omega) \in \Delta(\Omega)$. By the measurability condition, $B_i^p(E) = \{\omega \in \Omega \mid t_{B_i}(\omega)(E) \geq p\} \in \mathcal{D}$ for each $E \in \mathcal{D}$.

While the above conditions are slightly different from those in Samet (2000), these conditions axiomatize type mappings. The mapping t_{B_i} is well defined by Conditions (2a) and (2b). By (2c), each $t_{B_i}(\omega)$ is monotonic (i.e., $E \subseteq F$ implies $t_{B_i}(\omega)(E) \leq t_{B_i}(\omega)(F)$). Condition (2d) is a normalization: $t_{B_i}(\cdot)(\Omega) = 1$. Thus, by restricting attention to the first four conditions, one gets non-additive beliefs (or capacities).

By (2e), each $t_{B_i}(\omega)$ is super-additive: $t_{B_i}(\omega)(E \cap F) + t_{B_i}(\omega)(E \cap (\neg F)) \leq t_{B_i}(\omega)(E)$. Note that (2a) and (2e) imply (2c). By (2f), each $t_{B_i}(\omega)$ is sub-additive:

$t_{B_i}(\omega)(E) + t_{B_i}(\omega)(F) \leq t_{B_i}(\omega)(E \cup F)$. Thus, by restricting attention to the first six conditions, one gets finitely-additive beliefs. By (2g) or (2h), a finitely-additive probability measure $t_{B_i}(\omega)$ becomes countably additive. Since both (2g) and (2h) would typically be imposed on non-additive beliefs, I have presented both conditions.

Condition (2i) requires player i to be certain of her beliefs. The set $[t_{B_i}(\omega)] = \{\omega' \in \Omega \mid t_{B_i}(\omega') = t_{B_i}(\omega)\}$ consists of states ω' that player i cannot distinguish from ω based on her probabilistic beliefs. Thus, player i is certain of her beliefs in that she believes E with probability one if E is implied by $[t_{B_i}(\omega)]$.

A *(probabilistic belief) morphism* from $\overrightarrow{\Omega}$ to $\overrightarrow{\Omega'}$ is a measurable mapping $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ satisfying: (i) $\Theta' \circ \varphi = \Theta$; and (ii) $B_i^p(\varphi^{-1}(\cdot)) = \varphi^{-1}(B_i'^p(\cdot))$ for all $(i,p) \in I \times [0,1]$. A probabilistic belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{A}_S)$ is *universal* if, for any probabilistic belief space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{A}_S)$ there is a unique morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. The previous arguments apply to the existence of a universal probabilistic belief space. One can also establish the existence of a universal probabilistic belief space for various notions of beliefs by dropping corresponding conditions in Definition 11 (2).

Corollary 1 (Universal Probabilistic Belief Space)**.** *There exists a universal probabilistic belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{A}_S)$.*

5.2.2 Universal Conditional Belief Space

I consider conditional beliefs. Specifically, I establish the existence of a universal conditional belief space for conditional belief systems (CPSs) (Battigalli and Siniscalchi, 1999; Guarino, 2017).[33] To that end, call a triple $(\Omega, \mathcal{D}, \mathcal{C})$ a *conditional space* (similarly to Guarino (2017)) (i) if $(\Omega, \mathcal{D})$ is an $\aleph_1$-algebra; (ii) if $\mathcal{C}$ is a non-empty sub-collection of $\mathcal{D}$ with $\emptyset \notin \mathcal{C}$; and (iii) if there exists a *conditional probability system (CPS)* μ on $(\Omega, \mathcal{D}, \mathcal{C})$. A CPS $\mu(\cdot|\cdot) : \mathcal{D} \times \mathcal{C} \to [0,1]$ is a function with the following

[33] Two remarks are in order. First, as discussed in Footnote 32, a state space here is not restricted to a product space. Also the universal conditional belief space consists of transfinite hierarchies of conditional beliefs. On the other hand, the framework here does not presuppose any topological restriction on nature states or any cardinal restriction on conditioning events. Thus, the construction of the universal conditional belief space would be complementary to Battigalli and Siniscalchi (1999) and Guarino (2017). Second, it would be interesting to examine "lexicographic probability systems (LPSs)" or "hypothetical knowledge" within this framework. Tsakas (2014) defines formal equivalence between conditional and lexicographic belief hierarchies in respective type spaces (under some topological assumptions on nature states and beliefs), and establishes the existence of a universal lexicographic belief space from a universal conditional belief space. For a connection of conditional beliefs and hypothetical knowledge, see, for example, Di Tillio, Halpern, and Samet (2014).

three properties: (i) each $\mu(\cdot|C)$ is a countably-additive probability measure; (ii) Normality: $\mu(C|C) = 1$ for each $C \in \mathcal{C}$; and (iii) Chain Rule: $\mu(E|C) = \mu(E|D)\mu(D|C)$ for any $E \in \mathcal{D}$ and $C, D \in \mathcal{C}$ with $E \subseteq D \subseteq C$. Call each $C \in \mathcal{C}$ a *conditioning event* (or a *condition*, for short). Fix a conditional space $(S, \mathcal{A}_S, \mathcal{C}_S)$, where $(S, \mathcal{A}_S)$ is an $\aleph_1$-algebra of nature states.

Denote by $\Delta^{\mathcal{C}}(\Omega)$ the set of CPSs on $(\Omega, \mathcal{D}, \mathcal{C})$ endowed with the σ-algebra

$$\Sigma^{\mathcal{C}}_{\Delta} := \sigma(\{\{\mu \in \Delta^{\mathcal{C}}(\Omega) \mid \mu(E|C) \geq p\} \in \mathcal{P}(\Delta^{\mathcal{C}}(\Omega)) \mid (E, C, p) \in \mathcal{D} \times \mathcal{C} \times [0,1]\}).$$

A player i's *conditional type mapping* is a measurable map $t_i : (\Omega, \mathcal{D}) \to (\Delta^{\mathcal{C}}(\Omega), \Sigma^{\mathcal{C}}_{\Delta})$. I formulate a conditional belief space using "conditional" p-belief operators $B_i^p(\cdot|C)$ for each player i and each condition $C \in \mathcal{C}$.

Definition 12 (Conditional Belief Space)**.** *A conditional belief space of I on $(S, \mathcal{A}_S, \mathcal{C}_S)$ is a tuple $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}, \mathcal{C}), (B_i^p(\cdot|C))_{(i,p,C) \in I \times [0,1] \times \mathcal{C}}, \Theta\rangle$ with the following properties.*

1. *$(\Omega, \mathcal{D}, \mathcal{C})$ is a conditional space and $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_S)$ is a measurable mapping with $\mathcal{C} = \Theta^{-1}(\mathcal{C}_S)$.*

2. *For each $i \in I$, $(B_i^p(\cdot|C))_{(p,C) \in [0,1] \times \mathcal{C}}$ is player i's conditional p-belief operators $B_i^p(\cdot|C) : \mathcal{D} \to \mathcal{D}$ satisfying the following properties.*

    (a) For each $C \in \mathcal{C}$, $(B_i^p(\cdot|C))_{p \in [0,1]}$ satisfies the properties specified in Definition 11 (2).

    (b) Normality: $B_i^1(C|C) = \Omega$ for all $C \in \mathcal{C}$.

    (c) Chain Rule: $B_i^p(E|D) \cap B_i^q(D|C) \subseteq B_i^{pq}(E|C)$ for any $E \in \mathcal{D}$ and $C, D \in \mathcal{C}$ with $E \subseteq D \subseteq C$.

By the assumption $\mathcal{C} = \Theta^{-1}(\mathcal{C}_S)$ in (1), denote $B^p_{i,C_S}(\cdot) := B_i^p(\cdot|\Theta^{-1}(C_S))$ for each $(i, p, C_S) \in I \times [0,1] \times \mathcal{C}_S$. This means that conditions in each conditional belief space are exogenously given as in Battigalli and Siniscalchi (1999) and Guarino (2017). Thus, I remark that one's conditional belief may fail to be a condition (i.e., $B_i^p(E|C) \notin \mathcal{C}$).

The conditions specified in (2) characterize each player i's conditional type mapping as in Di Tillio, Halpern, and Samet (2014, Theorem 1) (see also Guarino (2017) for his axiomatization of CPSs). First, by (2a), a measurable mapping $t_{B_i} : (\Omega, \mathcal{D}) \to$

$(\Delta^{\mathcal{C}}(\Omega), \Sigma^{\mathcal{C}}_{\Delta})$ is well defined for each condition as in Section 5.2.1. By (2b), each $t_{B_i}(\omega)(\cdot|\cdot)$ satisfies Normality. Under (2a) and (2b), it can be seen that (2c) characterizes Chain Rule.

A *(conditional belief) morphism* from $\overrightarrow{\Omega}$ to $\overrightarrow{\Omega'}$ is a measurable mapping $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ satisfying: (i) $\Theta' \circ \varphi = \Theta$; and (ii) $B^p_{i,C_S}(\varphi^{-1}(\cdot)) = \varphi^{-1}(B'^p_{i,C_S}(\cdot))$ for all $(i, p, C_S) \in I \times [0,1] \times \mathcal{C}_S$. A conditional belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{A}_S, \mathcal{C}_S)$ is *universal* if, for any conditional belief space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{A}_S, \mathcal{C}_S)$, there is a unique morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. The previous arguments apply to the existence of a universal conditional belief space.

Corollary 2 (Universal Conditional Belief Space)**.** *There exists a universal conditional belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{A}_S, \mathcal{C}_S)$.*

5.2.3 Universal Dynamic Knowledge-Belief Space

Epistemic analyses of dynamic games often call for players' knowledge and beliefs. As in Battigalli and Bonanno (1997), consider players' knowledge and beliefs indexed by time. Let a nature state $(S, \mathcal{A}_S)$ be a κ-algebra. While a knowledge operator $K_{i,t} : \mathcal{D} \to \mathcal{D}$ represents player i's knowledge at time $t \in \mathbb{N}$, a belief operator $B_{i,t} : \mathcal{D} \to \mathcal{D}$ does her qualitative belief at time t.

Definition 13 (Dynamic Knowledge-Belief Space)**.** *A dynamic κ-knowledge-belief space of I on $(S, \mathcal{A}_S)$ is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (K_{i,t}, B_{i,t})_{(i,t) \in I \times \mathbb{N}}, \Theta \rangle$ with the following properties.*

1. *$(\Omega, \mathcal{D})$ is a κ-algebra and the mapping $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_S)$ is measurable.*

2. *Knowledge operators $K_{i,t} : \mathcal{D} \to \mathcal{D}$ satisfy Truth Axiom, Positive Introspection, Monotonicity, κ-Conjunction, Necessitation, and Negative Introspection.*

3. *Belief operators $B_{i,t} : \mathcal{D} \to \mathcal{D}$ satisfy Consistency, Positive Introspection, Monotonicity, κ-Conjunction, Necessitation, and Negative Introspection.*

4. *Knowledge and belief operators jointly satisfy: (i) $K_{i,t}(\cdot) \subseteq B_{i,t}(\cdot)$; (ii) $B_{i,t}(\cdot) \subseteq K_{i,t}B_{i,t}(\cdot)$; and (iii) $B_{i,t}(\cdot) = B_{i,t}B_{i,t+1}(\cdot)$.*

In (4), the first condition means that knowledge implies belief at each time. The second states that each player knows her own belief at each time. Note that

$(\neg B_{i,t})(\cdot) \subseteq K_{i,t}(\neg B_{i,t})(\cdot)$ holds, given Truth Axiom and Negative Introspection of knowledge. The third condition captures the idea of belief persistence by Battigalli and Bonanno (1997): player i believes E at time t iff she believes at t that she (will) believe E at $t+1$. Player i's knowledge satisfies *perfect recall* if $K_{i,t}(\cdot) \subseteq K_{i,t+1}(\cdot)$ for all $t \in \mathbb{N}$. A dynamic knowledge-belief space *with perfect recall* is a dynamic knowledge-belief space such that each player's knowledge satisfies perfect recall.

A dynamic knowledge-belief space is mathematically a belief space of $I \times \mathbb{N} \times \{0,1\}$, where "player $(i,t,0)$'s belief operator" is $K_{i,t}$ while "player $(i,t,1)$'s belief operator" is $B_{i,t}$, with the specified conditions. Thus, the previous arguments apply to the existence of a universal dynamic knowledge-belief space.

Corollary 3 (Universal Dynamic Knowledge-Belief Space)**.** *There exists a universal dynamic knowledge-belief space (with/without perfect recall)* $\overrightarrow{\Omega^*}$ *of* I *on* $(S, \mathcal{A}_S)$*.*

5.2.4 Universal Knowledge-Unawareness Space

A knowledge-unawareness space is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (K_i, U_i)_{i \in I}, \Theta \rangle$ where $K_i : \mathcal{D} \to \mathcal{D}$ is player i's knowledge operator and $U_i : \mathcal{D} \to \mathcal{D}$ is player i's unawareness operator. The previous arguments assert the existence of a universal knowledge-unawareness space under various assumptions on knowledge and unawareness.

Call the universal knowledge-unawareness space $\overrightarrow{\Omega^*} := \langle (\Omega^*, \mathcal{D}^*), (K_i^*, U_i^*)_{i \in I}, \Theta^* \rangle$ *non-trivial* if $U_i^*([e]) \neq \emptyset$ for some $(i,e) \in I \times \mathcal{L}$. This is equivalent to stating that there is a knowledge-unawareness space $\overrightarrow{\Omega}$ (within the category of knowledge-unawareness spaces at hand) such that $U_i([\![e]\!]_{\overrightarrow{\Omega}}) \neq \emptyset$ for some $(i,e) \in I \times \mathcal{L}$.

It would be an interesting research avenue for generalizing the framework of this paper to a generalized state space consisting of multiple state spaces as in Heifetz, Meier, and Schipper (2006, 2008) due to some limitation of standard state space models to describe richer notions of unawareness (e.g., Chen, Ely, and Luo (2012), Dekel, Lipman, and Rustichini (1998), Fukuda (2018), and Modica and Rustichini (1994)). While the framework of this paper requires the domain (the collection of events) $\mathcal{D}$ to be a κ-algebra of sets on underlying states Ω, the domain in the generalized state space has a more general lattice structure. I conjecture that the idea of this paper can be applied when players' knowledge and unawareness operators are defined on a κ-complete lattice.

A Appendix

Figure 1 illustrates the interrelations among the definitions and lemmas for the construction of a universal belief space (Theorem 1).

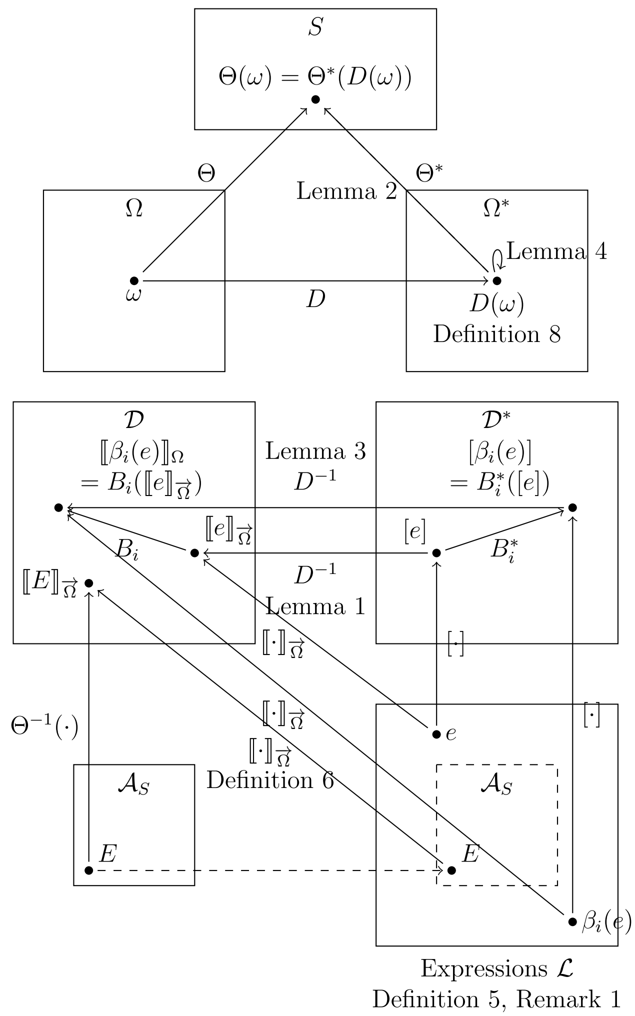

Figure 1: Interrelations among the Definitions and Lemmas for Theorem 1.

Proof of Remark 1. First, $\mathcal{L}_\lambda \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_S)$ follows because (i) $\mathcal{L}_0 \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_S)$; and (ii) if $\mathcal{L}_\beta \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_S)$ for all $\beta < \alpha$ then $\mathcal{L}_\alpha \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_S)$. Conversely, it can be seen that if $e \in \mathcal{L}_\lambda$ then there is $\alpha < \lambda$ such that $e \in \mathcal{L}_\alpha$. I show $\mathcal{L}^I_\lambda(\mathcal{A}_S) \subseteq \mathcal{L}_\lambda$. First, $\mathcal{A}_S \subseteq \mathcal{L}_\lambda$. Second, if $e \in \mathcal{L}_\lambda$ then $e \in \mathcal{L}_\alpha$ for some $\alpha < \lambda$ and thus $(\neg e) \in \mathcal{L}_{\alpha+1} \subseteq \mathcal{L}_\lambda$. Third, take $\mathcal{F} \subseteq \mathcal{L}_\lambda$ with $0 < |\mathcal{F}| < \lambda$. There is $\gamma < \lambda$ such that $e \in \mathcal{L}_\gamma \subseteq \mathcal{L}_\lambda$ for all $e \in \mathcal{F}$, and thus $\bigwedge \mathcal{F} \in \mathcal{L}_\lambda$. Hence, $\mathcal{L}^I_\lambda(\mathcal{A}_S) \subseteq \mathcal{L}_\lambda$. $\square$

Remark A.1 (Footnote 19)**.** Two states, possibly residing in different belief spaces, are identified when the descriptions are identical. I remark that this notion is related to behavioral equivalence (Kurz, 2000). Let $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ be belief spaces in a given category. States $(\omega, \omega') \in \Omega \times \Omega'$ are *behaviorally equivalent* if there are a belief space $\overrightarrow{\Omega''}$ and morphisms $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega''}$ and $\varphi' : \overrightarrow{\Omega'} \to \overrightarrow{\Omega''}$ such that $\varphi(\omega) = \varphi'(\omega')$.

I show that $(\omega, \omega') \in \Omega \times \Omega'$ is behaviorally equivalent iff $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$. This means that, in order to show that two states are identical in terms of players' hierarchies of beliefs, it suffices to show that they are behaviorally equivalent. The proof goes as follows. If $(\omega, \omega') \in \Omega \times \Omega'$ is behaviorally equivalent, then $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega''}}(\varphi(\omega)) = D_{\overrightarrow{\Omega''}}(\varphi'(\omega')) = D_{\overrightarrow{\Omega'}}(\omega')$. The converse holds once I show that the description map is a morphism.

Proof of Lemma 1. The proof consists of the following three steps. The first step establishes the following correspondence between syntactic and semantic operations.

1. $[(\neg e)] = \neg[e]$ for any $e \in \mathcal{L}$.

2. $[S] = \Omega^*$ and $[\emptyset] = \emptyset$. In other words, $[\bigwedge \emptyset] = \Omega^*$ and $[\bigvee \emptyset] = \emptyset$.

3. If $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$, then $\left[\bigwedge_{e \in \mathcal{E}} e\right] = \bigcap_{e \in \mathcal{E}} [e]$ and $\left[\bigvee_{e \in \mathcal{E}} e\right] = \bigcup_{e \in \mathcal{E}} [e]$.

To prove (1), fix $e \in \mathcal{L}$. Then, $\omega^* \in [(\neg e)]$ iff $(1, (\neg e)) \in \omega^* = D(\omega)$ iff $\omega \in [\![\neg e]\!]_{\overrightarrow{\Omega}} = \neg [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega \notin [\![e]\!]_{\overrightarrow{\Omega}}$ iff $(1, e) \notin D(\omega) = \omega^*$ iff $\omega^* \notin [e]$ iff $\omega^* \in (\neg[e])$, where a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. Thus, $[(\neg e)] = \neg[e]$.

To prove (2), if $\omega^* \in \Omega^*$ then $\omega^* = D(\omega)$ for some belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$. Since $\omega \in \Omega = \Theta^{-1}(S) = [\![S]\!]_{\overrightarrow{\Omega}}$, I get $(1, S) \in D(\omega) = \omega^*$ and thus $\omega^* \in [S]$. Hence, $\Omega^* \subseteq [S] (\subseteq \Omega^*)$. Now, $[\emptyset] = [\neg S] = \neg[S] = \emptyset$.

To show (3), take $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. It suffices to show $[\bigwedge \mathcal{E}] = \bigcap_{e \in \mathcal{E}} [e]$. Since $\omega^* \in [e]$ iff $(1, e) \in \omega^* = D(\omega)$ iff $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$, I have: $\omega^* \in [\bigwedge \mathcal{E}]$ iff $(1, \bigwedge \mathcal{E}) \in \omega^* = D(\omega)$ iff $\omega \in [\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}} = \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega^* \in \bigcap_{e \in \mathcal{E}} [e]$, where a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$.

The second step establishes that $\mathcal{D}^*$ is a κ-algebra on Ω^*. If $[e] \in \mathcal{D}^*$, then it follows from the first step and $(\neg e) \in \mathcal{L}$ that $\neg[e] = [(\neg e)] \in \mathcal{D}^*$. Next, take $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. It follows from the first step and $\bigwedge \mathcal{E} \in \mathcal{L}$ that $\bigcap_{e \in \mathcal{E}} [e] = [\bigwedge \mathcal{E}] \in \mathcal{D}^*$. For the case that $\mathcal{E} = \emptyset$, observe $\Omega^* = [S] \in \mathcal{D}^*$.

The third step establishes that, for any belief space $\overrightarrow{\Omega}$, the description map $D : \Omega \to \Omega^*$ satisfies $D^{-1}([\cdot]) = [\![\cdot]\!]_{\overrightarrow{\Omega}}$. For any $[e] \in \mathcal{D}^*$, $\omega \in D^{-1}([e])$ iff $D(\omega) \in [e]$ iff $(1, e) \in D(\omega)$ iff $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$. □

Proof of Lemma 2. For any $\omega^* \in \Omega^*$, choose a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = D(\omega)$ and define $\Theta^*(\omega^*) := \Theta(\omega)$, where $(0, \Theta(\omega)) \in D(\omega)$. I show that $\Theta^* : \Omega^* \to S$ is well defined (i.e., Θ^* does not depend on a particular choice of $\overrightarrow{\Omega}$ and ω with $\omega^* = D(\omega)$). Suppose that $\omega^* = D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$ for some $\omega \in \Omega$ and $\omega' \in \Omega'$. Then, $(0, \Theta(\omega)) = (0, \Theta'(\omega'))$, i.e., $\Theta(\omega) = \Theta'(\omega')$.

I show $(\Theta^*)^{-1}(E) = [E] \in \mathcal{D}^*$ for each $E \in \mathcal{A}_S$ as follows: $\omega^* \in (\Theta^*)^{-1}(E)$ iff $\Theta^*(\omega^*) = \Theta^*(D(\omega)) = \Theta(\omega) \in E$ iff $\omega \in \Theta^{-1}(E) = [\![E]\!]_{\overrightarrow{\Omega}}$ iff $(1, E) \in D(\omega) = \omega^*$ iff $\omega^* \in [E]$, where $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. □

Next, to establish Lemma 3, I provide Lemma A.1 below. Suppose that a certain property of beliefs is represented by operators $f_{\overrightarrow{\Omega}} : \mathcal{D} \to \mathcal{D}$ in each belief space $\overrightarrow{\Omega}$. Operators are generally generated by composing belief operators $(B_i)_{i \in I}$ and set-algebraic as well as constant and identity operations. For example, let $f_{\overrightarrow{\Omega}}(\cdot) = B_i(\cdot)$ and $g_{\overrightarrow{\Omega}}(\cdot) = B_i B_i(\cdot)$. Positive Introspection is characterized by $f_{\overrightarrow{\Omega}}(\cdot) \subseteq g_{\overrightarrow{\Omega}}(\cdot)$. Truth Axiom is characterized by $f_{\overrightarrow{\Omega}}(\cdot) \subseteq \mathrm{id}_{\mathcal{D}}(\cdot)$. Monotonicity is expressed as $f_{\overrightarrow{\Omega}}$ being *monotone*: $f_{\overrightarrow{\Omega}}(E) \subseteq f_{\overrightarrow{\Omega}}(F)$ for all $E, F \in \mathcal{D}$ with $E \subseteq F$. Likewise, Non-empty λ-Conjunction is expressed as $f_{\overrightarrow{\Omega}}$ satisfying *non-empty λ-conjunction*: $\bigcap_{E \in \mathcal{E}} f_{\overrightarrow{\Omega}}(E) \subseteq f_{\overrightarrow{\Omega}}(\bigcap \mathcal{E})$ for all $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$. Abusing the notation, denote by $f_{\overrightarrow{\Omega^*}}$ the corresponding operation in $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (B_i^*)_{i \in I}, \Theta^* \rangle$.

Lemma A.1 (Preservation of Properties of Beliefs)**.** *Suppose that $f_{\overrightarrow{\Omega}} : \mathcal{D} \to \mathcal{D}$ and $g_{\overrightarrow{\Omega}} : \mathcal{D} \to \mathcal{D}$ are defined in each κ-belief space $\overrightarrow{\Omega}$. Suppose further that if $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ is measurable then $\varphi^{-1} f_{\overrightarrow{\Omega'}}(\cdot) = f_{\overrightarrow{\Omega}} \varphi^{-1}(\cdot)$ and $g_{\overrightarrow{\Omega}} \varphi^{-1}(\cdot) = \varphi^{-1} g_{\overrightarrow{\Omega'}}(\cdot)$.*

1. *(a) If $f_{\overrightarrow{\Omega}}(\cdot) \subseteq g_{\overrightarrow{\Omega}}(\cdot)$ holds for every belief space $\overrightarrow{\Omega}$, then $f_{\overrightarrow{\Omega^*}}(\cdot) \subseteq g_{\overrightarrow{\Omega^*}}(\cdot)$.*

   (b) If $f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq g_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}})$ for some belief space $\overrightarrow{\Omega}$ and some $e \in \mathcal{L}$, then $f_{\overrightarrow{\Omega^*}}([e]) \not\subseteq g_{\overrightarrow{\Omega^*}}([e])$.

   (c) If there exits a surjective measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$, then $f_{\overrightarrow{\Omega}}(\cdot) \subseteq g_{\overrightarrow{\Omega}}(\cdot)$ implies $f_{\overrightarrow{\Omega'}}(\cdot) \subseteq g_{\overrightarrow{\Omega'}}(\cdot)$.

2. *(a) If $f_{\overrightarrow{\Omega}}$ is monotone for every belief space $\overrightarrow{\Omega}$, then so is $f_{\overrightarrow{\Omega^*}}$.*

(b) *Suppose* $f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq f_{\overrightarrow{\Omega}}([\![f]\!]_{\overrightarrow{\Omega}})$ *for some belief space* $\overrightarrow{\Omega}$ *and some* $e, f \in \mathcal{L}$ *with* $[\![e]\!]_{\overrightarrow{\Omega}} \subseteq [\![f]\!]_{\overrightarrow{\Omega}}$. *Then,* $f_{\overrightarrow{\Omega^*}}([e]) \not\subseteq f_{\overrightarrow{\Omega^*}}([f])$.

(c) *Suppose that there exits a surjective measurable map* $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$. *If* $f_{\overrightarrow{\Omega}}$ *is monotone, then so is* $f_{\overrightarrow{\Omega'}}$.

3. (a) *If* $f_{\overrightarrow{\Omega}}$ *satisfies non-empty* λ*-conjunction for every belief space* $\overrightarrow{\Omega}$, *then so does* $f_{\overrightarrow{\Omega^*}}$.

(b) *Suppose* $\bigcap_{e \in \mathcal{E}} f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq f_{\overrightarrow{\Omega}}(\bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}})$ *for some belief space* $\overrightarrow{\Omega}$ *and some* $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ *with* $|\mathcal{E}| < \lambda$. *Then,* $\bigcap_{e \in \mathcal{E}} f_{\overrightarrow{\Omega^*}}([e]) \not\subseteq f_{\overrightarrow{\Omega^*}}(\bigcap_{e \in \mathcal{E}} [e])$.

(c) *Suppose that there exits a surjective measurable map* $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$. *If* $f_{\overrightarrow{\Omega}}$ *satisfies non-empty* λ*-conjunction, so does* $f_{\overrightarrow{\Omega'}}$.

Proof of Lemma A.1. 1. I show that if $\omega \in g_{\overrightarrow{\Omega}}(E)$ for all $E \in \mathcal{D}$ with $\omega \in f_{\overrightarrow{\Omega}}(E)$ then, for any $E' \in \mathcal{D}'$, $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(E')$ implies $\varphi(\omega) \in g_{\overrightarrow{\Omega'}}(E')$. If $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(E')$, then $\omega \in \varphi^{-1}(f_{\overrightarrow{\Omega'}}(E')) = f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'))$. By supposition, $\omega \in g_{\overrightarrow{\Omega}}(\varphi^{-1}(E')) = \varphi^{-1}(g_{\overrightarrow{\Omega'}}(E'))$, i.e., $\varphi(\omega) \in g_{\overrightarrow{\Omega'}}(E')$.

(a) If $\omega^* \in f_{\overrightarrow{\Omega^*}}([e])$ then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = D(\omega)$. Now, $\omega^* = D(\omega) \in g_{\overrightarrow{\Omega^*}}([e])$.

(b) By hypothesis, there is $\omega \in \Omega$ such that $\omega \in f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1} f_{\overrightarrow{\Omega^*}}([e])$ and $\omega \notin g_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1} g_{\overrightarrow{\Omega^*}}([e])$.

(c) If $\omega' \in f_{\overrightarrow{\Omega'}}(E')$ then $\omega' = \varphi(\omega)$ for some $\omega \in \Omega$. Now, $\omega' = \varphi(\omega) \in g_{\overrightarrow{\Omega'}}(E')$.

2. I show that if $f_{\overrightarrow{\Omega}}$ is monotone, then $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(E')$ implies $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(F')$ for all $E', F' \in \mathcal{D}'$ with $E' \subseteq F'$. Indeed, if $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(E')$, then $\omega \in \varphi^{-1}(f_{\overrightarrow{\Omega'}}(E')) = f_{\overrightarrow{\Omega}}(\varphi^{-1}(E')) \subseteq f_{\overrightarrow{\Omega}}(\varphi^{-1}(F')) = \varphi^{-1}(f_{\overrightarrow{\Omega'}}(F'))$.

(a) Let $[e] \subseteq [f]$. If $\omega^* \in f_{\overrightarrow{\Omega^*}}([e])$ then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = D(\omega)$. Now, $\omega^* = D(\omega) \in f_{\overrightarrow{\Omega^*}}([f])$.

(b) By hypothesis, there is $\omega \in \Omega$ such that $\omega \in f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1} f_{\overrightarrow{\Omega^*}}([e])$ and $\omega \notin f_{\overrightarrow{\Omega}}([\![f]\!]_{\overrightarrow{\Omega}}) = D^{-1} f_{\overrightarrow{\Omega^*}}([f])$.

(c) Let $E' \subseteq F'$. If $\omega' \in f_{\overrightarrow{\Omega'}}(E')$ then there is $\omega \in \Omega$ with $\omega' = \varphi(\omega)$. Now, $\omega' = \varphi(\omega) \in f_{\overrightarrow{\Omega'}}(F')$.

3. I show that if $f_{\overrightarrow{\Omega}}$ satisfies non-empty λ-conjunction, then $\varphi(\omega) \in \bigcap_{E' \in \mathcal{E}'} f_{\overrightarrow{\Omega'}}(E')$ implies $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(\bigcap \mathcal{E}')$ for all $\mathcal{E}' \in \mathcal{P}(\mathcal{D}') \setminus \{\emptyset\}$ with $|\mathcal{E}'| < \lambda$. Indeed, if $\varphi(\omega) \in \bigcap_{E' \in \mathcal{E}'} f_{\overrightarrow{\Omega'}}(E')$, then $\omega \in \varphi^{-1}(f_{\overrightarrow{\Omega'}}(E')) = f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'))$ for all $E' \in \mathcal{E}'$, i.e., $\omega \in \bigcap_{E' \in \mathcal{E}'} f_{\overrightarrow{\Omega}}(\varphi^{-1}(E')) \subseteq f_{\overrightarrow{\Omega}}(\bigcap_{E' \in \mathcal{E}'} \varphi^{-1}(E')) = f_{\overrightarrow{\Omega}}(\varphi^{-1}(\bigcap \mathcal{E}')) = \varphi^{-1}(f_{\overrightarrow{\Omega'}}(\bigcap \mathcal{E}'))$.
   (a) Fix $\mathcal{E}^* \in \mathcal{P}(\mathcal{D}^*) \setminus \{\emptyset\}$ with $|\mathcal{E}^*| < \lambda$. If $\omega^* \in \bigcap_{[e] \in \mathcal{E}^*} f_{\overrightarrow{\Omega^*}}([e])$ then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = D(\omega)$. Now, $\omega^* = D(\omega) \in f_{\overrightarrow{\Omega^*}}(\bigcap \mathcal{E}^*)$.
   (b) By hypothesis, there is $\omega \in \Omega$ with $\omega \in \bigcap_{e \in \mathcal{E}} f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1}(\bigcap_{e \in \mathcal{E}} f_{\overrightarrow{\Omega^*}}([e]))$ and $\omega \notin f_{\overrightarrow{\Omega}}(\bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}}) = f_{\overrightarrow{\Omega}}([\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}}) = D^{-1}(f_{\overrightarrow{\Omega^*}}([\bigwedge \mathcal{E}])) = D^{-1}(f_{\overrightarrow{\Omega^*}}(\bigcap_{e \in \mathcal{E}} [e]))$.
   (c) Fix $\mathcal{E}' \in \mathcal{P}(\mathcal{D}') \setminus \{\emptyset\}$ with $|\mathcal{E}'| < \lambda$. If $\omega' \in \bigcap_{E' \in \mathcal{E}'} f_{\overrightarrow{\Omega'}}(E')$ then there is $\omega \in \Omega$ with $\omega' = \varphi(\omega)$. Now, $\omega' = \varphi(\omega) \in f_{\overrightarrow{\Omega'}}(\bigcap \mathcal{E}')$.

□

Two remarks on Lemma A.1 are in order. First, B_i^* violates some property of beliefs if there exists a belief space $\overrightarrow{\Omega}$ which violates the corresponding property with respect to $\mathcal{D}_\kappa = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$. Second, for belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$, if there is a surjective measurable mapping $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ such that $B_i \varphi^{-1}(\cdot) = \varphi^{-1} B_i'(\cdot)$, then B_i' inherits the properties of B_i. Now, I prove Lemma 3.

Proof of Lemma 3. Fix $i \in I$. I show that the operator $B_i^* : \mathcal{D}^* \to \mathcal{D}^*$ is well defined and inherits all the properties imposed in the given category of belief spaces. Once they are established, for any $[e] \in \mathcal{D}^*$, $B_i(D^{-1}([e])) = B_i([\![e]\!]_{\overrightarrow{\Omega}}) = [\![\beta_i(e)]\!]_{\overrightarrow{\Omega}} = D^{-1}([\beta_i(e)]) = D^{-1}(B_i^*([e]))$.

To show that B_i^* is well defined, let $e, f \in \mathcal{D}$ be such that $[e] = [f]$. If $\omega^* \in B_i^*([e]) = [\beta_i(e)]$, then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $D(\omega) \in [\beta_i(e)]$, i.e., $(1, \beta_i(e)) \in D(\omega)$. Thus, $\omega \in [\![\beta_i(e)]\!]_{\overrightarrow{\Omega}} = B_i([\![e]\!]_{\overrightarrow{\Omega}}) = B_i(D^{-1}([e]))$. Since $[e] = [f]$, I obtain $\omega \in [\![\beta_i(f)]\!]_{\overrightarrow{\Omega}}$. That is, $\omega^* = D(\omega) \in [\beta_i(f)] = B_i^*([f])$. By changing the role of e and f, I conclude $B_i^*([e]) = B_i^*([f])$.

Next, I show that each B_i^* inherits logical properties. For No-Contradiction, apply Lemma A.1 (1a) by taking $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i(\emptyset), \emptyset)$. For Consistency, apply Lemma A.1 (1a) by taking $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i(\cdot) \cap (\neg B_i)(\cdot), \emptyset)$. For Monotonicity, apply Lemma A.1 (2a) by taking $f_{\overrightarrow{\Omega}} = B_i$. For Necessitation, apply Lemma A.1 (1a) by taking $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (\Omega, B_i(\Omega))$. For Non-empty λ-Conjunction, apply Lemma A.1 (3a) by taking $f_{\overrightarrow{\Omega}} = B_i$.

Next, I apply Lemma A.1 (1a) to show that each B_i^* inherits introspective properties. For Truth Axiom, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i, \mathrm{id}_{\mathcal{D}})$. For Positive Introspection, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i, B_i B_i)$. For Negative Introspection, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i, B_i(\neg B_i))$.

Finally, consider the Kripke property. If $b^*_{B_i^*}(\omega^*) \subseteq [e]$ then $b_{B_i}(\omega) = \bigcap_{F \in \mathcal{D}: \omega \in B_i(F)} F \subseteq D^{-1} \bigcap_{[f] \in \mathcal{D}^*: \omega^* \in B_i^*[f]} [f] \subseteq D^{-1}[e]$, where a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. Since B_i satisfies the Kripke property, $\omega \in B_i D^{-1}[e] = D^{-1} B_i^*[e]$, as desired. □

Proof of Remark 4. First, $(1, e) \notin \omega^*$ iff $\omega^* \notin [e]$ iff $\omega^* \in \neg[e] = [\neg e]$ (recall the proof of Lemma 1) iff $(1, (\neg e)) \in \omega^*$. Second, suppose that $(1, e) \in \omega^*$ and $(1, (e \to f)) \in \omega^*$. Then, $\omega^* \in [e]$ and $\omega^* \in [e \to f] = [(\neg e) \vee f] = \neg[e] \cup [f]$. Thus, $\omega^* \in [f]$, i.e., $(1, f) \in \omega^*$. Third, $(1, \bigwedge \mathcal{E}) \in \omega^*$ iff $\omega^* \in [\bigwedge \mathcal{E}] = \bigcap_{e \in \mathcal{E}} [e]$ (recall the proof of Lemma 1) iff $\omega^* \in [e]$ for all $e \in \mathcal{E}$ iff $(1, e) \in \omega^*$ for all $e \in \mathcal{E}$. □

Proof of Proposition 1. The first part follows from Remark 4. The third part follows from Remark 4 and the fact that $(1, (\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg\beta_i)(e)) \in \omega^*$ iff $\omega^* \in (\neg B_i^*)([e]) \cap (\neg B_i^*)(\neg B_i^*)([e])$. Thus, I prove the second part.

Suppose that $(1, \beta_i(e)) \notin \omega^*$ and $(1, \beta_i(\neg e)) \notin \omega^*$. Then, $\omega^* \in [(\neg\beta_i)(e)] \cap [(\neg\beta_i)(\neg e)] = [(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e)]$. This implies $(1, (\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e)) \in \omega^*$.

Next, $(1, \beta_i(e)) \in \omega^*$ implies $(1, (\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e)) \notin \omega^*$. Likewise, $(1, \beta_i(\neg e)) \in \omega^*$ implies $(1, (\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e)) \notin \omega^*$. Also, $(1, (\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e)) \in \omega^*$ implies $(1, \beta_i(e)) \notin \omega^*$ and $(1, \beta_i(\neg e)) \notin \omega^*$. Thus, under Consistency on i's beliefs, it is enough to show that $(1, \beta_i(e)) \in \omega^*$ and $(1, \beta_i(\neg e)) \in \omega^*$ do not hold simultaneously. Indeed, if they hold simultaneously, then $\omega^* \in B_i^*([e]) \cap B_i^*(\neg[e]) = \emptyset$, a contradiction.

Conversely, assume that exactly one of the three conditions holds. If $\omega^* \in B_i^*([e])$ then $(1, \beta_i(e)) \in \omega^*$. Then, $(1, \beta_i(\neg e)) \notin \omega^*$, i.e., $\omega^* \in (\neg B_i^*)([\neg e]) = (\neg B_i^*)([e]^c)$, establishing Consistency. □

Proof of Lemma 4. First, I show that there is a unique $s \in S$ such that $\omega^* \cap (\{0\} \times S) = D(\omega^*) \cap (\{0\} \times S) = \{(0, s)\}$. Let $(0, s) \in \omega^*$ and $(0, s') \in D(\omega^*)$. There are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $s = \Theta(\omega) = \Theta^*(D(\omega)) = \Theta^*(\omega^*) = s'$. Note that the argument does not depend on a particular choice of belief spaces.

Second, in a similar way to Heifetz and Samet (1998b, Lemma 4.6) and Meier (2006, Lemma 6), I show by induction that $[\![\cdot]\!]_{\overrightarrow{\Omega^*}} = [\cdot]$. Once these assertions are established, $\omega^* = \{s\} \sqcup \{e \in \mathcal{L} \mid (1, e) \in \omega^*\} = \{s\} \sqcup \{e \in \mathcal{L} \mid \omega^* \in [e]\} = \{s\} \sqcup \{e \in \mathcal{L} \mid \omega^* \in [\![e]\!]_{\overrightarrow{\Omega^*}}\} = D(\omega^*)$ for any $\omega^* \in \Omega^*$.

To establish the second step, start from $E \in \mathcal{A}_S$. Then, $\omega^* \in [\![E]\!]_{\overrightarrow{\Omega^*}} = (\Theta^*)^{-1}(E)$ iff $\Theta^*(\omega^*) = \Theta^*(D(\omega)) = \Theta(\omega) \in E$ iff $\omega \in \Theta^{-1}(E) = [\![E]\!]_{\overrightarrow{\Omega}}$ iff $(1, E) \in D(\omega)$ iff $\omega^* = D(\omega) \in [E]$, where $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$.

Next, let $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. Assume the induction hypothesis that $[\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$ for all $e \in \mathcal{E}$. Then, $[\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega^*}} = \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega^*}} = \bigcap_{e \in \mathcal{E}} [e] = [\bigwedge \mathcal{E}]$, where the last equality follows from the proof of Lemma 1.

Next, assume the induction hypothesis that $[\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$. By definition, $[\beta_i(e)] = B_i^*([e]) = B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = [\![\beta_i(e)]\!]_{\overrightarrow{\Omega^*}}$. Also, $[(\neg e)] = \neg [e] = \neg [\![e]\!]_{\overrightarrow{\Omega^*}} = [\![\neg e]\!]_{\overrightarrow{\Omega^*}}$. □

Proof of Theorem 1. I have already shown that $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (B_i^*)_{i \in I}, \Theta^* \rangle$ is a belief space of I on $(S, \mathcal{A}_S)$ of the given category such that, for any belief space $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta \rangle$, the description map $D_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is a morphism. Thus, I need only to show that $D_{\overrightarrow{\Omega}}$ is a unique morphism. If $\varphi : \Omega \to \Omega^*$ is a morphism, then it follows from Remark 3 and Lemma 4 that $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega^*}}(\varphi(\omega)) = \varphi(\omega)$ for any $\omega \in \Omega$, i.e., $D_{\overrightarrow{\Omega}} = \varphi$. □

Proof of Proposition 2.

1. It suffices to show that if Φ is satisfiable then it is satisfiable in $\overrightarrow{\Omega^*}$. If there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega \in [\![f]\!]_{\overrightarrow{\Omega}} = D^{-1}([f])$ for all $f \in \Phi$, then $D(\omega) \in [f] = [\![f]\!]_{\overrightarrow{\Omega^*}}$ for all $f \in \Phi$.

2. For the first assertion, it is enough to show that $\Phi \models_{\overrightarrow{\Omega^*}} e$ implies $\Phi \models e$. Let $\overrightarrow{\Omega}$ be a belief space. If $\omega \in [\![f]\!]_{\overrightarrow{\Omega}} = D^{-1}([f])$ for all $f \in \Phi$, then $D(\omega) \in [f] = [\![f]\!]_{\overrightarrow{\Omega^*}}$ for all $f \in \Phi$. By assumption, $D(\omega) \in [\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$, i.e., $\omega \in D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}}$. Thus, $\Phi \models e$. Now, the second assertion can be seen as a special case of the first. Let $\overrightarrow{\Omega}$ be a belief space, and take $\omega \in \Omega$. Since $\Omega^* = [\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$, $\omega \in D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}}$. Thus, $\Omega = [\![e]\!]_{\overrightarrow{\Omega}}$.

□

Proof of Proposition 3. First, I show that the mapping defined in the proposition is surjective. Take any $(\Theta(\omega), (\{e \in \mathcal{L} \mid \omega \in B_i [\![e]\!]_{\overrightarrow{\Omega}}\})_{i \in I})$, where $\overrightarrow{\Omega}$ is a belief space and $\omega \in \Omega$. Then, it follows from Lemmas 3 and 4 that

$$\begin{aligned}(\Theta(\omega), (\{e \in \mathcal{L} \mid \omega \in B_i [\![e]\!]_{\overrightarrow{\Omega}}\})_{i \in I}) &= (\Theta^*(D(\omega)), (\{e \in \mathcal{L} \mid D(\omega) \in B_i^* [\![e]\!]_{\overrightarrow{\Omega^*}}\})_{i \in I}) \\ &= (\Theta^*(D(\omega)), (\{e \in \mathcal{L} \mid D(\omega) \in B_i^* [e]\})_{i \in I}).\end{aligned}$$

Second, I show that the mapping defined in the proposition is injective. Suppose that $\Theta^*(\omega^*) = \Theta^*(\tilde{\omega}^*) = s$ and $\{e \in \mathcal{L} \mid \omega^* \in B_i^*[e]\} = \{e \in \mathcal{L} \mid \tilde{\omega}^* \in B_i^*[e]\}$ for each $i \in I$ (recall $[\![\cdot]\!]_{\overrightarrow{\Omega^*}} = [\cdot]$). Since $(0, \Theta^*(\omega^*)) \in \omega^*$ and $(0, \Theta^*(\tilde{\omega}^*)) \in \tilde{\omega}^*$, states ω^* and $\tilde{\omega}^*$ contain the same unique nature state s. Now, I show by induction that ω^* and $\tilde{\omega}^*$ contain the same set of expressions. First, for any $E \in \mathcal{A}_S$, if $(1, E) \in \omega^*$ (i.e., $\omega^* \in [E] = (\Theta^*)^{-1}(E)$), then $\Theta^*(\tilde{\omega}^*) = \Theta^*(\omega^*) \in E$ and thus $(1, E) \in \tilde{\omega}^*$. The converse is also true. Second, $(1, \neg e) \in \omega^*$ iff $(1, e) \notin \omega^*$ iff $(1, e) \notin \tilde{\omega}^*$ iff $(1, \neg e) \in \tilde{\omega}^*$. Third, let $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. Then, $(1, \bigwedge \mathcal{E}) \in \omega^*$ iff $(1, e) \in \omega^*$ for all $e \in \mathcal{E}$ iff $(1, e) \in \tilde{\omega}^*$ for all $e \in \mathcal{E}$ iff $(1, \bigwedge \mathcal{E}) \in \tilde{\omega}^*$. Fourth, fix $i \in I$. Then, $(1, \beta_i(e)) \in \omega^*$ iff $\omega^* \in B_i^*[e]$ iff $\tilde{\omega}^* \in B_i^*[e]$ iff $(1, \beta_i(e)) \in \tilde{\omega}^*$. The induction is complete, and $\omega^* = \tilde{\omega}^*$. □

Proof of Theorem 2. Step 1. The proof consists of two steps. The first step shows that Ω^* is a coherent set of descriptions. Observe that Conditions (1b) to (1e) follow from Remark 4. For (1a), take $\omega^* \in \Omega^*$. There is a unique nature state $s = \Theta^*(\omega^*)$ with $(0, s) \in \omega^*$. For any $E \in \mathcal{A}_S$ with $s \in E$, I have $\omega^* \in (\Theta^*)^{-1}(E) = [E]$, i.e., $(1, E) \in \omega^*$.

Next, I show (1f). Fix $\omega^* \in \Omega^*$, and it is enough to show that each of the following expressions is valid in $\overrightarrow{\Omega^*}$. For No-Contradiction, since $[\![\emptyset \leftrightarrow \beta_i(\emptyset)]\!]_{\overrightarrow{\Omega^*}} = (\neg B_i^*)(\emptyset) = \Omega^*$, $(\emptyset \leftrightarrow \beta_i(\emptyset))$ is valid in $\overrightarrow{\Omega^*}$. For Consistency, consider $[\![\beta_i(e) \to (\neg\beta_i)(\neg e)]\!]_{\overrightarrow{\Omega^*}} = (\neg B_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup (\neg B_i^*)(\neg [\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*$. For Non-empty λ-Conjunction, consider:

$$\left[\!\!\left[(\bigwedge_{e \in \mathcal{E}} \beta_i(e)) \to \beta_i(\bigwedge \mathcal{E})\right]\!\!\right]_{\overrightarrow{\Omega^*}} = \left(\neg \bigcap_{e \in \mathcal{E}} B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}})\right) \cup B_i^*(\bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*.$$

For Necessitation, consider $[\![\beta_i(S)]\!]_{\overrightarrow{\Omega^*}} = B_i^*([\![S]\!]_{\overrightarrow{\Omega^*}}) = B_i^*(\Omega^*) = \Omega^*$.

For Truth Axiom, consider $[\![\beta_i(e) \to e]\!]_{\overrightarrow{\Omega^*}} = (\neg B_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup [\![e]\!]_{\overrightarrow{\Omega^*}} = \Omega^*$. For Positive Introspection, consider $[\![\beta_i(e) \to \beta_i\beta_i(e)]\!]_{\overrightarrow{\Omega^*}} = (\neg B_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup B_i^* B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*$. For Negative Introspection, consider $[\![(\neg\beta_i)(e) \to \beta_i(\neg\beta_i)(e)]\!]_{\overrightarrow{\Omega^*}} = B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup B_i^*(\neg B_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*$.

Next, I show that Ω^* satisfies (2a) to (2c). Consider (2a). If $(e \leftrightarrow f)$ is valid in $\overrightarrow{\Omega^*}$, then $[\![e]\!]_{\overrightarrow{\Omega^*}} = [\![f]\!]_{\overrightarrow{\Omega^*}}$ and $B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = B_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})$, i.e., $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega^*}} = [\![\beta_i(f)]\!]_{\overrightarrow{\Omega^*}}$. Then, it follows that $[\![\beta_i(e) \leftrightarrow \beta_i(f)]\!]_{\overrightarrow{\Omega}} = \Omega^*$, i.e., $(\beta_i(e) \leftrightarrow \beta_i(f))$ is valid in $\overrightarrow{\Omega^*}$.

For (2b), similarly to the above argument, if $(e \to f)$ is valid in $\overrightarrow{\Omega^*}$, then $[\![e]\!]_{\overrightarrow{\Omega^*}} \subseteq [\![f]\!]_{\overrightarrow{\Omega^*}}$ and thus $B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) \subseteq B_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})$, i.e., $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega^*}} \subseteq [\![\beta_i(f)]\!]_{\overrightarrow{\Omega^*}}$. Then, $[\![\beta_i(e) \to \beta_i(f)]\!]_{\overrightarrow{\Omega^*}} = \Omega^*$, i.e., $(\beta_i(e) \to \beta_i(f))$ is valid in $\overrightarrow{\Omega^*}$.

For (2c), by supposition, $\bigcap \{[\![f]\!]_{\overrightarrow{\Omega^*}} \in \mathcal{D}^* \mid \omega^* \in B_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})\} \subseteq [\![e]\!]_{\overrightarrow{\Omega^*}}$. By the Kripke property of $\overrightarrow{\Omega^*}$, $\omega^* \in B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = [\beta_i(e)]$, i.e., $(1, \beta_i(e)) \in \omega^*$.

Step 2. The second step shows that $\Omega \subseteq \Omega^*$ for any set Ω of coherent descriptions. To that end, I introduce a belief structure on Ω and show that the description map $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map.

Step 2.1. By slightly abusing the notation, let $[e]_{\overrightarrow{\Omega}} := \{\omega \in \Omega \mid (1, e) \in \omega\}$ for each $e \in \mathcal{L}$. Let $\mathcal{D} := \{[e]_{\overrightarrow{\Omega}} \in \mathcal{P}(\Omega) \mid e \in \mathcal{L}\}$. Note that $[\cdot]_{\overrightarrow{\Omega}} \in \mathcal{D}$ is different from $[\cdot] \in \mathcal{D}^*$. I show that $(\Omega, \mathcal{D})$ is a κ-algebra. First, I show that $[\emptyset]_{\overrightarrow{\Omega}} = \emptyset$. Suppose to the contrary that $\omega \in [\emptyset]_{\overrightarrow{\Omega}}$. Then, by definition, $(1, \emptyset) \in \omega$, which is impossible. Hence, $\emptyset = [\emptyset]_{\overrightarrow{\Omega}} \in \mathcal{D}$. Second, $[S]_{\overrightarrow{\Omega}} \in \mathcal{D}$ follows from $[S]_{\overrightarrow{\Omega}} = \Omega$. Third, I show that $[\neg e]_{\overrightarrow{\Omega}} = \neg [e]_{\overrightarrow{\Omega}}$. Indeed, $\omega \in [\neg e]_{\overrightarrow{\Omega}}$ iff $(1, (\neg e)) \in \omega$ iff $(1, e) \notin \omega$ iff $\omega \in \neg [e]_{\overrightarrow{\Omega}}$. Hence, $\mathcal{D}$ is closed under complementation. Fourth, I show that $[\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}} = \bigcap_{e \in \mathcal{E}} [e]_{\overrightarrow{\Omega}}$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. Indeed, $\omega \in \bigcap_{e \in \mathcal{E}} [e]_{\overrightarrow{\Omega}}$ iff $(1, e) \in \omega$ for all $e \in \mathcal{E}$ iff $(1, \bigwedge \mathcal{E}) \in \omega$ iff $\omega \in [\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}}$. Hence, $(\Omega, \mathcal{D})$ is a κ-algebra.

Step 2.2. Define the mapping $\Theta : \Omega \to S$ which associates, with each $\omega \in \Omega$, the unique nature state $s \in S$ with $(0, s) \in \omega$. The mapping $\Theta : \Omega \to S$ is a well-defined measurable map such that $(\Theta)^{-1}(E) = [E]_{\overrightarrow{\Omega}}$ for each $E \in \mathcal{A}_S$. Indeed, take $E \in \mathcal{A}_S$. If $\omega \in [E]_{\overrightarrow{\Omega}}$, then $(1, E) \in \omega$. Hence, $\Theta(\omega) \in E$, i.e., $\omega \in \Theta^{-1}(E)$. Conversely, if $\omega \in \Theta^{-1}(E)$ then $\Theta(\omega) \in E$, and thus $(1, E) \in \omega$. Hence, $\omega \in [E]_{\overrightarrow{\Omega}}$.

Step 2.3. Define players' belief operators on $(\Omega, \mathcal{D})$. Fix $i \in I$, and define $B_i : \mathcal{D} \to \mathcal{D}$ as $B_i([e]_{\overrightarrow{\Omega}}) := [\beta_i(e)]_{\overrightarrow{\Omega}}$ for each $[e] \in \mathcal{D}$. I first show that B_i is well defined. If $[e]_{\overrightarrow{\Omega}} = [f]_{\overrightarrow{\Omega}}$, then $[(e \leftrightarrow f)]_{\overrightarrow{\Omega}} = \Omega$. This implies that $[(\beta_i(e) \leftrightarrow \beta_i(f))]_{\overrightarrow{\Omega}} = \Omega$. Thus, $[\beta_i(e)] = [\beta_i(f)]$.

Next, I show that B_i reflects assumptions on players' beliefs. Consider the logical properties. For No-Contradiction, if $\omega \in B_i([\emptyset]_{\overrightarrow{\Omega}}) = [\beta_i(\emptyset)]_{\overrightarrow{\Omega}}$, then $(1, \beta_i(\emptyset)) \in \omega$. Since $(\emptyset \leftrightarrow \beta_i(\emptyset)) \in \omega$, it follows that $(1, \emptyset) \in \omega$, which is impossible. Thus, $B_i([\emptyset]_{\overrightarrow{\Omega}}) = \emptyset$. For Consistency, if $\omega \in B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$ then $(1, \beta_i(e)) \in \omega$. Since $(1, (\beta_i(e) \to (\neg\beta_i)(\neg e))) \in \omega$, it follows that $(1, (\neg\beta_i)(\neg e)) \in \omega$, i.e., $\omega \in [(\neg\beta_i)(\neg e)]_{\overrightarrow{\Omega}} = (\neg B_i)(\neg [e]_{\overrightarrow{\Omega}})$.

For Necessitation, $\Omega = [\beta_i(S)]_{\overrightarrow{\Omega}} = B_i([S]_{\overrightarrow{\Omega}}) = B_i(\Omega)$. For Monotonicity, take $[e]_{\overrightarrow{\Omega}}, [f]_{\overrightarrow{\Omega}} \in \mathcal{D}$ with $[e]_{\overrightarrow{\Omega}} \subseteq [f]_{\overrightarrow{\Omega}}$. Then, $[e \to f]_{\overrightarrow{\Omega}} = \Omega$. It follows that $[\beta_i(e) \to$

$\beta_i(f)]_{\overrightarrow{\Omega}} = \Omega$, i.e., $[\beta_i(e)]_{\overrightarrow{\Omega}} \subseteq [\beta_i(f)]_{\overrightarrow{\Omega}}$. Thus, $B_i([e]_{\overrightarrow{\Omega}}) \subseteq B_i([f]_{\overrightarrow{\Omega}})$.

For Non-empty λ-Conjunction, take $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$. If $\omega \in \bigcap_{e\in\mathcal{E}} B_i([e]_{\overrightarrow{\Omega}}) = [\bigwedge_{e\in\mathcal{E}} \beta_i(e)]_{\overrightarrow{\Omega}}$ then $(1, \bigwedge_{e\in\mathcal{E}} \beta_i(e)) \in \omega$. Since $(1, (\bigwedge_{e\in\mathcal{E}} \beta_i(e) \to \beta_i(\bigwedge \mathcal{E}))) \in \omega$, it follows that $(1, \beta_i(\bigwedge \mathcal{E})) \in \omega$, i.e., $\omega \in [\beta_i(\bigwedge \mathcal{E})]_{\overrightarrow{\Omega}} = B_i(\bigcap_{e\in\mathcal{E}} [e]_{\overrightarrow{\Omega}})$. For the Kripke property, $\omega \in B_i([e]_{\overrightarrow{\Omega}})$ for any $(\omega, [e]_{\overrightarrow{\Omega}}) \in \Omega \times \mathcal{D}$ such that $\bigcap\{[f]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid \omega \in B_i([f]_{\overrightarrow{\Omega}})\} \subseteq [e]_{\overrightarrow{\Omega}}$.

Moving on to the introspective properties, consider Truth Axiom. If $\omega \in B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$ then $(1, \beta_i(e)) \in \omega$. Since $(1, (\beta_i(e) \to e)) \in \omega$, it follows that $(1, e) \in \omega$, i.e., $\omega \in [e]_{\overrightarrow{\Omega}}$.

For Positive Introspection, if $\omega \in B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$ then $(1, \beta_i(e)) \in \omega$. Since $(1, (\beta_i(e) \to \beta_i\beta_i(e))) \in \omega$, it follows that $(1, \beta_i\beta_i(e)) \in \omega$, i.e., $\omega \in [\beta_i\beta_i(e)]_{\overrightarrow{\Omega}} = B_iB_i([e]_{\overrightarrow{\Omega}})$. For Negative Introspection, if $\omega \in (\neg B_i)([e]_{\overrightarrow{\Omega}}) = [(\neg\beta_i)(e)]_{\overrightarrow{\Omega}}$ then $(1, (\neg\beta_i)(e)) \in \omega$. Since $(1, ((\neg\beta_i)(e) \to \beta_i(\neg\beta_i)(e))) \in \omega$, it follows that $(1, \beta_i(\neg\beta_i)(e)) \in \omega$, i.e., $\omega \in [\beta_i(\neg\beta_i)(e)]_{\overrightarrow{\Omega}} = B_i(\neg B_i)([e]_{\overrightarrow{\Omega}})$.

Step 2.4. The above arguments establish that $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, \Theta\rangle$ is a belief space (of the given category). Finally, I show that the description map $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map so that $\Omega \subseteq \Omega^*$.

To that end, I establish by induction that $[\cdot]_{\overrightarrow{\Omega}} : \mathcal{L} \to \mathcal{D}$, viewed as a mapping, coincides with the semantic interpretation function $[\![\cdot]\!]_{\overrightarrow{\Omega}}$. First, fix $E \in \mathcal{A}_S$. Then, $\omega \in [\![E]\!]_{\overrightarrow{\Omega}} = \Theta^{-1}(E)$ iff $\Theta(\omega) \in E$ iff $\omega \in [E]_{\overrightarrow{\Omega}}$. Second, supposing $[\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$, I have $\omega \in [\![\neg e]\!]_{\overrightarrow{\Omega}} = \neg[\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega \notin [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega \notin [e]_{\overrightarrow{\Omega}}$ iff $\omega \in [\neg e]_{\overrightarrow{\Omega}}$. Third, suppose that $[\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$ for all $e \in \mathcal{E}$ with $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ and $|\mathcal{E}| < \kappa$. Then, $\omega \in [\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}} = \bigcap_{e\in\mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega \in \bigcap_{e\in\mathcal{E}} [e]_{\overrightarrow{\Omega}} = [\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}}$. Fourth, supposing $[\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$, I have $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega}} = B_i([\![e]\!]_{\overrightarrow{\Omega}}) = B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$. The induction is complete.

Now, I show that $D(\omega) = \omega$ for all $\omega \in \Omega$. First, $(1, e) \in D(\omega)$ iff $D(\omega) \in [e]$ iff $\omega \in D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$ iff $(1, e) \in \omega$. Second, if $(0, s) \in \omega$ then $s = \Theta(\omega) = \Theta^*(D(\omega))$, and thus $(0, s) \in D(\omega)$. Conversely, if $(0, s) \in D(\omega)$ then $s = \Theta^*(D(\omega)) = \Theta(\omega)$, and thus $(0, s) \in \omega$. This completes the proof of the statement that D is an inclusion map, and the proof of Theorem 2 is complete. □

Proof of Proposition 4. Part 1. Let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ be a morphism. In a similar way to Heifetz and Samet (1998a), I show by induction that $\mathcal{C}_\alpha = \varphi^{-1}(\mathcal{C}'_\alpha)$ for all α.

For $\alpha = 0$, $\mathcal{C}_0 = \varphi^{-1}(\mathcal{C}'_0)$ follows because $(\Theta)^{-1}(E) = \varphi^{-1}((\Theta')^{-1}(E))$ for any

$E \in \mathcal{A}_S$. Suppose that $\mathcal{C}_\beta = \varphi^{-1}(\mathcal{C}'_\beta)$ for all $\beta < \alpha$. Then,

$$\mathcal{C}_\alpha = \mathcal{A}_\kappa \left(\{\varphi^{-1}(E') \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\} \cup \bigcup_{i\in I} \{\varphi^{-1}(B'_i(E')) \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\} \right)$$
$$= \varphi^{-1} \left(\mathcal{A}_\kappa \left(\{E' \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\} \cup \bigcup_{i\in I} \{B'_i(E') \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\} \right)\right) = \varphi^{-1}(\mathcal{C}'_\alpha).$$

Thus, if $\mathcal{C}'_\alpha = \mathcal{C}'_{\alpha+1}$, then $\mathcal{C}_\alpha = \mathcal{C}_{\alpha+1}$. In other words, if the κ-rank of $\overrightarrow{\Omega'}$ is α then that of $\overrightarrow{\Omega}$ is at most α.

Part 2. Fix a κ-belief space $\overrightarrow{\Omega}$. Define $\mathcal{D}_\alpha := \mathcal{A}_\kappa(\{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}_\alpha\})$ for each $\alpha \leq \kappa$, where $\mathcal{L}_\alpha$ is defined in Remark 1. Note that $\mathcal{D}_\kappa = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$. I show that $\mathcal{D}_\alpha = \mathcal{C}_\alpha$ for all $\alpha \leq \kappa$. For $\alpha = 0$, $\mathcal{D}_0 = \{\Theta^{-1}(E) \in \mathcal{D} \mid E \in \mathcal{A}_S\} = \mathcal{C}_0$. Next, if $\mathcal{D}_\beta = \mathcal{C}_\beta$ for all $\beta < \alpha$, then

$$\mathcal{D}_\alpha = \mathcal{A}_\kappa \left(\left(\bigcup_{\beta<\alpha} \mathcal{D}_\beta \right) \cup \bigcup_{i\in I} \{B_i([\![e]\!]) \in \mathcal{D} \mid [\![e]\!] \in \bigcup_{\beta<\alpha} \mathcal{D}_\beta\} \right) = \mathcal{C}_\alpha.$$

Hence, $\mathcal{C}_\kappa = \mathcal{D}_\kappa = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$, implying $\mathcal{C}_\kappa = \mathcal{C}_{\kappa+1}$, i.e., the κ-rank of $\overrightarrow{\Omega}$ is at most κ. □

Proof of Corollary 1. Construct Ω^* by collecting all the probabilistic belief spaces as in the proof of Theorem 1. The set Ω^* is not empty (consider $\overrightarrow{\{s\}}$). The tuple $\overrightarrow{\Omega^*} := \langle(\Omega^*, \mathcal{D}^*), (B_i^{*p})_{(i,p)\in I\times[0,1]}, \Theta^*\rangle$ is defined as in the proof of Theorem 1. To see that $\overrightarrow{\Omega^*}$ is a probabilistic belief space, it suffices to show that the p-belief operators B_i^{*p} satisfy the specified properties of probabilistic beliefs in Definition 11 (2). Then, as in the proof of Theorem 1, $\overrightarrow{\Omega^*}$ is universal.

First, (2a), (2b), (2d), and (2h) follow from Lemma A.1 (1a). Next, (2c) follows from Lemma A.1 (2a). Next, (2g) follows from Lemma A.1 (2a) and (3a).

Next, (2e) and (2f) follow from the following variant of Lemma A.1 (1a). Let $f_{\overrightarrow{\Omega}} : \mathcal{D}^2 \to \mathcal{D}$ and $g_{\overrightarrow{\Omega}} : \mathcal{D}^2 \to \mathcal{D}$ are such that, for any measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$, $\varphi^{-1} f_{\overrightarrow{\Omega'}}(E', F') = f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'), \varphi^{-1}(F'))$ and $\varphi^{-1} g_{\overrightarrow{\Omega'}}(E', F') = g_{\overrightarrow{\Omega}}(\varphi^{-1}(E'), \varphi^{-1}(F'))$ for all $E', F' \in \mathcal{D}'$. If $f_{\overrightarrow{\Omega}}(E, F) \subseteq g_{\overrightarrow{\Omega}}(E, F)$ (for all $E', F' \in \mathcal{D}'$) holds for every belief space $\overrightarrow{\Omega}$, then $f_{\overrightarrow{\Omega^*}}([e], [f]) \subseteq g_{\overrightarrow{\Omega^*}}([e], [f])$ for all $[e], [f] \in \mathcal{D}^*$.

For (2i), if $[t^*_{B^*_i}(\omega^*)] \subseteq [e]$ then $[t_{B_i}(\omega)] \subseteq D^{-1}[e]$ and thus $\omega \in B^1_i(D^{-1}[e]) =$

$D^{-1}B_i^{*1}([e])$, where a probabilistic belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. $\square$

Proof of Corollary 2. Construct Ω^*, as in the proof of Theorem 1, by viewing the set of players in each conditional belief space as $\overline{I} := I \times [0,1] \times \mathcal{C}_S$. To see that Ω^* is not empty, take a CPS μ on $(S, \mathcal{A}_S, \mathcal{C}_S)$. Consider $\langle (S, \mathcal{A}_S, \mathcal{C}_S), (B^p_{i,C})_{(i,p,C)\in\overline{I}}, \mathrm{id}_S \rangle$, where for any $(i, p, C, E) \in \overline{I} \times \mathcal{D}$, (i) $B^p_{i,C}(E) := \emptyset$ if $\mu(E|C) < p$; and (ii) $B^p_{i,C}(E) := S$ if $\mu(E|C) \geq p$.

Next, as in the proof of Theorem 1, define $\mathcal{D}^*$, Θ^*, and an auxiliary collection of p-belief operators $(B^{*p}_{i,C_S})_{(i,p,C_S)\in\overline{I}}$ as $B^{*p}_{i,C_S}([e]) := [\beta^p_{i,C_S}(e)]$ for each $[e] \in \mathcal{D}^*$. By construction, $D^{-1}(B^{*p}_{i,C_S}([e])) = B^p_{i,C_S}(D^{-1}[e])$. Since $(\Theta^*)^{-1}(C_S) = [C_S] \in \mathcal{D}^*$, let $\mathcal{C}^* := \{[C_S] \in \mathcal{D}^* \mid C_S \in \mathcal{A}_S\}$. By construction, $\mathcal{C}^* \subseteq \mathcal{D}^*$, $(\Theta^*)^{-1}(\mathcal{C}_S) = \mathcal{C}^*$, and $\emptyset \notin \mathcal{C}^*$ (this is because Θ^* is surjective). Then, $(\Omega^*, \mathcal{D}^*, \mathcal{C}^*)$ is a conditional space, and $(B^{*p}_{i,C_S})_{(i,p,C_S)\in\overline{I}}$ is a well-defined collection of p-belief operators (observe that $B^{*p}_i(\cdot|[C_S]) = B^{*p}_{i,C_S}$). As in the proof of Corollary 1, the p-belief operators satisfy the specified properties, i.e., $\overrightarrow{\Omega^*} := \langle (\Omega^*, \mathcal{D}^*, \mathcal{C}^*), (B^{*p}_i(\cdot|[C_S]))_{(i,p,[C_S])\in I\times[0,1]\times\mathcal{C}^*}, \Theta^* \rangle$ is a conditional belief space. By construction, $\overrightarrow{\Omega^*}$ is universal. $\square$

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