Formalizing Common Belief with No Underlying Assumption on Individual Beliefs*

Satoshi Fukuda†

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Abstract

This paper formalizes common belief among players with no underlying assumption on their individual beliefs. Especially, players may not be logically omniscient, i.e., they may not believe logical consequences of their beliefs. The key idea is to use a novel concept of a common basis: it is an event such that, whenever it is true, every player believes its logical consequences. The common belief in an event obtains when a common basis implies the mutual belief in that event. If players' beliefs are assumed to be true, then common belief reduces to common knowledge. The formalization nests previous axiomatizations of common belief and common knowledge which have assumed players' logical monotonic reasoning. Under this formalization, unlike others, if players have common belief in rationality then their actions survive iterated elimination of strictly dominated actions even if their beliefs are not monotonic.

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1 Introduction

Notions of common belief and common knowledge play indispensable roles whenever multiple players interactively reason with each other. Informally, an event is common belief between Alice and Bob if they mutually believe it, they believe that they believe

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†Department of Decision Sciences and IGIER, Bocconi University, Milan 20136, Italy.

it, and so on *ad infinitum*. Common belief and common knowledge are scrutinized in diverse contexts: computer science (e.g., McCarthy et al. (1978)), game theory and social sciences (e.g., Aumann (1976) and Friedell (1969)), and logic and philosophy (e.g., Lewis (1969)). The foundational nature of these concepts lies in the very fact that a model of knowledge and belief itself is often implicitly assumed to be common belief (or common knowledge) among the players.

The purpose of this paper is to provide a formalization of common belief with no a priori assumption on individual beliefs. Individual beliefs can be contradictory, inconsistent, non-conjunctive, or non-monotonic. That is, a player may believe a contradiction, she may simultaneously believe an event and its negation, she can believe multiple events without believing its conjunction, or she fails to believe some logical consequences of her own beliefs. Especially, logical monotonicity is at the heart of what is referred to as the "logical omniscience" problem. A casual observation suggests that a person can know the rule of a zero-sum game like chess without knowing its optimal strategy. Even if the players of a game are implicitly assumed to commonly believe the structure of the game and rationality of the (other) players, it is not at all clear whether the players may understand logical implications of common belief in rationality such as eliminations of strictly dominated actions.

Yet, relaxing logical monotonicity while maintaining tractability is no easy task. Consider, for instance, a standard possibility correspondence model of interactive beliefs (e.g., Aumann (1976, 1999), Geanakoplos (1989), and Morris (1996)). Each player has her possibility correspondence, which associates, with each state of the world, the set of states that she considers possible. She believes an event at a state if the event includes the possibility set at that state. Mutual and common beliefs are also represented by way of players' possibility correspondences. The tractability of the model hinges on the premise that players are logical reasoners.

Toward a step to analyzing strategic reasoning among players who lack sophisticated logical inferential ability, say, to deduce or compute optimal strategies, this paper provides a framework for capturing interactive beliefs, especially common belief, among them. The formalization of common belief in this paper can also justify the informal sense in which the structure of a game is common belief (or common knowledge) among the players, even if they are not logically omniscient. This paper also clarifies how much logical sophistication individual players need to have in order for common belief to satisfy given properties. To the best of my knowledge, this is the first paper to study common belief that does not resort to the premise that players are logical reasoners.[1]

[1] In the literature on interactive beliefs, exceptions are Lismont and Mongin (1994a, 2003). They weaken players' logical monotonicity by proposing an axiom which they call quasi-monotonicity in their syntactic framework. In decision theory, a pioneering and exceptional work is Lipman (1999). Also, Morris (1996) studies a player's belief from her preferences. In his model, roughly, a player's preference relation is a complete ordering if and only if her belief is closed under arbitrary conjunction and logical monotonicity. This is exactly the condition under which her belief is induced by a possibility correspondence. For an overview of logical omniscience problems in computer science

I represent common belief in a set-theoretical (i.e., semantic) framework so as to accommodate various notions of beliefs. If individual players' beliefs are true, then common belief reduces to common knowledge. Beliefs can be probabilistic as well as qualitative. Both knowledge and belief can be analyzed at the same time.[2]

The framework has three components. The first is an underlying set of states of the world. Players reason about some aspects of the underlying state space.

The second is the collection of events, which are subsets of the state space. Since an event represents a property about the underlying states, the collection of events determines the language available to players when they interactively reason with each other. Especially, logical (set-algebraic) conditions on the collection capture assumptions on depths of players' reasoning. For example, if players engage in any finite depths of interactive reasoning, then the collection of events forms an algebra of subsets of the state space. If players' beliefs are probabilistic, then the collection of events is a σ-algebra. If there is no restriction on depths of reasoning as in possibility correspondence models, then the collection of events is closed under arbitrary set-algebraic operations. My formalization of common belief does not hinge on a particular assumption on players' depths of reasoning.

The third is each player's belief operator, which associates, with each event E, the event that the player believes E.[3] Qualitative and quantitative features of beliefs and knowledge are incorporated into the corresponding properties of belief operators.[4]

To formalize the common belief operator, I start with defining a new and auxiliary notion that I call common bases. A common basis is a particular type of event known to be publicly evident in the literature (e.g., Milgrom (1981)). An event E is publicly evident if everybody believes E whenever E is true. A common basis is an event E such that everybody believes any logical implication of E whenever E is true. If players' beliefs are (quasi-)monotonic, then a common basis and a publicly-evident event coincide with each other.

Following the terminology by Lewis (1969), I use the auxiliary notion of common

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and logic, see Fagin et al. (2003, Chapter 9) and the references therein.

[2]For example, in an extensive-form game with perfect information, each player forms her belief about the opponents' future plays while she has knowledge about past moves.

[3]The hidden assumption in such a semantic belief model is that if two events E and F coincide (i.e., $E = F$) then the beliefs in E and F coincide, even if the denotations of E and F may be different. This paper supposes no other restriction on individual players' beliefs. However, if the analysts introduce the collection of syntactic formulas that express exogenously given values (e.g., the set of action profiles) about which the players reason and players' interactive beliefs about them, then there exists a canonical semantic model based on the syntactic language in which the identification of events E and F are minimal: in the canonical model, if the denotations of E and F are different, then the events E and F are distinguished. The canonical model also admits common belief introduced in this paper. See Fukuda (2019b).

[4]A player's probabilistic beliefs can be represented by a collection of p-belief operators (Friedell, 1969; Monderer and Samet, 1989). A p-belief operator associates, with each event E, the event that the player assigns a degree of her belief at least p to E (she p-believes E).

bases literally as a "basis" for common belief.[5] Specifically, I define the common belief in E as the largest common basis implying the mutual belief in E (i.e., everybody believes E).[6] If the common basis F implies the mutual belief in E, then F implies the mutual belief in the mutual belief in E by itself.[7] In fact, F implies any higher-order mutual belief in E irrespective of properties of individual players' beliefs. I exploit the property of a common basis F that is defined as an event with the property that the players have the mutual belief of any order in any event implied by F. Not only does any chain of mutual beliefs hold, but also F implies the very fact that E is common belief. Consequently, if an event E is common belief, then it is common belief that E is common belief, that is, common belief satisfies positive introspection, without assuming logical monotonicity on individual players.

One of the main motivations of the literature behind axiomatizations of common belief (common knowledge) is to provide formalizations that satisfy positive introspection in that the iterative definition of the chain of mutual beliefs (mutual knowledge) may fail it (e.g., Barwise (1989), Heifetz (1999), and Lismont and Mongin (1994b, 1995)). Another is the informal sense in which the structure of a game can be common belief (common knowledge). Gilboa (1988), in his syntactic model, incorporates the statement that the model is common knowledge. Using the fact that the common knowledge of a statement implies the common knowledge of the common knowledge of the statement, he derives the sense in which the model is commonly known.

Positive introspection is in fact a consequence of the property which I call *Completeness*: if an event E is common belief and if the common belief in E (not E itself) implies an event F, then the event F is common belief. While common belief may be non-monotonic, I show it is always complete. Positive introspection is a particular case with F being the common belief in E itself. Conceptually, this result implies that if the structure of a game is commonly believed then the players commonly believe any statement derived from the common (and hence mutual) belief in the structure even if they are not fully logical. Also, if they commonly believe their rationality, then they commonly (and hence mutually) believe any logical consequence. The previous formalizations such as the iterative one may fail Completeness if the players are not logical. Theorem 1 shows my formalization of common belief is the most permissive

[5] Lewis (1969) introduces common knowledge by a notion which he calls a "basis for common knowledge." The basis for common knowledge F "indicates to everybody that everybody has reason to believe F," and such F provides players with the chain of mutual knowledge. For Lewis' account of common knowledge, see also Cubitt and Sugden (2003) and Vanderschraaf (1998).

[6] The idea behind maximality is to formulate common belief as the "infimum" of individual beliefs. Aumann (1976) defines common knowledge by the infimum of individual players' knowledge partitions. McCarthy et al. (1978) introduce a notion of "any fool" knows.

[7] Contrast this argument with the formalization of common belief by publicly-evident events. If a publicly-evident event F implies the mutual belief in E, then the analysts who assume logical monotonicity on the players can assert that the mutual belief in F (ensured by public evidence) implies the mutual belief in the mutual belief in E. The idea of common bases builds this inference into their definition without requiring players' actual inferences and consequently their logical abilities.

one satisfying Completeness and implying mutual belief. Theorem 2 shows that my definition and the iterative one coincide when the iterative one is complete (or the iterative common belief in an event is a common basis).

This paper does not claim that its formulation of common belief is the most sensible one for any possible situation in which players are not logically omniscient. As the first study of common belief that does not presuppose any property on players' beliefs, however, this paper shows that the formulation of common belief of this paper is the most permissive one that guarantees that common belief in rationality characterizes iterated eliminations of strictly dominated actions irrespective of assumptions on players' beliefs.[8] Specifically, Theorem 3 shows: (i) common belief in rationality leads to iterated eliminations of strictly dominated actions irrespective of properties of the players' beliefs; and (ii) iterative common belief in rationality, in contrast, may not lead to iterated eliminations if iterative common belief is not complete (e.g., if the players' beliefs are not monotonic). For the second assertion, if iterative common belief in rationality is a common basis (or if iterative common belief is complete), then iterative common belief in rationality reduces to common belief in rationality under the formulation of this paper. The concept of rationality I adopt is fairly standard: for no action a_i of player i, she believes playing a_i is strictly better than following her strategy given the opponents' strategies.

This paper generalizes the previous formalizations of common belief and common knowledge that have assumed logical monotonicity under weaker monotonicity conditions (Corollaries 1 and 2 and Proposition 1). The key observation is that an event E is common belief at a state ω if and only if there is a common basis F that is true at ω and that implies the mutual belief in E. When publicly-evident events are common bases, the formalization reduces to Monderer and Samet (1989). I also examine the "fixed point" characterization of the common belief in E as the maximal event F in a way such that F is true if and only if the conjunction of the mutual beliefs in E and F obtains. If every player's belief is (quasi-)monotonic and conjunctive (i.e., a player believes the conjunction of E and F whenever she believes E and F), then the formalization coincides with Friedell (1969), Halpern and Moses (1990), and Lismont and Mongin (1994a,b, 2003). If mutual belief is countably conjunctive (i.e., if everybody believes a countable number of events then everybody believes its conjunction) and if publicly-evident events are common bases, then common belief reduces to the iterative notion. Individual players' logical monotonicity is not necessary to characterize common belief by the iterative definition. Moreover, if players' beliefs are derived from possibility correspondences, then common belief is also induced from the transitive closure of the possibility correspondence that captures mutual belief.[9] The

[8]Implications of common belief in rationality are pioneered by, for instance, Brandenburger and Dekel (1987b), Stalnaker (1994), and Tan and Werlang (1988) in standard belief models.

[9]This is implicit in Aumann (1976)'s "reachability" condition. See also Green (2012), Halpern and Moses (1990), Hérves-Beloso and Monteiro (2013), and the references therein. Also, the transitivity of a possibility correspondence is associated with positive introspection (e.g., Binmore and

paper enables one to compare the relations among previous formalizations in terms of how each formalization presupposes players' logical abilities.

This paper also demonstrates how much logical sophistication each player needs to have in order for common belief to inherit or satisfy given properties of individual or mutual beliefs (Proposition 2). For example, if mutual belief is true (i.e., if everybody believes an event E then the event E is true), then common belief is also true. Especially, if every player's belief is assumed true (i.e., players' knowledge instead of their beliefs is analyzed), then common belief reduces to common knowledge. Under this assumption, an event E is common belief at a state ω if and only if there is a common basis that is true at ω and that implies E.[10] Thus, if mutual belief is true, then common belief (common knowledge) is monotonic irrespective of whether individual beliefs are monotonic. I also show if individual players' beliefs are not monotonic or true, then common belief may be non-monotonic.

The paper is structured as follows. Section 2 provides a basic environment. Sections 3.1 and 3.2 formalize and characterize common belief, respectively. Section 3.3 shows my definition generalizes the previous literature. Section 3.4 studies how common belief relates to individual beliefs. Section 4 shows that, under my formalization, common belief in rationality leads to iterated elimination of strictly dominated actions irrespective of such properties of players' beliefs as monotonicity. Section 5 provides concluding remarks. The proofs are relegated to Appendix A. Appendix B separately studies common knowledge.

2 Underlying Framework

I begin with technical preliminaries on set algebras on which players' beliefs are defined. For any infinite cardinal number κ and any set Ω, a subset $\mathcal{D}$ of the power set $\mathcal{P}(\Omega)$ is a *κ-complete algebra* (on Ω) if $\mathcal{D}$ is closed under complementation and under arbitrary union (and consequently intersection) of any sub-collection with cardinality less than κ. Denote the complement of E by E^c or $\neg E$. Let $\emptyset =: \bigcup \emptyset \in \mathcal{D}$ and $\Omega =: \bigcap \emptyset \in \mathcal{D}$. An $\aleph_0$-complete algebra is an algebra of sets, and an $\aleph_1$-complete algebra is a σ-algebra (where $\aleph_0$ is the least infinite cardinal and $\aleph_1$ is the least uncountable cardinal).[11] A subset $\mathcal{D}$ of $\mathcal{P}(\Omega)$ is an (∞-)*complete algebra* (on Ω) if $\mathcal{D}$ is closed under complementation and arbitrary union (and intersection). Here, ∞ is used as a symbol (not a cardinal) satisfying $\kappa < \infty$ for any cardinal κ.

Brandenburger (1990), Geanakoplos (1989), Morris (1996), and Shin (1993)).

[10]First, this generalizes Aumann (1999), Binmore and Brandenburger (1990), Brandenburger and Dekel (1987a), Geanakoplos (1989), Nielsen (1984), and Shin (1993). Second, Appendix B studies common knowledge separately from common belief by calling E *common knowledge* at ω if there is a common basis that is true at ω and that implies E.

[11]Technically, it is without loss to take an infinite regular cardinal κ (Meier, 2006, Remark 1). If an infinite cardinal κ is not regular then any κ-complete algebra is κ^+-complete, where the successor cardinal κ^+ is regular (assuming the axiom of choice). Note that $\aleph_0$ and $\aleph_1$ are regular.

I move on to providing the underlying framework for representing players' interactive beliefs. Let κ be an infinite cardinal or $\kappa = \infty$, and let I be a non-empty set of *players*. A *κ-belief space* (of I) is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, C \rangle$. First, the players are reasoning about some aspects of the *state space* Ω. Second, their objects of reasoning (i.e., the collection of events) are represented by a κ-complete algebra $\mathcal{D}$ on the state space, where κ stands for the limitation (or the maximum lower bound) on the players' depths of reasoning in that the analysts can always examine players' belief hierarchies up to (the ordinality of) κ. For example, if $\kappa = \aleph_0$, then the players can engage in finite depths of interactive reasoning. If $\kappa = \aleph_1$, then the players interactively reason about their countable depths of beliefs (e.g., they reason about the countable chain of mutual beliefs).[12] If $\kappa = \infty$ then there is no limitation on depths of reasoning. Third, $B_i : \mathcal{D} \to \mathcal{D}$ is player i's *belief operator* for each $i \in I$. For each *event* $E \in \mathcal{D}$, the set $B_i(E)$ denotes the event that (i.e., the set of states at which) player i believes E. A player i believes an event E at a state ω if $\omega \in B_i(E)$. Fourth, $C : \mathcal{D} \to \mathcal{D}$ is a *common belief operator* to be formally defined in Section 3.1.

Belief operators on a general set algebra allow one to analyze diverse forms of beliefs and knowledge in a unified way. A belief operator can represent qualitative belief and knowledge derived from a possibility correspondence. Players' probabilistic beliefs are also accommodated through p-belief operators $(B_i^p)_{(i,p) \in I \times [0,1]}$ on a σ-algebra (Friedell, 1969; Monderer and Samet, 1989). A player p-believes an event E at a state ω if her type at ω (a mapping from $\mathcal{D}$ into $[0,1]$) assigns a degree of her belief ("probability") at least $p \in [0,1]$ to the event E.[13] One can also introduce dynamic or conditional beliefs.[14]

Here, I consider the following nine properties of (qualitative) belief and knowledge. These are basic properties of possibility correspondence models, and also shed light on players' logical abilities. Fix $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I} \rangle$ and $i \in I$. Also, define the correspondence $b_{B_i} : \Omega \to \mathcal{P}(\Omega)$ by

$$b_{B_i}(\omega) := \{\omega' \in \Omega \mid \omega' \in E \text{ for all } E \in \mathcal{D} \text{ with } \omega \in B_i(E)\} = \bigcap \{E \in \mathcal{D} \mid \omega \in B_i(E)\}.$$

The set $b_{B_i}(\omega)$ is regarded as the set of states (not necessarily the event) player i considers possible at ω. Now:

[12] In the context of rationalizability, Lipman (1994) studies a game in which players may engage in transfinite levels of reasoning (eliminations of never-best-replies).

[13] Samet (2000, Theorem 2) provides conditions on p-belief operators under which they are derived from a type mapping from the state space to the set of (countably-additive) types (see also Gaifman (1988)). One can also consider finitely-additive types (or more general (especially, non-monotonic) set functions) on an ($\aleph_0$-complete) algebra $\mathcal{D}$. See, for instance, Meier (2006) and Zhou (2010).

[14] First, Battigalli and Bonanno (1997) study knowledge and qualitative beliefs indexed by time. Second, Di Tillio, Halpern, and Samet (2014) provide conditions on p-belief operators which induce a conditional probability system (CPS) as in Battigalli and Siniscalchi (1999). Thus, one could also accommodate lexicographic belief systems (LPSs) as in Blume, Brandenburger, and Dekel (1991a,b). See also Brandenburger, Friedenberg, and Keisler (2007), Halpern (2010), Tsakas (2014), and the references therein for relations between CPS and LPS.

1. *No-Contradiction*: $B_i(\emptyset) = \emptyset$.

2. *Consistency*: $B_i(E) \subseteq (\neg B_i)(E^c)$ for any $E \in \mathcal{D}$.

3. *Monotonicity*: $E \subseteq F$ implies $B_i(E) \subseteq B_i(F)$.

4. *Necessitation*: $B_i(\Omega) = \Omega$.

5. *Non-empty λ-Conjunction* (where λ is a fixed infinite cardinal with $\lambda \leq \kappa$ or $\lambda = \kappa = \infty$): $\bigcap_{E \in \mathcal{E}} B_i(E) \subseteq B_i(\bigcap \mathcal{E})$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$.

6. *Truth Axiom*: $B_i(E) \subseteq E$ for any $E \in \mathcal{D}$.

7. *Positive Introspection*: $B_i(\cdot) \subseteq B_i B_i(\cdot)$.

8. *Negative Introspection*: $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$.

9. *Kripke property*: for any $(\omega, E) \in \Omega \times \mathcal{D}$, $\omega \in B_i(E)$ if (and only if) $b_{B_i}(\omega) \subseteq E$.

First, No-Contradiction says that player i never believes a contradiction in the form of $\emptyset$. Second, Consistency states that if player i believes E then she does not believe the negation of E. Third, Monotonicity provides the players with their logical inference ability. If player i believes E and if E implies F, then she believes F. One of the main purposes of this paper is to study common belief when players fail Monotonicity. Fourth, Necessitation means that player i always believes a tautology in the form of Ω. Fifth, Non-empty λ-Conjunction states that if player i believes each of a non-empty collection of events (with cardinality less than λ) then she believes its conjunction. Empty Conjunction is identified as Necessitation. I also call Non-empty $\aleph_0$-Conjunction and Non-empty $\aleph_1$-Conjunction, respectively, *Finite Conjunction* and *Countable Conjunction*. In a (countably-additive) probabilistic environment, a probability 1-belief operator B_i^1 satisfies Countable Conjunction while a p-belief operator B_i^p with $p \in (0,1)$ may fail even Finite Conjunction.

Sixth, Truth Axiom distinguishes knowledge and belief: knowledge is assumed to satisfy Truth Axiom in the literature. It states if player i "knows" E at ω then E is true at ω. Seventh, Positive Introspection states if player i believes E then she believes that she believes E. Eighth, Negative Introspection states if player i does not believe E then she believes that she does not believe E.

Call an event E *self-evident to* player i if $E \subseteq B_i(E)$. That is, i believes E whenever E is true. Denote by $\mathcal{J}_{B_i} := \{E \in \mathcal{D} \mid E \subseteq B_i(E)\}$ the collection of events self-evident to i. Positive Introspection and Negative Introspection are characterized by $\{B_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\} \subseteq \mathcal{J}_{B_i}$ and $\{(\neg B_i)(E) \in \mathcal{D} \mid E \in \mathcal{D}\} \subseteq \mathcal{J}_{B_i}$, respectively.

Ninth, the Kripke property is the condition under which a player's belief is derived from the possibility correspondence b_{B_i}. It states that player i believes E at ω if (and only if) E includes the set of states considered possible at ω. Indeed, B_i satisfies the Kripke property if and only if (henceforth, abbreviated as iff) it is induced by some correspondence $b : \Omega \to \mathcal{P}(\Omega)$ (i.e., $B_i(E) = \{\omega \in \Omega \mid b(\omega) \subseteq E\}$ for all $E \in \mathcal{D}$) (Fukuda, 2019a). If $\mathcal{D}$ is a complete algebra, then the Kripke property is equivalent jointly to Monotonicity, Non-empty ∞-Conjunction, and Necessitation (e.g., Morris (1996, Theorem 1) when $\mathcal{D} = \mathcal{P}(\Omega)$).

The next section introduces the notion of common belief (and subsequently the common belief operator) among the players I in a κ-belief space, irrespective of assumptions on their beliefs. Henceforth, I assume that a given κ-belief space $\overrightarrow{\Omega}$ of I satisfies $|I| < \kappa$ so as to introduce the *mutual belief operator* $B_I : \mathcal{D} \to \mathcal{D}$ defined by $B_I(\cdot) := \bigcap_{i \in I} B_i(\cdot)$. The event $B_I(E)$ is the set of states at which every player in I believes E. The above nine properties of beliefs can be defined analogously for the mutual (and common) belief operators.

Define chains of mutual beliefs. Let $B_I^\alpha(\cdot) := \mathrm{id}_\mathcal{D}$ for $\alpha = 0$. For a successor ordinal $\alpha = \beta + 1$, define $B_I^\alpha := B_I \circ B_I^\beta$. For a non-zero limit ordinal α, let $B_I^\alpha(\cdot) := \bigcap_{\beta: 1 \leq \beta < \alpha} B_I^\beta(\cdot)$. In any κ-belief space, B_I^α is well defined if $|\alpha| < \kappa$.

In order to define the iterative common belief operator as the countable conjunction of mutual beliefs, consider a κ-belief space with $\kappa \geq \aleph_1$. To avoid the clash of notation between a generic state (ω) and the standard notation for the least infinite ordinal (which is also ω), denote by ϖ the least infinite ordinal throughout this paper.[15] Define the *iterative common belief* operator by B_I^ϖ: an event E is iterative common belief among I at ω, i.e., $\omega \in B_I^\varpi(E) = \bigcap_{n \in \mathbb{N}} B_I^n(E)$, if everybody (in I) believes E at ω, everybody believes that everybody believes E at ω, *ad infinitum*.

To conclude this section, I remark that specifying one's belief operator is equivalent to specifying the collection of events that one believes at each state. Such a collection is referred to as a neighborhood system (or a Montague-Scott structure).[16] The formal equivalence is presented in Remark A.1 in Appendix A.

3 Formalization of Common Belief

3.1 Formalization

Unless otherwise stated, fix $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I} \rangle$ throughout this section. I start with the auxiliary definition of a common basis. An event E is a *common basis* if everybody believes any logical implication of E whenever E is true. Formally:

Definition 1. Call $E \in \mathcal{D}$ a *common basis* (among I) if $E \subseteq B_I(F)$ for any $F \in \mathcal{D}$ with $E \subseteq F$. Denote by $\mathcal{J}_I$ the collection of common bases among I.

A common basis is a stronger form of a publicly-evident event (Milgrom, 1981), where an event E is *publicly evident* if it is self-evident to every i: $E \subseteq B_I(E)$,

[15]The symbol ϖ for the least infinite ordinal is used for B_I^ϖ (i.e., B_I^α with $\alpha = \varpi$) and for Table 4 in Section 4.5.

[16]See, for example, Fagin et al. (2003) and Pacuit (2017). Heifetz (1999) and Lismont and Mongin (1994a,b, 1995) use neighborhood systems in formalizing common belief and common knowledge.

or $E \in \bigcap_{i \in I} \mathcal{J}_{B_i}$.[17] While a common basis is publicly evident, the converse is not necessarily true (examples will be provided).

In the literature pioneered by Monderer and Samet (1989), an event E is common belief at a state ω if there is a publicly-evident event $F \in \bigcap_{i \in I} \mathcal{J}_{B_i}$ that is true at ω and that implies the mutual belief in E: $\omega \in F \subseteq B_I(E)$. If the players' beliefs are monotonic, then the publicly-evident event F implies the chain of mutual beliefs: $\omega \in F \subseteq B_I^{\varpi}(E)$. This is because $F \subseteq B_I^n(E)$ implies $F \subseteq B_I(F) \subseteq B_I^{n+1}(E)$ for each $n \in \mathbb{N}$.

In contrast, when players' beliefs may not be monotonic, I, as an analyst, exploit the property of a common basis that is defined as an event with the property that every player believes any of its logical consequences. If a common basis F implies the mutual belief $B_I(E)$ in an event E, then F implies the mutual belief in the mutual belief in E, i.e., $B_I^2(E)$. In fact, F implies the iteration of mutual beliefs $B_I^{\varpi}(E)$. Now, call an event E *common belief (commonly believed)* at a state ω if there is a common basis F that is true at ω and that implies the mutual belief in E.

Definition 2. For any $E \in \mathcal{D}$, the set of states at which E is common belief is:

$$\{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_I \text{ such that } \omega \in F \subseteq B_I(E)\}. \tag{1}$$

I introduce two definitions. First, I define the closure condition on the publicly-evident events: publicly-evident events are common bases (i.e., $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$). This is a weaker form of the monotonicity condition in that if the players' beliefs are monotinic, then the closure condition on the publicly-evident events is satisfied. I study its implications such as how common belief reflects properties of individual and mutual beliefs under this weaker monotonicity condition.

Definition 3. The collection of publicly-evident events $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is *closed (with respect to common bases)* if $\bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$.

Second, at this level of generality, there is no result asserting that the set of states at which an event E is common belief is itself an event. Especially, if $\mathcal{D}$ is an ($\aleph_0$-complete) algebra but fails to be an $\aleph_1$-complete algebra, then even the iterative common belief operator B_I^{ϖ} may not be well defined because the players cannot reason about limits of their mutual beliefs. Suppose, for example, that the analysts would like to study common belief syntactically by finitary languages. Or suppose that the players' beliefs are represented by finitely-additive (or non-additive) measures on an ($\aleph_0$-complete) algebra $\mathcal{D}$. In order for the players to reason about common belief, the analysts would need to introduce the common belief operator $C : \mathcal{D} \to \mathcal{D}$ as a primitive of a κ-belief space. In other words, I incorporate the common belief operator

[17] Publicly-evident events are termed as self-evident events (Aumann, 1999), common truisms (Binmore and Brandenburger, 1990), public events (Geanakoplos, 1989), belief closed events (Lismont and Mongin, 1994a,b, 1995, 2003; Mertens and Zamir, 1985; Vassilakis and Zamir, 1993), evident knowledge events (Monderer and Samet, 1989), common information (Nielsen, 1984), and so forth.

C into a primitive of the belief space so that I can separate how much (many rounds) the players can reason from the definition of common belief.

Thus, a given κ-belief space $\langle(\Omega, \mathcal{D}), (B_i)_{i \in I}, C\rangle$ is assumed to have a well-defined common belief operator $C : \mathcal{D} \to \mathcal{D}$ that maps each event E to the event that E is common belief consistently with Expression (1). This holds iff C satisfies

$$C(E) := \max\{F \in \mathcal{D} \mid F \subseteq B_I(E) \text{ and } F \in \mathcal{J}_I\} \text{ for each } E \in \mathcal{D}, \tag{2}$$

where "max" is taken with respect to the set inclusion (i.e., on a partially-ordered space $\langle \mathcal{D}, \subseteq \rangle$). In other words, $C(E)$ is a well-defined event iff $\mathcal{J}_I$ contains a $\subseteq$-maximal element included in $B_I(E)$. Since $\emptyset \in \mathcal{J}_I$, the set in the right-hand side of Expression (2) is always non-empty. Indeed, $\mathcal{J}_I = \{\emptyset\}$ iff $C(\cdot) = \emptyset$. Note that while I almost always restrict attention to the entire player set I, one can analogously define the common belief operator C_G among a (non-empty) subset G of players. By definition, if $G_1 \subseteq G_2 \subseteq I$ then $C_{G_2}(\cdot) \subseteq C_{G_1}(\cdot)$.

If $\mathcal{D}$ is a complete algebra then $C(E)$ is the union of all $F \in \mathcal{J}_I$ with $F \subseteq B_I(E)$, and thus the common belief operator C is always well defined without incorporating it into a given ∞-belief space as a primitive. I will also show that, under certain assumptions on players' beliefs, common belief is well defined in an $\aleph_1$-belief space (i.e., on a measurable space with players' belief operators).

In sum, a κ-belief space $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (B_i)_{i \in I}, C\rangle$ is assumed to satisfy the following: $(\Omega, \mathcal{D})$ is a κ-complete algebra, $|I| < \kappa$, each $B_i : \mathcal{D} \to \mathcal{D}$ is player i's belief operator, and $C : \mathcal{D} \to \mathcal{D}$ is the common belief operator satisfying Expression (2).

3.2 Characterization

I reformulate the common belief operator C. While C may be non-monotonic, it satisfies the following *Completeness* property with respect to its implications.

Definition 4. An operator $\tilde{C} : \mathcal{D} \to \mathcal{D}$ is *complete* if $\tilde{C}(E) \subseteq \tilde{C}(F)$ for any $E, F \in \mathcal{D}$ with $\tilde{C}(E) \subseteq F$.

The main theorem characterizes the common belief operator C as the maximum operator satisfying Completeness and implying mutual belief. I will demonstrate that Completeness, instead of the iteration of mutual beliefs, yields desirable properties of common belief. Especially, Section 4 shows: (i) common belief in rationality, under my formalization, leads to actions that survive iterative elimination of strictly dominated actions, irrespective of underlying individual beliefs; and (ii) iterative common belief in rationality may not lead to actions that survive iterative elimination of strictly dominated actions if iterative common belief is not complete.

Theorem 1. *1. The common belief operator C is complete.*

2. *If an operator $\tilde{C} : \mathcal{D} \to \mathcal{D}$ satisfies (i) Completeness and (ii) $\tilde{C}(\cdot) \subseteq B_I(\cdot)$, then $\tilde{C}(\cdot) \subseteq C(\cdot)$. If $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed, then Condition (i) (i.e., Completeness) can be replaced by Positive Introspection.*

In Part (1), by the definition of the common belief operator C, it is complete iff each $C(\cdot)$ is a common basis. Completeness of C yields Positive Introspection irrespective of individual beliefs: $C(\cdot) \subseteq C(\cdot)$ implies $C(\cdot) \subseteq CC(\cdot)$.[18] While Completeness and Positive Introspection are equivalent under Monotonicity, Completeness is typically stronger than Positive Introspection. The previous literature has shown that the iterative common belief may fail Positive Introspection (e.g., Barwise (1989), Heifetz (1999), and Lismont and Mongin (1994b, 1995)) and consequently Completeness.

Completeness of C implies $C(\cdot) \subseteq CB_I(\cdot)$: if an event E is common belief then it is common belief that everybody believes E. Since Completeness of C implies Positive Introspection and since common belief implies mutual belief, $C(\cdot) \subseteq B_I C(\cdot)$ also holds: if E is common belief then everybody believes E is common belief. Indeed, $C(\cdot) \subseteq B_I^{\alpha} C(\cdot)$ for any ordinal α with $1 \leq |\alpha| < \kappa$.

Under Positive Introspection, Completeness is weaker than Monotonicity. I will provide examples in which common belief may fail to be monotonic. Under Positive Introspection and Truth Axiom, Completeness and Monotonicity are equivalent. Proposition 2 (5) shows if mutual belief satisfies Truth Axiom then so does common belief, and consequently common belief (knowledge) is monotonic.

Theorem 1 (2) provides the sense in which common belief is the "infimum" of players' beliefs (satisfying Completeness). For the second statement, assume the closure condition $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$. Consider a hypothetical individual satisfying Positive Introspection and implying mutual belief. Then, any event believed by the hypothetical individual is common belief. The closure condition $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$ relates common bases to the infimum (intersection) of self-evident events to the players: $E \in \bigcap_{i \in I} \mathcal{J}_{B_i}$ iff $E \subseteq C(E)$. This is because $\mathcal{J}_I \subseteq \{E \in \mathcal{D} \mid E \subseteq C(E)\} \subseteq \bigcap_{i \in I} \mathcal{J}_{B_i}$ (especially, whenever a common basis E is true at ω, E is common belief at ω).

The rest of this subsection examines when common belief reduces to the iterative one B_I^{ϖ} or, more generally, $C = B_I^{\alpha}$ for some non-zero limit ordinal α with $|\alpha| < \kappa$.[19] I show: (i) the common belief in an event E implies the mutual belief in E up to any ordinal level α with $1 \leq |\alpha| < \kappa$; and (ii) if some chain of mutual beliefs forms a common basis then it induces common belief; especially, some chain of mutual beliefs characterizes common belief iff it is complete. Define $B_I^{\leq \alpha}(\cdot) := \bigcap_{\beta : 1 \leq \beta \leq \alpha} B_I^{\beta}(\cdot)$.

Theorem 2. *For any ordinal α with $1 \leq |\alpha| < \kappa$, $C(\cdot) \subseteq B_I^{\alpha}(\cdot)$. For each $E \in \mathcal{D}$, $C(E) = B_I^{\leq \alpha}(E)$ iff $B_I^{\leq \alpha}(E) \in \mathcal{J}_I$. Moreover, $C(\cdot) = B_I^{\leq \alpha}(\cdot)$ iff $B_I^{\leq \alpha}(\cdot)$ is complete.*

[18] Thus, a player's individual and "common" beliefs B_i and $C_{\{i\}}$ may differ. One can show $B_i = C_{\{i\}}$ iff i's beliefs are her common bases: $B_i(\cdot) \in \mathcal{J}_{\{i\}} := \{E \in \mathcal{D} \mid E \subseteq B_i(F)$ for any $F \in \mathcal{D}$ with $E \subseteq F\}$.

[19] Remark A.2 in Appendix A briefly discusses other iterative definitions of common belief.

E	$B_1(E)$	$B_2(E)$	$B_I(E)$	$B_I^{\varpi}(E)$	$C(E)$	$(\neg C)(E)$	$C(\neg C)(E)$
$\emptyset$	Ω	Ω	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$
$\{\omega_1\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$
$\{\omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$
$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$
Ω	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$

Table 1: C fails all but for Positive Introspection

If α is a non-zero limit ordinal then $B_I^{\leq\alpha}(\cdot) = B_I^{\alpha}(\cdot)$. Thus, Theorem 2 also characterizes when common belief reduces to some limit of mutual beliefs B_I^{α}. Consider an $\aleph_1$-belief space so that countable conjunctions of events (especially, B_I^{ϖ}) are well defined. The first part of Theorem 2 states that common belief $C(\cdot)$ implies iterative one $B_I^{\varpi}(\cdot)$. The second part states, for example, that iterative common belief $B_I^{\varpi}(\cdot)$ reduces to common belief $C(\cdot)$ when $B_I^{\varpi}(\cdot)$ itself is a common basis. The third part means $C = B_I^{\varpi}$ iff B_I^{ϖ} is complete.

Remark 1. I provide an example to illustrate Theorems 1 and 2. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. For each $i \in I = \{1, 2\}$, define B_i as in Table 1. First, common belief can fail every property discussed in Section 2 except for Positive Introspection. Second, B_I^{ϖ} fails Completeness so that $C(\cdot) \subsetneq B_I^{\varpi}(\cdot)$. Third, while B_I and C fail No-Contradiction, consider B'_i with $B'_i(\emptyset) = \emptyset$ and $B'_i(E) = B_i(E)$ for any $E \neq \emptyset$. Since $\emptyset$ is always a common basis and since $B'_I(E) = B_I(E)$ for any $E \neq \emptyset$, the resulting common belief operator C' is given by $C'(\emptyset) = \emptyset$ and $C'(E) = C(E)$ for any $E \neq \emptyset$. Especially, the second point does not hinge on No-Contradiction.

Next, if $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed, then common belief would be characterized by *some* limit of mutual beliefs. For ease of exposition, let $(\Omega, \mathcal{D})$ be an ∞-complete algebra. Fix $E \in \mathcal{D}$, and consider the ordinal sequence $(B_I^{\leq\alpha}(E))_{\alpha \geq 1}$. Since the sequence is decreasing, there is some limit ordinal α with $B_I^{\leq\alpha}(E) = B_I^{\leq\alpha+1}(E) \subseteq B_I(B_I^{\leq\alpha}(E))$, and $C(E) = B_I^{\leq\alpha}(E) = B_I^{\alpha}(E)$. However, Section 4.5 shows first that this α can be arbitrarily large, i.e., there is no pre-determined length at which iterations of mutual beliefs converge: for any non-zero limit ordinal (indeed, any non-zero ordinal) β, there is a belief space with $C(\cdot) \subsetneq B_I^{\beta}(\cdot)$. Second, even if the sequence $(B_I^{\leq\alpha}(E))_{\alpha \geq 1}$ converges, $B_I^{\leq\alpha}(E) = B_I^{\alpha}(E) \supsetneq C(E)$ when $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is *not* closed. Third, $B_I^{\alpha}(E)$ may possess undesirable properties: for instance, such iterative common belief in rationality may not characterize elimination of strictly dominated actions.

To conclude the discussions when $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed, I characterize conditions under which the iterative definition fully characterizes common belief.

Corollary 1. *Let $\overrightarrow{\Omega}$ be a κ-belief space with $\kappa \geq \aleph_1$ and $\bigcap_{i\in I} \mathcal{J}_{B_i}$ closed.*

1. *$C = B_I^{\varpi}$ iff B_I^{ϖ} satisfies Positive Introspection iff $B_I^{\varpi}(\cdot) \subseteq B_I B_I^{\varpi}(\cdot)$.*
2. *If B_I satisfies Countable Conjunction, then $C = B_I^{\varpi}$. If B_I additionally satisfies Truth Axiom, then $C(E) = B_I^{\varpi}(E) \cap E$ for all $E \in \mathcal{D}$.*

Four remarks on Corollary 1 are in order. First, both parts of the corollary imply that, under the preconditions, the players can possibly reason about common belief without introducing the common belief operator as a primitive of the κ-belief space. Second, the second part shows that, for the iterative definition to characterize common belief, monotonicity is not necessarily needed, i.e., the closure of $\bigcap_{i\in I} \mathcal{J}_{B_i}$ as well as Countable Conjunction of mutual belief are sufficient. Third, the closure condition cannot be dropped (e.g., while B_I^{ϖ} in Remark 1 satisfies Positive Introspection, $B_I^{\varpi}(\{\omega_1\}) \supsetneq C(\{\omega_1\})$). Fourth, the corollary holds for a non-zero limit ordinal α (for the second part, assume B_I^{α} satisfies Non-empty λ-Conjunction with $|\alpha| < \lambda \leq \kappa$).

Under the closure condition, the iterative definition B_I^{ϖ} fails to be the common belief operator C when it fails Positive Introspection $B_I^{\varpi}(\cdot) \subseteq B_I^{\varpi} B_I^{\varpi}(\cdot)(= B_I^{\varpi+\varpi}(\cdot))$ or public-evidence $B_I^{\varpi}(\cdot) \subseteq B_I B_I^{\varpi}(\cdot)(= B_I^{\varpi+1}(\cdot))$ by the very fact that the chain of mutual beliefs stops at the least infinite ordinal level.[20]

Lastly, suppose the mutual belief operator B_I satisfies the Kripke property on a κ-complete algebra $(\Omega, \mathcal{D})$ with $\kappa \geq \aleph_1$. Since the Kripke property implies Monotonicity and Countable Conjunction, $C = B_I^{\varpi}$. I show that the common belief operator $C = B_I^{\varpi}$ has the Kripke property. Especially, if the players' beliefs are induced from a possibility correspondence, then common belief is induced from the transitive closure of the players' possibility correspondences.

Note first that B_I has the Kripke property if each B_i does so because $b_{B_I}(\cdot) = \bigcup_{i\in I} b_{B_i}(\cdot)$. By induction, each B_I^n has the Kripke property: $b_{B_I^n}(\cdot) = b_{B_I}^n(\cdot)$, where $b_{B_I}^1 := b_{B_I}$ and $b_{B_I}^n(\cdot) := \bigcup_{\omega' \in b_{B_I}(\cdot)} b_{B_I}^{n-1}(\omega')$ for $n \geq 2$.

Let $b_C(\cdot) := \bigcup_{n\in\mathbb{N}} b_{B_I}^n(\cdot)$, i.e., $b_C(\cdot)$ is the transitive closure of $b_{B_I}(\cdot)$. If each B_i has the Kripke property, then $b_C(\omega)$ is the set of states reachable from ω (Aumann, 1976): ω' is *reachable* from ω if there are sequences $(\omega^j)_{j=1}^m$ of Ω and $(i^j)_{j=1}^{m-1}$ of I with $m \in \mathbb{N} \setminus \{1\}$ such that $\omega = \omega^1$, $\omega' = \omega^m$, and $\omega^{j+1} \in b_{B_{i^j}}(\omega^j)$ for all $j \in \{1, \ldots, m-1\}$.

Now, there is an operator $C : \mathcal{D} \to \mathcal{D}$ such that $C(E) = \{\omega \in \Omega \mid b_C(\omega) \subseteq E\} = B_I^{\varpi}(E)$. Moreover, C is the common belief operator with the Kripke property. Note that the transitivity of b_C is associated with Positive Introspection of C (under Monotonicity, Positive Introspection is equivalent to Completeness).[21] In sum:

[20] While the first infinite ordinal chain of mutual beliefs is already hard to check in reality (Monderer and Samet, 1989), the definition of common belief as the (first infinite ordinal) chain of mutual beliefs is in fact not strong enough to capture introspective properties of common belief (see, for example, Barwise (1989) and Lismont and Mongin (1994b, 1995)).

[21] On the one hand, the Kripke property of C is a sort of a "measurability" condition on $b_C : \Omega \to \mathcal{P}(\Omega) : C(E) = \{\omega \in \Omega \mid b_C(\omega) \subseteq E\} \in \mathcal{D}$ for any $E \in \mathcal{D}$. On the other hand, since C is a

Corollary 2. *Let $(\Omega, \mathcal{D})$ be a κ-complete algebra with $\kappa \geq \aleph_1$, and let $(B_i)_{i \in I}$ be such that B_I satisfies the Kripke property. Then, each B_I^n inherits the Kripke property and $b_{B_I^n} = b_{B_I}^n$. Moreover, $C : \mathcal{D} \to \mathcal{P}(\Omega)$ defined by $C(E) := \{\omega \in \Omega \mid b_C(\omega) \subseteq E\}$ is a well-defined common belief operator from $\mathcal{D}$ into itself satisfying the Kripke property, where $b_C(\cdot) := \bigcup_{n \in \mathbb{N}} b_{B_I}^n(\cdot)$. If each B_i satisfies the Kripke property, then $b_C(\omega) = \{\omega' \in \Omega \mid \omega'$ is reachable from $\omega\}$ for each $\omega \in \Omega$.*

3.3 Comparison with the Previous Literature

Proposition 1 shows my formalization of common belief nests the following previous characterizations roughly when the publicly-evident events are closed: the characterization by Monderer and Samet (1989) in terms of publicly-evident events (Expression (3)) and the "fixed-point" characterization by Friedell (1969), Halpern and Moses (1990), and Lismont and Mongin (1994a,b, 1995, 2003) (Expression (6)).

To that end, I introduce Quasi-Monotonicity (Lismont and Mongin, 1994a, 2003). An operator $B : \mathcal{D} \to \mathcal{D}$ satisfies *Quasi-Monotonicity* if $E \subseteq B(E) \cap F$ implies $B(E) \subseteq B(F)$ for any $E, F \in \mathcal{D}$. Monotonicity implies Quasi-Monotonicity. Also, if each B_i satisfies Quasi-Monotonicity then so does B_I.

If B_I satisfies Quasi-Monotonicity, then $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed. The converse, however, is not necessarily true.[22] Thus, the closure condition on publicly-evident events is weaker than Quasi-Monotonicity.

Proposition 1. *Let $\bigcap_{i \in I} \mathcal{J}_{B_i}$ be closed, and let $E \in \mathcal{D}$. First,*

$$C(E) = \{\omega \in \Omega \mid \text{there is } F \in \bigcap_{i \in I} \mathcal{J}_{B_i} \text{ with } \omega \in F \subseteq B_I(E)\} \tag{3}$$

$$= \max\{F \in \mathcal{D} \mid F \subseteq B_I(E) \cap B_I(F)\}. \tag{4}$$

Second, if B_I satisfies Quasi-Monotonicity, then

$$C(E) = \max\{F \in \mathcal{D} \mid F = B_I(E) \cap B_I(F)\}. \tag{5}$$

Third, if B_I satisfies Finite Conjunction as well as Quasi-Monotonicity, then

$$C(E) = \max\{F \in \mathcal{D} \mid F = B_I(E \cap F)\} = \max\{F \in \mathcal{D} \mid F \subseteq B_I(E \cap F)\}. \tag{6}$$

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primitive with which to analyze common belief, I allow each $b_C(\omega)$ not to be an event. In a model in which players' ($\aleph_1$-)measurable partitional possibility correspondences $b_i : \Omega \to \mathcal{D}$ are a primitive, Green (2012) shows the common knowledge partition b_C is universally measurable if $(\Omega, \mathcal{D})$ is a measurable space induced by some Polish topology (see also Hérves-Beloso and Monteiro (2013) and the references therein for the measurability of the common knowledge partition). Generally, for a given operator $B : \mathcal{D} \to \mathcal{D}$ on a κ-complete algebra satisfying the Kripke property, if each possibility set $b_B(\cdot)$ is measurable (i.e., $b_B : \Omega \to \mathcal{D}$), then any correspondence $b : \Omega \to \mathcal{P}(\Omega)$ that induces B (i.e., $B(E) = \{\omega \in \Omega \mid b(\omega) \subseteq E\}$ for every $E \in \mathcal{D}$) satisfies $b = b_B$ (Fukuda, 2019a).

[22] As a simple example, let $(\Omega, \mathcal{D}) = (\{\omega\}, \mathcal{P}(\Omega))$. Let $B_i(E) = E^c$ for each $(i, E) \in I \times \mathcal{D}$. While $\bigcap_{i \in I} \mathcal{J}_{B_i} = \{\emptyset\} = \mathcal{J}_I$, B_I violates Quasi-Monotonicity: $\emptyset \subseteq B_I(\emptyset) \cap \Omega$ and $B_I(\emptyset) \not\subseteq B_I(\Omega)$.

Proposition 1 elucidates the implicit assumptions on players' logical abilities in the previous literature. Specifically, together with Corollary 1, the iterative definition $B_I^{\overline{\omega}}$, the fixed-point definition as well as the one by publicly-evident events, and my definition (Expression (2)) all agree as long as mutual belief satisfies Quasi-Monotonicity and Countable Conjunction. However, Remark A.3 in Appendix A discusses counterexamples for the proposition when the preconditions (especially, the closure condition) are violated (there, even each "max" may not be well defined).

Generally, $C(E)$ is included in the right-hand side of Expression (3), which is the formalization by Monderer and Samet (1989, Proposition). When publicly-evident events are closed, Expression (3) reduces to Monderer and Samet (1989, Proposition) irrespective of (any other) assumptions on players' beliefs.

If B_I satisfies Quasi-Monotonicity, then $C(E)$ is characterized as the largest fixed point of $f_E(\cdot) := B_I(E) \cap B_I(\cdot)$. As the literature formulates common belief and common knowledge in terms of a fixed point using a variant of Tarski's fixed point theorem, the largest event F satisfying $F \subseteq f_E(F)$ satisfies $F = f_E(F)$. If mutual belief additionally satisfies Finite Conjunction, then common belief $C(E)$ is characterized as the greatest event F satisfying $F \subseteq B_I(E \cap F)$. Again, F turns out to be the largest fixed point $F = B_I(E \cap F)$ as in Halpern and Moses (1990): F holds (i.e., E is common belief) iff everybody believes the conjunction of E and F.

I also relate the third part of Proposition 1 to Lismont and Mongin (1994a,b, 1995, 2003). They define the set of states at which an event E is common belief as

$$\{\omega \in \Omega \mid \text{there is } F \in \bigcap_{i \in I} \mathcal{J}_{B_i} \text{ with } \omega \in B_I(F) \text{ and } F \subseteq E\}. \tag{7}$$

Lismont and Mongin (2003, Propositions 2) show that if B_I is quasi-monotonic then their definition of common belief reduces to the largest fixed point F satisfying $F = B_I(E \cap F)$. Thus, the third part of Proposition 1 also shows that if B_I satisfies Quasi-Monotonicity and Finite Conjunction then the formalizations of common belief by Lismont and Mongin (1994a,b, 1995, 2003), Monderer and Samet (1989), and this paper coincide.[23] Note that Expression (7) implicitly imposes Monotonicity of common belief irrespective of individual beliefs. Thus, if B_I satisfies Quasi-Monotonicity and Finite Conjunction then C satisfies Monotonicity.

3.4 Individual, Mutual, and Common Beliefs

I examine how common belief inherits properties of individual and mutual beliefs. I also clarify how logically sophisticated the players need to be in order for common

[23] Lismont and Mongin (2003, Proposition 3) show that, letting B_I be quasi-monotonic, if E is common belief according to their definition (Expression (7)) then E is common belief in the sense of Monderer and Samet (1989) (i.e., the right-hand side of Expression (3)). The converse holds when B_I satisfies Finite Conjunction.

belief to inherit properties of individual and mutual beliefs. The common belief operator C satisfies Completeness (and consequently Positive Introspection) irrespective of individual beliefs (Theorem 1), and it inherits the Kripke property from mutual or individual beliefs (Corollary 2). Now:

Proposition 2. *1. If B_I satisfies No-Contradiction (resp. Consistency), then so does C.*

2. If B_I satisfies Quasi-Monotonicity (resp. Monotonicity), then so does C.

3. Every B_i satisfies Necessitation iff B_I satisfies it iff C satisfies it.

4. If B_I satisfies Non-empty λ-Conjunction, then so does C.

5. Let B_I satisfy Truth Axiom. Then, for each $E \in \mathcal{D}$,

$$C(E) = \{\omega \in \Omega \mid \textit{there is } F \in \mathcal{J}_I \textit{ with } \omega \in F \subseteq E\}.$$

Especially, C satisfies Truth Axiom and Monotonicity.

6. Let $\bigcap_{i \in I} \mathcal{J}_{B_i}$ be closed, and let each B_i satisfy Negative Introspection. If $B_i C(\cdot) \subseteq C(\cdot)$ for each $i \in I$, then C satisfies Negative Introspection. The converse holds when each B_i satisfies Consistency.

In Part (1), if some B_i satisfies No-Contradiction and Consistency, respectively, then B_I and consequently C satisfy No-Contradiction and Consistency. Part (2) is closely related to Lismont and Mongin (2003, Proposition 4). In Part (3), while C may fail Necessitation, $C(\cdot) \subseteq B_I(\Omega)$ (as $C(\cdot) \subseteq \Omega$). Denote by RAT_I the event that the players are "rational" (formally studied in Section 4). Section 4 shows that the property $C(\mathrm{RAT}_I) \subseteq B_I(\Omega)$ prevents the players from choosing strictly dominated actions under common belief in rationality. As $B_I^{\varpi}(\mathrm{RAT}_I) \subseteq B_I(\Omega)$ may fail without Necessitation of B_I, the players may even take strictly dominated actions under iterative common belief in rationality (Section 4.4). In Part (4), if each B_i satisfies Non-empty λ-Conjunction then so does B_I.

In Part (5), common belief inherits Truth Axiom from mutual belief. It implies whether the common belief operator C satisfies Monotonicity hinges also on whether individual beliefs are true. Thus, C is monotonic if B_I is either (i) monotonic, (ii) true, or (iii) quasi-monotonic and finitely conjunctive (see Remark 1 for a counterexample). If any one player's belief satisfies Truth Axiom, then C satisfies Truth Axiom, Monotonicity, and Positive Introspection. Especially, if players' beliefs are assumed to satisfy Truth Axiom (i.e., knowledge instead of belief is analyzed), then common belief turns out to be common knowledge.[24]

[24]Part (5) generalizes the previous literature on common knowledge and common certainty (i.e., common p-belief with probability $p = 1$) such as: Aumann (1999), Binmore and Brandenburger (1990), Brandenburger and Dekel (1987a), Geanakoplos (1989), Nielsen (1984), Shin (1993), and Vassilakis and Zamir (1993). Lismont and Mongin (2003) also study a notion of common knowledge imposing Quasi-Monotonicity on individual players' beliefs.

Generally, if the mutual belief in an event E is true, i.e., $B_I(E) \subseteq E$, then the common belief in E is true: $C(E) \subseteq E$. Thus, if the players have the correct mutual belief in the common belief in E, i.e., $B_I C(E) \subseteq C(E)$, then $C(E) = CC(E)$. Section 4 studies the role of correct mutual belief in rationality, $B_I(\mathrm{RAT}_I) \subseteq \mathrm{RAT}_I$, in the statement that common belief in rationality leads to action profiles that survive elimination of strictly dominated actions.[25]

Next, I turn to Negative Introspection. Generally, Negative Introspection of individual beliefs does not necessarily imply that of C (e.g., Colombetti (1993)). Part (6) generalizes Bonanno and Nehring (2000) in that common belief inherits Negative Introspection if the publicly-evident events are closed and each player has correct belief in common belief.[26]

4 Implications of Common Belief in Rationality

This section establishes that, under the definition of common belief C, common belief in rationality leads to actions that survive iterated elimination of strictly dominated actions (IESDA) irrespective of assumptions on beliefs (Theorem 3). Importantly, iterative common belief in rationality may not necessarily lead to actions that survive IESDA if B_I^{ϖ} is not complete (e.g., if the players' beliefs do not satisfy Monotonicity).

Section 4.1 defines an epistemic model of a game that describes the players' actions based on their interactive beliefs, rationality, and a process of IESDA. Then, Sections 4.2 and 4.3 establish and discuss Theorem 3, respectively. Section 4.4 provides an example in which iterative common belief B_I^{ϖ} in rationality may not characterize IESDA. Section 4.5 demonstrates that, for any non-zero limit ordinal α, iterative common belief B_I^{α} in rationality may not characterize IESDA.

4.1 An Epistemic Model of a Game and IESDA

A *(strategic) game* is a tuple $\Gamma = \langle (A_i)_{i \in I}, (\succcurlyeq_i)_{i \in I} \rangle$: A_i is a non-empty set of actions available to player i, and $\succcurlyeq_i$ is i's (complete and transitive) preference ordering on $A := \bigtimes_{i \in I} A_i$. Denote by $\sim_i$ and $\succ_i$ the indifference and strict preference orderings, respectively.

Fix a game Γ, and fix an infinite cardinal κ with $\max(|A|, |I|) < \kappa$. A *model* of the game Γ is a tuple $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, C, (\sigma_i)_{i \in I} \rangle$ (abusing the notation, denote it by $\overrightarrow{\Omega}$) with the following three properties. First, $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, C \rangle$ is a κ-belief space. Second, $\sigma_i : \Omega \to A_i$ is a *strategy (function)* of player i jointly satisfying the

[25] Bonanno and Tsakas (2018) demonstrate the role of Truth Axiom of C in implications of common belief in "weak-dominance rationality."

[26] The assumption that the publicly-evident events are closed cannot be dropped. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. For each $i \in I$, let $B_i(E) = E$ if $E \in \{\{\omega_2\}, \{\omega_1, \omega_3\}, \Omega\}$ and $B_i(E) = \emptyset$ otherwise. While B_i satisfies Truth Axiom and Negative Introspection, $\bigcap_{i \in I} \mathcal{J}_{B_i}$ fails to be closed. If $E = \{\omega_1, \omega_3\}$ then $(\neg C)(E) = \{\omega_2\} \not\subseteq \emptyset = C(\neg C)(E)$.

measurability condition that $[a_i' \succsim_i a_i] := \{\omega \in \Omega \mid (a_i', \sigma_{-i}(\omega)) \succsim_i (a_i, \sigma_{-i}(\omega))\} \in \mathcal{D}$ for any $a_i, a_i' \in A_i$. In words, $[a_i' \succsim_i a_i]$ is the event that player i prefers taking action a_i' to a_i given the opponents' strategies σ_{-i}. Define $[a_i' \succ_i a_i]$ and $[a_i' \sim_i a_i]$ analogously.[27] Third, the set RAT_i of states at which player i is rational (see, e.g., Bonanno (2008, 2015) and Chen, Long, and Luo (2007)) is assumed to be an event:

$$\mathrm{RAT}_i := \{\omega \in \Omega \mid \text{there is no } a_i' \in A_i \text{ with } \omega \in B_i([a_i' \succ_i \sigma_i(\omega)])\} \in \mathcal{D}.$$

Let $\mathrm{RAT}_I := \bigcap_{i \in I} \mathrm{RAT}_i$. Player i is *rational* at $\omega \in \Omega$ if there is no action $a_i' \in A_i$ such that player i believes that playing a_i' is strictly better than playing $\sigma_i(\omega)$ given the opponents' strategies σ_{-i}. In other words, player i is rational at ω if, for any action a_i', she always considers it possible that playing $\sigma_i(\omega)$ is at least as good as playing a_i' given the opponents' strategies σ_{-i}: $\omega \in (\neg B_i)(\neg[\sigma_i(\omega) \succsim_i a_i'])$ for any $a_i' \in A_i$.[28]

To define a process of iterated elimination of strictly dominated actions (IESDA), fix a game Γ and identify any subset $A' = \bigtimes_{i \in I} A_i'$ of A with a (sub-)game. A process of *iterated elimination of strictly dominated actions (IESDA)* is an ordinal sequence of $A^\alpha = \bigtimes_{i \in I} A_i^\alpha$ (with $|\alpha| \leq |A|$) defined as follows: (i) $A^0 = A$; (ii) for an ordinal $\alpha > 0$, if the game $\bigcap_{\beta < \alpha} A^\beta$ has strictly dominated actions then A^α is obtained by eliminating *at least one* such action from $\bigcap_{\beta < \alpha} A^\beta$; and (iii) if the game $\bigcap_{\beta < \alpha} A^\beta$ does not contain any strictly dominated action then let $A^\alpha = \bigcap_{\beta < \alpha} A^\beta$. Since $(A^\alpha)_\alpha$ is a decreasing sequence, there exists the least ordinal α (with $|\alpha| \leq |A|$) with $A^\alpha = \bigcap_{\beta < \alpha} A^\beta$. Define $A^{\mathrm{IESDA}} := A^\alpha$. An action profile $a \in A$ *survives* the process of IESDA if $a \in A^{\mathrm{IESDA}}$. As is well known, the process of IESDA may require a transfinite number of eliminations and may depend on the order of eliminations, for example, for an infinite action space and discontinuous utility functions that represent the players' preferences (e.g., Dufwenberg and Stegeman (2002)).

4.2 Characterization of Common Belief in Rationality

Now, I establish the main result of this section: common belief in rationality characterizes the solution concept of IESDA. The main part states: if the players have common belief and correct mutual belief in each players' rationality, then the resulting play survives a process of IESDA, irrespective of assumptions on the players' beliefs. The second part states: any action profile that survives a process of IESDA can be obtained as an implication of common belief in rationality for some model. Completeness of common belief C is crucial for the theorem. Sections 4.4 and 4.5 demonstrate iterative common belief may *not* characterize IESDA if it is not complete.

[27] While not necessary, one can incorporate the assumption that each player is certain of her own strategy under another (stronger) measurability condition: $\sigma_i^{-1}(\{a_i\}) \in \mathcal{D}$ for all $a_i \in A_i$. Letting $[\sigma_i(\omega)] := \sigma_i^{-1}(\{\sigma_i(\omega)\}) \in \mathcal{D}$, the assumption that player i is *certain of her own strategy* amounts to: $[\sigma_i(\omega)] \subseteq B_i([\sigma_i(\omega)])$ for every $\omega \in \Omega$. If B_i satisfies Consistency and Monotonicity, then the assumption implies $B_i([\sigma_i(\cdot)]) = [\sigma_i(\cdot)]$, $B_i([\sigma_i(\cdot)]^c) = [\sigma_i(\cdot)]^c$, and Necessitation $B_i(\Omega) = \Omega$.

[28] Under the Kripke property of B_i, $\omega \in \mathrm{RAT}_i$ iff, for any $a_i' \in A_i$, there is a state ω' considered possible at ω (i.e., $\omega' \in b_{B_i}(\omega)$) such that $(\sigma_i(\omega), \sigma_{-i}(\omega')) \succsim_i (a_i', \sigma_{-i}(\omega'))$.

Theorem 3. *Fix a game* Γ*, a process of IESDA, and an infinite cardinal* κ *with* $\kappa > \max(|A|, |I|)$*.*

1. *Let* $\overrightarrow{\Omega}$ *be a model of* Γ *with* $B_I(\mathrm{RAT}_i) \subseteq \mathrm{RAT}_i$ *for all* $i \in I$*. If* $\omega \in \bigcap_{i\in I} C(\mathrm{RAT}_i)$*, then* $\sigma(\omega) \in A^{\mathrm{IESDA}}$*.*

2. *For any* $a \in A^{\mathrm{IESDA}}$*, there exist a model* $\overrightarrow{\Omega}$ *of* Γ *and a state* $\omega \in \Omega$ *with: (i)* $a = \sigma(\omega)$*; (ii)* $B_I(\mathrm{RAT}_i) \subseteq \mathrm{RAT}_i$ *for all* $i \in I$*; and (iii)* $\omega \in \bigcap_{i\in I} C(\mathrm{RAT}_i)$*.*

The proof of the first part is by induction. Since the players are assumed to have correct mutual belief in rationality, common belief in rationality implies rationality (recall Proposition 2 (5)). First, common belief in rationality implies that no player takes a strictly dominated action as, otherwise, it would violate rationality. This argument does not hinge on Necessitation of the players' beliefs. In contrast, Section 4.4 demonstrates that, under correct iterative common belief in rationality, the players may take strictly dominated actions because they may fail to believe their actions are strictly dominated by the failure of Necessitation. Note that if B_i satisfies Necessitation, then player i is not rational at ω at which $\sigma_i(\omega)$ is strictly dominated. Thus, under Necessitation, the players never take strictly dominated actions under correct iterative common belief in rationality.

Second, since common belief in rationality is a common basis, the players believe that no player takes a strictly dominated action. By this mutual belief, no player takes a strictly dominated action in the subgame in which some strictly dominated actions are eliminated. In contrast, since iterative common belief in rationality may not be a common basis, it may not necessarily imply that the players believe that no player takes a strictly dominated action. Consequently, some player may take a strictly dominated action in the subgame in which some strictly dominated actions are eliminated in the first step.

In this way, common belief in rationality implies that no player takes a strictly dominated action in each subgame generated by the process of IESDA. If common belief in rationality implies no player takes a strictly dominated action in the λ-th round, then, since common belief in rationality is a common basis, the players believe that no player takes a strictly dominated action in the λ-th round of eliminations. Then, no player takes a strictly dominated action in the $(\lambda+1)$-th round as, otherwise, it would violate rationality.

If iterative common belief in each player's rationality is a common basis, then at any state consistent with iterative common belief in rationality, the resulting actions survive IESDA. However, this is exactly when iterative common belief in rationality reduces to common belief in rationality (see Theorem 2). Thus, the chain of mutual beliefs in rationality and the process of elimination of strictly dominated actions correspond with each other when iterative common belief in rationality is a common basis (i.e., iterative common belief in rationality coincides with common belief in rationality).

4.3 Discussions on Theorem 3

Before presenting examples, seven remarks on Theorem 3 are in order. First, as discussed, Completeness plays a crucial role. Theorem 3 continues to hold for any operator $\tilde{C}$ satisfying (i) Completeness and (ii) $\tilde{C}(\cdot) \subseteq B_I(\cdot)$. Section 4.4 will demonstrate that the theorem does not hold under iterative common belief B_I^{ϖ} because B_I^{ϖ} may fail Completeness "at RAT_i" (or, $B_I^{\varpi}(\text{RAT}_i)$ is not a common basis). On the other hand, if $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed, then some $B_I^{\alpha} = C$ captures the prediction under common belief in rationality. However, Section 4.5 shows no iterative common belief B_I^{α} may capture the prediction under common belief in rationality precisely because B_I^{α} is not complete (or, $B_I^{\alpha}(\text{RAT}_i)$ is not a common basis). Taking it for granted that a notion of common belief implies mutual belief and recalling Theorem 1, the operator C is the most permissive common belief operator guaranteeing Theorem 3.

Second, while Theorem 3 assumes that, for every player i, her rationality is common belief at ω, one can formulate an alternative assumption on the common belief that all the players are rational. Let $\overrightarrow{\Omega}$ be a model of Γ such that $B_I(\text{RAT}_I) \subseteq \text{RAT}_I$: the players have correct mutual belief in their rationality. Then, $\omega \in C(\text{RAT}_I)$ implies $\sigma(\omega) \in A^{\text{IESDA}}$. This is because $C(\text{RAT}_I) \subseteq \text{RAT}_I \subseteq \text{RAT}_i$ for each $i \in I$ implies $C(\text{RAT}_I) \subseteq \bigcap_{i \in I} C(\text{RAT}_i)$. In the standard case in which every B_i is induced by a possibility correspondence, these two formulations of the theorem coincide with each other.[29] In either case, Theorem 3 holds when the players have correct common belief in rationality: $\bigcap_{i \in I} C(\text{RAT}_i) \subseteq \text{RAT}_I$ or $C(\text{RAT}_I) \subseteq \text{RAT}_I$.

Third, the theorem assumes correct mutual belief in each player's rationality $B_I(\text{RAT}_i) \subseteq \text{RAT}_i$. If player i herself correctly believes her own rationality (i.e., $B_i(\text{RAT}_i) \subseteq \text{RAT}_i$), then the assumption is satisfied. If B_i is induced by her serial and transitive possibility correspondence (i.e., Consistency and Positive Introspection as well as the Kripke property are assumed) and if she is certain of her own strategy (see footnote 27), then it can be seen that she correctly believes her own rationality.

Fourth, rationality and common belief in rationality (RCBR) alone (without imposing correct common belief) may not suffice. Section 4.4 demonstrates $\omega \in \text{RAT}_I \cap C(\text{RAT}_I)$ (or $\omega \in \bigcap_{i \in I}(\text{RAT}_i \cap C(\text{RAT}_i))$) may not necessarily imply $\sigma(\omega) \in A^{\text{IESDA}}$. This is because the operator defined by $E \mapsto C(E) \cap E$ may fail Completeness. In contrast, Appendix B defines the common knowledge operator C^* that satisfies Truth Axiom, Completeness, and $C^*(\cdot) \subseteq B_I(\cdot)$ (consequently, $C^*(E) \subseteq C(E) \cap E$). Then, common knowledge of rationality leads to actions that survive IESDA: $\omega \in \bigcap_{i \in I} C^*(\text{RAT}_i)$ implies $\sigma(\omega) \in A^{\text{IESDA}}$. Note that $C^*(\text{RAT}_I) \subseteq \bigcap_{i \in I} C^*(\text{RAT}_i)$ follows since C^* satisfies Monotonicity.

Fifth, on the other hand, if B_I satisfies Finite Conjunction and if $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed, then RCBR lead to actions that survive IESDA: $\omega \in C(\text{RAT}_I) \cap \text{RAT}_I$ implies $\sigma(\omega) \in A^{\text{IESDA}}$. This is because $C(\text{RAT}_I) \cap \text{RAT}_I$ itself becomes a common

[29] If each B_i satisfies Monotonicity then $C(\text{RAT}_I) \subseteq \bigcap_{i \in I} C(\text{RAT}_i)$. If each B_i satisfies Non-empty λ-Conjunction with $\lambda > |I|$, then $\bigcap_{i \in I} C(\text{RAT}_i) \subseteq C(\text{RAT}_I)$.

basis. Thus, if B_I satisfies Countable Conjunction and if $\bigcap_{i\in I}\mathcal{J}_{B_i}$ is closed (e.g., countably-additive probability 1-belief operators $(B_i^1)_{i\in I}$ satisfy Countable Conjunction and Monotonicity), then (i) $C = B_I^{\varpi}$ (recall Corollary 1 (2)) and (ii) RCBR lead to actions that survive IESDA.

Sixth, Theorem 3 does not hinge on contradictory beliefs because $C(E)$ is independent of $C(\emptyset)$ for any non-empty E (Remark 1). As the first case, suppose $\mathrm{RAT}_i = \emptyset$ for some player i. Suppose further that C violates No-Contradiction. The assumption that the players have correct mutual belief in i's rationality implies $C(\mathrm{RAT}_i) \subseteq \mathrm{RAT}_i = \emptyset$, and the statement is vacuous.[30] As the second case, suppose $\mathrm{RAT}_i \neq \emptyset$ for every i. Then, since none of $C(\mathrm{RAT}_i)$ depends on $C(\emptyset)$, the violation of No-Contradiction $C(\emptyset) \neq \emptyset$ does not play any role.

Seventh, Theorem 3 holds under any stronger notion of rationality. Let $\mathrm{RAT}_i^* \subseteq \mathrm{RAT}_i$ for each $i \in I$. If $B_I(\mathrm{RAT}_i^*) \subseteq \mathrm{RAT}_i^*$ for each $i \in I$ as in the assumption of the theorem, $\omega \in \bigcap_{i\in I} C(\mathrm{RAT}_i^*)$ implies $\sigma(\omega) \in A^{\mathrm{IESDA}}$. This is because $C(\mathrm{RAT}_i^*)$ is a common basis implying RAT_i and consequently $C(\mathrm{RAT}_i)$.

4.4 First Counterexample for Iterative Common Belief

I provide an example in which, under iterative common belief in rationality, the players take actions that do not survive a process of IESDA. Let Γ be the two-player game represented by Table 2 in the usual way. By inspection, $A^{\mathrm{IESDA}} = \{(c, f)\}$.

	d	e	f
a	$0,0$	$0,1$	$0,0$
b	$1,0$	$1,1$	$1,2$
c	$0,0$	$2,1$	$2,2$

Table 2: A Two-player Strategic Game

Define individual and common beliefs (B_1, B_2, C) on $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3, \omega_4\}, \mathcal{P}(\Omega))$ as in Table 3. By inspection, $\bigcap_{i\in I}\mathcal{J}_{B_i}$ is not closed and $C(\cdot) \subsetneq B_I^{\varpi}(\cdot)$. I have assumed No-Contradiction and Necessitation for B_I and consequently C.

Denote player i's strategy by $\sigma_i = (\sigma_i(\omega_1), \sigma_i(\omega_2), \sigma_i(\omega_3), \sigma_i(\omega_4))$. The rest of this subsection provides six strategy profiles $\sigma = (\sigma_1, \sigma_2)$. The first two profiles demonstrate that iterative common belief in rationality does not lead to actions that survive IESDA. The other four profiles supplement the discussions on Theorem 3 (its assumptions and the roles of Consistency and Necessitation).

First, let $\sigma_1 = (a, c, c, b)$ and $\sigma_2 = (d, e, f, e)$. Then, $\mathrm{RAT}_i = \{\omega_2, \omega_3\}$. Consider, for example, player $i = 2$. At ω_1, player 2 believes that taking $a_2' = e$ is strictly better than following $\sigma_2(\omega_1) = d$. Next, player 2 chooses $e = \sigma_2(\omega)$ at $\omega \in \{\omega_2, \omega_4\}$.

[30] Likewise, the alternative assumption that the players have correct mutual belief in their rationality implies $C(\mathrm{RAT}_I) \subseteq \mathrm{RAT}_I = \emptyset$, and the statement is again vacuous.

E	$B_i(E) = B_I(E)$	$B_I^{\varpi}(E)$	$C(E)$	$C(E) \cap E$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\emptyset$	$\emptyset$
$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$	$\emptyset$	$\emptyset$
$\{\omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_4\}$	$\{\omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$	$\{\omega_4\}$
$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_4\}$	$\{\omega_4\}$	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$
$\{\omega_1, \omega_4\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\emptyset$
$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_2, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_4\}$
$\{\omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$
$\{\omega_1, \omega_2, \omega_3\}$	$\{\omega_1, \omega_2, \omega_3\}$	$\{\omega_1, \omega_2, \omega_3\}$	$\{\omega_1, \omega_2, \omega_3\}$	$\{\omega_1, \omega_2, \omega_3\}$
$\{\omega_1, \omega_2, \omega_4\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$
$\{\omega_2, \omega_3, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_1, \omega_3, \omega_4\}$	$\{\omega_3, \omega_4\}$
Ω	Ω	Ω	Ω	Ω

Table 3: Players' Interactive Beliefs on $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3, \omega_4\}, \mathcal{P}(\Omega))$

Player 2 believes that taking $a_2' = f$ is strictly better than following e at ω_1, ω_3, or ω_4. Thus, while player 2 is not rational at ω_4, she is rational at ω_2 (she does not believe taking d is better than following e at ω_2 either). At ω_3, player 2 does not believe that deviating to d or e is strictly better than following $\sigma_2(\omega_3) = f$. While $\bigcap_{i \in I} C(\text{RAT}_i) = C(\text{RAT}_I) = \{\omega_3\}$, $\bigcap_{i \in I} B_I^{\varpi}(\text{RAT}_i) = B_I^{\varpi}(\text{RAT}_I) = \{\omega_2, \omega_3\} \in (\bigcap_{i \in I} \mathcal{J}_{B_i}) \setminus \mathcal{J}_I$. While $\sigma(\omega_3) = (c, f) \in A^{\text{IESDA}}$, $\sigma(\omega_2) = (c, e) \notin A^{\text{IESDA}}$.

To see why implications of common belief in rationality across two notions differ, observe first that no player takes a strictly dominated action under both notions of common belief in rationality. By Necessitation, rationality prevents the players from taking a strictly dominated action, and the players have correct common belief in rationality: $\bigcap_{i \in I} C(\text{RAT}_i) \subseteq \text{RAT}_I \subseteq \{\omega_2, \omega_3, \omega_4\}$ and $\bigcap_{i \in I} B_I^{\varpi}(\text{RAT}_i) \subseteq \text{RAT}_I \subseteq \{\omega_2, \omega_3, \omega_4\}$. However, iterative common belief in rationality may not lead to the mutual belief that no player takes a strictly dominated action: $B_I^{\varpi}(\text{RAT}_i) \not\subseteq B_I(\{\omega_2, \omega_3, \omega_4\}) = \{\omega_1, \omega_3, \omega_4\}$. In other words, although iterative common belief in rationality implies that the players never take strictly dominated actions, it does not imply that *they believe* that they never take strictly dominated actions. Put differently, $B_I^{\varpi}(\text{RAT}_i)$ is not a common basis/complete "at the event $\{\omega_2, \omega_3, \omega_4\}$ that no player takes a strictly dominated action." The other characterizations of common belief in Proposition 1 are not complete "at the event $\{\omega_2, \omega_3, \omega_4\}$" either.

Second, let $\sigma_1 = (c, b, a, a)$ and $\sigma_2 = (e, e, d, d)$. Then, $\text{RAT}_i = \{\omega_1\}$. While $\bigcap_{i \in I} C(\text{RAT}_i) = C(\text{RAT}_I) = \emptyset$, $\bigcap_{i \in I} B_I^{\varpi}(\text{RAT}_i) = B_I^{\varpi}(\text{RAT}_I) = \{\omega_1\} \in (\bigcap_{i \in I} \mathcal{J}_{B_i}) \setminus$

$\mathcal{J}_I$. Observe $\sigma(\omega_1) = (c, e) \notin A^{\text{IESDA}}$. While rationality prevents the players from taking strictly dominated actions under both notions of common belief in rationality, again, iterative common belief in rationality does not lead to the mutual belief that no player takes a strictly dominated action: $B_I^{\varpi}(\text{RAT}_i) \nsubseteq B_I(\{\omega_1, \omega_2\}) = \{\omega_2, \omega_3\}$. In other words, $B_I^{\varpi}(\text{RAT}_i)$ is not a common basis.

Third, to see that Theorem 3 holds as long as $\bigcap_{i \in I} C(\text{RAT}_i) \subseteq \text{RAT}_I$ or $C(\text{RAT}_I) \subseteq \text{RAT}_I$, let $\sigma_1 = (b, a, c, b)$ and $\sigma_2 = (d, f, f, e)$. Then, $\text{RAT}_i = \{\omega_3\}$. While the assumption of Theorem 3 is not satisfied (i.e., $B_I(\text{RAT}_i) \nsubseteq \text{RAT}_i$), $\bigcap_{i \in I} C(\text{RAT}_i) \subseteq \text{RAT}_I$. Indeed, $\{\omega_3\} = \bigcap_{i \in I} C(\text{RAT}_i)$ and $\sigma(\omega_3) = (c, f) \in A^{\text{IESDA}}$.

Fourth, RCBR may not necessarily lead to actions that survive IESDA (i.e., $\omega \in C(\text{RAT}_I) \cap \text{RAT}_I$ does not necessarily imply $\sigma(\omega) \in A^{\text{IESDA}}$) when the operator defined by $E \mapsto C(E) \cap E$ is not complete. Let $\sigma_1 = (b, c, a, b)$ and $\sigma_2 = (e, f, d, e)$. Then, $\text{RAT}_i = \{\omega_4\}$. The assumption of Theorem 3 does not hold: even $C(\text{RAT}_I) = \{\omega_3, \omega_4\} \nsubseteq \text{RAT}_I$. At state $\omega_4 \in C(\text{RAT}_I) \cap \text{RAT}_I$, $\sigma(\omega_4) = (b, e) \notin A^{\text{IESDA}}$. At ω_4, the players are rational and have common belief in rationality. While they never take strictly dominated actions at ω_4 (by rationality), they do not believe that they never take strictly dominated actions: $C(\text{RAT}_I) \cap \text{RAT}_I = \{\omega_4\} \nsubseteq B_I(\{\omega_1, \omega_2, \omega_4\})$.

Fifth, while the common belief operator C violates Consistency in the example, Theorem 3 and the counterexample (i.e., the contrast between C and B_I^{ϖ}) do not hinge on Consistency. For example, redefine the players' beliefs by $B_i(E) \cap E$ in the example. By construction, the redefined operators B_I and C satisfy Truth Axiom and thus Consistency and No-Contradiction. The collection of common bases does not change. Thus, $B_I^{\varpi}(\{\omega_2, \omega_3\}) = \{\omega_2, \omega_3\}$ and $C(\{\omega_2, \omega_3\}) = \{\omega_3\}$ continue to hold. As in the original example, let $\sigma_1 = (a, c, c, b)$ and $\sigma_2 = (d, e, f, e)$. Then, $\text{RAT}_i = \{\omega_2, \omega_3\}$. While $\bigcap_{i \in I} C(\text{RAT}_i) = C(\text{RAT}_I) = \{\omega_3\}$, $\bigcap_{i \in I} B_I^{\varpi}(\text{RAT}_i) = B_I^{\varpi}(\text{RAT}_I) = \{\omega_2, \omega_3\}$. While $\sigma(\omega_3) = (c, f) \in A^{\text{IESDA}}$, $\sigma(\omega_2) = (c, e) \notin A^{\text{IESDA}}$.

Sixth, I demonstrate that, absent Necessitation, players may even take strictly dominated actions under iterative common belief in rationality (note, however, the counterexample in this subsection has already shown that players with iterative common belief in rationality may take actions that do not survive a process of IESDA without resorting to such misunderstanding). Intuitively, this is because players who violate Necessitation may even misunderstand objective features of the game about which they are reasoning. As an example, there exists a model of the Prisoners' Dilemma in which two players play cooperation under iterative common belief in rationality by the failure of Necessitation.[31]

However, I examine the role of Necessitation in detail by modifying the exam-

[31] Let $A_i = \{c, d\}$ for each $i \in I = \{1, 2\}$, and let $(d, c) \succ_i (c, c) \succ_i (d, d) \succ_i (c, d)$. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$, and let $B_i(E) = E$ for any $E \neq \Omega$ and $B_i(\Omega) = \{\omega_1, \omega_2\}$. Hence, at $\omega \in \{\omega_1, \omega_2\}$, player i believes an event E at ω iff E holds true at ω. At ω_3, however, player i does not believe a tautology Ω (except for the tautology Ω, player i believes $E(\neq \Omega)$ at ω_3 iff E holds true at ω_3). Let $\sigma_i = (c, d, c)$. Then, $\text{RAT}_i = \{\omega_2, \omega_3\}$, $B_I^{\varpi}(\{\omega_2, \omega_3\}) = \{\omega_2, \omega_3\}$, and $C(\{\omega_2, \omega_3\}) = \{\omega_2\}$. At $\omega_3 \in \bigcap_{i \in I} B_I^{\varpi}(\text{RAT}_i)$, the players choose $\sigma(\omega_3) = (c, c) \notin A^{\text{IESDA}} = \{(d, d)\}$.

ple of this subsection. Redefine $B_i(\Omega) = \{\omega_1, \omega_2, \omega_3\}$.[32] Let $\sigma_1 = (a, a, c, a)$ and $\sigma_2 = (d, d, f, d)$. Then, $\mathrm{RAT}_i = \{\omega_3, \omega_4\}$. Each player i is rational at ω_4: for example, player 1 does not believe at ω_4 that a is strictly dominated by b as her belief violates Necessitation. While $\bigcap_{i \in I} C(\mathrm{RAT}_i) = C(\mathrm{RAT}_I) = \{\omega_3\}$, $\bigcap_{i \in I} B_I^{\varpi}(\mathrm{RAT}_i) = B_I^{\varpi}(\mathrm{RAT}_I) = \{\omega_3, \omega_4\}$. While $\sigma(\omega_3) = (c, f) \in A^{\mathrm{IESDA}}$, the players take a strictly dominated action at ω_4: $\sigma(\omega_4) = (a, d) \notin A^{\mathrm{IESDA}}$.

On the one hand, under common belief C, RCBR prevent each player i from playing a strictly dominated action a_i. If action a_i is strictly dominated by a_i', then $[a_i' \succ_i a_i] = \Omega$. Let $\omega \in \Omega$ be with $\sigma_i(\omega) = a_i$ and $\omega \in C(\mathrm{RAT}_I) \cap \mathrm{RAT}_i$ (or $\omega \in C(\mathrm{RAT}_i) \cap \mathrm{RAT}_i$). Then, $\omega \in \mathrm{RAT}_i$ and $\omega \in C(\mathrm{RAT}_I) \subseteq B_i([a_i' \succ_i a_i])$, which is a contradiction. On the other hand, since iterative common belief may not be complete, iterative common belief in rationality (e.g., $\omega_4 \in B_I^{\varpi}(\mathrm{RAT}_i)$) does not necessarily imply that player i believes that a_i is strictly dominated. Hence, her action $\sigma_i(\omega)$ (e.g., $a = \sigma_1(\omega_4)$ or $d = \sigma_2(\omega_4)$) does not contradict her rationality.

4.5 Second Counterexample for Iterative Common Belief

The previous subsection shows that, even if the chain of mutual beliefs B_I^{ϖ} converges, it may not characterize IESDA when it is not complete. The example of this subsection shows that B_I^{α} does not even converge for any given non-zero limit ordinal α (including $\alpha = \varpi$). Moreover, while the chain of mutual beliefs $B_I^{\leq \alpha+1}(\mathrm{RAT}_I)$ in rationality converges (so that $B_I^{\alpha+\varpi}(\mathrm{RAT}_I)$ is publicly evident), the prediction under such iterative common belief does not capture IESDA due to the failure of completeness.

This suggests two points for the assertion that iterations of mutual (monotonic) beliefs may need to be arbitrarily long to capture common belief (e.g., Barwise (1989), Heifetz (1999), and Lismont and Mongin (1994b, 1995)). First, some iterative (possibly non-monotonic) mutual beliefs can still capture common belief as long as the publicly-evident events are closed. Second, when the publicly-evident events are *not* closed, this coincidence may totally break down: arbitrarily-long iterative mutual beliefs in rationality may lead to actions that do not survive IESDA.

I define a two-player strategic game Γ. Let α be a non-zero limit ordinal, and let $A_i := \alpha + 2$ for each $i \in I := \{1, 2\}$. Since the players can reason about their actions, let κ be an infinite cardinal with $\kappa > |A|$ or $\kappa = \infty$.

Define each $\succcurlyeq_i$ as follows: for any action profiles $(\beta, \gamma), (\beta', \gamma') \in A_i \times A_{-i}$,

$$(\beta, \gamma) \succcurlyeq_i (\beta', \gamma') \text{ iff } u_i(\beta, \gamma) \geq u_i(\beta', \gamma'),$$

where an auxiliary function $u_i : A_i \times A_{-i} \to \alpha + 2$ is defined as follows. First, if $\beta = 0$

[32] By this change, the common bases $\{\omega_3, \omega_4\}$, $\{\omega_1, \omega_3, \omega_4\}$, and Ω that contain ω_4 in the original example are no longer common bases. Thus, replace any $C(E) = \{\omega_3, \omega_4\}$ with $\{\omega_3\}$, any $C(E) = \{\omega_1, \omega_3, \omega_4\}$ with $\{\omega_1, \omega_3\}$, and $C(\Omega) = \{\omega_1, \omega_2, \omega_3\}$. In this new example, B_I^{ϖ} remains the same except for $B_I^{\varpi}(\Omega) = \{\omega_1, \omega_2, \omega_3\}$.

or β is a non-zero limit ordinal with $\beta(\leq \alpha)$, then

$$u_i(\beta, \gamma) := \begin{cases} 0 & \text{if } \beta > \gamma \\ \beta & \text{if } \beta \leq \gamma \end{cases}.$$

Second, if β is a successor ordinal $(1 \leq)\beta = \delta + 1(\leq \alpha + 1)$, then

$$u_i(\beta, \gamma) := \begin{cases} \beta & \text{if } \delta \leq \gamma \\ 0 & \text{if } \delta > \gamma \end{cases}.$$

Table 4 depicts the case with $\alpha = \varpi 2$ (also, the north-west $(\varpi + 2) \times (\varpi + 2)$ matrix represents the case with $\alpha = \varpi$). An action profile $(\alpha + 1, \alpha + 1)$ is the unique one that survives any process of IESDA. Samet (2015) considers a related game with $A_i = \varpi + 2$ in which the unique prediction is obtained for one more round of elimination after which all finite rounds of eliminations have been made.

	0	1	2	3	4	⋯⋯⋯	ϖ	$\varpi+1$	$\varpi+2$	$\varpi+3$	$\varpi+4$	⋯⋯⋯	$\varpi 2$	$\varpi 2+1$
0	0	0	0	0	0	⋯⋯⋯	0	0	0	0	0	⋯⋯⋯	0	0
1	1	1	1	1	1	⋯⋯⋯	1	1	1	1	1	⋯⋯⋯	1	1
2	0	2	2	2	2	⋯⋯⋯	2	2	2	2	2	⋯⋯⋯	2	2
3	0	0	3	3	3	⋯⋯⋯	3	3	3	3	3	⋯⋯⋯	3	3
4	0	0	0	4	4	⋯⋯⋯	4	4	4	4	4	⋯⋯⋯	4	4
⋮	⋮	⋮	⋮	⋮	⋮	⋯⋯⋯	⋮	⋮	⋮	⋮	⋮	⋯⋯⋯	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋯⋯⋯	⋮	⋮	⋮	⋮	⋮	⋯⋯⋯	⋮	⋮
ϖ	0	0	0	0	0	⋯⋯⋯	ϖ	ϖ	ϖ	ϖ	ϖ	⋯⋯⋯	ϖ	ϖ
$\varpi+1$	0	0	0	0	0	⋯⋯⋯	$\varpi+1$	$\varpi+1$	$\varpi+1$	$\varpi+1$	$\varpi+1$	⋯⋯⋯	$\varpi+1$	$\varpi+1$
$\varpi+2$	0	0	0	0	0	⋯⋯⋯	0	$\varpi+2$	$\varpi+2$	$\varpi+2$	$\varpi+2$	⋯⋯⋯	$\varpi+2$	$\varpi+2$
$\varpi+3$	0	0	0	0	0	⋯⋯⋯	0	0	$\varpi+3$	$\varpi+3$	$\varpi+3$	⋯⋯⋯	$\varpi+3$	$\varpi+3$
$\varpi+4$	0	0	0	0	0	⋯⋯⋯	0	0	0	$\varpi+4$	$\varpi+4$	⋯⋯⋯	$\varpi+4$	$\varpi+4$
⋮	⋮	⋮	⋮	⋮	⋮	⋯⋯⋯	⋮	⋮	⋮	⋮	⋮	⋯⋯⋯	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋯⋯⋯	⋮	⋮	⋮	⋮	⋮	⋯⋯⋯	⋮	⋮
$\varpi 2$	0	0	0	0	0	⋯⋯⋯	0	0	0	0	0	⋯⋯⋯	$\varpi 2$	$\varpi 2$
$\varpi 2+1$	0	0	0	0	0	⋯⋯⋯	0	0	0	0	0	⋯⋯⋯	$\varpi 2+1$	$\varpi 2+1$

Table 4: The specification of u_i for $\alpha = \varpi 2$

Since the game Γ has been defined, I move onto a model $\langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, C, (\sigma_i)_{i\in I}\rangle$ of Γ. Define $(\Omega, \mathcal{D}) := (\alpha + 3, \mathcal{P}(\Omega))$ (note that the state space Ω is not the least uncountable ordinal). For any $\beta \in \Omega$, denote by $[\beta, \alpha + 2]$ the set of ordinals γ with $\beta \leq \gamma \leq \alpha + 2$. Define B_i as

$$B_i(E) := \begin{cases} E & \text{if } E \in \{\emptyset, \{\alpha+2\}, \{\alpha+1, \alpha+2\}, \Omega\} \\ \{\alpha+2\} & \text{if } E = \{0, \alpha+1, \alpha+2\} \\ [\min(E)+1, \sup(E)] & \text{otherwise} \end{cases}.$$

Note that, for any non-empty subset E of Ω, $\min(E)$ and $\sup(E)$ are well defined. Since $(\Omega, \mathcal{D})$ is a complete algebra, the common belief operator C is well defined by Expression (2). Note that $\mathcal{J}_I = \{\emptyset, \{\alpha+2\}, \Omega\} \subsetneq \{\emptyset, \{\alpha+2\}, \{\alpha+1, \alpha+2\}, \Omega\} = \bigcap_{i \in I} \mathcal{J}_{B_i}$, i.e., the publicly-evident events are not closed. Define σ_i as

$$\sigma_i(\omega) := \begin{cases} \alpha+1 & \text{if } \omega \in \{0, \alpha+2\} \\ \alpha & \text{otherwise} \end{cases}.$$

Then, $\mathrm{RAT}_i = \{0, \alpha+2\}$. On the one hand, $C(\mathrm{RAT}_i) = \{\alpha+2\} \subseteq \mathrm{RAT}_i$. On the other hand, for any ordinal β with $1 \leq |\beta| < \kappa$,

$$B_I^{\beta}(\mathrm{RAT}_i) = \begin{cases} [\beta, \alpha+2] & \text{if } \beta < \alpha+1 \\ \{\alpha+1, \alpha+2\} & \text{if } \beta \geq \alpha+1 \end{cases}.$$

Now, for the given non-zero limit ordinal α, iterative common belief $B_I^{\alpha}(\mathrm{RAT}_i) = [\alpha, \alpha+2]$ is a proper superset of $C(\mathrm{RAT}_i)$. In the next round, the chain of mutual beliefs in RAT_i eventually converges to $B_I^{\leq \alpha+1}(\mathrm{RAT}_i) = \{\alpha+1, \alpha+2\}$, i.e., $B_I^{\alpha+\varpi}(\mathrm{RAT}_i) = \{\alpha+1, \alpha+2\}$. However, this is again a proper superset of $C(\mathrm{RAT}_i)$ due to the fact that the publicly-evident events are not closed. Moreover, the prediction under convergent iterative common belief in rationality does not capture IESDA. On the one hand, common belief in rationality $\bigcap_{i \in I} C(\mathrm{RAT}_i)$ (or $C(\mathrm{RAT}_I)$) captures IESDA: $\sigma(\alpha+2) = (\alpha+1, \alpha+1) \in A^{\mathrm{IESDA}}$. On the other hand, at state $\alpha+1 \in \bigcap_{i \in I} B_I^{\alpha+\varpi}(\mathrm{RAT}_i) = B_I^{\alpha+\varpi}(\mathrm{RAT}_I)$, the players choose $\sigma(\alpha+1) = (\alpha, \alpha) \notin A^{\mathrm{IESDA}}$.

5 Concluding Remarks

This paper formalized common belief without assuming any property on individual beliefs such as monotonicity. Beliefs can be qualitative as well as quantitative, on a general set algebra. If beliefs are assumed correct, common belief becomes common knowledge. When individual beliefs are (quasi-)monotonic, my formalization reduces to the past literature. The paper studied how much logical sophistication the players need to have in order for a given property of common belief to hold. For example, the paper provided conditions under which common belief can be taken as the chain of mutual beliefs even if individual beliefs are not monotonic. The key idea is to define common belief from common bases. The paper investigated the relation among individual, mutual, and common beliefs. While common belief may not be monotonic, it is complete, i.e., closed under its logical implication. Under my formalization, unlike the iterative one, common belief in rationality leads to actions that survive iterated elimination of strictly dominated actions even if the players' beliefs are not monotonic.

For avenues for further research, first, this paper would be useful for studying how various results regarding common knowledge and common belief hinge on (or are robust to) players' logical omniscience issues, beyond implications of common belief in rationality in strategic games.

Second, it is interesting to implement the idea of this paper in the context of a logical (precisely, syntactical) system to formalize common knowledge and common belief without imposing individual players' monotonic reasoning.[33] A lattice-theoretical structure on the set of propositions would be a key to defining common belief.

The third is to formulate common belief in enriched domains where each player has collections of (p-)belief operators. Examples of enriched domains (where players retain probabilistic sophistication) include ambiguous beliefs (as in Ahn (2007), Bewley (1986), and Gilboa and Schmeidler (1989)) and lexicographic beliefs. Another instance would be non-standard state space models of (un)awareness as in Heifetz, Meier, and Schipper (2006, 2013). Possibility correspondences on their generalized state space induce knowledge operators on events that satisfy logical and introspective properties. Thus, the question is to formalize common belief from players' belief operators on such a generalized state space that do not necessarily satisfy logical or introspective properties. Their generalized state space consists of multiple sub-spaces, ranked by the degree to which each sub-space can describe different aspects of the world, and thus each event has an additional component indicating the subspace in which the event is described. Since events are ordered (by set inclusion and the ranking of subspaces), common bases could naturally be defined.

A Proofs

Remark A.1. A *neighborhood system* is a mapping $\mathcal{B}_i : \Omega \to \mathcal{P}(\mathcal{D})$ with $B_{\mathcal{B}_i}(E) := \{\omega \in \Omega \mid E \in \mathcal{B}_i(\omega)\} \in \mathcal{D}$ for all $E \in \mathcal{D}$. A belief operator B_i induces the neighborhood system defined by $\mathcal{B}_{B_i}(\omega) := \{E \in \mathcal{D} \mid \omega \in B_i(E)\}$. Conversely, a neighborhood system $\mathcal{B}_i : \Omega \to \mathcal{P}(\mathcal{D})$ induces the belief operator $B_{\mathcal{B}_i}$. Belief operators and neighborhood systems are equivalent in the sense that $\mathcal{B}_{B_{\mathcal{B}_i}} = \mathcal{B}_i$ and $B_i = B_{\mathcal{B}_{B_i}}$.

Proof of Theorem 1. For Part (1), if $C(E) \subseteq F$, then $C(E) \in \mathcal{J}_I$ implies $C(E) \subseteq B_I(F)$, and thus $C(E) \subseteq C(F)$. For Part (2), fix $E \in \mathcal{D}$. If $\tilde{C}(E) \subseteq F$, then $\tilde{C}(E) \subseteq \tilde{C}(F) \subseteq B_I(F)$, i.e., $\tilde{C}(E) \in \mathcal{J}_I$. Since $\tilde{C}(E) \subseteq B_I(E)$, $\tilde{C}(E) \subseteq C(E)$. Next, Positive Introspection of $\tilde{C}$ and Condition (ii) imply $\tilde{C}(\cdot) \subseteq \tilde{C}\tilde{C}(\cdot) \subseteq B_I\tilde{C}(\cdot)$. Then, $\tilde{C}(\cdot) \in \bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$ implies $\tilde{C}(\cdot) \subseteq C(\cdot)$. □

Remark A.2. Consider the following two iterative definitions of common belief. First, $\omega \in C(E)$ implies $\omega \in (B_{i_n} \circ \cdots \circ B_{i_2} \circ B_{i_1})(E)$ for any finite sequence of players $(i_1, i_2, \ldots, i_n)$: player i_n believes ... player i_2 believes player i_1 believes E at ω. This

[33] Some syntactic formalizations of common belief and common knowledge requiring Monotonicity are: Bonanno (1996), Halpern and Moses (1992), and Lismont and Mongin (1994b, 1995). Heifetz (1999) allows infinitary languages as I allow $\mathcal{D}$ to be a κ-complete algebra. Lismont and Mongin (1994a, 2003) require Quasi-Monotonicity instead of Monotonicity.

iterative definition of common belief coincides with B_I^{ϖ} if B_I satisfies Monotonicity and Non-empty λ-Conjunction with $\lambda > |I|$.

Second, let $\omega \in C(E)$, and let B_I satisfy Finite Conjunction or $B_I(E) \subseteq E$. Then, at ω, everybody believes E (i.e., $\omega \in B_I E$), everybody believes that E and everybody believes E at ω (i.e., $\omega \in B_I(E \cap B_I E)$), everybody believes that E and everybody believes that E and everybody believes E at ω (i.e., $\omega \in B_I(E \cap B_I(E \cap B_I E))$), and so forth *ad infinitum*. This is because, for any $F \in \mathcal{D}$, $C(E) \subseteq F$ implies $C(E) \subseteq B_I(E \cap F)$. If B_I satisfies Finite Conjunction and Monotonicity, then this alternative iterative common belief coincides with B_I^{ϖ}.

Proof of Theorem 2. By definition, $C(\cdot) \subseteq B_I^1(\cdot)$. Suppose $C(\cdot) \subseteq B_I^{\beta}(\cdot)$. If $\omega \in C(E)$, then $\omega \in F \subseteq C(E) \subseteq B_I^{\beta}(E)$ for some $F \in \mathcal{J}_I$, and thus $\omega \in F \subseteq B_I^{\beta+1}(E)$. If $C(E) \subseteq B_I^{\beta}(E)$ for all $\beta < \alpha$, then $C(E) \subseteq \bigcap_{\beta:1 \leq \beta < \alpha} B_I^{\beta}(E) = B_I^{\alpha}(E)$.

Next, $B_I^{\leq\alpha}(\cdot) \subseteq B_I(\cdot)$. If $B_I^{\leq\alpha}(E) \in \mathcal{J}_I$ then $C(E) \subseteq B_I^{\leq\alpha}(E) \subseteq C(E)$. If $B_I^{\leq\alpha}$ is complete, then it follows from the previous part and Theorem 1 (2) that $C(\cdot) \subseteq B_I^{\leq\alpha}(\cdot) \subseteq C(\cdot)$. □

Proof of Corollary 1.
1. If $C = B_I^{\varpi}$ then B_I^{ϖ} satisfies Positive Introspection. If B_I^{ϖ} satisfies Positive Introspection, then $B_I^{\varpi}(\cdot) \subseteq B_I B_I^{\varpi}(\cdot)$. If $B_I^{\varpi}(\cdot) \subseteq B_I B_I^{\varpi}(\cdot)$, then $B_I^{\varpi}(\cdot) \in \bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$. Since $B_I^{\varpi}(\cdot) \subseteq B_I(\cdot)$, $C(\cdot) \subseteq B_I^{\varpi}(\cdot) \subseteq C(\cdot)$.

2. Fix $E \in \mathcal{D}$. By Theorem 2, $C(\cdot) \subseteq B_I^{\varpi}(\cdot)$. Conversely, since B_I satisfies Countable Conjunction, $B_I^{\varpi}(\cdot) \subseteq \bigcap_{n \in \mathbb{N}} B_I(B_I^n(\cdot)) \subseteq B_I(B_I^{\varpi}(\cdot))$. By the first part, $B_I^{\varpi} = C$. If B_I satisfies Truth Axiom, then $B_I^{\varpi}(E) = B_I^{\varpi}(E) \cap E$.

□

Proof of Proposition 1. First, Expression (3) follows because $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$. Next, I show Expression (4). Take any $F \in \mathcal{D}$ with $F \subseteq B_I(E) \cap B_I(F)$. Then, $F \in \bigcap_{i \in I} \mathcal{J}_{B_i} = \mathcal{J}_I$ and $F \subseteq B_I(E)$. Thus, $F \subseteq C(E)$. The converse set inclusion holds because $C(E) \subseteq B_I(E)$ and $C(E) \subseteq CC(E) \subseteq B_I(C(E))$.

Second, I show that Quasi-Monotonicity of B_I implies Expression (5). This can be seen as a variant of Tarski's fixed point theorem, stating that the greatest event F satisfying $F \subseteq f_E(F) := B_I(E) \cap B_I(F)$, given that it exists, is the greatest fixed point of $f_E(\cdot)$. See also Lismont and Mongin (2003, Proposition 1).

I start with showing Quasi-Monotonicity of f_E. If $F \subseteq f_E(F) \cap F'$, then it follows from $F \subseteq f_E(F) \cap F' \subseteq B_I(F) \cap F'$ and Quasi-Monotonicity of B_I that $B_I(F) \subseteq B_I(F')$. Thus, $f_E(F) = B_I(E) \cap B_I(F) \subseteq B_I(E) \cap B_I(F') = f_E(F')$.

Now, if $F \subseteq f_E(F) = f_E(F) \cap f_E(F)$ then $f_E(F) \subseteq f_E(f_E(F))$. If F is the largest event satisfying $F \subseteq f_E(F)$, then $f_E(F) \subseteq f_E(f_E(F))$ implies $f_E(F) \subseteq F$.

Third, Finite Conjunction of B_I implies $B_I(E) \cap B_I(F) \subseteq B_I(E \cap F)$. I show that, by Quasi-Monotonicity of B_I, if $F \subseteq B_I(E \cap F)$ then $B_I(E \cap F) \subseteq B_I(E) \cap B_I(F)$. Let $F \subseteq B_I(E \cap F)$. Since $E \cap F \subseteq B_I(E \cap F) \cap E$, I get $B_I(E \cap F) \subseteq B_I(E)$. Also, $E \cap F \subseteq B_I(E \cap F) \cap F$ yields $B_I(E \cap F) \subseteq B_I(F)$. □

Remark A.3. I remark on counterexamples for Proposition 1. For the first part, Remark 1 provides a counterexample due to $\mathcal{J}_I \neq \bigcap_{i \in I} \mathcal{J}_{B_i}$. In fact, even the equivalence between Expressions (3) and (4) fails because Expression (4) is not well defined for $E = \emptyset$. For the second part, let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2\}, \mathcal{P}(\Omega))$, and consider: $B_I(\emptyset) = \{\omega_1\}$, $B_I(\{\omega_1\}) = B_I(\{\omega_2\}) = \{\omega_2\}$, and $B_I(\Omega) = \Omega$. Then, $C(\emptyset) = \emptyset$, $C(\{\omega_1\}) = C(\{\omega_2\}) = \{\omega_2\}$, and $C(\Omega) = \Omega$. While $C(E) = \max\{F \in \mathcal{D} \mid F \subseteq B_I(E) \cap B_I(F)\}$ for every $E \in \mathcal{D}$, the event $\max\{F \in \mathcal{D} \mid F = B_I(E) \cap B_I(F)\}$ is not well defined when $E = \emptyset$. For the third part, Remark B.1 in Appendix B (i.e., Table B.1) provides counterexamples.

Proof of Proposition 2.

1. First, by No-Contradiction of B_I, $C(\emptyset) \subseteq B_I(\emptyset) = \emptyset$. Second, by Consistency of B_I, $C(E) \cap C(E^c) \subseteq B_I(E) \cap B_I(E^c) = \emptyset$.

2. First, assume Quasi-Monotonicity on B_I. Let $E \subseteq C(E) \cap F$, and let $\omega \in C(E)$. Since $E \subseteq C(E) \cap F \subseteq B_I(E) \cap F$, $\omega \in E' \subseteq B_I(E) \subseteq B_I(F)$ for some $E' \in \mathcal{J}_I$. Hence, $\omega \in C(F)$. Second, assume Monotonicity on B_I, and let $E \subseteq F$. If $\omega \in C(E)$ then $\omega \in E' \subseteq B_I(E) \subseteq B_I(F)$ for some $E' \in \mathcal{J}_I$. Then, $\omega \in C(F)$.

3. If each B_i satisfies Necessitation, then B_I satisfies Necessitation. Now, $\Omega \in \mathcal{J}_I$ and $B_I(\Omega) = \Omega$, leading to $C(\Omega) = \Omega$. If C satisfies Necessitation then $\Omega \subseteq C(\Omega) \subseteq B_I(\Omega) \subseteq B_i(\Omega)$ for all $i \in I$.

4. Suppose $\omega \in \bigcap_{\lambda \in \Lambda} C(E_\lambda)$, where $0 < |\Lambda| < \kappa$. For each $\lambda \in \Lambda$, there is $F_\lambda \in \mathcal{J}_I$ with $\omega \in F_\lambda \subseteq B_I(E_\lambda)$. Since $\omega \in \bigcap_{\lambda \in \Lambda} F_\lambda \subseteq \bigcap_{\lambda \in \Lambda} B_I(E_\lambda) \subseteq B_I(\bigcap_{\lambda \in \Lambda} E_\lambda)$, it suffices to show $\bigcap_{\lambda \in \Lambda} F_\lambda \in \mathcal{J}_I$. Let $\bigcap_{\lambda \in \Lambda} F_\lambda \subseteq F$. For each $\lambda \in \Lambda$, letting $\tilde{F}_\lambda := F_\lambda \cap (\bigcup_{\mu \neq \lambda} F_\mu^c)$, $F_\lambda = (\bigcap_{\mu \in \Lambda} F_\mu) \cup \tilde{F}_\lambda \subseteq F \cup \tilde{F}_\lambda$. Since $F_\lambda \in \mathcal{J}_I$, $F_\lambda \subseteq B_I(F \cup \tilde{F}_\lambda)$. Thus, $\bigcap_{\lambda \in \Lambda} F_\lambda \subseteq \bigcap_{\lambda \in \Lambda} B_I(F \cup \tilde{F}_\lambda) \subseteq B_I(\bigcap_{\lambda \in \Lambda}(F \cup \tilde{F}_\lambda)) = B_I(F)$.

5. If $\omega \in F \subseteq E$ for some $F \in \mathcal{J}_I$, then $\omega \in F \subseteq B_I(E)$, and thus $\omega \in C(E)$. This follows without Truth Axiom of B_I. Conversely, if $\omega \in C(E)$ then $\omega \in F \subseteq B_I(E) \subseteq E$ for some $F \in \mathcal{J}_I$, where the last set inclusion follows from Truth Axiom of B_I. Finally, Truth Axiom and Monotonicity follow from $C(E) = \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_I \text{ with } \omega \in F \subseteq E\}$.

6. First, it suffices to show $(\neg C)(\cdot) \in \mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$, i.e., $(\neg C)(\cdot) \subseteq B_i(\neg C)(\cdot)$ for each $i \in I$. Since $B_i C(\cdot) \subseteq C(E) \subseteq B_i C(\cdot)$ and since B_i satisfies Negative Introspection, $(\neg C)(\cdot) = (\neg B_i) C(\cdot) \subseteq B_i(\neg B_i) C(\cdot) = B_i(\neg C)(\cdot)$. Conversely, $B_i C(\cdot) \subseteq (\neg B_i)(\neg C)(\cdot) \subseteq (\neg C)(\neg C)(\cdot) \subseteq C(\cdot)$.

□

Proof of Theorem 3. Part (1). Let $\omega \in \bigcap_{i \in I} C(\mathrm{RAT}_i)$. For each $i \in I$, there is a common basis $F_i \in \mathcal{J}_I$ with $\omega \in F_i \subseteq B_I(\mathrm{RAT}_i) \subseteq \mathrm{RAT}_i$.

For each ordinal α in the given process of IESDA, denote by D_i^α the set of actions in A_i^α which are eliminated by the process (note that D_i^α may be empty). I show

$\sigma_i(\omega') \notin D_i^\alpha$ for each $i \in I$, $\omega' \in F_i$, and α. Let $\alpha = 0$. If $\sigma_i(\omega') \in D_i^0$ for some (i, ω'), then there is $a_i' \in A_i$ that strictly dominates $\sigma_i(\omega')$. Thus, $F_i \subseteq [a_i' \succ_i \sigma_i(\omega')]$. Since F_i is a common basis, $\omega' \in F_i \subseteq B_i([a_i' \succ_i \sigma_i(\omega')])$. Hence, $\omega' \notin \mathrm{RAT}_i$, a contradiction. Let α be a non-zero ordinal, and suppose that $\sigma_i(\omega') \notin D_i^\gamma$ for all $\gamma < \alpha$, $i \in I$, and $\omega' \in F_i$. Then, $\sigma_i(\omega') \in A_i^\alpha$ for all $i \in I$ and $\omega' \in F_i$. If $\sigma_i(\omega') \in D_i^\alpha$ for some (i, ω'), then there is $a_i' \in A_i^\alpha$ such that $(a_i', a_{-i}) \succ_i (\sigma_i(\omega'), a_{-i})$ for all $a_{-i} \in A_{-i}^\alpha$. Thus, $F_i \subseteq [a_i' \succ_i \sigma_i(\omega')]$. Since F_i is a common basis, $\omega' \in F_i \subseteq B_i([a_i' \succ_i \sigma_i(\omega')])$. Thus, $\omega' \notin \mathrm{RAT}_i$, a contradiction. The induction is complete. Now, for each $i \in I$, $\omega \in F_i$ implies $\sigma_i(\omega) \notin D_i^\alpha$ for all α. It follows $\sigma(\omega) \in A^{\mathrm{IESDA}}$.

Part (2). The proof of this part is standard (e.g., Bonanno (2008, 2015)). Take $a \in A^{\mathrm{IESDA}}$. Let $\Omega = A^{\mathrm{IESDA}}$ and $\mathcal{D} = \mathcal{P}(\Omega)$. Denote $\omega = (\omega_j)_{j \in I} \in \Omega$. For every $(i, \omega) \in I \times \Omega$, let $b_i(\omega) = \{\omega' \in \Omega \mid \omega_i' = \omega_i\}$. Every b_i is a partitional possibility correspondence. For every $i \in I$, define $B_i(E) := \{\omega \in \Omega \mid b_i(\omega) \subseteq E\}$ and $\sigma_i(\omega) := \omega_i$. By construction, $\mathrm{RAT}_i = \Omega$ and $B_i(\mathrm{RAT}_i) = \mathrm{RAT}_i$. Letting $\omega = a$, $\omega \in \bigcap_{i \in I} C(\mathrm{RAT}_i)$ and $a = \sigma(\omega) \in A^{\mathrm{IESDA}}$. □

B Formalization of Common Knowledge

The main text (Proposition 2 (5)) has shown that common belief inherits Truth Axiom from mutual belief. Also, the resulting notion of common belief reduces to common knowledge in the previous literature. To better understand the properties and relations of common belief and common knowledge, this appendix formalizes common knowledge separately from common belief.

The structure of this appendix parallels the main text. I begin with defining common knowledge. Next, I study its properties. Finally, I relate my formalization to the previous literature. The proofs are relegated to Section B.1.

I introduce the notion of common knowledge as knowledge induced by common bases $\mathcal{J}_I$. Call an event E *common knowledge* (among I) at a state ω if there is a common basis $F \in \mathcal{J}_I$ that is true at ω and that implies E (equivalently, $B_I(E) \cap E$). Thus, the set of states at which an event E is common knowledge is:

$$\{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_I \text{ such that } \omega \in F \subseteq E\}. \tag{B.1}$$

I make two observations on why this definition captures common knowledge. First, Expression (B.1) captures a notion of knowledge derived from information $\mathcal{J}_I$. At state ω, there is some information $F \in \mathcal{J}_I$ such that F is true at ω and that E is implied (or "proved") by F in the sense that $F \subseteq E$. Second, by the definition of common bases, the common knowledge of E entails the common belief in E. Proposition 2 (5) states that if the mutual belief operator B_I satisfies Truth Axiom then Expression (B.1) coincides with $C(E)$.

I justify the definition of common knowledge by Expression (B.1) on several grounds. To do so, I first characterize the common knowledge operator. As in the main text, suppose that a given space $\langle(\Omega,\mathcal{D}),(B_i)_{i\in I}\rangle$ satisfies, for each $E\in\mathcal{D}$,

$$C^*(E) := \max\{F\in\mathcal{D} \mid F\in\mathcal{J}_I \text{ and } F\subseteq E\}\in\mathcal{D}. \tag{B.2}$$

Then, $C^*(E)$ coincides with Expression (B.1). Henceforth, I incorporate C^* as a primitive, i.e., I consider $\langle(\Omega,\mathcal{D}),(B_i)_{i\in I},C^*\rangle$ such that C^* defined by Expression (B.2) is a well-defined operator from $\mathcal{D}$ into itself. The common knowledge operator $C^*:\mathcal{D}\to\mathcal{D}$ can be axiomatized by the maximal event with the property that if E is common knowledge at ω then there is a common basis F that is true at ω and that implies E. Thus, $C^*(E)$ is the maximal common basis included in E. This implies that if C^* is well defined then so is C. If the publicly-evident events are closed then this observation generalizes the idea in the past literature that the common knowledge of E is the maximal publicly-evident event included in E.

I examine how the common knowledge operator C^* inherits properties of individual and mutual beliefs.

Proposition B.1. *1. C^* satisfies Positive Introspection, Monotonicity, and Truth Axiom.*

2. C^* satisfies Necessitation iff every B_i satisfies Necessitation.

3. If B_I satisfies Non-empty λ-Conjunction, then so does C^*.

4. (a) If $K:\mathcal{D}\to\mathcal{D}$ satisfies (i) Positive Introspection, (ii) Monotonicity, (iii) Truth Axiom, and (iv) $K(\cdot)\subseteq B_I(\cdot)$, then $K(\cdot)\subseteq C^*(\cdot)$.

(b) If $K:\mathcal{D}\to\mathcal{D}$ satisfies (i) Completeness, (ii) Truth Axiom, and (iii) $K(\cdot)\subseteq B_I(\cdot)$, then $K(\cdot)\subseteq C^*(\cdot)$.

In Proposition B.1 (1), Positive Introspection, Monotonicity, and Truth Axiom are jointly equivalent to Completeness and Truth Axiom. By Truth Axiom and Positive Introspection, $C^*(\cdot)=C^*C^*(\cdot)$. The fact that C^* satisfies these properties comes from the way in which (common) knowledge is defined as a logical deduction from a collection of events (here, the common bases $\mathcal{J}_I$) through Expression (B.1) (see Fukuda (2019a)). The second and third parts say that, common knowledge inherits Necessitation and (Non-empty λ-) Conjunction from individual beliefs as common belief does.

I discuss two implications of the fourth part. First, it formalizes the sense in which common knowledge is the "infimum" of players' knowledge. Consider a hypothetical individual satisfying Positive Introspection, Monotonicity, and Truth Axiom. Suppose further that every event "known" by the hypothetical individual is believed by every player. Then, any event the hypothetical individual "knows" is common knowledge.

This fourth part, therefore, justifies the definition of common knowledge in the sense that C^* is the strongest knowledge operator that entails common belief. Let $K:\mathcal{D}\to\mathcal{D}$ be a (knowledge) operator satisfying Positive Introspection, Monotonicity, and Truth Axiom. If $K(\cdot)\subseteq C(\cdot)$, then $K(\cdot)\subseteq C^*(\cdot)$.

Second, the fourth part states that the common knowledge operator C^* is the operator that satisfies Truth Axiom in addition to the properties of the common belief operator C (characterized in Theorem 1). Completeness and Truth Axiom in Part (4b) (equivalently, Positive Introspection, Monotonicity, and Truth Axiom) are reformulated as: $K(E) \subseteq F$ iff $K(E) \subseteq K(F)$.

Three additional remarks are in order. First, consider Negative Introspection. If each B_i satisfies Truth Axiom and Negative Introspection, then it follows from Proposition 2 (5) and (6) that $C^* = C$ satisfies Negative Introspection, provided $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed. Second, Truth Axiom implies No-Contradiction and Consistency. Third, the collection of common bases and common knowledge induced by common bases are related through $\mathcal{J}_I = \{C^*(E) \in \mathcal{D} \mid E \in \mathcal{D}\} = \{E \in \mathcal{D} \mid E \subseteq C^*(E)\}$. Conceptually, while C^* is induced from $\mathcal{J}_I$, the operator C^*, in turn, reproduces $\mathcal{J}_I$.

Next, I compare common knowledge C^* with mutual beliefs B_I^α on a κ-complete algebra with $\kappa \geq \aleph_1$. Namely, $C^*(E) \subseteq C(E) \cap E \subseteq B_I^\alpha(E)$ for any ordinal α with $0 \leq |\alpha| < \kappa$. If B_I satisfies Truth Axiom and Countable Conjunction and if $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed, then $C^*(E) = C(E) = B_I^\varpi(E) \cap E$.

Next, I re-write the common knowledge operator C^* in terms of the largest fixed point of an operator $g_E(F) := B_I(F) \cap E$. If $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed, then

$$C^*(E) = \max\{F \in \mathcal{D} \mid F \subseteq B_I(F) \cap E\}. \tag{B.3}$$

If B_I satisfies Quasi-Monotonicity, then $C^*(E) = \max\{F \in \mathcal{D} \mid F = B_I(F) \cap E\}$. Again, the largest event F satisfying $F \subseteq g_E(F)$ is the largest fixed point of g_E.

When does common knowledge coincide with true common belief (i.e., $C^*(E) = C(E) \cap E$ for all E)? On the one hand, $C^*(E) \subseteq C(E) \cap E$ for all $E \in \mathcal{D}$. The next proposition shows that the converse set inclusion also obtains if $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed and B_I satisfies Finite Conjunction. Also, Remark B.1 provides counterexamples when B_I fails Finite Conjunction or $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is not closed.

Proposition B.2. *If $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed and B_I satisfies Finite Conjunction, then $C^*(E) = C(E) \cap E$ for all $E \in \mathcal{D}$.*

Finally, suppose that the mutual belief operator B_I satisfies the Kripke property on a κ-complete algebra with $\kappa \geq \aleph_1$. I show that the reflexive-transitive closure of b_{B_I} is the possibility correspondence that induces C^*.

Corollary B.1. *Let $(\Omega, \mathcal{D})$ be a κ-complete algebra with $\kappa \geq \aleph_1$, and let $(B_i)_{i \in I}$ be such that B_I satisfies the Kripke property. Then, $C^* : \mathcal{D} \to \mathcal{P}(\Omega)$ defined by $C^*(E) := \{\omega \in \Omega \mid b_{C^*}(\omega) \subseteq E\}$ is a well-defined common knowledge operator from $\mathcal{D}$ into itself satisfying the Kripke property, where $b_{C^*}(\omega) := b_C(\omega) \cup \{\omega\}$ and $b_C(\omega) := \bigcup_{n \in \mathbb{N}} b_{B_I}^n(\omega)$ for each $\omega \in \Omega$.*

Corollary B.1 follows because $\{\omega \in \Omega \mid b_{C^*}(\omega) \subseteq E\} = \{\omega \in \Omega \mid b_C(\omega) \subseteq E\} \cap E = C(E) \cap E \in \mathcal{D}$ for each $E \in \mathcal{D}$.

E	$B_i(E) = C(E)$	$C^*(E)$	$C^{\text{LM}}(E)$	$B_i(E) = C(E)$	$C^*(E)$	$C^{\text{LM}}(E)$
$\emptyset$	$\{\omega_1\}$	$\emptyset$	$\{\omega_1\}$	Ω	$\emptyset$	Ω
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2\}$	$\emptyset$	Ω
$\{\omega_2\}$	$\{\omega_1, \omega_3\}$	$\emptyset$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	Ω
$\{\omega_3\}$	$\{\omega_1\}$	$\emptyset$	$\{\omega_1\}$	Ω	$\emptyset$	Ω
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	Ω
$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2\}$	$\emptyset$	Ω
$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\emptyset$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	Ω
Ω	Ω	Ω	Ω	Ω	Ω	Ω

Table B.1: Common Belief and Common Knowledge Operators C and C^*

B.1 Proofs

Proof of Proposition B.1.

1. Truth Axiom and Monotonicity hold by construction. I show Completeness (and consequently Positive Introspection). Let $C^*(E) \subseteq F$. If $\omega \in C^*(E)$ then $\omega \in E' \subseteq E$ for some $E' \in \mathcal{J}_I$. Then, $\omega \in E' \subseteq C^*(E') \subseteq C^*(E) \subseteq F$. Thus, $\omega \in C^*(F)$.
2. If C^* satisfies Necessitation then $\Omega \subseteq C^*(\Omega) \subseteq B_i(\Omega)$ for all $i \in I$. If every B_i satisfies Necessitation, then $\Omega \in \mathcal{J}_I$. Thus, $C^*(\Omega) = \Omega$.
3. Let $\omega \in \bigcap_{\lambda \in \Lambda} C^*(E_\lambda)$, where $0 < |\Lambda| < \kappa$. For each $\lambda \in \Lambda$, there is $F_\lambda \in \mathcal{J}_I$ with $\omega \in F_\lambda \subseteq E_\lambda$. Now, $\omega \in \bigcap_{\lambda \in \Lambda} F_\lambda \subseteq \bigcap_{\lambda \in \Lambda} E_\lambda$, and it follows from the proof of Proposition 2 (4) that $\bigcap_{\lambda \in \Lambda} F_\lambda \in \mathcal{J}_I$.
4. Since both characterizations are equivalent, I show only the first one. First, $K(E) \in \mathcal{J}_I$ because, if $K(E) \subseteq F$ then $K(E) \subseteq KK(E) \subseteq K(F) \subseteq B_I(F)$. Next, if $\omega \in K(E)$ then $\omega \in K(E) \subseteq E$. Thus, $\omega \in C^*(E)$.

□

Proof of Proposition B.2. Fix $E \in \mathcal{D}$. First, $C^*(E) \subseteq C(E) \cap E$. Conversely, observe

$$C(E) \cap E \subseteq B_I(C(E)) \cap B_I(E) \cap E \subseteq B_I(C(E) \cap E) \cap E,$$

where the second set inclusion follows from Finite Conjunction. Since $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed, Expression (B.3) yields $C(E) \cap E \subseteq C^*(E)$. □

Remark B.1. I provide two examples where $C^*(E) \subsetneq C(E) \cap E$ for some $E \in \mathcal{D}$. While B_I and C violate No-Contradiction in the two examples, the arguments do not hinge on No-Contradiction (recall Remark 1). For both examples, let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$.

In the first example, define B_i as in the second column of Table B.1 for each $i \in I$. The mutual belief operator B_I in this example is from the proof of Lismont and Mongin (2003, Proposition 3). The operator B_I satisfies Monotonicity and thus $\bigcap_{i \in I} \mathcal{J}_{B_i} =$

$\{\emptyset, \{\omega_1\}, \{\omega_1, \omega_3\}, \Omega\}$ is closed. The operator B_I, however, fails Finite Conjunction. For example, $B_I(\{\omega_1, \omega_2\}) \cap B_I(\{\omega_1, \omega_3\}) = \{\omega_1, \omega_3\} \not\subseteq \{\omega_1\} = B_I(\{\omega_1\})$.

For $E = \{\omega_2, \omega_3\}$, $C^*(E) = \emptyset \subsetneq \{\omega_3\} = C(E) \cap E$. I also remark that $C \neq C^{\mathrm{LM}}$, where C^{LM} is defined as

$$C^{\mathrm{LM}}(E) := \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_I \text{ such that } \omega \in B_I(F) \text{ and } F \subseteq E\}. \quad \text{(B.4)}$$

Note that, since $\mathcal{J}_I = \bigcap_{i \in I} \mathcal{J}_{B_i}$, the event $C^{\mathrm{LM}}(E)$ coincides with Expression (7). This is also a counterexample for the third part of Proposition 1 due to the failure of Finite Conjunction. For $E = \{\omega_2\}$, $C(E) \neq \max\{F \in \mathcal{D} \mid F = B_I(E \cap F)\}$.

In the second example, let B_i be as in the fifth column of Table B.1 for each $i \in I$. While B_I satisfies Finite Conjunction, $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is not closed. For $E = \{\omega_3\}$, $C^*(E) = \emptyset \subsetneq \{\omega_3\} = C(E) \cap E$. I also remark that Expressions (7) and (B.4) satisfy $\Omega = B_I(\emptyset) \subseteq C^{\mathrm{LM}}(\cdot)$.

Incidentally, this is also a counterexample for the third part of Proposition 1. For example, $\max\{F \in \mathcal{D} \mid F \subseteq B_I(\{\omega_2\} \cap F)\}$ and $\max\{F \in \mathcal{D} \mid F = B_I(\{\omega_1, \omega_3\} \cap F)\}$ are not well defined.

Finally, I remark on relation between C^{LM} and C. The following can be shown (without assuming $C^{\mathrm{LM}}(\cdot) \in \mathcal{D}$). If B_I satisfies Truth Axiom then $C = C^* = C^{\mathrm{LM}}$. If B_I satisfies Quasi-Monotonicity then $C^{\mathrm{LM}}(E) \subseteq C(E)$. If B_I satisfies Finite Conjunction and $\bigcap_{i \in I} \mathcal{J}_{B_i}$ is closed, then $C(\cdot) \subseteq C^{\mathrm{LM}}(\cdot)$.

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