Epistemic Foundations for Set-algebraic Representations of Knowledge*

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Abstract

This paper formalizes an informal idea that an agent's knowledge is characterized by a collection of sets such as a σ-algebra within the framework of a state space model. The paper fully characterizes why the agent's knowledge takes (or does not take) such a set algebra as a σ-algebra or a topology, depending on logical and introspective properties of knowledge and on the underlying structure of the state space. The agent's knowledge is summarized by a collection of events if and only if she can only know what is true, she knows any logical implication of what she knows, and she is introspective about what she knows. In this case, for any event, the collection that represents knowledge has the maximal event included in the original event. When the underlying space is a measurable space, the collection becomes a σ-algebra if and only if the agent is additionally introspective about what she does not know.

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1 Introduction

Economic agents base their decisions on their knowledge about uncertainty that they face. Where does their knowledge come from? Researchers often represent an agent's knowledge by a (sub-)σ-algebra on an underlying state space. For each element E of such σ-algebra, the agent is supposed to "know" whether the set E obtains at a realized state ω (i.e., ω lies in E) in an informal sense. This informal understanding of

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the σ-algebra as representing the agent's knowledge helps researchers to connect various measure- and probability-theoretical formal apparatus and their informal ideas behind economic problems at hand.

The main objective of this paper is to provide epistemic foundations for representing an agent's knowledge by a collection of sets in terms of the agent's logical and introspective properties of knowledge and of the structure of an underlying state space. Conceptually, this paper fully characterizes the hidden assumptions on knowledge when researchers use a particular form of a set algebra. The paper also clarifies the formal sense in which a given collection of subsets of the underlying states has an informational content. Especially, it fully answers the question of when a σ-algebra is an adequate tool for representing one's knowledge.

To fix an idea, consider an agent named Ashley. The underlying space of uncertainty is a pair of a set of states and a collection of subsets of the states (i.e., events) about which she reasons. An event describes a certain aspect of the states. To describe her knowledge most generally within state space models, represent it by her knowledge operator that maps each event E to the event that she knows E.

Under what conditions on Ashley's knowledge, if possible, can the analysts represent her knowledge by a set algebra such as a σ-algebra or a topology? The main result of the paper (Theorem 1) shows that one can represent Ashley's knowledge by a set algebra if and only if her knowledge satisfies the *basic conditions*: Truth Axiom, Monotonicity, and Positive Introspection. By Truth Axiom, if Ashley knows an event at a state, then the event is true at that state. Thus, Ashley's knowledge is truthful in contrast to her beliefs, as she may believe something false. By Monotonicity, if Ashley knows an event E and if E implies an event F, then she knows F. By Positive Introspection, if Ashley knows an event E, then she knows that she knows E.

Roughly, the main result is stated as follows. Under the basic conditions, the collection of events E which Ashley knows whenever E obtains (i.e., E is self-evident) satisfies the "maximality property:" for any event E, the self-evident collection contains the maximal event included in E. Conversely, let $\mathcal{J}$ be a collection of events with the maximality property. Then, one can construct a knowledge operator satisfying the basic conditions and whose self-evident collection coincides with $\mathcal{J}$ as follows: an event E is known at a state ω if and only if there is an event $F \in \mathcal{J}$ which is true at ω and which implies E (i.e., $\omega \in F \subseteq E$). If $\mathcal{J}$ is taken as Ashley's self-evident events, then it recovers her original knowledge operator.

Moreover, additional properties of knowledge are represented as set-algebraic properties of self-evident events. I fully characterize when Ashley's knowledge forms such a set algebra as a σ-algebra or a topology depending on her additional properties and on the structure of the underlying states. When the underlying space is sufficiently rich, Ashley's knowledge forms a topology if and only if she additionally knows a tautology and conjunctions of what she knows. Her knowledge forms a subalgebra of the underlying space (e.g., a sub-σ-algebra of a measurable space) if and only if her knowledge additionally satisfies Negative Introspection: if she does not know an event then she knows that she does not know it. Interestingly, Negative Introspection, together with Truth Axiom and Monotonicity, implies the conjunction property (Corollary 1). This representation is convenient as one can introduce fully-introspective knowledge on a probability space in a well-defined manner. For example, epistemic analyses of dynamic games often call for both knowledge and probabilistic beliefs (e.g., Dekel and Gul (1997)). However, standard partitional knowledge, defined for any subset of states, may not always be measurable.

The maximality property is originally studied by Salonen (2009) and Samet (2010), which examine the relation between an information partition and a fully-introspective knowledge operator on an algebra of sets. The technical contribution of the main result is mild. Conceptually, however, it fully identifies: (i) the minimum conditions for knowledge to be represented by a collection of events (i.e., Truth Axiom, Monotonicity, and Positive Introspection) irrespective of the structure of underlying states; and (ii) when knowledge can be summarized by such a different set-algebra as a σ-algebra or a topology, depending on properties of knowledge and underlying states. Further discussions will be given in the next subsection.

Does there exist the smallest collection of events $\mathcal{J}(\mathcal{D}')$ including a given collection $\mathcal{D}'$ and satisfying the maximality property? Call it the *information closure* of $\mathcal{D}'$. Proposition 2 provides the condition under which a given collection has an informational content in that its information closure exists. This proposition makes it possible to compare collections of events according to informativeness (Corollary 2).

To demonstrate that information closures preserve informational contents of collections, consider a collection of information sets. Proposition 1 characterizes when Ashley's knowledge is induced from her information sets on an arbitrary state space such as a measurable space. Her information set associated with a given state is the set of states that she considers possible at that state, even though she cannot identify the realized state. Thus, Ashley knows an event E at a state if her information set at that state implies (i.e., is included in) the event E.

Proposition 3 states that Ashley's self-evident events coincide with the information closure of her information sets. That is, her knowledge is induced by information sets if and only if the information sets have the informational content. Also, the information closure turns out to be the smallest σ-algebra or topology depending on assumptions on knowledge (i.e., the nature of information sets) and the state space. For example, information sets form a partition if and only if knowledge satisfies Negative Introspection in addition to the basic conditions. In this case, the information closure forms a sub-algebra (e.g., a sub-σ-algebra of a measurable space). If the information partition is countable, then its information closure coincides with the smallest σ-algebra. Moreover, Corollary 3 states that the finer the information sets are, the larger (or, the more informative) the information closure is. To complete the discussion that information closures preserve informativeness, Proposition 4 establishes the Blackwell-type relation, as in Dubra and Echenique (2004), between preferences over signals (i.e., mappings defined on underlying states) and information closures generated by signals.

The paper is organized as follows. The rest of this section discusses the technical contributions of this paper, especially, how it resolves the problems documented in the literature on the sense in which σ-algebras represent knowledge. Section 2 sets up a framework. Section 3 presents the main results. Section 3.1 establishes the equivalence between a knowledge operator and a collection of events. Section 3.2 studies the informational content of a given collection. Section 3.3 studies its implications such as how a collection that represents knowledge is generated from information sets. Section 4 provides concluding remarks. Proofs are relegated to Appendix A.

1.1 Technical Overview

Partitions and σ-algebras. It has been argued by Billingsley (2012) and Dubra and Echenique (2004) that the σ-algebras generated from partitions may not necessarily preserve informational contents of partitions, as such σ-algebras may not always be closed under arbitrary unions of partitions (see also Nielsen (1984) and Tobias (2019)). Subsequently, Hérves-Beloso and Monteiro (2013) and Lee (2018) study the technical condition on an underlying space under which countable measurable partitions and generated σ-algebras equivalently represent information. Lee (2018) goes on further to show that the equivalence fails without the technical condition.

Instead of taking the smallest σ-algebra $\sigma(\cdot)$ as *given*, this paper proposes the information closure $\mathcal{J}(\cdot)$ by answering a conceptual question: what does it mean by a set algebra representing knowledge, and how does the answer depend on properties of knowledge and underlying states? The paper shows that partitions and the information closures of the partitions are equivalent and that the latter form σ-algebras satisfying the maximality property.

This paper provides the following unified view (i) on the sense in which a σ-algebra captures an agent's knowledge and (ii) on the relation between an information partition and a σ-algebra on a measurable space of uncertainty, solely based on properties of knowledge that researchers intend to assume and on the structure of the underlying space. The key is that the information closure of a partition depends on the structure of an underlying space, because the maximality property is defined with respect to the ambient structure.

On the one hand, under the technical condition on an underlying measurable space by Hérves-Beloso and Monteiro (2013) and Lee (2018), the information closure $\mathcal{J}(\cdot)$ reduces to the smallest σ-algebra $\sigma(\cdot)$, which generates a sub-σ-algebra of the underlying measurable space. Thus, my result replicates their positive equivalence result under their condition on the ambient structure, solely from the properties of knowledge: Ashley's self-evident events turn out to be the σ-algebra generated by her countable information partition if and only if her knowledge is induced by her countable information partition.

On the other hand, the maximality property confirms Dubra and Echenique (2004) in the following two ways. First, the maximality property implies a particular form of closure under union: an arbitrary union is in the information closure as long as it is a well-defined event of the underlying space. However, such arbitrary union is a valid object only when it is an event, as Lee (2018) points out that the general non-equivalence between partitions and generated σ-algebras comes from the fact that an arbitrary union of partition cells may not be measurable on the underlying measurable space. Thus, for example, the finest partition consisting of singleton sets generates the entire collection of events (which is not necessarily the power set). Second, Dubra and Echenique (2004) implicitly assume that an ambient structure is the power set algebra. If an underlying structure is the power set (precisely, complete) algebra, then the maximality property reduces exactly to the closure under arbitrary union. Thus, for a power set algebra, this paper provides an epistemic foundation for Dubra and Echenique (2004): a set algebra representing knowledge is closed under arbitrary union if and only if knowledge satisfies the basic conditions.

Proposition 3 and Corollary 3 show that partitions and the information closures of partitions (which are σ-algebras satisfying the maximality property) are equivalent. While it has been recognized that not every σ-algebra has an informational content, what is new is that the paper fully characterizes the class of σ-algebras that have informational contents. As a result, whenever the technical condition is met for an underlying structure, the σ-algebra generated by a countable partition is derived as its information closure. The subtlety lies in the fact that the information closure depends on the structure of an ambient space.

σ-algebras and Topologies. Here, I compare σ-algebras and topologies in terms of assumptions implicitly put on knowledge. Various strands of economics literature use σ-algebras to represent agents' information. In Wilson (1978), for instance, an event E is in an agent's set algebra if and only if she knows whether a prevailing state is in E or E^c.[1] At the same time, researchers in various fields (e.g., Shin (1993) in economics) also represent knowledge using a topology, especially the one closed under arbitrary intersection.[2]

The crucial difference is Negative Introspection. Take a self-evident event E to an

[1]See also Radner (1968) and Yannelis (1991) for representing agents' information as σ-algebras in general equilibrium theory. Hérves-Beloso and Monteiro (2013) and Lee (2018) study Wilson (1978)'s idea by information partitions.

[2]In computer science, logic, mathematics, and philosophy, see, for example, Barwise and Etchemendy (1990), Chagrov and Zakharyaschev (1997), Pacuit (2017), Vickers (1989), and the references therein. While this paper does not aim at surveying a topology closed under arbitrary intersection, such a topology is termed as an Alexsandroff topology (e.g., Pacuit (2017), Vickers (1989), and the references therein), a complete ring (Birkhoff, 1987), a saturated topology (Lorrain, 1969), a principal topology (Steiner, 1966), and so on. Such a topology is also characterized as the one that has the minimum open neighborhood (i.e., the information set) at each state. Also, such a topology has a one-to-one correspondence with a pre-order (a reflexive-and-transitive relation) often called a specialization pre-order.

agent. Under Negative Introspection (and Truth Axiom), the agent knows whether a prevailing state is in E or E^c. Without Negative Introspection, she may not necessarily know the negation E^c at a state at which E is false. While the agent can discern whether E is true or not under Negative Introspection, she can only know E is true whenever E obtains without Negative Introspection. This intuition is exactly what non-partitional knowledge models aim to deliver: agents take information only at face value.[3] Indeed, non-partitional knowledge models are equivalently characterized by topologies closed under arbitrary intersection.

While Proposition 3 implies that the information closure of an information partition on a measurable space is a σ-algebra, it states that the information closure of non-partitional information sets forms a topology on a rich underlying structure. Thus, while the previous point has argued that the information closure of a partition depends on an underlying space, this point makes it clear that the information closure of information sets also depends on a property of knowledge (Negative Introspection in this case) that researchers intend to assume. In either case, information closures preserve informational contents of information sets.

2 Framework

This section introduces a framework which represents agents' knowledge by knowledge operators on a "sample space." I start with three technical definitions on collections of sets. Fix a set Ω. First, for any infinite cardinal number κ, a subset $\mathcal{D}$ of the power set $\mathcal{P}(\Omega)$ is a *κ-complete algebra* (on Ω) if $\mathcal{D}$ is closed under complementation and under arbitrary union (and consequently intersection) of any sub-collection with cardinality less than κ (i.e., *κ-union* and *κ-intersection*). I follow the conventions that $\emptyset = \bigcup \emptyset \in \mathcal{D}$ and that, with Ω being an underlying set, $\Omega = \bigcap \emptyset \in \mathcal{D}$. For example, denoting by $\aleph_0$ the least infinite cardinal, an $\aleph_0$-complete algebra is an algebra of sets. Denoting by $\aleph_1$ the least uncountable cardinal, an $\aleph_1$-complete algebra is a σ-algebra.[4] Second, a subset $\mathcal{D}$ of $\mathcal{P}(\Omega)$ is an (∞-)*complete algebra* (on Ω) if $\mathcal{D}$ is closed under complementation and under arbitrary union and intersection. Throughout, κ denotes an infinite cardinal or the symbol ∞. Call $(\Omega, \mathcal{D})$ a κ-complete algebra if $\mathcal{D}$ is a κ-complete algebra on Ω.

[3] Non-partitional knowledge models study (i) implications on interactive knowledge and solution concepts in games and (ii) information processing leading to the violation of Negative Introspection. See, for instance, Bacharach (1985), Binmore and Brandenburger (1990), Brandenburger, Dekel, and Geanakoplos (1992), Dekel and Gul (1997), Geanakoplos (1989), Morris (1996), Samet (1990), and Shin (1993). Also, the violation of Negative Introspection is related to a notion of unawareness in that an agent does not know an event and she does not know that she does not know it. For unawareness, see, for example, Dekel, Lipman, and Rustichini (1998), Fagin and Halpern (1987), Modica and Rustichini (1994, 1999), and Schipper (2015). Fukuda (2018b) studies non-trivial unawareness by an agent whose knowledge satisfies the basic conditions.

[4] As in Meier (2006, Remark 1), κ can be assumed to be an infinite regular cardinal. Note that $\aleph_0$ and $\aleph_1$ are regular.

2.1 Knowledge Representation by a Knowledge Operator

Let I be a non-empty set of agents. A (κ-)*knowledge space* (of I) is a tuple $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (K_i)_{i \in I}\rangle$ such that $(\Omega, \mathcal{D})$ is a κ-complete algebra and that $K_i : \mathcal{D} \to \mathcal{D}$ is agent i's knowledge operator. Each element ω of Ω is a *state*, $\mathcal{D}$ is the *domain*, and each element E of $\mathcal{D}$ is an *event*. An operator $K_i : \mathcal{D} \to \mathcal{D}$ is a *knowledge operator* if it satisfies the *basic conditions*: (i) *Truth Axiom* ($K_i(E) \subseteq E$ for any $E \in \mathcal{D}$); (ii) *Monotonicity* ($E \subseteq F$ implies $K_i(E) \subseteq K_i(F)$); and (iii) *Positive Introspection* ($K_i(\cdot) \subseteq K_i K_i(\cdot)$). An agent i knows an event E at a state ω if $\omega \in K_i(E)$. Thus, $K_i(E)$ denotes the event that agent i knows E.

The rest of this subsection discusses the components of the κ-knowledge space. I start with the domain $\mathcal{D}$. While standard possibility correspondence models often take the power set of underlying states as its domain, I allow the domain to be a κ-complete algebra for the following two reasons. First, this paper shows that the informational content of a collection of events, or the properties of knowledge that the given collection intends to represent, depends on the underlying structure of a domain, i.e., a choice of κ. In the literature examining the sense in which σ-algebras capture agents' information, for example, Dubra and Echenique (2004) argue that the σ-algebra generated from a partition may fail to capture the informational content of the partition as it may not be closed under arbitrary unions of the partition. As they consider the power set algebra as an underlying space, Theorem 1 in Section 3.1 shows that, on an ∞-complete algebra (but not necessarily an $\aleph_1$-algebra), the collection of events that captures knowledge is closed under arbitrary union if and only if (hereafter, abbreviated as iff) knowledge satisfies the basic conditions.

Second, I expand the class of knowledge spaces so that both knowledge and belief can be defined on the same domain. Epistemic analyses of dynamic games often call for both knowledge and probabilistic beliefs. In the current framework, knowledge can be introduced as objects of probabilistic beliefs on a probability space. Even for knowledge alone, it would be necessary to specify $\mathcal{D}$ as a κ-complete algebra for logical foundations, as events generated by some logical system may not necessarily form the power set.[5] In so doing, κ determines possible depths of agents' reasoning. An $\aleph_0$-complete algebra can accommodate finite depths of reasoning such as "Alice knows that Bob knows that it is raining." An $\aleph_1$-complete algebra can accommodate countable depths of reasoning such as the iterative definition of common knowledge.[6]

Henceforth, since I almost always deal with the knowledge of a single agent, fix a generic agent i and drop the subscript unless otherwise stated. Now, I discuss the assumptions on knowledge. Truth Axiom says that the agent can only know what is true, as opposed to beliefs that can be false. It implies *Consistency*: $K(E) \subseteq$

[5]Such semantic knowledge models include Aumann (1999), Bacharach (1985), Samet (1990), and Shin (1993). A knowledge operator in Samet (2010) is defined on an ($\aleph_0$-complete) algebra.

[6]In a related context, rationalizability would call for transfinite levels of reasoning in the form of eliminations of never-best-replies. See, for instance, Chen, Luo, and Qu (2016) and Lipman (1994).

$(\neg K)(E^c)$ for any $E \in \mathcal{D}$. It means that, if the agent knows an event E then she *considers E possible* in that she does not know its negation E^c. This notion of possibility is the dual of knowledge in standard state space models. Define the *possibility operator* $L_K : \mathcal{D} \to \mathcal{D}$ by $L_K(E) := (\neg K)(E^c)$. Consistency then means that knowledge implies possibility. Next, Monotonicity says that if the agent knows some event then she knows any of its logical consequences. Positive Introspection states that if the agent knows some event then she knows that she knows it.

Next, I introduce three additional logical and introspective properties of knowledge. First, *Necessitation* refers to $K(\Omega) = \Omega$: the agent knows any tautology such as $E \cup E^c$. Second, letting λ be a fixed infinite cardinal with $\lambda \leq \kappa$ or letting $\lambda = \kappa = \infty$, *Non-empty λ-Conjunction* refers to $\bigcap_{E \in \mathcal{E}} K(E) \subseteq K(\bigcap \mathcal{E})$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$. It states that the agent knows any non-empty conjunction of events with cardinality less than λ if she knows each. The converse set inclusion follows from (indeed, is equivalent to) Monotonicity. By identifying Necessitation as the empty conjunction, *λ-Conjunction* refers jointly to Non-empty λ-Conjunction and Necessitation. I refer to (Non-empty) $\aleph_0$- and $\aleph_1$-Conjunction, respectively, as (*Non-empty*) *Finite* and *Countable Conjunction*.

Third, *Negative Introspection* refers to $(\neg K)(\cdot) \subseteq K(\neg K)(\cdot)$: if the agent does not know some event then she knows that she does not know it. Together with Truth Axiom, Negative Introspection holds with equality ($(\neg K) = K(\neg K)$), which, in turn, implies Positive Introspection (e.g., Aumann (1999, p. 270)). While standard partitional knowledge models presuppose Negative Introspection, non-partitional knowledge models (see Footnote 3) dispense with Negative Introspection on the ground that it entails a strong form of rationality. Corollary 1 in Section 3.1 shows that Negative Introspection, Monotonicity, and Truth Axiom imply κ-Conjunction.

2.2 Knowledge Representation by Information Sets

Before I represent the agent's knowledge by a set algebra in Section 3, here I examine the condition under which her knowledge is derived from a possibility correspondence: it associates, with each state, the set of states that the agent considers possible. That is, I ask the condition under which a κ-knowledge space is identified as a possibility correspondence model of knowledge on a κ-complete algebra. Throughout the subsection, fix a κ-knowledge space $\langle(\Omega, \mathcal{D}), K\rangle$.

I define a notion of possibility between states (i.e., a possibility relation). A state ω' is considered *possible* at a state ω if, for any event $E \in \mathcal{D}$ which the agent knows at ω, the event E is true at ω'. Define $b_K : \Omega \to \mathcal{P}(\Omega)$ so that $b_K(\omega)$, the *information set* at ω, is the set of states that the agent considers possible at ω:

$$b_K(\omega) := \{\omega' \in \Omega \mid \omega' \in E \text{ for all } E \in \mathcal{D} \text{ with } \omega \in K(E)\} = \bigcap\{E \in \mathcal{D} \mid \omega \in K(E)\}.$$

Note that $b_K(\omega)$ may not necessarily be an event when $\mathcal{D}$ is not a complete algebra.

Now, K satisfies the *Kripke property* if $\omega \in K(E)$ if (and only if) $b_K(\omega) \subseteq E$ for all $(\omega, E) \in \Omega \times \mathcal{D}$. It means that the agent knows an event E at a state ω if(f) E contains any state considered possible at ω. The Kripke property implies Monotonicity and κ-Conjunction.

How can one characterize the Kripke property? If $\kappa = \infty$, the Kripke property is equivalent to Monotonicity and κ-Conjunction (e.g., Morris (1996) when $\mathcal{D} = \mathcal{P}(\Omega)$). On a general κ-complete algebra, however, the converse is not necessarily true. Samet (2010) provides specific examples where a knowledge operator on an ($\aleph_0$-complete) algebra satisfies the "S5" properties (i.e., Monotonicity, Finite Conjunction, Truth Axiom, Positive Introspection, and Negative Introspection) but is not derived from partitional information sets.

The following proposition characterizes the Kripke property. It generalizes Samet (2010, Theorem)'s condition under which "S5" knowledge is derived from partitional information sets in an $\aleph_0$-knowledge space. In the case of $\kappa = \aleph_1$, the proposition identifies the condition under which the agent's knowledge on a probabilistic-belief space can be induced by a possibility correspondence.[7]

Proposition 1. *The following characterizes the Kripke property of K. If $(\omega, F) \in \Omega \times \mathcal{D}$ satisfies $E \cap F \neq \emptyset$ for all $E \in \mathcal{D}$ with $\omega \in K(E)$, then $b_K(\omega) \cap F \neq \emptyset$.*

Three remarks are in order. First, Proposition 1 does not hinge on Truth Axiom or Positive Introspection. The only requirement is Monotonicity. Thus, the proposition can be used to check whether any given monotonic operator (e.g., a (common) belief operator that may fail Truth Axiom) satisfies the Kripke property. Also, the proposition shows that the Kripke property on K under which it is derived from information sets turns out to be different from the basic conditions on K (under which it is summarized by a set algebra).

When a general operator K satisfies the Kripke property, the standard characterizations of the introspective properties of K in terms of b_K extends to any κ-algebra $(\Omega, \mathcal{D})$. For example, K satisfies Truth Axiom and Positive Introspection iff b_K is reflexive (i.e., $\omega \in b_K(\omega)$) and transitive (i.e., $\omega' \in b_K(\omega)$ implies $b_K(\omega') \subseteq b_K(\omega)$). Likewise, K satisfies Truth Axiom, Positive Introspection, and Negative Introspection iff b_K yields a partition on Ω.

Second, Proposition 1 characterizes the condition under which K is induced by *some* possibility correspondence (not necessarily b_K).

Remark 1. There exists $b : \Omega \to \mathcal{P}(\Omega)$ with $K(E) = \{\omega \in \Omega \mid b(\omega) \subseteq E\}$ for all $E \in \mathcal{D}$ iff K satisfies the Kripke property (i.e., $K(E) = \{\omega \in \Omega \mid b_K(\omega) \subseteq E\}$ for all $E \in \mathcal{D}$). In this case, (i) $b(\cdot) \subseteq b_K(\cdot)$ and (ii) if $b(\cdot) \in \mathcal{D}$ then $b = b_K$.

[7] While game-theoretical analyses would call for both knowledge and beliefs, the characterization of the Kripke property on a σ-algebra is non-trivial because the knowledge of an event is written as the union of all information sets contained in the event. For example, Meier (2008, p. 56) puts: "We do not know of any natural, and not too restrictive condition on the partitions of the players that would guarantee that the derived knowledge operators send measurable sets to measurable sets."

The claim (i) states that b_K is the coarsest one inducing K. This is because $b(\omega) \subseteq E$ whenever $\omega \in K(E)$. The claim (ii) states that if b is $\mathcal{D}$-valued (i.e., $b : \Omega \to \mathcal{D}$), then K is induced by the unique possibility correspondence $b_K = b$. This is because $\omega \in K(b(\omega))$. Hence, if b_K is not $\mathcal{D}$-valued, then there is no $\mathcal{D}$-valued possibility correspondence. Generally, it is not easy to check if a possibility correspondence on a κ-complete algebra is $\mathcal{D}$-valued (e.g., Green (2012) and Hérves-Beloso and Monteiro (2013) when $\kappa = \aleph_1$). The claim (ii) suggests that one can restrict attention to b_K to examine the existence of a $\mathcal{D}$-valued possibility correspondence.

Third, the possibility in terms of $\omega' \in b_K(\omega)$ and the possibility in the sense of $\omega \in L_K(\{\omega'\})$ are equivalent if every singleton set $\{\omega'\}$ is an event by Monotonicity of K: $b_K(\omega) = \{\omega' \in \Omega \mid \omega \in L_K(\{\omega'\})\}$. For example, Morris (1996) introduces a possibility correspondence from the possibility operator L_K.

3 Main Results

I provide a set-algebraic representation of knowledge, which embodies the intuition that the content of knowledge can be captured by a sub-collection of a given domain. To that end, I introduce the following three definitions. First, an event E is *self-evident* (to the agent) if $E \subseteq K(E)$. In words, E is self-evident if the agent knows E whenever E obtains. Denote by $\mathcal{J}_K := \{E \in \mathcal{D} \mid E \subseteq K(E)\}$ the collection of self-evident events, and call it the *self-evident collection*.

Second, a given sub-collection $\mathcal{J}$ of $\mathcal{D}$ satisfies the *maximality property* (with respect to (henceforth abbreviated as w.r.t.) $\mathcal{D}$) if $\emptyset \in \mathcal{J}$ and for any event E, the $\subseteq$-largest element of $\mathcal{J}$ included in E (the largest in the sense of set inclusion), $\max\{F \in \mathcal{D} \mid F \in \mathcal{J} \text{ and } F \subseteq E\}$, exists in $\mathcal{J}$. It can be seen that $\mathcal{J}$ satisfies the maximality property w.r.t. $\mathcal{D}$ iff $\{\omega \in \Omega \mid \text{there is } F \in \mathcal{J} \text{ such that } \omega \in F \subseteq E\} \in \mathcal{J}$ for each $E \in \mathcal{D}$. The maximality property is extended from Salonen (2009) and Samet (2010), which define it for a sub-algebra $\mathcal{J}$ of an ($\aleph_0$-complete) algebra $\mathcal{D}$. As Theorem 1 and Corollary 1 demonstrate, this corresponds to the knowledge (defined on an algebra) which satisfies Truth Axiom, Monotonicity, (Positive Introspection, Finite Conjunction), and Negative Introspection.

Third, if $\mathcal{J}$ satisfies the maximality property w.r.t. $\mathcal{D}$, then define an operator $K_{\mathcal{J}} : \mathcal{D} \to \mathcal{D}$ as, for each $E \in \mathcal{D}$,

$$K_{\mathcal{J}}(E) := \max\{F \in \mathcal{J} \mid F \subseteq E\} = \{\omega \in \Omega \mid \omega \in F \subseteq E \text{ for some } F \in \mathcal{J}\}. \quad (1)$$

With these three definitions in mind, Section 3.1 shows the equivalence between a knowledge operator and a self-evident collection. Section 3.2 identifies when a given sub-collection of events has an informational content, i.e., when its information closure exists. Section 3.3 applies the concept of information closures. It shows that a knowledge operator satisfies the Kripke property iff the self-evident collection is the information closure of the information sets.

3.1 Knowledge Representation by a Set Algebra

The main result establishes the equivalence between a knowledge operator and a collection of events satisfying the maximality property. The self-evident collection of a knowledge operator satisfies the maximality property. Conversely, for any collection with the maximality property, there is a knowledge operator whose self-evident collection coincides with the given collection. Thus, the minimum set of assumptions behind a set-algebraic knowledge representation is Truth Axiom, Monotonicity, and Positive Introspection. Other properties of knowledge (e.g., Negative Introspection) are characterized by set-algebraic properties of the collection (e.g., the closure under complementation). Throughout this subsection, fix a κ-complete algebra $(\Omega, \mathcal{D})$.

Theorem 1. *1. For a knowledge operator K, the self-evident collection $\mathcal{J}_K$ satisfies the maximality property and $K = K_{\mathcal{J}_K}$: $K(E) = \max\{F \in \mathcal{J}_K \mid F \subseteq E\} \in \mathcal{J}_K$ for each $E \in \mathcal{D}$. Conversely, for any $\mathcal{J} \in \mathcal{P}(\mathcal{D})$ satisfying the maximality property, $K_{\mathcal{J}}$ is a knowledge operator and $\mathcal{J} = \mathcal{J}_{K_{\mathcal{J}}}$. Moreover:*

- *(a) K (resp. $K_{\mathcal{J}}$) satisfies Non-empty λ-Conjunction iff $\mathcal{J}_K$ (resp. $\mathcal{J}$) is closed under non-empty λ-intersection.*
- *(b) K (resp. $K_{\mathcal{J}}$) satisfies Necessitation iff $\Omega \in \mathcal{J}_K$ (resp. $\Omega \in \mathcal{J}$).*
- *(c) K (resp. $K_{\mathcal{J}}$) satisfies Negative Introspection iff $\mathcal{J}_K$ (resp. $\mathcal{J}$) is closed under complementation.*

2. If $\mathcal{J} \in \mathcal{P}(\mathcal{D})$ satisfies the maximality property, then $\bigcup \mathcal{E} \in \mathcal{J}$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{J})$ with $\bigcup \mathcal{E} \in \mathcal{D}$. Especially, $\mathcal{J}$ is closed under κ-union. If $\kappa = \infty$, $\mathcal{J} \in \mathcal{P}(\mathcal{D})$ satisfies the maximality property iff it is closed under arbitrary union.

A corollary of Theorem 1 is that Negative Introspection, together with Truth Axiom and Monotonicity, implies all the other properties of knowledge postulated in Section 2.1. Recalling that Truth Axiom and Negative Introspection imply Positive Introspection, it follows from Theorem 1 that K satisfies κ-Conjunction (including Necessitation) iff $\mathcal{J}_K$ is closed under κ-intersection. Now, by Theorem 1, $\mathcal{J}_K$ is closed under κ-union and complementation. Thus, it is closed under κ-intersection. To the best of my knowledge, it has not been established in the previous literature that Negative Introspection (the introspective property on the lack of knowledge), under a minimum set of assumptions on knowledge (i.e., Truth Axiom and Monotonicity), implies Conjunction (the logical property on the conjunction of the knowledge). The idea that knowledge can be summarized by a set algebra establishes the corollary in a simple way.

Corollary 1. *Truth Axiom, Monotonicity, and Negative Introspection imply κ-Conjunction including Necessitation (as well as Consistency and Positive Introspection).*

Six additional remarks on Theorem 1 are in order. First, in Part (1), all of Truth Axiom, Positive Introspection, and Monotonicity are essential. If any one condition is absent, then there is a simple example where $K \neq K'$ and $\mathcal{J}_K = \mathcal{J}_{K'}$.[8]

Second, by Truth Axiom and Positive Introspection, $\mathcal{J}_K = \{K(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$. Thus, the self-evident collection comprises solely of the largest self-evident event $K(E)$ included in $E \in \mathcal{D}$. Consequently, $K(E) = \bigcup\{F \in \mathcal{J}_K \mid F \subseteq E\} \in \mathcal{D}$ for each $E \in \mathcal{D}$, although $\mathcal{J}_K$ and $\mathcal{D}$ may not necessarily be closed under arbitrary union.[9] Also, the basic conditions reduce to the following single property: for any $E, F \in \mathcal{D}$, $K(F) \subseteq K(E)$ iff $K(F) \subseteq E$.

Third, Part (2) clarifies what it means by a collection $\mathcal{J}$ being closed under arbitrary union whenever it is well-defined on a general κ-complete algebra: it is a consequence of the maximality property, the condition under which $\mathcal{J}$ induces a knowledge operator. On a complete algebra, $\mathcal{J}$ is closed under arbitrary union iff it induces a knowledge operator.

Fourth, the theorem enables one to compare assumptions implicitly put on knowledge and on an underlying space among different set-algebraic knowledge representations under the common framework. When knowledge is represented by a topology, the domain $\mathcal{D}$ is implicitly assumed to be a complete algebra (usually, the power set) and knowledge is assumed to satisfy Finite Conjunction as well as the basic conditions. In particular, knowledge induced by a reflexive-and-transitive possibility correspondence is summarized as a topology closed under arbitrary intersection (see Footnote 3 for references). In these models, a self-evident collection may not necessarily be closed under complementation in the absence of Negative Introspection. If a self-evident event E satisfies $E^c \notin \mathcal{J}_K$, it is not the case that the agent knows E is false (E^c is true) whenever E is false. That is, the agent cannot conclude that E is false just by the fact that she does not observe E. Such non-partitional knowledge models study agents who process information at face value.

In contrast, under Negative Introspection, the self-evident collection becomes a sub-κ-complete algebra. For any $E \in \mathcal{J}_K$, the agent knows whether E is true or not at any state: $\Omega = K(E) \cup K(E^c)$. Indeed, K satisfies Negative Introspection iff $\mathcal{J}_K = \{E \in \mathcal{D} \mid K(E) \cup K(E^c) = \Omega\}$.[10] This formalizes the (common) sense in which knowledge is interpreted as a sub-algebra of a given domain (e.g., Hérves-Beloso and

[8]Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2\}, \mathcal{P}(\Omega))$. First, define K, K_T, and K_M as: $K(\emptyset) = K(\{\omega_2\}) = \emptyset$ and $K(\{\omega_1\}) = K(\Omega) = \{\omega_1\}$; $K_T(\emptyset) = \emptyset$ and $K_T(E) = \{\omega_1\}$ otherwise; and $K_M(\{\omega_1\}) = \{\omega_1\}$ and $K_M(E) = \emptyset$ otherwise. Then, K_T and K_M violate Truth Axiom and Monotonicity, respectively. For each $K' \in \{K_T, K_M\}$, $K \neq K'$ but $\mathcal{J}_K = \mathcal{J}_{K'}$. Second, let $K(\cdot) = \emptyset$ and $K'(\Omega) = \{\omega_1\}$ and $K'(E) = \emptyset$ otherwise. Then, K' violates Positive Introspection, $K \neq K'$, and $\mathcal{J}_K = \mathcal{J}_{K'}$.

[9]One can always extend a knowledge operator $K : \mathcal{D} \to \mathcal{D}$ to $K^* : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ by $K^*(E) := \bigcup\{F \in \mathcal{J}_K \mid F \subseteq E\}$ for each $E \in \mathcal{P}(\Omega)$. By construction, $K^*|_{\mathcal{D}} = K$. This is analogous to the inner measure induced by a probability measure.

[10]Lee (2018, Section 4) studies a generic equivalence between probability-one belief B that satisfies Truth Axiom almost surely and a sub-σ-algebra $\{E \in \mathcal{D} \mid B(E) \cup B(E^c) = \Omega\}$ on a measurable space $(\Omega, \mathcal{D})$. See also Hérves-Beloso and Monteiro (2013) and Stinchcombe (1990).

Monteiro (2013), Lee (2018), Stinchcombe (1990), and Wilson (1978)). Indeed, a self-evident event coincides with an event which Hérves-Beloso and Monteiro (2013) call an informed set (see also Lee (2018, Lemma 3)): letting K be induced by a partition b_K, an event is self-evident iff it is an informed set.

Fifth, letting $\overline{\mathcal{J}_K} := \{F \in \mathcal{D} \mid F^c \in \mathcal{J}_K\}$, the possibility operator satisfies $L_K(E) = \min\{F \in \overline{\mathcal{J}_K} \mid E \subseteq F\} \in \overline{\mathcal{J}_K}$ for each $E \in \mathcal{D}$. That is, $\overline{\mathcal{J}_K}$ always has the minimum event including $E \in \mathcal{D}$, and it coincides with $L_K(E)$. Since $\mathcal{J}_K$ is not necessarily closed under complementation, however, $\mathcal{J}_K$ does not necessarily satisfy $L_K(E) = \min\{F \in \mathcal{J}_K \mid E \subseteq F\}$.[11] Without Negative Introspection of K, L_K is not characterized by the "minimality" property of $\mathcal{J}_K$.

Sixth, set-inclusion between self-evident collections naturally gives "knowledgeability" relation. Let K_i and K_j be knowledge operators. Then,

$$\mathcal{J}_{K_i} \subseteq \mathcal{J}_{K_j} \textit{ if and only if } K_i(\cdot) \subseteq K_j(\cdot). \tag{2}$$

The first condition implies the second through the maximality property. Conversely, the second condition implies the first because $E \subseteq K_i(E) \subseteq K_j(E)$ for any $E \in \mathcal{J}_{K_i}$.

3.2 Information Closures

I ask when a given collection $\mathcal{D}' \in \mathcal{P}(\mathcal{D})$ has an informational content. I examine the condition under which there exists the smallest collection including $\mathcal{D}'$ and satisfying the maximality property. Throughout the subsection, fix a κ-complete algebra $(\Omega, \mathcal{D})$.

Call a sub-collection $\mathcal{D}'$ of $\mathcal{D}$ an *information basis* if, for every $E \in \mathcal{D}$,

$$K_{\mathcal{D}'}(E) := \{\omega \in \Omega \mid \text{there is } F \in \mathcal{D}' \text{ such that } \omega \in F \subseteq E\} \in \mathcal{D}.$$

In words, the sub-collection $\mathcal{D}'$ is an information basis if, for each event E, the set of states at which E can be supported by some $F \in \mathcal{D}'$ forms an event.

Two remarks are in order. First, the definition of $K_{\mathcal{D}'}$ is a weakening of $K_{\mathcal{J}}$ in Expression (1) in that $\mathcal{D}'$ may not satisfy the maximality property. If $\mathcal{D}'$ itself satisfies the maximality property then this definition coincides with Expression (1). Second, any $\mathcal{D}'$ with $|\mathcal{D}'| < \kappa$ is an information basis because $K_{\mathcal{D}'}(E) = \bigcup\{F \in \mathcal{D}' \mid F \subseteq E\} \in \mathcal{D}$. Thus, any finite $\mathcal{D}'$ is an information basis. Any countable $\mathcal{D}'$ on a measurable space is an information basis.

For an information basis $\mathcal{D}'$, define the *information closure* $\mathcal{J}(\mathcal{D}')$ of $\mathcal{D}'$: $\mathcal{J}(\mathcal{D}') := \{K_{\mathcal{D}'}(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$. The next proposition establishes the following. First, for an information basis $\mathcal{D}'$, the information closure $\mathcal{J}(\mathcal{D}')$ is the smallest collection including

[11] As an example, consider $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{D} = \mathcal{P}(\Omega)$, and $\mathcal{J}_K = \{\emptyset, \{\omega_1\}\}$. Then, $L_K(\{\omega_1\}) = \min\{F \in \overline{\mathcal{J}_K} \mid \{\omega_1\} \subseteq F\} = \Omega \neq \{\omega_1\} = \min\{F \in \mathcal{J}_K \mid E \subseteq F\}$. This happens because $\{\omega_1\}^c$ is not self-evident. This means that the maximality property of a self-evident collection is generally different from the "relative completeness" of a sub-Boolean algebra (Halmos, 1955, Section 4) (which could be seen as the "minimality" property) due to the failure of Negative Introspection.

$\mathcal{D}'$ and satisfying the maximality property. Second, I study additional set-algebraic properties of information closures. Third, let $\kappa = \infty$. For any $\mathcal{D}'$, the information closure is the smallest collection including $\mathcal{D}'$ and closed under arbitrary union.

Proposition 2. *Let $\mathcal{D}'$ be an information basis.*

1. *The information closure $\mathcal{J}(\mathcal{D}')$ is the smallest collection including $\mathcal{D}'$ and satisfying the maximality property. Moreover, $K_{\mathcal{D}'} = K_{\mathcal{J}(\mathcal{D}')}$ and*

$$\mathcal{J}(\mathcal{D}') = \{E \in \mathcal{D} \mid \textit{if } \omega \in E \textit{ then there is } F \in \mathcal{D}' \textit{ with } \omega \in F \subseteq E\}. \tag{3}$$

2. *(a) $\mathcal{D}' \cup \{\Omega\}$ is an information basis, and $\mathcal{J}(\mathcal{D}' \cup \{\Omega\}) = \mathcal{J}(\mathcal{D}') \cup \{\Omega\}$.*
   (b) If $\hat{\mathcal{D}}' := \{\bigcap \mathcal{E} \in \mathcal{D} \mid \mathcal{E} \subseteq \mathcal{D}' \text{ with } 0 < |\mathcal{E}| < \lambda\}$ is an information basis, then $\mathcal{J}(\hat{\mathcal{D}}')$ is the smallest collection including $\mathcal{D}'$, satisfying the maximality property, and closed under non-empty λ-intersection. Consequently, if $\mathcal{D}'$ is closed under non-empty λ-intersection, so is $\mathcal{J}(\mathcal{D}')$.
   (c) The collection $\mathcal{J}(\mathcal{D}')$ does not necessarily inherit the closure under complementation from $\mathcal{D}'$. The following are equivalent: (i) $\mathcal{J}(\mathcal{D}')$ is closed under complementation; (ii) $\mathcal{J}(\mathcal{D}')$ is a sub-κ-complete algebra; and (iii) $K_{\mathcal{D}'}$ satisfies Negative Introspection.

3. *Suppose $|\mathcal{D}'| < \kappa$ (no restriction is imposed on $\mathcal{D}'$ if $\kappa = \infty$). Then, $\mathcal{J}(\mathcal{D}') = \bigcap\{\mathcal{J}' \in \mathcal{P}(\mathcal{D}) \mid \mathcal{D}' \subseteq \mathcal{J}' \text{ and } \mathcal{J}' \text{ is closed under } \kappa\text{-union}\}$. Also, $\mathcal{J}(\mathcal{D}')$ is closed under complementation (if and) only if it is the smallest κ-complete algebra including $\mathcal{D}'$.*

Despite Proposition 2 (2c), Proposition 3 in Section 3.3 shows that if $\mathcal{D}'$ is an information partition then $\mathcal{J}(\mathcal{D}')$ becomes a sub-κ-complete algebra of $\mathcal{D}$. Especially, if the partition $\mathcal{D}'$ is countable on a measurable space then $\mathcal{J}(\mathcal{D}')$ is the σ-algebra generated by $\mathcal{D}'$.

In Proposition 2 (3), if arbitrary unions of $\mathcal{D}'$ are events, then $\mathcal{J}(\mathcal{D}')$ is the smallest collection including $\mathcal{D}'$ and being closed under κ-union. Also, the smallest collection satisfying the maximality property and closed under complementation is the smallest κ-complete algebra including $\mathcal{D}'$.

The next subsection examines the sense in which $\mathcal{J}(\cdot)$ preserves informational contents of sub-collections. To that end, letting $\mathcal{E}$ and $\mathcal{E}'$ be information bases, $\mathcal{E}'$ is *at least as informative as* $\mathcal{E}$ if $\mathcal{J}(\mathcal{E}) \subseteq \mathcal{J}(\mathcal{E}')$ (equivalently, $K_{\mathcal{E}}(\cdot) \subseteq K_{\mathcal{E}'}(\cdot)$).

Corollary 2. *An information basis $\mathcal{E}'$ is at least as informative as an information basis $\mathcal{E}'$ iff for any $E \in \mathcal{E}$ and $\omega \in E$, there is $E' \in \mathcal{E}'$ with $\omega \in E' \subseteq E$.*

3.3 Applications to Information Sets and Signals

I apply the notion of information closures to study the relation between a self-evident collection $\mathcal{J}_K$ and information sets $\mathcal{B}_K := \{b_K(\omega)\}_{\omega \in \Omega}$. I also examine the relation between informational contents of and preferences over signals.

Proposition 3. *Let $\overrightarrow{\Omega}$ be a κ-knowledge space.*

1. $b_K(\omega) = \bigcap\{E \in \mathcal{J}_K \mid \omega \in E\}$ *for each* $\omega \in \Omega$.
2. *Suppose $b_K(\cdot) \in \mathcal{D}$. The following characterizes the Kripke property of K.*
   (a) *The information sets $\mathcal{B}_K$ form an information basis with $\mathcal{J}_K = \mathcal{J}(\mathcal{B}_K)$.*
   (b) $\mathcal{J}_K = \left\{E \in \mathcal{D} \mid E = \bigcup_{\omega \in F} b_K(\omega) \text{ for some } F \in \mathcal{P}(\Omega)\right\}$.
3. *Suppose: (i) $b_K(\cdot) \in \mathcal{D}$; (ii) K satisfies the Kripke property; and (iii) $\mathcal{B}_K$ forms a partition (i.e., K additionally satisfies Negative Introspection). Then, $\mathcal{J}_K = \mathcal{J}(\mathcal{B}_K)$ is a sub-κ-complete algebra.*

Part (1) holds regardless of the Kripke property, as the key is the equivalence between K and $\mathcal{J}_K$. It shows that each $b_K(\omega)$ is the "atom" of $\mathcal{J}_K$ at ω. Thus, the possibility relation is stated in terms of $\mathcal{J}_K$ as follows: ω' is considered possible at ω if $\omega' \in E$ for any $E \in \mathcal{J}_K$ with $\omega \in E$.

There are two ways to look at Part (2). First, the Kripke property is the exact condition that provides the information sets $\mathcal{B}_K$ with an informational content. Second, under the Kripke property, the information sets and the self-evident collection are equivalent through Parts (1) and (2). For example, on a complete algebra $\mathcal{D}$, $b_K(\omega)$ is the minimum open neighborhood of ω (see Footnote 2) in a topological space $(\Omega, \mathcal{J}_K)$, and $\mathcal{J}_K = \mathcal{J}(\mathcal{B}_K)$ is the smallest Alexsandroff topology (the smallest topology closed under arbitrary intersection; see Footnote 2) including $\mathcal{B}_K$.

The Kripke property implies (2b) without assuming that b_K is $\mathcal{D}$-valued: the self-evident collection $\mathcal{J}_K$ consists of well-defined events formed by arbitrary unions of information sets. This part can be written as $\mathcal{J}_K = \{\bigcup_{\omega \in F} b_K(\omega) \in \mathcal{P}(\Omega) \mid F \in \mathcal{P}(\Omega)\} \cap \mathcal{D}$, and generalizes Lee (2018, Lemma 3) and Noguchi (2018, Lemmas 15 and 16) (see also Tobias (2019)).

As in (3), suppose that K additionally satisfies the Kripke property and Negative Introspection. It states that $\mathcal{J}_K = \mathcal{J}(\mathcal{B}_K)$ is a sub-κ-complete algebra, generalizing Hérves-Beloso and Monteiro (2013, Theorem 1). If $\mathcal{B}_K$ is a countable sub-collection of a σ-algebra $\mathcal{D}$, then it follows from Proposition 2 (3) that $\mathcal{J}_K$ is the smallest σ-algebra $\sigma(\mathcal{B}_K)$ including $\mathcal{B}_K$. As Hérves-Beloso and Monteiro (2013, Proposition 4) and Lee (2018, Theorem 1) show the equivalence of a countably generated σ-algebra and a countable measurable information partition, $\mathcal{J}(\cdot)$ generates the smallest σ-algebra $\sigma(\cdot)$ for countable information partitions. Thus, for example, if each information set has a positive measure on a probability space as in a standard setting, then the self-evident collection coincides with the σ-algebra generated by the information sets.[12]

At the same time, if $\mathcal{D}$ is a complete algebra, then $\mathcal{J}_K = \mathcal{J}(\mathcal{B}_K)$ is a sub-complete algebra and is generated by arbitrary unions of partition cells, providing a foundation for Dubra and Echenique (2004). Proposition 3 nests the previous results in a consistent manner.

The next result provides the sense in which $\mathcal{J}(\cdot)$ preserves information sets. By Expression (2), Corollary 2, and Proposition 3, information sets are weakly finer iff self-evident collections are weakly larger.

Corollary 3. *Let $\overrightarrow{\Omega}$ be a κ-knowledge space of $\{i, j\}$ such that K_i and K_j satisfy the Kripke property. Let $b_{K_i}, b_{K_j} : \Omega \to \mathcal{P}(\Omega)$ be their possibility correspondences. The following are all equivalent.*

1. $b_{K_j}(\cdot) \subseteq b_{K_i}(\cdot)$. *2.* $K_i(\cdot) \subseteq K_j(\cdot)$. *3.* $\mathcal{J}_{K_i} \subseteq \mathcal{J}_{K_j}$.
4. For any $\omega \in \Omega$ and $\omega' \in b_{K_i}(\omega)$, there is $\omega'' \in \Omega$ with $\omega' \in b_{K_j}(\omega'') \subseteq b_{K_i}(\omega)$.

Three remarks on the corollary are in order. First, Corollary 3, together with Proposition 3, provides the sense in which $\mathcal{J}(\cdot)$ preserves the relation between information sets and self-evident collections in a way that nests the previous literature. On a complete algebra $\mathcal{D}$, the corollary generalizes Dubra and Echenique (2004, Parts 2 and 5 of Theorem A) and Hérves-Beloso and Monteiro (2013, Proposition 1): $b_{K_j}(\cdot) \subseteq b_{K_i}(\cdot)$ iff the collection of events that are formed by arbitrary unions of information sets $\mathcal{B}_{K_i}$ is included in the collection of events that are formed by arbitrary unions of $\mathcal{B}_{K_j}$. In contrast, let $\mathcal{D}$ be a σ-algebra. Assume that $\mathcal{B}_{K_i}$ and $\mathcal{B}_{K_j}$ are countable and that $b_{K_i}, b_{K_j} : \Omega \to \mathcal{D}$. Then, $b_{K_j}(\cdot) \subseteq b_{K_i}(\cdot)$ iff $\sigma(\mathcal{B}_{K_i}) = \mathcal{J}(\mathcal{B}_{K_i}) \subseteq \mathcal{J}(\mathcal{B}_{K_j}) = \sigma(\mathcal{B}_{K_j})$.

Second, Corollary 3 sheds light on common knowledge (e.g., Aumann (1976) and Friedell (1969)) as the infimum of the individual knowledge. Let K_i be the knowledge operator of agent $i \in I$. Call an event E *common knowledge* among the agents I at a state ω if there is a (publicly-evident) event $F \in \bigcap_{i \in I} \mathcal{J}_{K_i}$ such that $\omega \in F \subseteq E$ (e.g., Monderer and Samet (1989)). Technically, it may not be easy to assert the existence of a common knowledge operator C, or, to show that the set of states at which an event E is common knowledge forms an event (e.g., Green (2012)). Conceptually, Proposition 2 states that the common knowledge operator C is well-defined iff $\bigcap_{i \in I} \mathcal{J}_{K_i}$ is an information basis. (Also, Proposition 1 states whether C satisfies the Kripke property.) If each K_i satisfies Countable Conjunction (assuming $\kappa \geq \aleph_1$), then $\bigcap_{i \in I} \mathcal{J}_{K_i}$ turns out to be an information basis. Moreover, if each

[12]While this paper does not study probability-theoretical aspects of self-evident collections on a probability space, if the collection of information sets is uncountable on a probability space, then some information sets have probability zero so that one has to take care of Bayesian updating on such a null set if the posterior belief is derived from a prior distribution. See, for example, Brandenburger and Dekel (1987), Lee (2018), and Noguchi (2018) for the use of regular conditional probability measures or Kajii and Morris (1997) and Nielsen (1984) for enriching the underlying state space.

K_i satisfies the Kripke property, then C satisfies the Kripke property and common knowledge is the infimum of players' knowledge in the sense of Corollary 3.

Third, Dubra and Echenique (2004, Theorem A) and Hérves-Beloso and Monteiro (2013, Propositions 1 and 5) study the Blackwell-type relation among preferences over signals, partitions generated by signals, and set algebras generated from arbitrary unions of partitions. So far, I have established the relation between information sets and generated set-algebras. I complete the discussion that information closures preserve informational contents by establishing the Blackwell-type result.

Let a state space $(\Omega, \mathcal{D})$ be a κ-complete algebra. A *signal* is a pair $(f : \Omega \to Y_f, \mathcal{Y}_f)$ such that $\mathcal{Y}_f \subseteq \mathcal{P}(Y_f)$, $f^{-1}(\mathcal{Y}_f) \subseteq \mathcal{D}$, and that $f^{-1}(\mathcal{Y}_f)$ is an information basis. Thus, the hypothetical agent endowed with the informational content of f with respect to $\mathcal{Y}_f$ would know an event E at a state ω whenever there is $E_f \in \mathcal{Y}_f$ with $\omega \in f^{-1}(E_f) \subseteq E$. Dubra and Echenique (2004) implicitly assume $\mathcal{Y}_f = \{f(\omega)\}_{\omega \in \Omega}$.[13] I specify the collection $\mathcal{Y}_f$ as I do not assume that every singleton $\{f(\omega)\}$ (or every subset of Y_f) is "observable." That is, by the informational content of the function f, I do not always restrict attention to the partition $(f^{-1}(\{f(\omega)\}))_{\omega \in \Omega}$ generated by f. Instead, I specify $\mathcal{Y}_f$ with which to consider the informational content of f.[14]

Let $(Z, \mathcal{D}_Z)$ be a κ-complete algebra of *consequences*. Assume that there are at least two elements $x, y \in Z$ distinguished by some $E_Z \in \mathcal{D}_Z$: $x \in E_Z$ and $y \notin E_Z$. An *act* is a measurable function $a : (\Omega, \mathcal{D}) \to (Z, \mathcal{D}_Z)$. Denote by $\mathcal{A}$ the set of acts. A *decision-maker* is a preference (i.e., complete-and-transitive) relation $\succeq$ on acts $\mathcal{A}$.

For any $\mathcal{Z} \in \mathcal{P}(\mathcal{D}_Z)$, call an act $a \in \mathcal{A}$ a *$\mathcal{Z}$-act* if $(a, \mathcal{Z})$ is a signal, i.e., if $a^{-1}(\mathcal{Z}) \in \mathcal{P}(\mathcal{D})$ is an information basis. The *$\mathcal{Z}$-act* a produces its informational content with respect to the observational constraint $\mathcal{Z}$. Dubra and Echenique (2004) implicitly restrict attention to $\mathcal{D}_Z = \mathcal{P}(Z)$ and $\mathcal{Z} = \{a(\omega)\}_{\omega \in \Omega}$ for each $a \in \mathcal{A}$.

A $\mathcal{Z}$-act a is $(f, \mathcal{Y}_f)$-*feasible* if $f^{-1}(\mathcal{Y}_f)$ is at least as informative as $a^{-1}(\mathcal{Z})$. Note that a $\{a(\omega)\}_{\omega \in \Omega}$-act a is $(f, \{f(\omega)\}_{\omega \in \Omega})$-feasible if $a(\omega) = a(\omega')$ whenever $f(\omega) = f(\omega')$. A preference relation $\succeq$ on acts $\mathcal{A}$ induces an ordering for each pair of signals. The decision-maker $\succeq$ *prefers* a signal $(f, \mathcal{Y}_f)$ to $(g, \mathcal{Y}_g)$ if, for any $(g, \mathcal{Y}_g)$-feasible $\mathcal{Z}$-act a, there is an $(f, \mathcal{Y}_f)$-feasible $\mathcal{Z}'$-act a' with $a' \succeq a$.

Proposition 4. *Let $(f, \mathcal{Y}_f)$ and $(g, \mathcal{Y}_g)$ be signals. Every decision-maker prefers $(f, \mathcal{Y}_f)$ to $(g, \mathcal{Y}_g)$ iff $f^{-1}(\mathcal{Y}_f)$ is at least as informative as $g^{-1}(\mathcal{Y}_g)$.*

One can apply Proposition 4 to a pair of a $\mathcal{Z}$-act a and a $\mathcal{Z}'$-act a': every decision-maker prefers $(a, \mathcal{Z})$ to $(a', \mathcal{Z}')$ iff $a^{-1}(\mathcal{Z})$ is at least as informative as $(a')^{-1}(\mathcal{Z}')$.

Proposition 4 establishes the Blackwell-type result for quite a large class of information and consequently preferences. First, let $\mathcal{D} = \mathcal{P}(\Omega)$ and $\mathcal{D}_Z = \mathcal{P}(Z)$. As

[13] Generally, for any $\mathcal{Y}_f \supseteq \{f(\omega)\}_{\omega \in \Omega}$, $\mathcal{J}(f^{-1}(\mathcal{Y}_f)) = \mathcal{J}(f^{-1}(\mathcal{P}(Y_f)))$ by Corollary 2.

[14] In a related but different context, the idea that not every singleton $\{f(\omega)\}$ is observable can also be seen in the decision-theory literature on non-additive expected utility representations such as Ghirardato (2001) and Mukerji (1997). In their papers, a signal (or an act) is defined not as a function $f : \Omega \to Y_f$ but as a correspondence $f : \Omega \rightrightarrows Y_f$.

in Dubra and Echenique (2004, Theorem A) and Hérves-Beloso and Monteiro (2013, Proposition 1), the proposition holds when signals and acts are restricted to $(f, \mathcal{Y}_f) = (f, \{f(\omega)\}_{\omega\in\Omega})$ and $(a, \mathcal{Z}) = (a, \{a(\omega)\}_{\omega\in\Omega})$, respectively. Then, $\mathcal{J}(f^{-1}(\mathcal{Y}_f))$ coincides with the collection of arbitrary unions of the partition generated by f (recall Proposition 3 (2)). Second, in Hérves-Beloso and Monteiro (2013, Proposition 5), $(\Omega, \mathcal{D})$ is a measurable space and each signal f is a measurable function from $(\Omega, \mathcal{D})$ into a measurable space $(Y_f, \mathcal{Y}_f)$. Given their assumption on measurable spaces, the information closure $\mathcal{J}(\cdot)$ is equal to the smallest σ-algebra $\sigma(\cdot)$.

4 Concluding Remarks

This paper fully characterized the class of set algebras that represent knowledge in terms of the properties of knowledge and the structure of underlying states. Implicit in any representation are Truth Axiom, Positive Introspection, and Monotonicity. The class of σ-algebras satisfying the maximality property characterizes fully-introspective knowledge on a measurable space. The paper also formalized the sense in which a given sub-collection of events has an informational content. The paper then established the equivalence between information sets and the self-evident collection generated by (i.e., the information closure of) the information sets.

This paper provided a common framework for various set-algebraic knowledge representations. While I discussed how a topology closed under arbitrary intersection characterizes a non-partitional model, the framework can be applied to the seemingly-different knowledge representation in mathematical psychology referred to as the "knowledge space theory" (Doignon and Falmagne, 1985, 2016; Falmagne and Doignon, 2011). It aims to theoretically and empirically assess an agent's knowledge on an academic subject.[15] The knowledge of the agent is modeled as a "knowledge structure" $(\Omega, \mathcal{J})$ such that $\emptyset, \Omega \in \mathcal{J} \subseteq \mathcal{P}(\Omega)$ and that $\mathcal{J}$ is closed under arbitrary union. While Falmagne and Doignon (2011, p. 38) mention that the closure under arbitrary union "may not be convincing for an educator, *a priori*," it follows that knowledge $\mathcal{J}$ satisfies the maximality property w.r.t. $\mathcal{D} = \mathcal{P}(\Omega)$ (hence, the basic conditions) as well as Necessitation. Also, they construct the knowledge structure $(\Omega, \mathcal{J}_{\preccurlyeq})$ from a pre-order $\preccurlyeq$ that they call a surmise relation. Define $\mathcal{J}_{\preccurlyeq}$ so that each $E \in \mathcal{J}_{\preccurlyeq}$ is a $\preccurlyeq$-upper set (i.e., $\omega' \in E$ for all $\omega \in E$ and $\omega \preccurlyeq \omega'$). Then, $\mathcal{J}_{\preccurlyeq}$ is a topology closed under arbitrary intersection, and $\preccurlyeq$-upper contour sets $b_{\preccurlyeq}(\omega) = \{\omega' \in \Omega \mid \omega \preccurlyeq \omega'\}$ define a reflexive-and-transitive possibility correspondence. The collection $\mathcal{J}_{\preccurlyeq}$ is a self-evident collection because $b_{\preccurlyeq}(\omega) \subseteq E$ (i.e., $\omega \in K_{b_{\preccurlyeq}}(E)$) for all $\omega \in E$.

One interesting avenue for future research is to study structures of the family of self-evident collections on a given state space. Such pioneering papers as Allen (1983),

[15] This mathematical-psychology literature provides an alternative knowledge representation referred to as a "surmise system." Fukuda (2018a) provides a generalization that encompasses a possibility correspondence model, the surmise system, and a "local reasoning" model of Fagin and Halpern (1987) in computer science.

Cotter (1986), and Stinchcombe (1990) introduce topological structures on the family of sub-σ-algebras on a given probability space in order to study the continuous dependence of economic variables on information.[16] It would be interesting to study some metric between self-evident collections to examine such continuity. Another interesting avenue for future research is to study the extent to which qualitative or probability p-beliefs (Monderer and Samet, 1989) can be summarized by set algebras. In doing so, as this paper characterized the failure of Negative Introspection as that of the closure under complementation, it would be interesting to explore a connection between an agent's probabilistic sophistication and her logical or introspective properties of beliefs (e.g., characterizations of non-additive beliefs by Ghirardato (2001) and Mukerji (1997) in terms of an agent's bounded perception).

A Appendix

Proof of Proposition 1. Assume the Kripke property. If $b_K(\omega) \cap F = \emptyset$, then $b_K(\omega) \subseteq F^c$ (i.e., $\omega \in K(F^c)$) and $F^c \cap F = \emptyset$. Conversely, suppose $b_K(\omega) \subseteq E'$ and $\omega \notin K(E')$ for some $(\omega, E') \in \Omega \times \mathcal{D}$. Since $b_K(\omega) \cap (E')^c = \emptyset$, there is $E \in \mathcal{D}$ with $\omega \in K(E)$ and $E \cap (E')^c = \emptyset$ (i.e., $E \subseteq E'$), contradicting Monotonicity of K. □

Proof of Theorem 1. For the first part of (1), I show $K(E) = \max\{F \in \mathcal{J}_K \mid F \subseteq E\} \in \mathcal{J}_K$. For any $F \in \mathcal{J}_K$ with $F \subseteq E$, Monotonicity implies $F \subseteq K(F) \subseteq K(E)$. By Positive Introspection and Truth Axiom, $K(E) \in \{F \in \mathcal{J}_K \mid F \subseteq E\}$. Conversely, $K_{\mathcal{J}}$ satisfies Truth Axiom because $K_{\mathcal{J}}(E) = \max\{F \in \mathcal{J} \mid F \subseteq E\} \subseteq E$. For Positive Introspection, observe $K_{\mathcal{J}}(E) \in \{F \in \mathcal{J} \mid F \subseteq K_{\mathcal{J}}(E)\}$. For Monotonicity, observe that if $E \subseteq F$ then $K_{\mathcal{J}}(E) \in \{F' \in \mathcal{J} \mid F' \subseteq F\}$. Next, I show $\mathcal{J} = \mathcal{J}_{K_{\mathcal{J}}}$. If $E \in \mathcal{J}$, then $E = K_{\mathcal{J}}(E)$ and thus $E \in \mathcal{J}_{K_{\mathcal{J}}}$. If $E \in \mathcal{J}_{K_{\mathcal{J}}}$, then $E = K_{\mathcal{J}}(E) \in \mathcal{J}$.

It suffices to prove the rest of Part (1) with respect to K and $\mathcal{J}_K$. For the "only-if" part of (1a), if $\mathcal{E} \in \mathcal{P}(\mathcal{J}_K) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$, then $\bigcap \mathcal{E} \subseteq \bigcap_{E \in \mathcal{E}} K(E) \subseteq K(\bigcap \mathcal{E})$. Conversely, if $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$, then $\bigcap_{E \in \mathcal{E}} K(E) \subseteq K(\bigcap_{E \in \mathcal{E}} K(E)) \subseteq K(\bigcap \mathcal{E})$. By inspection, (1b) holds. For the "only-if" part of (1c), if $E \in \mathcal{J}_K$ then $E^c = (\neg K)(E) = K(\neg K)(E) = K(E^c)$. Conversely, $(\neg K)(E) \in \mathcal{J}_K$ implies $(\neg K)(E) \subseteq \max\{F \in \mathcal{J}_K \mid F \subseteq (\neg K)(E)\} = K(\neg K)(E)$.

Consider Part (2). If $\mathcal{E} \subseteq \mathcal{J}$ and $\bigcup \mathcal{E} \in \mathcal{D}$, then $\bigcup \mathcal{E} = \max\{F \in \mathcal{J} \mid F \subseteq \bigcup \mathcal{E}\} \in \mathcal{J}$. Next, let $\kappa = \infty$. If $\mathcal{J}$ is closed under arbitrary union, then it satisfies the maximality property because $\emptyset = \bigcup \emptyset \in \mathcal{J}$ and $\max\{F \in \mathcal{J} \mid F \subseteq E\} = \bigcup\{F \in \mathcal{J} \mid F \subseteq E\} \in \mathcal{J}$ for each $E \in \mathcal{D}$. □

Proof of Proposition 2. For Part (1), first, $\mathcal{D}' \subseteq \mathcal{J}(\mathcal{D}')$ holds by construction. Next, I show $\mathcal{J}(\mathcal{D}')$ satisfies the maximality property. By definition, $\emptyset = K_{\mathcal{D}'}(\emptyset) \in \mathcal{J}(\mathcal{D}')$. For each $E \in \mathcal{D}$, $\max\{F \in \mathcal{J}(\mathcal{D}') \mid F \subseteq E\} = K_{\mathcal{D}'}(E) \in \mathcal{J}(\mathcal{D}')$. Next, I show that if

[16] Applications to game and general equilibrium theory include Correia-da-Silva and Hervés-Beloso (2007) and Monderer and Samet (1996) (see also the references therein).

a collection $\mathcal{J}'$ includes $\mathcal{D}'$ and satisfies the maximality property then $\mathcal{J}(\mathcal{D}') \subseteq \mathcal{J}'$. Let $E \in \mathcal{J}(\mathcal{D}')$. For any $\omega \in E$, there is $F \in \mathcal{D}' \subseteq \mathcal{J}'$ with $\omega \in F \subseteq E$. Hence, $E = \{\omega \in \Omega \mid \omega \in F \subseteq E \text{ for some } F \in \mathcal{J}'\} = \max\{F \in \mathcal{J}' \mid F \subseteq E\} \in \mathcal{J}'$. Finally, Expression (3) follows from $K_{\mathcal{J}(\mathcal{D}')}(E) = \max\{F \in \mathcal{J}(\mathcal{D}') \mid F \subseteq E\} = K_{\mathcal{D}'}(E)$.

By inspection, (2a) holds. For (2b), it suffices to show that $\mathcal{J}(\hat{\mathcal{D}}')$ is closed under non-empty λ-intersection. Take $\mathcal{E} \subseteq \mathcal{J}(\hat{\mathcal{D}}')$ with $0 < |\mathcal{E}| < \lambda$. Let $\omega \in \bigcap \mathcal{E}$. For each $E \in \mathcal{E}$, there is $F_E \in \hat{\mathcal{D}}'$ with $\omega \in F_E \subseteq E$. Then, $\omega \in \bigcap_{E \in \mathcal{E}} F_E \subseteq \bigcap \mathcal{E}$ and $\bigcap_{E \in \mathcal{E}} F_E \in \hat{\mathcal{D}}'$. For the first statement of (2c), consider the following (counter-)example. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. While $\mathcal{D}' = \{\emptyset, \{\omega_1\}, \{\omega_3\}, \{w_1, \omega_2\}, \{\omega_2, \omega_3\}, \Omega\}$ is closed under complementation, $\mathcal{J}(\mathcal{D}') = \mathcal{P}(\Omega) \setminus \{\{\omega_2\}\}$ is not. The second statement follows from Theorem 1 and Part (1) of this proposition.

For Part (3), it is sufficient to prove the first statement. Observe first that

$$\bigcap \{\mathcal{J}' \in \mathcal{P}(\mathcal{D}) \mid \mathcal{D}' \subseteq \mathcal{J}' \text{ and } \mathcal{J}' \text{ is closed under } \kappa\text{-union}\} = \{\bigcup \mathcal{E} \in \mathcal{D} \mid \mathcal{E} \subseteq \mathcal{D}'\}.$$

Next, it can be seen that this collection satisfies the maximality property and includes $\mathcal{D}'$. Thus, this collection includes $\mathcal{J}(\mathcal{D}')$. To get the converse set inclusion, observe that $\mathcal{J}(\mathcal{D}')$ includes $\mathcal{D}'$ and that $\mathcal{J}(\mathcal{D}')$ is closed under κ-union (Theorem 1 (2)). □

Proof of Proposition 3. Part (1) follows because $b_K(\omega) = \bigcap\{E \in \mathcal{D} \mid \omega \in K(E)\} \subseteq \bigcap\{E \in \mathcal{J}_K \mid \omega \in E\} \subseteq K(F) \subseteq F$ for each $F \in \mathcal{D}$ with $\omega \in K(F)$.

For Part (2), let $\mathcal{J} := \{\bigcup_{\omega \in F} b_K(\omega) \in \mathcal{D} \mid F \in \mathcal{P}(\Omega)\}$. First, it follows from the Kripke property, Truth Axiom, and Positive Introspection that $\mathcal{B}_K$ is an information basis. Indeed, $\{\omega \in \Omega \mid \text{there is } \omega' \in \Omega \text{ such that } \omega \in b_K(\omega') \subseteq E\} = K(E) \in \mathcal{J}_K$ for each $E \in \mathcal{D}$. Consequently, $\mathcal{J}(\mathcal{B}_K) \subseteq \mathcal{J}_K$. By Theorem 1 (2) and Truth Axiom, $\mathcal{J}_K \subseteq \mathcal{J}(\mathcal{B}_K)$. Second, (2a) and Theorem 1 (2) imply $\mathcal{J} \subseteq \mathcal{J}(\mathcal{B}_K) = \mathcal{J}_K$. By Truth Axiom, $\mathcal{J}_K \subseteq \mathcal{J}$. Third, by (2b), Truth Axiom, and Positive Introspection, $\mathcal{B}_K \subseteq \mathcal{J}_K$. Thus, $\omega \in b_K(\omega) \subseteq K(b_K(\omega))$. By Monotonicity, $\omega \in K(E)$ if(f) $b_K(\omega) \subseteq E$.

Part (3) follows from Theorem 1 and Part (2) of this proposition. □

Proof of Proposition 4. The "if" part follows because any $(g, \mathcal{Y}_g)$-feasible $\mathcal{Z}$-act a is $(f, \mathcal{Y}_f)$-feasible under the supposition. Conversely, suppose there is $E \in \mathcal{J}(g^{-1}(\mathcal{Y}_g))$ with $E \notin \mathcal{J}(f^{-1}(\mathcal{Y}_f))$. Define $\{E_Z\}$-act a as: (i) $a(\omega) = x$ if $\omega \in E$; and (ii) $a(\omega) = y$ if $\omega \notin E$. Since $a^{-1}(E_Z) = E$, $\{E_Z\}$-act a is $(g, \mathcal{Y}_g)$-feasible but not $(f, \mathcal{Y}_f)$-feasible. Let $\succeq$ be such that $a \succ a'$ for all $a' \in \mathcal{A} \setminus \{a\}$. By construction, $\succeq$ does not prefer $(f, \mathcal{Y}_f)$ to $(g, \mathcal{Y}_g)$. □

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