The Existence of Universal Qualitative Belief Spaces*

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March 21, 2020

Abstract

This paper establishes the existence of a canonical representation of players' interactive beliefs with a number of desirable features. Players' beliefs can be qualitative, truthful (i.e., knowledge), or probabilistic (e.g., countably-additive, finitely-additive, or non-additive). Players' logical and introspective properties can be specified one by one. The canonical model is the "largest" interactive belief model to which any particular model can be mapped in a unique belief-preserving way. The canonical model incorporates all possible ways in which players' interactive beliefs are described. Each state of the canonical model encodes players' interactive beliefs at that state within itself in a coherent manner.

JEL Classification: C70, D83

Keywords: Interactive Beliefs, Universal Belief Space, Universal Knowledge Space

*This paper is based on part of the first chapter of my Ph.D. thesis, "The Existence of Universal Knowledge Spaces," submitted to the University of California at Berkeley. I would like to thank David Ahn, William Fuchs, and Chris Shannon for their encouragement, support, and guidance; Robert Anderson, Eric Auerbach, Yu Awaya, Ivan Balbuzanov, Paulo Barelli, Pierpaolo Battigalli, Benjamin Brooks, Christopher Chambers, Aluma Dembo, Eduardo Faingold, Amanda Friedenberg, Drew Fudenberg, Itzhak Gilboa, Benjamin Golub, Sander Heinsalu, Ryuichiro Ishikawa, Yuichiro Kamada, Michihiro Kandori, Shachar Kariv, Ali Khan, Narayana Kocherlakota, Massimo Marinacci, Stephan Morris, Motty Perry, Herakles Polemarchakis, Joseph Root, Burkhard Schipper, Mikkel Sølvsten, Kenji Tsukada, and various conference and seminar participants for comments and discussions. All remaining errors are mine.

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1 Introduction

Consider a group of players who reason interactively about unknown external values, *states of nature* S, such as the payoffs and strategies in a game. Each player must reason about states of nature and about each other's beliefs about states of nature, and so on. This paper constructs the first formal framework general enough to represent any conceivable form of interactive beliefs irrespective of nature of beliefs. In particular, beliefs can be probabilistic or qualitative including knowledge. An arbitrary model of beliefs will capture some possible aspects of players' interactive reasoning but will generally exclude others. I construct a canonical model that includes all possible forms of reasoning about beliefs. The construction enables analysis of formal questions regarding beliefs without ad hoc restrictions on the nature of reasoning. Claims regarding beliefs can be logically disassociated from extraneous structure that is not immediately related to interactive beliefs.

A model of beliefs (a belief space) consists of the following three ingredients. The first ingredient is a set Ω. Each element $\omega \in \Omega$ is a list of possible specifications of the prevailing nature state $s \in S$ and players' interactive beliefs regarding nature states S (i.e., their beliefs about nature states S, their beliefs about their beliefs about S, and so on). Call each specification ω a state (of the world).

The second ingredient is the set of statements about which the players can reason. These statements, specified as subsets of states of the world Ω, are referred to as events. A belief space requires a description of the language available to the players, modeled as a collection of subsets of Ω which I call the domain.

The third ingredient is players' belief operators defined on the domain. For each event E, player i's belief operator assigns the set of states at which she believes E, i.e., the event that i believes E. Iterative applications of this operator generates higher-order interactive reasoning. One can represent various notions of qualitative or probabilistic beliefs by imposing properties of those beliefs on belief operators.

In applications, typically a specific model of beliefs is assumed a priori. This leaves open the possibility that some relevant aspects of reasoning are excluded. To address this, the main result of the paper (Theorem 1 in Section 3) demonstrates the existence of a universal (precisely, terminal) belief space into which any belief space is embedded in a unique manner that maintains all the structure of that smaller space. I formally show the extent to which any form of reasoning in the smaller space can be retrieved in the universal space: for any logical statement generated by nature states and players' interactive beliefs about them, the statement is true at some state of some belief space if and only if it is true at some state of the universal space; and the statement is true at all states of the universal space if and only if it is true at all states of any belief space (Proposition 2). Moreover, each state of the universal space specifies players' interactive beliefs at that state within itself (Corollary 1). The space is complete in that it includes all possible forms of reasoning (Proposition 1). Thus, the universal space leaves no relevant aspect of players' interactive beliefs unspecified as long as nature states and players' interactive beliefs about them are concerned.

I construct a universal belief space under a variety of assumptions on players' logical and introspective abilities. My result is theoretically interesting in that the existence of a universal belief space is unrelated to assumptions on players' beliefs. For example, my paper reconciles the previous existence results on canonical probabilistic belief structures and the previous non-existence results on canonical knowledge (or more general qualitative belief) structures. At the same time, it is substantively interesting because I establish the canonical representation of beliefs even when players are less than "perfectly rational" in terms of their logical or introspective abilities.

My framework nests partitional (Aumann, 1976) and non-partitional possibility correspondence models of knowledge and qualitative beliefs by identifying the conditions on players' belief operators under which their beliefs are induced from information sets on the underlying states of the world. Each player's information set associated with a state represents the set of states she considers possible at that state. While a player in a partitional model is logically omniscient and is fully introspective about what she knows and what she does not know, a player in a non-partitional model may, for example, fail Negative Introspection—she does not know a certain event, and she does not know that she does not know it.[1] My framework also nests other forms of possibility correspondence models of qualitative beliefs which may fail to be truthful.[2] I can further relax players' logical reasoning abilities inherent in

[1]Non-partitional models are motivated in part by notions of unawareness (e.g., Fagin and Halpern (1987), Modica and Rustichini (1994, 1999), and Schipper (2015)). The study of non-partitional models ranges from implications of common knowledge and common belief (e.g., Agreement theorems (Aumann, 1976)) to solution concepts in game theory. See, for example, Bacharach (1985), Binmore and Brandenburger (1990), Brandenburger, Dekel, and Geanakoplos (1992), Dekel and Gul (1997), Geanakoplos (1989), Morris (1996), Samet (1990), and Shin (1993).

[2]In the literature, knowledge is distinguished from belief in that a player can only know what is true while she can believe something false.

possibility correspondence models. For example, players may fail to believe logical consequences of their beliefs.

Qualitative beliefs may play an important role in characterizing solution concepts such as common belief in rationality in a game (Stalnaker, 1994) or especially in a game with ordinal payoff structures that do not admit probabilistic beliefs (Bonanno and Tsakas, 2018). One can also introduce qualitative beliefs as in Bjorndahl, Halpern, and Pass (2013) in the context of psychological games (Battigalli and Dufwenberg, 2009; Geanakoplos, Pearce, and Stacchetti, 1989) where players' interactive beliefs themselves enter into their preferences.

I establish the existence of a universal belief space in a way such that beliefs can be probabilistic as in type spaces (Harsanyi, 1967-1968): each player has a type mapping that associates, with each state, a probability distribution on the underlying states.[3] As Samet (2000) demonstrates the correspondence between a type mapping and a collection of p-belief operators (Monderer and Samet, 1989), the main result of this paper also asserts the existence of a universal probabilistic (e.g., countably-/finitely-/non-additive) belief space. Probabilistic beliefs can reduce to whether a player believes an event with probability at least p or not. Technically, my construction of a universal qualitative belief space follows the topology-free construction of a universal (countably-additive) type space by Heifetz and Samet (1998b).

My framework can endow players with both knowledge and belief on a general domain.[4] Indeed, the consideration of the domain of knowledge has often been neglected. Standard possibility correspondence models of knowledge allow any subset of states Ω to be an object of knowledge. Under such power-set specification, knowledge and probabilistic beliefs may be incompatible with each other when the knowledge of an event is not in the domain of the probability space. This incompatibility has been one of the issues that have hampered the epistemic analyses of players' knowledge and beliefs. Not only is my framework capable of capturing both knowledge and belief,

[3]The existence of a universal type/belief space is pioneered by Armbruster and Böge (1979), Böge and Eisele (1979), and Mertens and Zamir (1985). Their topological constructions are extended by, for example, Brandenburger and Dekel (1993) and Pintér (2005).

[4]Such consideration would be needed for analyzing: (i) players' knowledge about their own strategy and their beliefs about opponents' strategies (e.g., Dekel and Gul (1997)) or (ii) players' knowledge about their past-observed moves and their beliefs about past-unobserved and future moves in an extensive-form game (e.g., Battigalli and Bonanno (1997)). Knowledge and probability-one belief would differ in a continuous model: a player may believe with probability one that a random draw from the interval $[0, 1]$ is irrational while she does not know it (Monderer and Samet, 1989).

but also the framework admits a universal space.

In sum, I construct a universal belief space within the class of belief spaces that satisfy given assumptions on players' logical and introspective abilities. Beliefs can be probabilistic (e.g., countably-/finitely-/non-additive) and/or qualitative including (fully-/not-necessarily-introspective) knowledge. Special cases include previous belief models such as possibility correspondences and type mappings.

The paper is organized as follows. The rest of this section provides a technical overview of the main result. Section 2 defines a belief space, properties of beliefs, and a universal belief space. Section 3 constructs a universal belief space (Theorem 1). Section 4 characterizes the universal belief space as the "largest" set describing players' interactive beliefs in a coherent manner (Theorem 2). Section 5 discusses further applications such as probabilistic beliefs. Section 6 compares the existence result of a universal knowledge space with the previous non-existence results. Section 7 provides concluding remarks. Proofs are relegated to Appendix A.

1.1 Technical Overview

Heifetz and Samet (1998a) demonstrate that a universal standard partitional knowledge space generically does not exist, where a standard partitional knowledge space allows any subset of underlying states Ω to be an object of knowledge. They show that, unlike σ-additive probabilistic beliefs, a non-trivial sequence of interactive knowledge can develop beyond any given ordinal. The negative results are also obtained by Fagin (1994), Fagin et al. (1999), Fagin, Halpern, and Vardi (1991), and Heifetz and Samet (1999). Moreover, Meier (2005) shows that there is no universal qualitative belief (including knowledge) space represented by a more general non-partitional possibility correspondence (i.e., a general Kripke frame). If there were a universal space in a class of such general qualitative belief spaces, then one could construct a universal partitional knowledge space from the given class, which is impossible.

How do my positive results reconcile with the negative results? What plays a crucial role in establishing a universal knowledge (i.e., truthful-belief) space is to specify a set algebra as objects of players' knowledge, i.e., a specification of the language that the players are allowed to use in their reasoning.

To see this point, let κ be an infinite cardinal number. Call a collection of subsets of underlying states Ω a *κ-algebra* (a shorthand for a κ-complete algebra) if it is closed under complementation and under union (and consequently intersection) of any subcollection with cardinality less than κ. The power set of Ω is always κ-complete. For example, a κ-algebra subsumes an algebra of sets if κ is the least infinite cardinal $\aleph_0$. A κ-algebra subsumes a σ-algebra if κ is the least uncountable cardinal $\aleph_1$. Call a knowledge space (a belief space with players' beliefs truthful) a κ-knowledge space if its domain is a κ-algebra.

Specifying the domain of each knowledge space by a κ-algebra amounts to determining the language available to the players in reasoning about their interactive knowledge. Any κ-knowledge space can capture players' interactive knowledge of a form, player i knows that player j knows that ..., up to the level of κ. For example, any $\aleph_0$-knowledge space can capture any finite level of players' interactive knowledge as $\kappa = \aleph_0$ is the least infinite cardinal. Likewise, any $\aleph_1$-knowledge space can capture any countable level of players' interactive knowledge. Thus, a κ-knowledge space can accommodate an infinite knowledge hierarchy of the form, Alice knows that Bob knows that Alice knows..., which would naturally emerge when one considers common knowledge among players (e.g., Aumann (1976, 1999)). Transfinite numbers of reasoning would also be necessary if one considers implications of common knowledge of rationality in a general infinite game.[5]

The main result (Theorem 1 in Section 3) establishes that, for each fixed κ, there is a universal κ-belief space in each class of κ-belief spaces that respect some given assumptions on players' beliefs by taking care of all the (possibly transfinite) levels of interactive beliefs up to κ. In particular, a universal κ-knowledge space exists within a class of truthful κ-belief spaces.

The construction circumvents the previous non-existence results by explicitly specifying a domain of qualitative beliefs (or knowledge) as a κ-algebra. On the one hand, the previous negative results imply that a sequence of interactive knowledge can generally develop beyond any depth of reasoning in a discontinuous way if any subset of states of the world is an object of knowledge. On the other hand, once I specify the language available to the players as a κ-algebra, any κ-knowledge space can gener-

[5]See, for example, Lipman (1994). For any infinite cardinal κ, Fukuda (2020) examines a strategic game in which a unique prediction based on common belief in rationality requires interactive reasoning (i.e., iterative deletion of strictly dominated actions) of ordinal depth κ. Section 5.4 shows that any state of a model at which players commonly believe in their rationality is mapped to the corresponding state of the universal space at which players commonly believe in their rationality in the unique belief-preserving manner.

ally take into consideration players' interactive knowledge up to the ordinality of κ. Since κ can be arbitrarily fixed by the outside analysts, choose κ large enough, e.g., $\kappa > |S|$ where nature states S consist of action profiles in a given strategic game. The universal κ-qualitative-belief (or κ-knowledge) space can accommodate any possible interactive reasoning about the players' actions up to κ.

Thus, I turn the previously mentioned negative results into the positive one in the following two ways. First, I enlarge a class of knowledge spaces by allowing the domain of a knowledge space to be a κ-algebra. Second, I find a universal κ-knowledge space by keeping track of all possible forms of reasoning up to the depth of κ attained in the given class of κ-knowledge spaces. Thus, unlike a universal (σ-additive) type space, my universal knowledge (or generally, qualitative belief) space usually has transfinite (precisely, κ) hierarchies of interactive knowledge (beliefs) incorporating all possible forms of interactive reasoning up to the depth of κ.[6]

This paper shows existence hinges on the specification of a domain (i.e., the depth of reasoning) rather than on assumptions on players' beliefs. Observe the following analogy with σ-additive beliefs. The domain of a type space is implicitly assumed to be a σ-algebra because a σ-additive probability measure may not necessarily be defined on the power set. The domain of a type space (σ-algebra) is the language available to the players in reasoning up to countable probabilistic belief hierarchies. The domain of any $\aleph_1$-qualitative-belief space (σ-algebra) is the language available to the players in reasoning up to countable qualitative belief hierarchies. While I construct a universal belief space by keeping track of belief hierarchies up to the ordinality of $\aleph_1$, the continuity of σ-additive beliefs (precisely, the continuity of the operation "Δ") guarantees that the least infinite depth of interactive beliefs can determine any subsequent countable order in a universal type space (e.g., Fagin et al. (1999) and Heifetz and Samet (1998b)).[7] Furthermore, Meier (2006) shows, while a universal finitely-additive belief space does not exist if all subsets are measurable (see also Fagin et al. (1999)), it exists once players' beliefs are defined on a κ-algebra.

[6] In a somewhat related but different context of decision theory, Lipman (1991) studies a canonical set consisting of transfinite sequences of decision procedures to pick decision procedures.

[7] Moss and Viglizzo (2004, 2006) reformulate σ-additive type spaces as coalgebras for a certain endofunctor F, which is related to the functor Δ. They show that a universal type space (i.e., a terminal coalgebra) is expressed as the set of descriptions of each point (type profile together with a state of nature) in all coalgebras, endowed with measurable and coalgebra structures. Since a terminal coalgebra T is isomorphic to $F(T)$, it is also "(belief-)complete" (see Brandenburger (2003) and Brandenburger and Keisler (2006) for (belief-)completeness).

My existence result on a universal κ-knowledge space is related to the previous two positive results. First, Meier (2008) constructs a universal knowledge-belief space in which players' knowledge operators operate only on a σ-algebra on which players' probabilistic beliefs are defined. My framework nests Meier (2008) as a special class of $\aleph_1$-knowledge(-belief) spaces under his assumptions on players' knowledge, which may not necessarily be induced from possibility correspondences. Thus, in addition to nesting his result, this paper shows the existence of a universal κ-knowledge space for such models as partitional (or non-partitional) possibility correspondences. More generally, this paper shows the existence of a universal belief space is unrelated to specific nature of beliefs.

Second, Aumann (1999) constructs what he calls a canonical knowledge system (of a finitary epistemic $S5$ logic), where each state of the world is a "complete and coherent" set of formulas describing finite levels of players' interactive knowledge.[8] Theorem 2 in Section 4 reformulates a universal qualitative belief (or knowledge) space by generalizing and modifying the idea of Aumann (1999)'s canonical knowledge system for any combination of assumptions on players' beliefs and for any domain (i.e., for any κ). Thus, in a particular case in which players with fully-introspective knowledge reason about their finite interactive knowledge, Theorem 2 formally proves that Aumann (1999)'s canonical space can be taken as a universal space within the particular class. Generally, Theorem 2 shows that the universal κ-belief space constructed in Theorem 1 is the largest set (i) consisting of "complete and coherent" sets of formulas describing the players' belief hierarchies; (ii) satisfying the "coherency" condition on the entire space that induces the players' beliefs in a well-defined manner; and (iii) respecting given assumptions on their beliefs.

2 Belief Spaces

Throughout the paper, denote by I a non-empty set of players. Let S be a non-empty set of *states of nature*, endowed with a sub-collection $\mathcal{S}$ of the power set $\mathcal{P}(S)$. An element of S is regarded as a specification of the exogenous values that are relevant to the strategic interactions among the players. For example, $(S, \mathcal{S})$ is the set of

[8] Meier (2012) axiomatizes classes of belief/type spaces and shows that the space of all maximally consistent sets of formulas of his infinitary probability logic (i.e., the canonical space) is a universal space, which is isomorphic to the universal type space constructed by Heifetz and Samet (1998b). Zhou (2010) studies a canonical infinitary finitely-additive probability logic.

strategies or payoff functions endowed with a topological or measurable structure.

Throughout the paper, I explicitly assume the axiom of choice. Then, associate, with an (infinite) cardinal κ, the least ordinal $\overline{\kappa}$ (called the initial ordinal of κ) with its cardinality $|\overline{\kappa}| = \kappa$. That is, the cardinal κ is also identified with the ordinal $\overline{\kappa}$.

Next, I introduce technical definitions. Let κ be an infinite cardinal. Call a collection $\mathcal{D}$ of subsets of a set Ω (or a pair $(\Omega, \mathcal{D})$ itself) a *κ-complete algebra* (*κ-algebra*, for short) if $\mathcal{D}$ is closed under complementation and is closed under arbitrary union (and consequently intersection) of any sub-collection with cardinality less than κ (i.e, closed under *κ-union* and *κ-intersection*). Likewise, by defining a symbol ∞ (which is not a cardinal) as informally satisfying $\lambda < \infty$ for any cardinal λ, call the sub-collection $\mathcal{D}$ of $\mathcal{P}(\Omega)$ (or the pair $(\Omega, \mathcal{D})$) a *complete algebra* (*∞-algebra*, for short) if $\mathcal{D}$ is closed under complementation and is closed under arbitrary union and intersection. Note that $\emptyset =: \bigcup \emptyset \in \mathcal{D}$ and $\Omega =: \bigcap \emptyset \in \mathcal{D}$. For example, an $\aleph_0$-algebra is an algebra of sets, because $\aleph_0$ is the least infinite cardinal. An $\aleph_1$-algebra is a σ-algebra, because $\aleph_1$ is the least uncountable cardinal. Throughout the paper, κ refers to an infinite cardinal or the symbol ∞. Denote by $\mathcal{A}_\kappa(\cdot)$ the smallest κ-algebra (i.e., the intersection of all κ-algebras) including a given collection. For example, $\mathcal{A}_{\aleph_1}(\cdot) = \sigma(\cdot)$ generates the smallest $\aleph_1$-algebra.

Now, in order to specify a language the players are allowed to use in making inferences about nature states S and their interactive beliefs, endow $\mathcal{S}$ with a "logical" (precisely, a set-algebraic) structure. Letting κ be an infinite cardinal chosen by the outside analysts, call $E \in \mathcal{A}_\kappa(\mathcal{S})$ an *event of nature*. Each $E \in \mathcal{A}_\kappa(\mathcal{S})$ plays a role of a "proposition" regarding nature states S about which players interactively reason. Hence, if E is a nature event, then so is its complement E^c (also denote it by $\neg E$); if each $E \in \mathcal{E}$ with $|\mathcal{E}| < \kappa$ is a nature event, then so are its union $\bigcup \mathcal{E} = \bigcup_{E \in \mathcal{E}} E$ and its intersection $\bigcap \mathcal{E} = \bigcap_{E \in \mathcal{E}} E$. As I will discuss later, κ turns out to determine possible depth of players' reasoning.

I remark that, as mentioned in Meier (2006, Remark 1), it is without loss to assume an infinite cardinal κ to be regular. For any infinite cardinal κ which is not regular, $(S, \mathcal{A}_\kappa(\mathcal{S}))$ is indeed a κ^+-algebra, where the successor cardinal κ^+ is known to be regular by the axiom of choice. Hence, if the outside analysts take a non-regular (i.e., singular) infinite cardinal κ, then they are implicitly taking a regular infinite cardinal κ^+. Note that $\aleph_0$ and $\aleph_1$ are regular.

2.1 Belief Spaces

I define a model of players' beliefs in which belief operators on some "sample space" induce players' interactive beliefs regarding nature states $(S, \mathcal{S})$.

Definition 1 (Belief Space)**.** *A κ-belief space of I on $(S, \mathcal{S})$ (a belief space, for short) is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta \rangle$ with the following three properties.*

1. *$(\Omega, \mathcal{D})$ is a κ-algebra. Call Ω the set of states of the world (the state space). Call each $E \in \mathcal{D}$ an event (of the world).*

2. *For each $i \in I$, $B_i : \mathcal{D} \to \mathcal{D}$ is player i's belief operator. For each $E \in \mathcal{D}$, $B_i(E)$ denotes the event that player i believes E. A player $i \in I$ believes an event $E \in \mathcal{D}$ at a state $\omega \in \Omega$ if $\omega \in B_i(E)$.*

3. *$\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_\kappa(\mathcal{S}))$ is a measurable map: $\Theta^{-1}(E) \in \mathcal{D}$ for any $E \in \mathcal{S}$.*

In Condition (3), since $(\Omega, \mathcal{D})$ is a κ-algebra, Θ is measurable as long as $\Theta^{-1}(\mathcal{S}) \subseteq \mathcal{D}$. By this condition, any set-algebraic ("logical") operations in $\mathcal{A}_\kappa(\mathcal{S})$ are preserved in the domain $\mathcal{D}$. The mapping Θ can be regarded as a pair of mappings ($\Theta : \Omega \to S, \Theta^{-1} : \mathcal{A}_\kappa(\mathcal{S}) \to \mathcal{D}$) defining whether a nature event E is true at a state of the world ω: $\omega \in \Theta^{-1}(E)$ in $(\Omega, \mathcal{D})$ if and only if (iff, for short) $\Theta(\omega) \in E$ in $(S, \mathcal{A}_\kappa(\mathcal{S}))$.

While a standard partitional model assumes any subset of underlying states Ω to be an object of beliefs (i.e., $\mathcal{D} = \mathcal{P}(\Omega)$), my framework is more general and explicit about what the players can reason. First, this paper allows for treating both knowledge and beliefs together on a κ-algebra (primarily, σ-algebra) without assuming that players' partitions are at most countable (e.g., Aumann (1976)). Second, in the literature on logical foundations of state space models, events are generated by some logical system, and thus the domain may only form a κ-algebra for some κ, depending on the given logical system (e.g., Aumann (1999), Bacharach (1985), Samet (1990, 2010), and Shin (1993)). In Section 3, the domain of the universal belief space turns out to be a κ-algebra generated by events corresponding to an "infinitary language" defined by κ, nature states, and assumptions on players' beliefs.

Next, I define properties of qualitative beliefs. Theorem 1 (in Section 3) constructs a universal κ-belief space within a given class of κ-belief spaces satisfying an arbitrary combination of properties specified below. As will be seen, the following list of properties covers various classes of possibility correspondence models of (introspective/non-introspective) knowledge and qualitative beliefs. For example, if a belief operator B_i in a κ-belief space is induced by a partition, then B_i satisfies all the properties below (the converse also holds with some redundancies).[9]

Definition 2 (Propeties of Beliefs)**.** *Let* $\langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, \Theta\rangle$ *be a* κ*-belief space. Fix* $i \in I$*.*

1. *Monotonicity:* $B_i(E) \subseteq B_i(F)$ *for any* $E, F \in \mathcal{D}$ *with* $E \subseteq F$*.*
2. *Necessitation:* $B_i(\Omega) = \Omega$*.*
3. *Non-empty* λ*-Conjunction (*$\lambda \leq \kappa$ *is a fixed infinite cardinal or* $\lambda = \kappa = \infty$*):* $\bigcap_{E\in\mathcal{E}} B_i(E) \subseteq B_i(\bigcap \mathcal{E})$ *for any* $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ *with* $|\mathcal{E}| < \lambda$*.*
4. *The Kripke property: for each* $(\omega, E) \in \Omega \times \mathcal{D}$*,* $\omega \in B_i(E)$ *if(f)* $E \supseteq b_{B_i}(\omega) := \bigcap\{E \in \mathcal{D} \mid \omega \in B_i(E)\}$*.*
5. *Consistency:* $B_i(E) \subseteq (\neg B_i)(E^c)$ *for any* $E \in \mathcal{D}$*.*
6. *Truth Axiom:* $B_i(E) \subseteq E$ *for any* $E \in \mathcal{D}$*.*
7. *Positive Introspection:* $B_i(\cdot) \subseteq B_iB_i(\cdot)$*.*
8. *Negative Introspection:* $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$*.*

Monotonicity states that if a player believes some event then she believes any of its logical consequences. Necessitation means that a player believes any form of tautology such as $E \cup E^c$ expressed as Ω. Non-empty λ-Conjunction says that a player believes any non-empty conjunction of events (with cardinality less than λ) if she believes each event. Necessitation is identified as the empty conjunction since $\Omega = \bigcap\emptyset$. For example, if $B_i(E)$ denotes the event that player i believes E with probability one (assume countable additivity), then B_i satisfies Monotonicity, Necessitation, and Non-empty λ-Conjunction for $\lambda = \aleph_1$ but not necessarily for $\lambda > \aleph_1$.

The Kripke property is the condition under which B_i is induced from the possibility correspondence $b_{B_i} : \Omega \to \mathcal{P}(\Omega)$.[10] The information (or possibility) set $b_{B_i}(\omega) =$

[9] Theorem 1 extends to a class of κ-belief spaces in which belief operators satisfy general set-theoretical properties (given in Lemma A.1 in Appendix A) beyond Definition 2. Using this result, Section 5 constructs a universal probabilistic belief space for various notions of probabilistic beliefs.

[10] If there is $b : \Omega \to \mathcal{P}(\Omega)$ with $B_i(E) = \{\omega \in \Omega \mid b(\omega) \subseteq E\}$ for all $E \in \mathcal{D}$, then B_i satisfies the Kripke property, i.e., $B_i(E) = \{\omega \in \Omega \mid b_{B_i}(\omega) \subseteq E\}$ for all E (the converse trivially holds). See Fukuda (2019, Remark 1).

$\{\omega' \in \Omega \mid \omega' \in E$ for all $E \in \mathcal{D}$ with $\omega \in B_i(E)\}$ consists of states at which i considers possible at ω. The Kripke property implies Monotonicity, Necessitation, and Non-empty λ-Conjunction.[11]

Consistency means that, if a player believes an event E then she does not believe its negation E^c. Truth Axiom says that a player can only "know" what is true. Truth Axiom distinguishes belief and knowledge in that belief can be false while knowledge has to be true. Truth Axiom implies Consistency. Positive Introspection states that if a player believes some event then she believes that she believes it. Negative Introspection states that if a player does not believe some event then she believes that she does not believe it.

Three remarks are in order. First, one can assume different properties of beliefs for different players. Players may also have multiple kinds of "belief" operators.[12]

Second, my framework nests possibility correspondence models. Assume the Kripke property (and consequently Monotonicity, Necessitation, and Non-empty λ-Conjunction). The other properties can be expressed in terms of the possibility correspondence. First, B_i satisfies Consistency iff b_{B_i} is serial (i.e., $b_{B_i}(\cdot) \neq \emptyset$). Second, B_i satisfies Truth Axiom iff b_{B_i} is reflexive (i.e., $\omega \in b_{B_i}(\omega)$ for all $\omega \in \Omega$). Third, B_i satisfies Positive Introspection iff b_{B_i} is transitive (i.e., $\omega' \in b_{B_i}(\omega)$ implies $b_{B_i}(\omega') \subseteq b_{B_i}(\omega)$). Fourth, B_i satisfies Negative Introspection iff b_{B_i} is Euclidean (i.e., $\omega' \in b_{B_i}(\omega)$ implies $b_{B_i}(\omega) \subseteq b_{B_i}(\omega')$). Thus, b_{B_i} forms a partition iff B_i satisfies Truth Axiom, Positive Introspection, and Negative Introspection (note that Negative Introspection and Truth Axiom imply Positive Introspection). Likewise, one can capture non-partitional models: b_{B_i} is reflexive and transitive iff B_i satisfies Truth Axiom and Positive Introspection. Also, one can capture qualitative beliefs: b_{B_i} is serial, transitive, and Euclidean iff B_i satisfies Consistency, Positive Introspection, and Negative Introspection. Hence, one can exactly identify various classes of possibility correspondence models on a κ-algebra $(\Omega, \mathcal{D})$.

Third, in order to accommodate Truth Axiom, the state space $(\Omega, \mathcal{D})$ may not necessarily be the product κ-algebra of the nature states $(S, \mathcal{A}_\kappa(\mathcal{S}))$ and the players' type spaces $((T_i, \mathcal{T}_i))_{i \in I}$ (all of which form a κ-algebra). Assume player i's beliefs depend only on her own type. Then, B_i would violate Truth Axiom because, for any

[11] The converse may not necessarily hold unless $\lambda = \kappa = \infty$ (e.g., Samet (2010) when $\lambda = \kappa = \aleph_0$).

[12] Extend the set of players to $\{0, 1\} \times I$, where player i's knowledge operator (which satisfies Truth Axiom) is given by $K_i := B_{(0,i)}$ while her qualitative belief operator is given by $B_i := B_{(1,i)}$.

$E \in \mathcal{D}$, $B_i(E) = S \times E^i \times \prod_{j \in I \setminus \{i\}} T_j$ for some $E^i \in \mathcal{T}_i$.

2.2 A Terminal Belief Space

The main result (Theorem 1 in Section 3) constructs a universal κ-belief space for any infinite (regular) cardinal κ and for any combination of assumptions on players' beliefs specified in Definition 2. To that end, I define a universal belief space in a given class of belief spaces as a terminal belief space in the class. It is a belief space to which every belief space in the given class is uniquely mapped in a belief-preserving manner. In Section 3, the terminal belief space constructed in Theorem 1 (i) encodes players' interactive beliefs about the space itself (Proposition 1) and (ii) is the largest "coherent" set describing players' belief hierarchies (Theorem 2). I start by formalizing the notion of a belief-preserving map, a belief morphism.

Definition 3 (Belief Morphism). *Let* $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta \rangle$ *and* $\overrightarrow{\Omega'} = \langle (\Omega', \mathcal{D}'), (B'_i)_{i \in I}, \Theta' \rangle$ *be belief spaces of a given class. A (belief) morphism* $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ *is a measurable map* $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ *satisfying: (i)* $\Theta' \circ \varphi = \Theta$*; and (ii)* $B_i(\varphi^{-1}(E')) = \varphi^{-1}(B'_i(E'))$ *for each* $(i, E') \in I \times \mathcal{D}'$.

Condition (i) requires that the same nature state prevail for two associated belief spaces. By Condition (ii), players' beliefs are preserved from one space to another in that player i believes an event E' at $\varphi(\omega)$ iff she believes $\varphi^{-1}(E')$ at ω.

For any belief space $\overrightarrow{\Omega}$, the identity map id_Ω on Ω is a morphism from $\overrightarrow{\Omega}$ into itself. Denote by $\mathrm{id}_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega}$ the identity (belief) morphism. Next, call a belief morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ a *(belief) isomorphism*, if there is a morphism $\psi : \overrightarrow{\Omega'} \to \overrightarrow{\Omega}$ with $\psi \circ \varphi = \mathrm{id}_{\overrightarrow{\Omega}}$ and $\varphi \circ \psi = \mathrm{id}_{\overrightarrow{\Omega'}}$ (that is, φ is bijective and its inverse φ^{-1} is a morphism). If φ is an isomorphism then its inverse φ^{-1} is unique. Belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ are *isomorphic*, if there is an isomorphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$.

Now, I define a terminal belief space. It "includes" all belief spaces in that any belief space can be mapped to the terminal space by a unique morphism.

Definition 4 (Terminal Belief Space). *Fix a class of* κ*-belief spaces of* I *on* $(S, \mathcal{S})$*. A* κ*-belief space* $\overrightarrow{\Omega^*}$ *in the class is terminal if, for any* κ*-belief space* $\overrightarrow{\Omega}$ *in the class, there is a unique morphism* $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$.

Fix a non-empty set of players I, a set of nature states $(S, \mathcal{S})$, an infinite cardinal κ (or the symbol $\kappa = \infty$), and assumptions on players' beliefs. These inputs determine the class of κ-belief spaces. Then, any composite of two morphisms is a morphism; composites of morphisms are associative; and an identity morphism satisfies the identity law. Thus, the given class of κ-belief spaces of I on $(S, \mathcal{S})$ forms a *category*, where a belief space $\overrightarrow{\Omega}$ is an *object* and a belief morphism is a *morphism*. In the language of category theory, a terminal κ-belief space in the class is a terminal (final) object in the category of belief spaces. As is well known in category theory, a terminal belief space (in a given class) is unique up to belief isomorphism.

3 Construction of a Terminal Belief Space

Throughout this section, fix a non-empty set I of players, a set $(S, \mathcal{S})$ of nature states, an infinite regular cardinal κ, and assumptions on players' beliefs. A belief space refers to a κ-belief space of I on $(S, \mathcal{S})$ in the given category.

I construct a terminal (κ-)belief space by employing the "expressions-descriptions" approach (Heifetz and Samet (1998b) and Meier (2006, 2008)). I do so without imposing any restriction on $(S, \mathcal{S})$.[13]

The construction of a terminal belief space consists of six steps (Figure 1 in Appendix A illustrates the role that the definitions and lemmas in this section play in establishing Theorem 1). The first step is to inductively define *expressions*, syntactic formulas that express events defined solely in terms of nature $(S, \mathcal{A}_\kappa(\mathcal{S}))$ and interactive beliefs. Since any nature event $E \in \mathcal{A}_\kappa(\mathcal{S})$ is an object of beliefs, treat such E as a proposition and call it an expression. Since objects of beliefs are closed under conjunction, disjunction, negation, and the players' beliefs, define the corresponding syntactic (not set-theoretical) operations for expressions. For a set of expressions $\mathcal{E}$, define $(\bigwedge \mathcal{E})$ as a (syntactic) expression denoting the conjunction of expressions $\mathcal{E}$. For an expression e, let $(\neg e)$ be the (syntactic) expression denoting the negation of e, and let $(\beta_i(e))$ be the (syntactic) expression denoting that player i believes e. Formally:

Definition 5 (Expressions)**.** *Let λ be an infinite regular cardinal with $\lambda \leq \kappa$. The set of all λ-expressions $\mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$ is the smallest set satisfying the following.*

[13] Meier (2006, 2008) assumes the following "separative" condition on $(S, \mathcal{S})$: for any distinct $s, s' \in S$, there is $E \in \mathcal{S}$ with ($s \in E$ and $s' \notin E$) or ($s' \in E$ and $s \notin E$) (equivalently, there is $E \in \mathcal{A}_\kappa(\mathcal{S})$ with $s \in E$ and $s' \notin E$). Then, $\{s\} = \bigcap\{E \in \mathcal{A}_\kappa(\mathcal{S}) \mid s \in E\}$ for each $s \in S$, though it may be the case that $\{s\} \notin \mathcal{A}_\kappa(\mathcal{S})$.

1. *Every* $E \in \mathcal{A}_\kappa(\mathcal{S})$ *is a* λ*-expression.*

2. *If* $\mathcal{E}$ *is a set of* λ*-expressions with* $|\mathcal{E}| < \lambda$*, then so is* $(\bigwedge \mathcal{E})$*, where* $S := \bigwedge \emptyset$ *and identify* $\bigwedge \mathcal{E} := \bigcap \mathcal{E}$ *if* $\mathcal{E}$ *is a subset of* $\mathcal{A}_\kappa(\mathcal{S})$ *with* $|\mathcal{E}| < \lambda$*.*

3. *If* e *is a* λ*-expression then so is* $(\neg e)$*, where identify* $(\neg E) := E^c$ *for all* $E \in \mathcal{A}_\kappa(\mathcal{S})$*.*

4. *If* e *is a* λ*-expression, then so is* $(\beta_i(e))$ *for each* $i \in I$*.*

For $\lambda = \kappa$*, call each* κ*-expression an expression, and denote* $\mathcal{L} = \mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S}))$*.*

Remarks are in order. First, since κ is fixed, the (smallest) set $\mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$ is well defined by induction. Alternatively, Remark 1 below shows that $\mathcal{L}$ exactly consists of logical formulas regarding nature states and interactive beliefs about nature states of "depth at most $\overline{\kappa}$" (since κ is fixed, $\mathcal{L}$ and consequently $\mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$ are well defined).

Second, I consider κ-expressions beyond $\lambda = \aleph_0$ because a chain of finite qualitative belief hierarchies does not necessarily uniquely extend to the limit. Proposition 4 in Section 5.1 shows that, for countably-additive probabilistic beliefs on $\aleph_1$-algebras, $\aleph_0$-expressions $\lambda_{\aleph_0}(\mathcal{A}_{\aleph_1}(\mathcal{S}))$ suffice to capture countable belief hierarchies.

Third, since Definition 5 treats every $E \in \mathcal{A}_\kappa(\mathcal{S})$ as a logical formula (i.e., a λ-expression) and since $\mathcal{A}_\kappa(\mathcal{S})$ is closed under λ-intersection, identify λ-conjunction (a syntactic operation) with λ-intersection (a semantic operation) in $\mathcal{A}_\kappa(\mathcal{S})$. Similarly, identify negation and complementation in $\mathcal{A}_\kappa(\mathcal{S})$. For example, the conjunction $\bigwedge\{E, F\}$ $(E, F \in \mathcal{A}_\kappa(\mathcal{S}))$ is identified with the intersection $E \cap F$.

Fourth, for ease of notation, I often add or omit parentheses in denoting expressions (and in other occurrences). If $\mathcal{E}$ is a set of expressions with $|\mathcal{E}| < \kappa$, then let $(\bigvee \mathcal{E}) := \neg(\bigwedge\{(\neg e) \in \mathcal{L} \mid e \in \mathcal{E}\})$ and identify $\bigvee \emptyset := \emptyset$. Thus, $\bigvee \mathcal{E}$ denotes the disjunction of $\mathcal{E}$. Also, I interchangeably denote, for instance, $e_1 \wedge e_2 = \bigwedge\{e_1, e_2\}$ and $e_1 \vee e_2 = \bigvee\{e_1, e_2\}$.[14] I interchangeably denote $\bigwedge_{j \in J} e_j = \bigwedge\{e_j \mid j \in J\}$ and $\bigvee_{j \in J} e_j = \bigvee\{e_j \mid j \in J\}$ when expressions are indexed by some set J. Denote $(e \to f) := ((\neg e) \vee f)$ and $(e \leftrightarrow f) := ((e \to f) \wedge (f \to e))$.

The set $\mathcal{L}$ incorporates all the hierarchies of interactive beliefs regarding nature states up to the ordinality of κ (i.e., the ordinal depth $\overline{\kappa}$). The following remark

[14] For example, I simply do not distinguish $e_1 \vee e_2$ and $e_2 \vee e_1$. Similarly, since $\{e, e\} = \{e\}$, I simply identify $(e \wedge e)$ as e. These could be augmented by defining $(\bigwedge \mathcal{F})$ for an ordinal sequence of expressions $\mathcal{F}$ instead of a set of expressions.

shows how the set of expressions $\mathcal{L}$ is inductively generated from the nature states $(S, \mathcal{A}_\kappa(\mathcal{S}))$ in $\overline{\kappa}$ steps.

Remark 1 (Restatement of Expressions $\mathcal{L}$)**.** Let λ be an infinite regular cardinal with $\lambda \leq \kappa$. The following auxiliary ordinal sequence $(\mathcal{L}_\alpha)_{\alpha=0}^{\overline{\lambda}}$ generates $\mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S})) = \mathcal{L}_{\overline{\lambda}}$. In particular, if $\lambda = \kappa$ then $\mathcal{L} = \mathcal{L}_{\overline{\kappa}}$. Namely, let $\mathcal{L}_0 := \mathcal{A}_\kappa(\mathcal{S})$. For any ordinal α with $0 < \alpha \leq \overline{\lambda}$, define

$$\mathcal{L}_\alpha := \mathcal{L}'_\alpha \cup \{(\neg e) \mid e \in \mathcal{L}'_\alpha\} \cup \left\{ \bigwedge \mathcal{F} \,\middle|\, \mathcal{F} \subseteq \mathcal{L}'_\alpha \text{ and } 0 < |\mathcal{F}| < \lambda \right\}, \text{ where}$$

$$\mathcal{L}'_\alpha := \left(\bigcup_{\beta < \alpha} \mathcal{L}_\beta \right) \cup \bigcup_{i \in I} \{\beta_i(e) \mid e \in \bigcup_{\beta < \alpha} \mathcal{L}_\beta\}.$$

Intuitively, Remark 1 states that $\mathcal{L}$ consists of expressions of "depth at most $\overline{\kappa}$," i.e., logical formulas expressing interactive beliefs regarding $(S, \mathcal{A}_\kappa(\mathcal{S}))$ (indeed, $(S, \mathcal{S})$) up to the ordinality of κ.

While expressions themselves are defined independently of any particular belief space, for any belief space $\overrightarrow{\Omega}$, I can recursively identify each expression e with the corresponding event $[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D}$ so that $[\![e]\!]_{\overrightarrow{\Omega}}$ is the set of states of the world in which the expression e holds. Thus, the event $[\![e]\!]_{\overrightarrow{\Omega}}$ gives the semantic meaning of the expression e. Specifically, recalling the discussion on Definition 1 (3), $[\![E]\!]_{\overrightarrow{\Omega}} := \Theta^{-1}(E) \in \mathcal{D}$ is the set of states at which $E \in \mathcal{A}_S$ is true. The set of states $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega}}$ at which an expression $\beta_i(e)$ is true ("i believes e") is inductively given by $B_i([\![e]\!]_{\overrightarrow{\Omega}})$. Formally:

Definition 6 (Expressions Identified as Events)**.** *Fix a κ-belief space $\overrightarrow{\Omega}$. Inductively define the map $[\![\cdot]\!]_{\overrightarrow{\Omega}} : \mathcal{L} \to \mathcal{D}$, the semantic interpretation function of $\overrightarrow{\Omega}$, as follows.*

1. $[\![E]\!]_{\overrightarrow{\Omega}} := \Theta^{-1}(E)$ *for every* $E \in \mathcal{A}_\kappa(\mathcal{S})$.
2. $[\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}} := \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}}$ *for any* $\mathcal{E} \in \mathcal{P}(\mathcal{L})$ *with* $|\mathcal{E}| < \kappa$.
3. $[\![\neg e]\!]_{\overrightarrow{\Omega}} := \neg [\![e]\!]_{\overrightarrow{\Omega}} \left(= ([\![e]\!]_{\overrightarrow{\Omega}})^c\right)$ *for each expression* e.
4. $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega}} := B_i([\![e]\!]_{\overrightarrow{\Omega}})$ *for each* $i \in I$ *and expression* e.

Call $[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D}$ *the denotation of* e *in* $\overrightarrow{\Omega}$.

The semantic interpretation function of a given belief space is, by recursion, uniquely extended from Θ^{-1}.[15] By recursion, a morphism preserves semantics.

[15] While I do not discuss implications of finite-depth reasoning, one could analyze players' finite depth/level-α reasoning in any belief space $\overrightarrow{\Omega}$ by restricting attention to events $[\![e]\!]_{\overrightarrow{\Omega}}$ with $e \in \mathcal{L}_\alpha$.

Remark 2 (Morphism Preserves the Meaning of an Expression)**.** If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism, then $[\![\cdot]\!]_{\overrightarrow{\Omega}} = \varphi^{-1}([\![\cdot]\!]_{\overrightarrow{\Omega'}})$.

The second step is to define *descriptions* by the set of expressions and the nature state that obtain at each state of each belief space. Since nature states and expressions reside in different spaces, define a description to be a subset of the disjoint union $S \sqcup \mathcal{L} := \{(0, s) \in \{0\} \times S \mid s \in S\} \cup \{(1, e) \in \{1\} \times \mathcal{L} \mid e \in \mathcal{L}\}$. While this definition of the description is different from the one in the previous literature, this definition uniquely identifies the corresponding nature state for each description without any condition (e.g., the separative condition in Footnote 13) on $(S, \mathcal{S})$.

Definition 7 (Descriptions)**.** *For any belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$, define the description $D(\omega)$ of ω by $D(\omega) := \{\Theta(\omega)\} \sqcup \{e \in \mathcal{L} \mid \omega \in [\![e]\!]_{\overrightarrow{\Omega}}\}$.*

Each description $D(\omega) = \{(0, \Theta(\omega))\} \cup \{(1, e) \in \{1\} \times \mathcal{L} \mid \omega \in [\![e]\!]_{\overrightarrow{\Omega}}\}$ contains the unique nature state $\Theta(\omega) \in S$ associated with ω and the expressions e which are true at ω. For ease of notation, write $s \in_0 D(\omega)$ for $(0, s) \in D(\omega)$. Also, write $e \in_1 D(\omega)$ for $(1, e) \in D(\omega)$. The reader could even read "$s \in D(\omega)$" and "$e \in D(\omega)$" by disregarding the subscripts 0 and 1.

Descriptions have two roles in constructing a terminal belief space. First, I will construct the terminal belief space so that the underlying states Ω^* consist of all descriptions of states of the world ranged over all belief spaces in the given category:

$$\Omega^* := \{\omega^* \in \mathcal{P}(S \sqcup \mathcal{L}) \mid \omega^* = D(\omega) \text{ for some } \overrightarrow{\Omega} \text{ and } \omega \in \Omega\}. \tag{1}$$

Second, regard D as a mapping $D : \Omega \to \Omega^*$ (or $D_{\overrightarrow{\Omega}} : \Omega \to \Omega^*$ to stress its domain) for any belief space $\overrightarrow{\Omega}$, and call D the *description map*. The description map D turns out to be a unique morphism.

Two remarks are in order. First, Ω^* is not empty because there is a belief space $\overrightarrow{\Omega}$ with $\Omega \neq \emptyset$ in the given category. Consider a belief space $\overrightarrow{\{s\}} := \langle(\{s\}, \mathcal{P}(\{s\})), (\mathrm{id}_{\mathcal{P}(\{s\})})_{i \in I}, \Theta\rangle$ where $s \in S$ and $\Theta : \{s\} \ni s \mapsto s \in S$. Each $B_i = \mathrm{id}_{\mathcal{P}(\{s\})}$ satisfies all the properties of beliefs in Definition 2.

Second, Ω^* depends on the choice of a category of belief spaces. Consider any two categories of belief spaces where assumptions on players' beliefs in the first are also imposed in the second. Denoting by Ω^{1*} and Ω^{2*} the spaces constructed according to Equation (1), $\Omega^{2*} \subseteq \Omega^{1*}$ holds by construction.

To show that the description map D is a unique morphism, I remark that a morphism preserves the descriptions, i.e., nature states and belief hierarchies.

Remark 3 (Morphism Preserves Descriptions)**.** If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism, then $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \varphi$.

To see this, fix belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ and $(\omega, \omega') \in \Omega \times \Omega'$. Then, $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$ iff (i) $\Theta(\omega) = \Theta'(\omega')$ and (ii) $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega' \in [\![e]\!]_{\overrightarrow{\Omega'}}$ for all $e \in \mathcal{L}$. Thus, $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$ means that the outside analysts would consider states ω and ω' to be equivalent in terms of a prevailing nature state and prevailing expressions, abstracting away from physical representations of $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ (see Remark A.1 in Appendix A for further discussions). By Remark 2, both conditions are met for any $(\omega, \varphi(\omega)) \in \Omega \times \Omega'$ where $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism. Thus, for any state $\omega \in \Omega$, there is a state $\varphi(\omega) \in \Omega'$ such that ω and $\varphi(\omega)$ induce the same belief hierarchy together with the corresponding nature state.

I discuss two implications of Remark 3. First, call a belief space $\overrightarrow{\Omega}$ *non-redundant* (Mertens and Zamir, 1985, Definition 2.4) (or *non-flabby* (Fagin, 1994)) if its description map D is injective. In other words, for any distinct ω and ω', either $\Theta(\omega) \neq \Theta(\omega')$ or they are separated by (a sub-κ-algebra) $\mathcal{D}_{\overline{\kappa}} := \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$ (it can be expressed solely by the primitives of the belief space by Definition 13 in Section 6).

Second, Remark 3 implies that if $\overrightarrow{\Omega'}$ is non-redundant then there is at most one morphism from a given space $\overrightarrow{\Omega}$ into $\overrightarrow{\Omega'}$.[16] I will show that $D_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is a unique morphism by demonstrating that $D_{\overrightarrow{\Omega^*}}$ is the identity.

The third step is to define the domain $\mathcal{D}^*$ of the candidate terminal belief space Ω^*. Since each expression e corresponds to an object of beliefs, define the set $[e]$ of descriptions that make e true (i.e., contain e) to be objects of players' beliefs on Ω^*. Formally, for each $e \in \mathcal{L}$, define the set of descriptions $[e] := \{\omega^* \in \Omega^* \mid e \in_1 \omega^*\}$. I show that $\mathcal{D}^* := \{[e] \in \mathcal{P}(\Omega^*) \mid e \in \mathcal{L}\}$ is a legitimate domain.

Lemma 1 (Domain $\mathcal{D}^*$)**.** $(\Omega^*, \mathcal{D}^*)$ *is a κ-algebra. For any belief space $\overrightarrow{\Omega}$, the description map $D : (\Omega, \mathcal{D}) \to (\Omega^*, \mathcal{D}^*)$ is a measurable map with $D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}}$ for any $e \in \mathcal{L}$.*

The property that $D^{-1}_{\overrightarrow{\Omega}}([\cdot]) = [\![\cdot]\!]_{\overrightarrow{\Omega}}$ exhibits duality between the semantic interpretation function $[\![\cdot]\!]_{\overrightarrow{\Omega}} : \mathcal{L} \to \mathcal{D}$ (which, by recursion, is unique) and the description map

[16] *Proof.* If $\varphi, \psi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ are morphisms then $D_{\overrightarrow{\Omega'}} \circ \varphi = D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \psi$. Since $D_{\overrightarrow{\Omega'}}$ is injective, $\varphi = \psi$.

$D_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ (which turns out to be a unique morphism). Note also that the sub-κ-algebra $\mathcal{D}_{\overline{\kappa}}$ is the one induced by $D_{\overrightarrow{\Omega}}$: $\mathcal{D}_{\overline{\kappa}} = D_{\overrightarrow{\Omega}}^{-1}(\mathcal{D}^*) = \{D_{\overrightarrow{\Omega}}^{-1}([e]) \in \mathcal{D} \mid e \in \mathcal{L}\}$. Call $\overrightarrow{\Omega}$ *minimal* (Di Tillio, 2008) if $\mathcal{D}_{\overline{\kappa}} = \mathcal{D}$.[17]

The fourth step is to construct the mapping $\Theta^* : \Omega^* \to S$ that associates with each state $\omega^* \in \Omega^*$ the unique nature state s contained in ω^* (i.e., $s \in_0 \omega^*$).

Lemma 2 (Mapping Θ^*)**.** *There is a measurable map $\Theta^* : (\Omega^*, \mathcal{D}^*) \to (S, \mathcal{A}_\kappa(\mathcal{S}))$ with the following two properties: (i) $\Theta^*(D(\omega)) = \Theta(\omega)$ for any belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$; and (ii) $(\Theta^*)^{-1}(E) = [E] \in \mathcal{D}^*$ for all $E \in \mathcal{A}_\kappa(\mathcal{S})$.*

The fifth step is to introduce the players' belief operators on $\mathcal{D}^*$ in a way such that player i believes an event $[e]$ at a state ω^* iff ω^* contains $\beta_i(e)$ (i.e., $\beta_i(e) \in_1 \omega^*$). I show that this is well defined: if expressions e and f are equivalent in the sense that $e \in_1 \omega^*$ iff $f \in_1 \omega^*$, then $\beta_i(e)$ and $\beta_i(f)$ are equivalent in the same sense.[18]

Lemma 3 (Belief Operators B_i^*)**.** *Fix $i \in I$. Define $B_i^* : \mathcal{D}^* \to \mathcal{D}^*$ by $B_i^*([e]) := [\beta_i(e)]$ for each $e \in \mathcal{L}$. Then, B_i^* is a well-defined belief operator which inherits the properties of beliefs imposed in the given category. Moreover, for any belief space $\overrightarrow{\Omega}$, $D^{-1}(B_i^*([e])) = B_i(D^{-1}([e]))$ for all $[e] \in \mathcal{D}^*$.*

Lemma A.1 in Appendix A shows that each B_i^* inherits various properties satisfied in the given category beyond Definition 2. Using this result, Section 5 extends the construction of a terminal belief space to such notions as probabilistic beliefs.

I remark on two additional results proved in Lemma A.1. First, if there is a belief space $\overrightarrow{\Omega}$ such that B_i fails a given property with respect to $[\![e]\!]_{\overrightarrow{\Omega}}$, then B_i^* fails that property with respect to $[e]$. Thus, B_i^* satisfies the properties of beliefs for player i that are common among all the belief spaces in the given category. That is, B_i^* satisfies the properties that the outside analysts exactly would like to impose.[19]

[17]Friedenberg and Meier (2011) call it *strongly measurable* in a related context (recall $\mathcal{D}_{\overline{\kappa}} = D_{\overrightarrow{\Omega}}^{-1}(\mathcal{D}^*)$).

[18]First, this equivalence depends on assumptions on beliefs. For example, if Positive Introspection and Truth Axiom are imposed on player i, then $\beta_i(e)$ and $\beta_i\beta_i(e)$ are equivalent. Section 3.3 examines how assumptions on beliefs are encoded within Ω^* itself. Second, Proposition 2 in Section 3.2 shows if there are a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $[\![e]\!]_{\overrightarrow{\Omega}} \neq [\![f]\!]_{\overrightarrow{\Omega}}$ then e and f are not equivalent (i.e., $[e] \neq [f]$). Thus, such identifications (equivalences) of expressions are minimum in $\overrightarrow{\Omega^*}$.

[19]Roy and Pacuit (2013) define "substantive" and "structural" assumptions in a syntactic interactive epistemic model. In their framework, a terminal structure, if it exists, minimizes substantive assumptions and validates only structural assumptions.

Second, for belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ which may reside in different categories, if $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ is a surjective measurable map with $B_i \varphi^{-1}(\cdot) = \varphi^{-1} B'_i(\cdot)$, then B'_i inherits the properties of B_i.

So far, $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (B^*_i)_{i \in I}, \Theta^* \rangle$ is a belief space of the given category such that, for any belief space $\overrightarrow{\Omega}$, the description map $D : \Omega \to \Omega^*$ is a morphism.

The sixth step finally establishes that the description map D is a unique morphism. To that end, I show that the description map from $\overrightarrow{\Omega^*}$ into itself is the identity map.

Lemma 4 (Description Map $D_{\overrightarrow{\Omega^*}}$). *The description map $D_{\overrightarrow{\Omega^*}} : \overrightarrow{\Omega^*} \to \overrightarrow{\Omega^*}$ is the identity morphism.*

I prove Lemma 4 by showing $[\cdot] = [\![\cdot]\!]_{\overrightarrow{\Omega^*}}$, which means whether an expression e is true at ω^* is determined solely by whether $e \in_1 \omega^*$ within ω^*. The lemma implies that the belief space $\overrightarrow{\Omega^*}$ is non-redundant. Moreover, it implies $D_{\overrightarrow{\Omega}}$ is a unique morphism: if $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is a morphism then $D_{\overrightarrow{\Omega}}(\cdot) = D_{\overrightarrow{\Omega^*}}(\varphi(\cdot)) = \varphi(\cdot)$. Thus:

Theorem 1 ($\overrightarrow{\Omega^*}$ is Terminal). *The space $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (B^*_i)_{i \in I}, \Theta^* \rangle$ is a terminal κ-belief space of I on $(S, \mathcal{S})$ for a given category of κ-belief spaces.*

As discussed in Section 2.2, a terminal belief space exists uniquely up to isomorphism. Generally, $\overrightarrow{\Omega}$ is terminal iff the description map $D_{\overrightarrow{\Omega}}$ is an isomorphism. I discuss how the belief space Ω^* "includes" all belief spaces.

First, for any state ω of any particular belief space $\overrightarrow{\Omega}$, states $\omega \in \Omega$ and $D(\omega) \in \Omega^*$ are equivalent in the sense that the same state of nature $\Theta(\omega) = \Theta^*(D(\omega))$ prevails and the same set of expressions regarding nature and interactive beliefs obtains. This is because $D(\omega) = D_{\overrightarrow{\Omega^*}}(D(\omega))$ (recall discussions after Remark 3). To restate, for any representation $\overrightarrow{\Omega}$ of interactive beliefs regarding $(S, \mathcal{S})$ and for any realization $\omega \in \Omega$, the prevailing nature state and the prevailing set of expressions at ω are encoded in the state $D(\omega)$ in $\overrightarrow{\Omega^*}$. Hence, $\overrightarrow{\Omega^*}$ is also terminal in the sense of Friedenberg (2010): for any state $\omega \in \Omega$ of any belief space $\overrightarrow{\Omega}$, there is a unique state $\omega^* = D(\omega)$ in $\overrightarrow{\Omega^*}$ such that ω and ω^* induce the same belief hierarchies. Especially, the expressions $\{e \in \mathcal{L} \mid \omega \in B_i([\![e]\!]_{\overrightarrow{\Omega}})\}$ that player i believes at ω coincide with those $\{e \in \mathcal{L} \mid \omega^* \in B^*_i([e])\}$ that she believes at ω^*.

Second, a non-redundant belief space $\overrightarrow{\Omega}$ is, by definition, embedded into $\overrightarrow{\Omega^*}$: there is a belief (sub-)space $\overrightarrow{D(\Omega)} := \langle (D(\Omega), \mathcal{D}^* \cap D(\Omega)), (B'_i)_{i \in I}, \Theta^*|_{D(\Omega)} \rangle$ such that $D : \overrightarrow{\Omega} \to \overrightarrow{D(\Omega)}$ is a bijective morphism, where $\mathcal{D}^* \cap D(\Omega) := \{[e] \cap D(\Omega) \mid [e] \in \mathcal{D}^*\}$

and $B'_i([e] \cap D(\Omega)) := B^*_i([e]) \cap D(\Omega)$. If, in addition, $\overrightarrow{\Omega}$ is minimal, then $\overrightarrow{\Omega}$ and $\overrightarrow{D(\Omega)}$ are isomorphic (i.e., the inverse D^{-1} is also a morphism) because any $E \in \mathcal{D}$ is associated with some $[e] \in \mathcal{D}^*$ through $[e] = D^{-1}(E)$. Indeed, $\overrightarrow{\Omega}$ and $\overrightarrow{D(\Omega)}$ are isomorphic iff $\overrightarrow{\Omega}$ is non-redundant and minimal (if this is the case, then a κ-algebra $\mathcal{D}$ is typically not the power set).

3.1 Completeness

The rest of this section discusses properties of the terminal space $\overrightarrow{\Omega^*}$. I start with showing that the space Ω^* is (belief-)complete in that each state ω^* is in a one-to-one relation with the state of nature $s = \Theta^*(\omega^*)$ and the profile of sets of expressions that individual players believe at the state ω^*.[20] Formally, letting

$$\Omega^{**} := \{(s, \Psi) \in S \times \mathcal{P}(\mathcal{L})^I \mid (s, \Psi) = (\Theta(\omega), (\{e \in \mathcal{L} \mid \omega \in B_i[\![e]\!]_{\overrightarrow{\Omega}}\})_{i \in I}) \text{ for some belief space } \overrightarrow{\Omega} \text{ and } \omega \in \Omega\},$$

the following proposition states that the belief space $\overrightarrow{\Omega^*}$ exhausts nature and the players' beliefs in that the bijection exists between Ω^* and Ω^{**}.

Proposition 1 (Ω^* Exhausts Interactive Beliefs)**.** *The mapping* $\Omega^* \ni \omega^* \mapsto (\Theta^*(\omega^*), (\{e \in \mathcal{L} \mid \omega^* \in B^*_i[e]\})_{i \in I}) \in \Omega^{**}$ *is bijective. In particular,*

$$\Omega^{**} = \{(\Theta^*(\omega^*), (\{e \in \mathcal{L} \mid \omega^* \in B^*_i[e]\})_{i \in I}) \in S \times \mathcal{P}(\mathcal{L})^I \mid \omega^* \in \Omega^*\}. \quad (2)$$

The space Ω^* exhausts all possible forms of interactive beliefs that can realize at some state of some belief space in a way that any two distinct states in Ω^* describe different belief hierarchies. Thus, as Equation (2) shows, Ω^{**} is obtained by restricting attention to the terminal space. For each player i, each state ω^* contains all the relevant information about i's beliefs at ω^* within itself because, for any expression e, whether i believes $[e]$ or not at ω^* is well defined according to $\{e \in \mathcal{L} \mid \omega^* \in B^*_i[e]\}$.

Note that the space $\Omega^{**} (\subseteq S \times \mathcal{P}(\mathcal{L})^I)$ would not be a product space under Truth Axiom. In contrast, if one would restrict attention to product state spaces Ω and if each player's beliefs would depend only on her "types" (consequently, Truth Axiom is ruled out), then Ω^{**} would be written as a product space.

[20]Brandenburger and Keisler (2006) define a notion of (belief-)completeness in terms of a language.

3.2 Informational Robustness

Next, I show: (i) for any set of expressions that hold at some state of some belief space, the set of expressions hold at the corresponding state of the terminal space $\overrightarrow{\Omega^*}$; and (ii) if there is a belief space that distinguishes expressions e and f, then so does the terminal space $\overrightarrow{\Omega^*}$. To make the claims formal, I introduce:

Definition 8 (Semantic Properties)**.** *1. An expression $e \in \mathcal{L}$ is valid in a belief space $\overrightarrow{\Omega}$ (written $\models_{\overrightarrow{\Omega}} e$) if $[\![e]\!]_{\overrightarrow{\Omega}} = \Omega$. If e is valid in any belief space (of the given category), then e is valid (in the given category) (written $\models e$).*

2. *A set of expressions $\Phi \in \mathcal{P}(\mathcal{L})$ is satisfiable in $\overrightarrow{\Omega}$ if there is $\omega \in \Omega$ with $\omega \in [\![f]\!]_{\overrightarrow{\Omega}}$ for all $f \in \Phi$. Call Φ satisfiable if Φ is satisfiable in some belief space $\overrightarrow{\Omega}$.*

3. *An expression $e \in \mathcal{L}$ is a semantic consequence of $\Phi \in \mathcal{P}(\mathcal{L})$ in $\overrightarrow{\Omega}$ (written $\Phi \models_{\overrightarrow{\Omega}} e$) if, $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$ holds whenever $\omega \in [\![f]\!]_{\overrightarrow{\Omega}}$ for all $f \in \Phi$. If $\Phi \models_{\overrightarrow{\Omega}} e$ for any belief space $\overrightarrow{\Omega}$, then $e \in \mathcal{L}$ is a semantic consequence of Φ (written $\Phi \models e$).*

Let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ be a morphism. By Remark 2, any valid expression e in $\overrightarrow{\Omega'}$ is also valid in $\overrightarrow{\Omega}$. If Φ is satisfiable in $\overrightarrow{\Omega}$, then so is it in $\overrightarrow{\Omega'}$. Suppose further that φ is surjective. If e is a semantic consequence of Φ in $\overrightarrow{\Omega}$, then so is it in $\overrightarrow{\Omega'}$.

The following proposition shows that such semantic notions as satisfiability, semantic consequence, and validness in $\overrightarrow{\Omega^*}$ are informationally robust in the sense that they do not depend on particular belief spaces.

Proposition 2 (Informational Robustness)**.** *Let $e \in \mathcal{L}$ and $\Phi \in \mathcal{P}(\mathcal{L})$.*

1. *Φ is satisfiable iff Φ is satisfiable in $\overrightarrow{\Omega^*}$.*

2. *e is a semantic consequence of Φ iff e is a semantic consequence of Φ in $\overrightarrow{\Omega^*}$. Also, e is valid iff e is valid in $\overrightarrow{\Omega^*}$.*

Proposition 2 (1) states that the space $\overrightarrow{\Omega^*}$ exhausts all possible sets of satisfiable expressions (in some belief space $\overrightarrow{\Omega}$) within $\overrightarrow{\Omega^*}$. Put differently, if expressions Φ hold at some state ω in some belief space $\overrightarrow{\Omega}$, then the expressions Φ hold at $D(\omega)$ in $\overrightarrow{\Omega^*}$. This result reflects the insight by Moss and Viglizzo (2004, 2006) that their terminal object (for a "measure polynomial functor") consists of all satisfied theories (descriptions) of all points in all objects.

Proposition 2 (2) implies that if expressions e and f satisfy $[\![e]\!]_{\overrightarrow{\Omega}} \neq [\![f]\!]_{\overrightarrow{\Omega}}$ for some belief space $\overrightarrow{\Omega}$, then $([e] =)[\![e]\!]_{\overrightarrow{\Omega^*}} \neq [\![f]\!]_{\overrightarrow{\Omega^*}}(= [f])$. Suppose for instance that expressions e and $\beta_i(f)$ happen to satisfy $[\![e]\!]_{\overrightarrow{\Omega'}} = B'_i[\![f]\!]_{\overrightarrow{\Omega'}}$ in a particular representation (or a particular "context") $\overrightarrow{\Omega'}$. If another belief space $\overrightarrow{\Omega}$ distinguishes these expressions (i.e., $[\![e]\!]_{\overrightarrow{\Omega}} \neq B_i[\![f]\!]_{\overrightarrow{\Omega}}$), it follows in the belief space $\overrightarrow{\Omega^*}$ that $[\![e]\!]_{\overrightarrow{\Omega^*}} \neq B^*_i[\![f]\!]_{\overrightarrow{\Omega^*}}$. Put differently, if $[\![e]\!]_{\overrightarrow{\Omega^*}} = B^*_i[\![f]\!]_{\overrightarrow{\Omega^*}}$, then it is always the case in any belief space $\overrightarrow{\Omega}$ that $[\![e]\!]_{\overrightarrow{\Omega}} = B_i[\![f]\!]_{\overrightarrow{\Omega}}$. Thus, the space $\overrightarrow{\Omega^*}$ makes the minimum assumptions on how expressions $\mathcal{L}$ (i.e., natures states and interactive beliefs) are interpreted and identified (Recall the discussions in the fifth step of the construction of $\overrightarrow{\Omega^*}$ in Section 3).

Proposition 2 provides a formal sense in which any form of reasoning in a (smaller) belief space can be extended to the terminal space. In the context of Bayesian games, Friedenberg and Meier (2017) nevertheless show that a (smaller) type space may have a Bayesian equilibrium that cannot be extended to a canonical (larger) type space. The difference would come from the following features of my framework: (i) in a sense, the "strategy" choices Θ are part of the primitives of any belief space $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, \Theta\rangle$; and (ii) the "strategies" Θ are always assumed to be measurable.[21] Thus, for any description of the players' beliefs $\langle(\Omega, \mathcal{D}), (B_i)_{i\in I}\rangle$ and a measurable map $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_\kappa(\mathcal{S}))$, the canonical description $\langle(\Omega^*, \mathcal{D}^*), (B^*_i)_{i\in I}\rangle$ and the measurable map $\Theta^* : (\Omega^*, \mathcal{D}^*) \to (S, \mathcal{A}_\kappa(\mathcal{S}))$ (uniquely) preserve beliefs $B_iD^{-1} = D^{-1}B^*_i$ and nature states (e.g., actions) $\Theta = \Theta^* \circ D$ (accordingly, any expressions $[\![\cdot]\!]_{\overrightarrow{\Omega}} = D^{-1}[\![\cdot]\!]_{\overrightarrow{\Omega^*}} = D^{-1}[\cdot]$) through D.

3.3 Self-Reference

Each state $\omega^* \in \Omega^*$ contains expressions that hold at ω^* (i.e., $e \in_1 \omega^*$ iff $\omega^* \in [e]$) as well as the corresponding nature state $s = \Theta^*(\omega^*)$. I examine how each ω^* describes the nature state and players' beliefs in the following two senses. The first is its logical structure, following Aumann (1999). Each ω^* is *coherent*: if ω^* contains an expression e then it does not contain $(\neg e)$. Each ω^* is *complete*: if ω^* does not contain e then it contains $(\neg e)$. Every ω^* is logically closed. Especially, it contains such expressions as S that hold in any belief space of the given class.

[21] For the first point, letting $S = \prod_{i\in I} A_i$ be the action profiles in a strategic game, $\Theta = (\Theta_i)_{i\in I} : \Omega \to S$ is regarded as a strategy profile (see Sections 5.1 and 5.4). In contrast, a product type space $\Omega = S \times \prod_{i\in I} T_i$ has S in itself, and "Θ" is implicitly given by the projection from Ω into S. Thus, when it comes to the type space *alone*, a player's strategy choice (a mapping from her types into her actions) is given separately.

Proposition 3 (Logical Properties of Each State)**.** *Fix* $\omega^* \in \Omega^*$*,* $e \in \mathcal{L}$*,* $f \in \mathcal{L}$*, and* $\mathcal{E} \in \mathcal{P}(\mathcal{L})$ *with* $|\mathcal{E}| < \kappa$*.*

1. *Coherence and Completeness:* $e \notin_1 \omega^*$ *iff* $(\neg e) \in_1 \omega^*$*.*

2. *Closure under Implication: If* $e \in_1 \omega$ *and* $(e \to f) \in_1 \omega$*, then* $f \in_1 \omega$*.*

3. *Closure under Conjunction:* $\bigwedge \mathcal{E} \in_1 \omega^*$ *iff* $e \in_1 \omega^*$ *for all* $e \in \mathcal{E}$*.*

Second, the belief space $\overrightarrow{\Omega^*}$ resolves the following form of self-reference: players' beliefs are defined on states while states are supposed to completely describe the world. Recall each player's beliefs at each state ω^* are built within ω^* itself: $\omega^* \in B_i^*([e])$ iff $\beta_i(e) \in_1 \omega^*$. Since each ω^* satisfies the logical requirement postulated in Proposition 3 and since B_i^* inherits the properties of beliefs assumed in a given category, as is shown in the following corollary, the players' beliefs at each state are encoded within the state itself.

Corollary 1 (Beliefs within Each State)**.** *Fix* $\omega^* \in \Omega^*$*,* $i \in I$*, and* $e \in \mathcal{L}$*.*

1. *Exactly one of* $\beta_i(e) \in_1 \omega^*$ *or* $(\neg\beta_i)(e) \in_1 \omega^*$ *holds.*

2. *At least one of* $\beta_i(e) \in_1 \omega^*$*,* $\beta_i(\neg e) \in_1 \omega^*$*, or* $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e) \in_1 \omega^*$ *holds. Exactly one of them always holds (for any* (e, ω^*)*) iff Consistency holds for* i*.*

3. *Exactly one of* $\beta_i(e) \in_1 \omega^*$*,* $(\neg\beta_i)(e) \wedge \beta_i(\neg\beta_i)(e) \in_1 \omega^*$*, or* $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg\beta_i)(e) \in_1 \omega^*$ *holds. The third condition never occurs (for any* (e, ω^*)*) iff Negative Introspection holds for* i*.*

In the first part of Corollary 1, each ω^* completely describes i's beliefs: for any $e \in \mathcal{L}$, the state ω^* contains exactly one of the two expressions denoting "i believes e" or "i does not believe e." The second and third parts characterize how Ω^* encodes such properties as Consistency and Negative Introspection. Put differently, the space Ω^* itself encodes whether the assumptions on such properties of i' beliefs are made. Corollary 1 is related to some consistency conditions of Gilboa (1988) for a state to completely describe the world. Section 4 characterizes how states in Ω^* encode all the properties of beliefs specified in Definition 2 by demonstrating that properties imposed on player i's beliefs by the outside analysts are expressed within Ω^*.

4 Largest Coherent Set of Descriptions Ω^*

Each state in the terminal belief space $\overrightarrow{\Omega^*}$ consists of a set of expressions (together with a state of nature) satisfiable at some state of some belief space. For an infinite regular cardinal $\kappa > \aleph_0$ where infinitary operations are allowed, states in the terminal κ-belief space would generally be different from the collection of "maximally consistent" sets of expressions (together with a state of nature) in some syntax system, which are often used to prove the "completeness" theorem of the syntax system.[22]

The question arises as to how (or whether) one can characterize each state ω^* and the set $\overrightarrow{\Omega^*}$ in an explicit way. I characterize the space Ω^* as the largest *coherent set of descriptions*. Call a subset Ω of $\mathcal{P}(S \sqcup \mathcal{L})$ a coherent set of descriptions if: (i) each $\omega \in \Omega$ is a complete and coherent set of expressions together with a unique nature state; (ii) the set Ω as a whole induces the players' beliefs in a well-defined manner; and (iii) each ω reflects assumptions on players' beliefs. This characterization holds irrespective of a given cardinal κ and assumptions on beliefs. Formally:

Definition 9 (Coherent Set of Descriptions)**.** *Call a subset Ω of $\mathcal{P}(S \sqcup \mathcal{L})$ a coherent set of descriptions if it satisfies the following conditions.*

1. *Each $\omega \in \Omega$ satisfies the following.*
   (a) *There is a unique $s \in S$ with $s \in_0 \omega$. Moreover, $E \in_1 \omega$ for all $E \in \mathcal{A}_\kappa(\mathcal{S})$ with $s \in E$.*
   (b) *Closure under Implication: If $e \in_1 \omega$ and $(e \to f) \in_1 \omega$ then $f \in_1 \omega$.*
   (c) *Closure under Conjunction: For any $\mathcal{E} \in \mathcal{P}(\mathcal{L})$ with $|\mathcal{E}| < \kappa$, $\bigwedge \mathcal{E} \in_1 \omega$ iff $e \in_1 \omega$ for all $e \in \mathcal{E}$.*
   (d) *Coherency: For each $e \in \mathcal{L}$, if $(\neg e) \in_1 \omega$ then $e \notin_1 \omega$.*
   (e) *Completeness: For each $e \in \mathcal{L}$, if $e \notin_1 \omega$ then $(\neg e) \in_1 \omega$.*
2. *The set Ω satisfies the following ((2b) and (2c) depend on assumptions on beliefs).*
   (a) *If $e, f \in \mathcal{L}$ satisfy $(e \leftrightarrow f) \in_1 \omega$ for all $\omega \in \Omega$, then $(\beta_i(e) \leftrightarrow \beta_i(f)) \in_1 \omega$ for all $\omega \in \Omega$.*

[22] The discrepancy between a semantic notion of satisfiability and a syntactic notion of maximal consistency would emerge when infinitary operations are allowed (Karp, 1964). See also Heifetz (1997), Meier (2012), Moss and Viglizzo (2004, 2006), and Zhou (2010).

(b) *Let Monotonicity be assumed for player* i*. If* $e, f \in \mathcal{L}$ *satisfy* $(e \to f) \in_1 \omega$ *for all* $\omega \in \Omega$*, then* $(\beta_i(e) \to \beta_i(f)) \in_1 \omega$ *for all* $\omega \in \Omega$*.*

(c) *Let the Kripke property be assumed for player* i*. Then,* $\beta_i(e) \in_1 \omega$ *for any* $(e, \omega) \in \mathcal{L} \times \Omega$ *with the following condition: if* $\omega' \in \Omega$ *satisfies* $f \in_1 \omega'$ *for all* $f \in \mathcal{L}$ *with* $\beta_i(f) \in_1 \omega$*, then* $e \in_1 \omega'$*.*

3. *Depending on assumptions on beliefs, each* $\omega \in \Omega$ *contains any instance of the following expressions.*

   (a) *Necessitation:* $\beta_i(S)$*.*

   (b) *Non-empty* λ*-Conjunction:* $((\bigwedge_{e \in \mathcal{E}} \beta_i(e)) \to \beta_i(\bigwedge \mathcal{E}))$ *with* $0 < |\mathcal{E}| < \lambda$*.*

   (c) *Consistency:* $(\beta_i(e) \to (\neg\beta_i)(\neg e))$*.*

   (d) *Truth Axiom:* $(\beta_i(e) \to e)$*.*

   (e) *Positive Introspection:* $(\beta_i(e) \to \beta_i\beta_i(e))$*.*

   (f) *Negative Introspection:* $((\neg\beta_i)(e) \to \beta_i(\neg\beta_i)(e))$*.*

First, Condition (1a) states that each state of the world ω describes a corresponding nature state s in a well-defined manner. Also, the state ω contains those nature events $E \in \mathcal{A}_\kappa(\mathcal{S})$ that are true at s (i.e., $s \in E$).[23] Conditions (1b) through (1e) are logical requirements (including coherency and completeness) on each state of the world (recall Proposition 3 for each $\omega^* \in \Omega^*$).

Second, Condition (2a) requires that if two expressions e and f are equivalent in the sense that every ω contains $(e \leftrightarrow f)$ then expressions $\beta_i(e)$ and $\beta_i(f)$ are equivalent in the same sense. This condition allows one to define the players' belief operators in a way such that if two expressions e and f correspond to the same event then the events associated with the beliefs in e and f are the same.

Third, each condition in (2b), (2c), and (3) describes how states Ω describe the properties of beliefs. Corollary 1 has provided related characterizations for Consistency and Negative Introspection for Ω^*. Assumptions on beliefs are encoded within Ω itself in that the resulting belief operators satisfy given assumptions.

Now, I restate the terminal space Ω^* as the largest set of coherent descriptions.

[23]By Condition (1d), Condition (1a) implies that, for a unique $s \in S$ with $s \in_0 \omega$ and for any $E \in \mathcal{A}_\kappa(\mathcal{S})$, $E \in_1 \omega$ iff $s \in E$. Especially, $S \in_1 \omega$ and $\emptyset \notin_1 \omega$.

Theorem 2 (Ω^* as Largest Coherent Set of Descriptions)**.** *The set Ω^* constructed in Section 3 is the largest coherent set of descriptions: for any set Ω of coherent descriptions, there is a (non-redundant and minimal) belief space $\overrightarrow{\Omega}$ such that its description map $D_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map and thus $\Omega \subseteq \Omega^*$.*

5 Probabilistic Beliefs and Other Applications

The framework of this paper (especially, the existence and characterizations of a terminal belief space) applies to richer forms of beliefs such as probabilistic beliefs. Section 5.1 constructs a terminal space for countably-additive, finitely-additive, and non-additive (not-necessarily-additive) beliefs. Conceptually, I establish the existence of a terminal belief space under the same condition (i.e., the domain specification) irrespective of whether beliefs are probabilistic or qualitative (or knowledge). Technically, the framework nests, for example, Heifetz and Samet (1998b), Meier (2006), and Pintér (2012), and establishes the existence of a terminal non-additive belief space irrespective of any continuity property on beliefs.[24] Moreover, in the terminal countably-additive belief space, a chain of finite belief hierarchies uniquely extends to the limit belief hierarchy.

Section 5.2 discusses a terminal space for conditional probability systems (CPSs) (Battigalli and Siniscalchi, 1999; Guarino, 2017) using conditional p-belief operators (Di Tillio, Halpern, and Samet, 2014). In Section 5.3, players' knowledge and qualitative beliefs are indexed by time as in Battigalli and Bonanno (1997). Note that one can also combine knowledge and probabilistic beliefs as in Meier (2008). Section 5.4 incorporates common belief (e.g., Aumann (1976), Friedell (1969), and Monderer and Samet (1989)). Finally, Section 5.5 briefly discusses further possible applications, namely, terminal knowledge-unawareness and preference spaces.

[24] Heifetz and Samet (1998b) employ a product of players' type spaces while this paper does a single non-product state space Ω (if each player is always "certain" of her beliefs, then the non-product terminal belief space is isomorphic to the product of nature states and players' type spaces; see, for example, Mertens and Zamir (1985)). In the literature, the latter non-product structure is referred to as a "belief space" (Mertens and Zamir, 1985). These remarks also apply to a terminal space for conditional probability systems (CPSs) in Section 5.2.

5.1 Terminal Probabilistic Belief Space

I formulate a probabilistic belief space in terms of p-belief operators using the equivalence between a type mapping and p-belief operators established by Samet (2000). To accommodate countable probabilistic belief hierarchies, let $\kappa = \aleph_1$. Denote by $\Delta(\Omega)$ the set of countably-additive probability measures on an $\aleph_1$-algebra $(\Omega, \mathcal{D})$. Let Σ_Δ be the $\aleph_1$-algebra on $\Delta(\Omega)$ generated by $\{\{\mu \in \Delta(\Omega) \mid \mu(E) \geq p\} \in \mathcal{P}(\Delta(\Omega)) \mid (E, p) \in \mathcal{D} \times [0, 1]\}$ (Heifetz and Samet, 1998b).

Definition 10 (Probabilistic Belief Space)**.** *A probabilistic belief space of I on $(S, \mathcal{S})$ is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (B_i^p)_{(i,p) \in I \times [0,1]}, \Theta \rangle$ with the following properties.*

1. *$(\Omega, \mathcal{D})$ is an $\aleph_1$-algebra and the map $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_{\aleph_1}(\mathcal{S}))$ is measurable.*
2. *Player i's p-belief operators $B_i^p : \mathcal{D} \to \mathcal{D}$ satisfy the properties below. For each $E \in \mathcal{D}$, $B_i^p(E)$ is the event that player i believes E with probability at least p (i.e., she p-believes E).*
   - *(a) $B_i^0(\cdot) = \Omega$.*
   - *(b) If $p_n \uparrow p$ then $B_i^{p_n}(\cdot) \downarrow B_i^p(\cdot)$.*
   - *(c) Monotonicity: If $E \subseteq F$ then $B_i^p(E) \subseteq B_i^p(F)$.*
   - *(d) Normalization: $B_i^1(\Omega) = \Omega$.*
   - *(e) Super-additivity: $B_i^p(E \cap F) \cap B_i^q(E \cap (\neg F)) \subseteq B_i^{p+q}(E)$ for $p + q \leq 1$.*
   - *(f) Sub-additivity: $(\neg B_i^p)(E) \cap (\neg B_i^q)(F) \subseteq (\neg B_i^{p+q})(E \cup F)$ for $p + q \leq 1$.*
   - *(g) Continuity-from-above: If $E_n \downarrow E \in \mathcal{D}$ then $B_i^p(E_n) \downarrow B_i^p(E)$.*
   - *(h) Continuity-from-below: If $E_n \uparrow E \in \mathcal{D}$ then $B_i^p(E) = \bigcap_{r \in \mathbb{N}: p - \frac{1}{r} \geq 0} \bigcup_{n \in \mathbb{N}} B_i^{p - \frac{1}{r}}(E_n)$.*
   - *(i) Certainty-of-Beliefs: For any $(\omega, E) \in \Omega \times \mathcal{D}$, $[t_{B_i}(\omega)] \subseteq E$ implies $\omega \in B_i^1(E)$, where*

$$[t_{B_i}(\omega)] := \left(\bigcap_{(p,E) \in [0,1] \times \mathcal{D} : \omega \in B_i^p(E)} B_i^p(E) \right) \cap \left(\bigcap_{(p,E) \in [0,1] \times \mathcal{D} : \omega \in (\neg B_i^p)(E)} (\neg B_i^p)(E) \right).$$

In a probabilistic belief space, player i's p-belief operators induce her *type mapping* $t_{B_i} : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \Sigma_\Delta)$, a measurable map defined by $t_{B_i}(\omega)(E) := \sup\{p \in [0, 1] \mid \omega \in B_i^p(E)\}$ for each $(\omega, E) \in \Omega \times \mathcal{D}$. At each state ω, a countably-additive probability measure $t_{B_i}(\omega)$ dictates i's beliefs at that state. Since t_{B_i} is measurable, $B_i^p(E) = \{\omega \in \Omega \mid t_{B_i}(\omega)(E) \geq p\} \in \mathcal{D}$ for each $E \in \mathcal{D}$.

Conditions (2), slightly different from Samet (2000), axiomatize type mappings. By (2a) and (2b), the map t_{B_i} is well defined. By (2c), each $t_{B_i}(\omega)$ is monotonic (i.e., $E \subseteq F$ implies $t_{B_i}(\omega)(E) \leq t_{B_i}(\omega)(F)$). Condition (2d) is a normalization: $t_{B_i}(\cdot)(\Omega) = 1$. Thus, (2a)-(2d) yield non-additive beliefs (or capacities).

By (2e), each $t_{B_i}(\omega)$ is super-additive: $t_{B_i}(\omega)(E \cap F) + t_{B_i}(\omega)(E \cap (\neg F)) \leq t_{B_i}(\omega)(E)$. Note that (2a) and (2e) imply (2c). By (2f), each $t_{B_i}(\omega)$ is sub-additive: $t_{B_i}(\omega)(E \cup F) \leq t_{B_i}(\omega)(E) + t_{B_i}(\omega)(F)$. Thus, (2a)-(2f) yield finitely-additive beliefs. By (2g) or (2h), a finitely-additive probability measure $t_{B_i}(\omega)$ becomes countably additive. I have presented both (2g) and (2h) to accommodate non-additive beliefs.

Condition (2i) requires player i to be certain of her beliefs. The set $[t_{B_i}(\omega)] = \{\omega' \in \Omega \mid t_{B_i}(\omega') = t_{B_i}(\omega)\}$ consists of states ω' that player i cannot distinguish from ω based on her probabilistic beliefs. Thus, player i is certain of her beliefs in that she believes E with probability one if E is implied by $[t_{B_i}(\omega)]$.

A *(probabilistic belief) morphism* from $\overrightarrow{\Omega}$ to $\overrightarrow{\Omega'}$ is a measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ satisfying: (i) $\Theta' \circ \varphi = \Theta$; and (ii) $B_i^p(\varphi^{-1}(\cdot)) = \varphi^{-1}(B_i'^p(\cdot))$ for all $(i, p) \in I \times [0, 1]$. A probabilistic belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S})$ is *terminal* if, for any probabilistic belief space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{S})$, there is a unique morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. By Theorem 1, a terminal probabilistic belief space exists. One can also extend it to various notions of beliefs by dropping corresponding conditions in Definition 10 (2).

Corollary 2 (Terminal Probabilistic Belief Space)**.** *There exists a terminal probabilistic belief space* $\overrightarrow{\Omega^*}$ *of* I *on* $(S, \mathcal{S})$.

In the construction of the terminal belief space for a generic class of belief spaces in Section 3, recall that the set $\mathcal{L} = \mathcal{L}_\kappa^I(\mathcal{A}_\kappa(\mathcal{S}))$ of syntactic formulas describing nature and interactive beliefs is infinitary when $\kappa > \aleph_0$. Here, I show that, as in the previous literature, one can construct the terminal probabilistic belief space by taking care only of finite belief hierarchies. By the continuity (countable-additivity) of beliefs, finite belief hierarchies uniquely extend to countable ones.

Letting $(\kappa, \lambda) = (\aleph_1, \aleph_0)$ and recalling Definition 5 and Remark 1, consider λ-expressions $\mathcal{L}_\lambda^I(\mathcal{A}_\kappa(\mathcal{S}))$: syntactic formulas that express nature and finite belief hierarchies. While the players reason about countable belief hierarchies, the language available to the players is finite. I show:

Proposition 4 (Extesion of Finitary Language)**.** $\mathcal{D}^* = \sigma(\{[e] \in \mathcal{D}^* \mid e \in \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))\})$.

The proposition implies that finite belief hierarchies suffice. Once the outside analysts specify finite belief hierarchies by the language $\mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$, finite belief hierarchies uniquely extend to countable ones. Applying the construction of a generic terminal space in Section 3 to the current context, Lemma 1 implies that $(\Omega^*, \{[e] \in \mathcal{D}^* \mid e \in \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))\})$ is a λ-algebra. One can define each player's p-belief operator B^{*p}_i from $\{[e] \in \mathcal{D}^* \mid e \in \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))\}$ into itself. By countable additivity, each B^{*p}_i admits a unique extension to $\mathcal{D}^* = \sigma(\{[e] \in \mathcal{D}^* \mid e \in \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))\})$.

Next, in applications, the players' type mappings in a probabilistic belief space often admit a common prior. A *probabilistic belief space of I on $(S, \mathcal{S})$ with a common prior* is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}, \mu), (B^p_i)_{(i,p) \in I \times [0,1]}, \Theta \rangle$ such that: (i) $\langle (\Omega, \mathcal{D}), (B^p_i)_{(i,p) \in I \times [0,1]}, \Theta \rangle$ is a probabilistic belief space in the sense of Definition 10; and that (ii) μ is a (countably-additive) probability measure, called a *common prior*, satisfying

$$\mu(E) = \int_\Omega t_{B_i}(\omega)(E)\mu(d\omega) \text{ for each } (i, E) \in I \times \mathcal{D}. \tag{3}$$

Equation (3) says the "prior" probability of E is equal to the expectation of every player's "posterior" probabilities $t_{B_i}(\omega)(E)$ with respect to μ (e.g., Mertens and Zamir (1985)).

For probabilistic belief spaces with a common prior $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$, $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a *(probabilistic belief) morphism* if φ is a morphism between probabilistic belief spaces (i.e., $\Theta' \circ \varphi = \Theta$ and $B^p_i \varphi^{-1} = \varphi^{-1} B'^p_i$) and $\mu' = \mu \circ \varphi^{-1}$.[25] The last condition states the prior probabilities are preserved. A probabilistic belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S})$ with a common prior is *terminal* if, for any probabilistic belief space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{S})$ with a common prior, there is a unique morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. By Theorem 1:

Corollary 3 (Terminal Probabilistic Belief Space with a Common Prior)**.** *There exists a terminal probabilistic belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S})$ with a common prior.*

Finally, I discuss correlated equilibria (e.g., Aumann (1974)). Let $\langle (A_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be a strategic game with the following properties. The set A_i of i's actions is endowed with the smallest $\aleph_1$-algebra $\mathcal{S}_i$ generated by singleton actions: $\mathcal{S}_i := \mathcal{A}_{\aleph_1}(\{\{a_i\} \in$

[25] Let $\overrightarrow{\Omega}$ be a probabilistic belief space with a common prior, and let $\varphi : \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta \rangle \to \langle (\Omega', \mathcal{D}'), (B'_i)_{i \in I}, \Theta' \rangle$ be a morphism in the sense of Section 5.1. Letting $\overrightarrow{\Omega'}$ with $\mu' := \mu \circ \varphi^{-1}$, $\overrightarrow{\Omega'}$ is a probabilistic belief space with a common prior and $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism.

$\mathcal{P}(A_i) \mid a_i \in A_i\})$. Let the set $A := \prod_{i\in I} A_i$ of action profiles be endowed with the product $\aleph_1$-algebra $\mathcal{S} := \mathcal{A}_{\aleph_1}(\{\{a_i\} \times A_{-i} \in \mathcal{P}(A) \mid i \in I \text{ and } a_i \in A_i\})$. Let $u_i : A \to \mathbb{R}$ be i's bounded measurable utility function. Denote by $\pi_i : A \to A_i$ the projection. Any measurable function Θ from an $\aleph_1$-algebra $(\Omega, \mathcal{D})$ into $(A, \mathcal{S})$ can be decomposed into $\Theta = (\Theta_i)_{i\in I}$ such that each $\Theta_i = \pi_i \circ \Theta$ is measurable.

A *correlated equilibrium* is an $\aleph_1$-belief space $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}, \mu), (B_i^p)_{(i,p)\in I\times[0,1]}, \Theta\rangle$ with a common prior satisfying the following two properties. First, $\Theta_i^{-1}(\{a_i\}) \subseteq B_i^1(\Theta_i^{-1}(\{a_i\}))$ for each $a_i \in A_i$. Thus, whenever player i takes action a_i, she believes with probability one that she takes a_i. The second is the optimality condition. For any player i and for any measurable function $\tau_i : (\Omega, \mathcal{D}) \to (A_i, \mathcal{S}_i)$,

$$\int_\Omega u_i(\Theta_i(\omega), \Theta_{-i}(\omega))\mu(d\omega) \geq \int_\Omega u_i(\tau_i(\omega), \Theta_{-i}(\omega))\mu(d\omega).$$

Note that this definition of correlated equilibrium hinges not on knowledge (induced by a partition) but on probability-one beliefs.

Now, I construct a terminal correlated equilibrium—a terminal $\aleph_1$-belief space with a common prior satisfying the two requirements to be a correlated equilibrium—$\overrightarrow{\Omega^*} = \langle(\Omega^*, \mathcal{D}^*, \mu^*), (B_i^{*p})_{(i,p)\in I\times[0,1]}, \Theta^*\rangle$ such that, for any correlated equilibrium $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}, \mu), (B_i^p)_{(i,p)\in I\times[0,1]}, \Theta\rangle$, there exists a unique morphism $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. Indeed, if there exists some $\aleph_1$-belief space with a common prior satisfying the requirements of correlated equilibria within the given category, then the candidate terminal $\aleph_1$-belief space with a common prior $\overrightarrow{\Omega^*}$ (constructed as in Section 3) satisfies the two conditions to be a correlated equilibrium. Thus, $\overrightarrow{\Omega^*}$ is terminal.

5.2 Terminal Conditional Belief Space

I construct a terminal space for conditional belief systems (CPSs) (Battigalli and Siniscalchi, 1999; Guarino, 2017).[26] Call a triple $(\Omega, \mathcal{D}, \mathcal{C})$ a *conditional space* if: (i)

[26] Two remarks are in order. First, as discussed in Footnote 24, a state space here is not restricted to a product space. The framework here does not presuppose any topological restriction on nature states or any cardinal restriction on conditioning events, either. Thus, the construction of the terminal conditional belief space would be complementary to Battigalli and Siniscalchi (1999) and Guarino (2017). Second, it would be interesting to examine "lexicographic probability systems (LPSs)" or "hypothetical knowledge" within this framework. Tsakas (2014) defines a formal equivalence between conditional and lexicographic belief hierarchies in respective type spaces (under some topological assumptions on nature states and beliefs), and establishes the existence of a terminal lexicographic belief space from a terminal conditional belief space. For a connection of conditional beliefs and

$(\Omega, \mathcal{D})$ is an $\aleph_1$-algebra; (ii) $\mathcal{C}$ is a non-empty sub-collection of $\mathcal{D}$ with $\emptyset \notin \mathcal{C}$; and (iii) there exists a *conditional probability system* (*CPS*) μ on $(\Omega, \mathcal{D}, \mathcal{C})$. A function $\mu(\cdot|\cdot) : \mathcal{D} \times \mathcal{C} \to [0,1]$ is a CPS if: (i) each $\mu(\cdot|C)$ is a countably-additive probability measure; (ii) Normality: $\mu(C|C) = 1$ for each $C \in \mathcal{C}$; and (iii) Chain Rule: $\mu(E|C) = \mu(E|D)\mu(D|C)$ for any $E \in \mathcal{D}$ and $C, D \in \mathcal{C}$ with $E \subseteq D \subseteq C$. Call each $C \in \mathcal{C}$ a *conditioning event* (or a *condition*, for short). Fix a conditional space $(S, \mathcal{A}_{\aleph_1}(\mathcal{S}), \mathcal{C}_S)$, where $(S, \mathcal{S})$ is the set of nature states.

Denote by $\Delta^{\mathcal{C}}(\Omega)$ the set of CPSs on $(\Omega, \mathcal{D}, \mathcal{C})$ endowed with the $\aleph_1$-algebra

$$\Sigma^{\mathcal{C}}_{\Delta} := \mathcal{A}_{\aleph_1}(\{\{\mu \in \Delta^{\mathcal{C}}(\Omega) \mid \mu(E|C) \geq p\} \in \mathcal{P}(\Delta^{\mathcal{C}}(\Omega)) \mid (E, C, p) \in \mathcal{D} \times \mathcal{C} \times [0,1]\}).$$

A player i's *conditional type mapping* is a measurable map $t_i : (\Omega, \mathcal{D}) \to (\Delta^{\mathcal{C}}(\Omega), \Sigma^{\mathcal{C}}_{\Delta})$. I formulate a conditional belief space using conditional p-belief operators $B^p_i(\cdot|C)$ for each player i and each condition C, so that i's conditional p-belief operators induce her conditional type mapping.

Definition 11 (Conditional Belief Space)**.** *A conditional belief space of I on $(S, \mathcal{S}, \mathcal{C}_S)$ is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}, \mathcal{C}), (B^p_i(\cdot|C))_{(i,p,C) \in I \times [0,1] \times \mathcal{C}}, \Theta \rangle$ with the following properties.*

1. *$(\Omega, \mathcal{D}, \mathcal{C})$ is a conditional space and $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_{\aleph_1}(\mathcal{S}))$ is a measurable map with $\mathcal{C} = \Theta^{-1}(\mathcal{C}_S)$.*
2. *For each $i \in I$, player i's conditional p-belief operators $B^p_i(\cdot|C) : \mathcal{D} \to \mathcal{D}$ satisfy the following.*
   (a) *For each $C \in \mathcal{C}$, $(B^p_i(\cdot|C))_{p \in [0,1]}$ satisfies Definition 10 (2).*
   (b) *Normality: $B^1_i(C|C) = \Omega$ for all $C \in \mathcal{C}$.*
   (c) *Chain Rule: $B^p_i(E|D) \cap B^q_i(D|C) \subseteq B^{pq}_i(E|C)$ for any $E \in \mathcal{D}$ and $C, D \in \mathcal{C}$ with $E \subseteq D \subseteq C$.*

By the assumption $\mathcal{C} = \Theta^{-1}(\mathcal{C}_S)$ in (1), denote $B^p_{i,C_S}(\cdot) := B^p_i(\cdot|\Theta^{-1}(C_S))$ for each $(i, p, C_S) \in I \times [0,1] \times \mathcal{C}_S$. This means that conditions in each conditional belief space are exogenously given as in Battigalli and Siniscalchi (1999) and Guarino (2017). Thus, one's conditional belief may fail to be a condition (i.e., $B^p_i(E|C) \notin \mathcal{C}$).

Conditions (2) characterize each player i's conditional type mapping as in Di Tillio, Halpern, and Samet (2014, Theorem 1) (see also Guarino (2017) for an axiomatization

hypothetical knowledge, see, for example, Di Tillio, Halpern, and Samet (2014).

of CPSs). First, by (2a), a measurable map $t_{B_i} : (\Omega, \mathcal{D}) \to (\Delta^{\mathcal{C}}(\Omega), \Sigma_{\Delta}^{\mathcal{C}})$ is well defined for each condition as in Section 5.1. By (2b), each $t_{B_i}(\omega)(\cdot|\cdot)$ satisfies Normality. Under (2a) and (2b), it can be seen that (2c) characterizes Chain Rule.

A *(conditional belief) morphism* from $\overrightarrow{\Omega}$ to $\overrightarrow{\Omega'}$ is a measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ satisfying: (i) $\Theta' \circ \varphi = \Theta$; and (ii) $B_{i,C_S}^{p}(\varphi^{-1}(\cdot)) = \varphi^{-1}(B_{i,C_S}^{\prime p}(\cdot))$ for all $(i, p, C_S) \in I \times [0, 1] \times \mathcal{C}_S$. A conditional belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S}, \mathcal{C}_S)$ is *terminal* if, for any conditional belief space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{S}, \mathcal{C}_S)$, there is a unique morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. By Theorem 1:

Corollary 4 (Terminal Conditional Belief Space)**.** *There exists a terminal conditional belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S}, \mathcal{C}_S)$.*

5.3 Terminal Dynamic Knowledge-Belief Space

Epistemic analyses of dynamic games often call for players' knowledge and beliefs. As in Battigalli and Bonanno (1997), consider players' knowledge and beliefs indexed by time. While a knowledge operator $K_{i,t} : \mathcal{D} \to \mathcal{D}$ represents player i's knowledge at time $t \in \mathbb{N}$, a belief operator $B_{i,t} : \mathcal{D} \to \mathcal{D}$ does her qualitative beliefs at time t.

Definition 12 (Dynamic Knowledge-Belief Space)**.** *A dynamic κ-knowledge-belief space of I on $(S, \mathcal{S})$ is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (K_{i,t}, B_{i,t})_{(i,t) \in I \times \mathbb{N}}, \Theta \rangle$ with the following properties.*

1. *$(\Omega, \mathcal{D})$ is a κ-algebra and the map $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_{\kappa}(\mathcal{S}))$ is measurable.*

2. *Knowledge operators $K_{i,t} : \mathcal{D} \to \mathcal{D}$ satisfy Truth Axiom, (Positive Introspection,) Negative Introspection, and the Kripke property. Belief operators $B_{i,t} : \mathcal{D} \to \mathcal{D}$ satisfy Consistency, Positive Introspection, Negative Introspection, and the Kripke property.*

3. *Knowledge and belief operators jointly satisfy: (i) $K_{i,t}(\cdot) \subseteq B_{i,t}(\cdot)$; (ii) $B_{i,t}(\cdot) \subseteq K_{i,t}B_{i,t}(\cdot)$; and (iii) $B_{i,t}(\cdot) = B_{i,t}B_{i,t+1}(\cdot)$.*

In (2), for ease of illustration, I have assumed (i) both knowledge and qualitative belief are fully introspective, (ii) knowledge is truthful while qualitative belief is consistent, and (iii) both knowledge and qualitative belief are represented by possibility correspondences. The existence of a terminal space does not hinge on assumptions on knowledge and qualitative belief.

In (3), the first condition means that knowledge implies belief at each time. The second states that each player knows her own belief at each time. Note that $(\neg B_{i,t})(\cdot) \subseteq K_{i,t}(\neg B_{i,t})(\cdot)$ follows from Truth Axiom and Negative Introspection of knowledge. The third captures the idea of belief persistence by Battigalli and Bonanno (1997): player i believes E at time t iff she believes at t that she (will) believe E at $t+1$. Player i's knowledge satisfies *perfect recall* if $K_{i,t}(\cdot) \subseteq K_{i,t+1}(\cdot)$ for all $t \in \mathbb{N}$. A dynamic knowledge-belief space *with perfect recall* is a dynamic knowledge-belief space such that each player's knowledge satisfies perfect recall.

A dynamic knowledge-belief space is mathematically a belief space of $I \times \mathbb{N} \times \{0, 1\}$, where "player $(i, t, 0)$'s belief operator" is $K_{i,t}$ while "player $(i, t, 1)$'s operator" is $B_{i,t}$, with the specified conditions. Thus:

Corollary 5 (Terminal Dynamic Knowledge-Belief Space)**.** *There exists a terminal dynamic κ-knowledge-belief space (with/without perfect recall) $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S})$.*

5.4 Common Belief Operator in a Terminal Belief Space

I incorporate the notion of common belief, irrespective of a choice of κ and assumptions on beliefs, following Fukuda (2020). The definition of common belief does not resort to the chain of mutual beliefs. Thus, one can analyze players who fail logical reasoning (e.g., Monotonicity or Non-empty λ-Conjunction) or players who reason only about finite levels of interactive beliefs.

Fix a non-empty set I of players, and let κ be an infinite regular cardinal with $\kappa > |I|$. By this assumption, in any κ-belief space $\langle(\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta\rangle$, define the *mutual belief operator* $B_I : \mathcal{D} \to \mathcal{D}$ by $B_I(\cdot) = \bigcap_{i \in I} B_i(\cdot)$.

Call an event $E \in \mathcal{D}$ a *common basis* if everybody believes any logical implication of E whenever E is true: $E \subseteq F$ implies $E \subseteq B_I(F)$ (Fukuda, 2020). If the mutual belief operator B_I satisfies Monotonicity, then E is a common basis if(f) it is publicly-evident: $E \subseteq B_I(E)$. Denote by $\mathcal{J}_I$ the collection of common bases. Now, an event $E \in \mathcal{D}$ is *common belief* at a state $\omega \in \Omega$ if there is a common basis $F \in \mathcal{J}_I$ which is true at ω and which implies the mutual belief in E: $\omega \in F \subseteq B_I(E)$. If B_I satisfies Monotonicity, then this definition of common belief reduces to Monderer and Samet (1989). Observe the common belief in E at a state ω implies the chain of mutual beliefs in E at that state: at ω, everybody believes E (i.e., $\omega \in B_I(E)$), everybody believes that everybody believes E (i.e., $\omega \in B_I B_I(E)$), and so on *ad infinitum*. The converse holds (i.e., common belief reduces to the chain of mutual beliefs) when B_I satisfies Monotonicity and Non-empty $\aleph_1$-Conjunction (Fukuda, 2020).

A *κ-belief space with a common belief operator* is a tuple $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, C, \Theta\rangle$ such (i) that $\langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, \Theta\rangle$ is a κ-belief space and (ii) that $C : \mathcal{D} \to \mathcal{D}$ satisfies $C(E) = \max\{F \in \mathcal{J}_I \mid F \subseteq B_I(E)\}$ for each $E \in \mathcal{D}$, where "max" is taken with respect to the set inclusion. I show a terminal space exists within the class of κ-belief spaces with a common belief operator irrespective of κ and assumptions on beliefs.

Corollary 6 (Terminal Belief Space with a Common Belief Operator)**.** *There exists a terminal κ-belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S})$ with a common belief operator.*

Generally, any property of beliefs satisfied in a belief space $\overrightarrow{\Omega}$ holds in $\overrightarrow{D(\Omega)} = \langle(D(\Omega), \mathcal{D}^* \cap D(\Omega)), (B'_i)_{i\in I}, \Theta^*|_{D(\Omega)}\rangle$ (defined in Section 3). If $\overrightarrow{\Omega}$ admits a common belief operator, then so does $\overrightarrow{D(\Omega)}$. If Necessitation holds for every player, $\overrightarrow{D(\Omega)}$ is *belief-closed* (Mertens and Zamir, 1985): $B'_i(D(\Omega)) = D(\Omega)$ for all $i \in I$. Consequently, $D(\Omega)$ is commonly believed in $\overrightarrow{D(\Omega)}$.

Finally, I show: within a given class of belief spaces with a common belief operator, (i) the players' rationality is well defined in the terminal space, and (ii) common belief in rationality in any given belief space is preserved in the terminal space. Fix a strategic game $\langle(A_i)_{i\in I}, (\succsim_i)_{i\in I}\rangle$: each A_i is a player i's (non-empty) action set, and each $\succsim_i$ is player i's (complete and transitive) preference relation on $A := \prod_{i\in I} A_i$. Denote by $\sim_i$ and $\succ_i$ the indifference and strict preference relations, respectively. Since the players reason about their actions, let $S := A$. For simplicity, let κ be an infinite regular cardinal with $\max(|I|, |A|) < \kappa$. Each player is able to reason about her own actions, so assume $\{a_i\} \times A_{-i} \in \mathcal{S}$ for each $i \in I$ and $a_i \in A_i$. Since $\kappa > |A|$, the assumption amounts to $\mathcal{A}_\kappa(\mathcal{S}) = \mathcal{P}(S)$.

In a κ-belief space with a common belief operator $\langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, C, \Theta\rangle$, the measurable map Θ is decomposed into $\Theta = (\Theta_i)_{i\in I}$, where $\Theta_i = \pi_i \circ \Theta$ and $\pi_i : A \to A_i$ is the projection. For each $a_i \in A_i$, $\Theta_i^{-1}(\{a_i\}) = [\![\{a_i\} \times A_{-i}]\!]_{\overrightarrow{\Omega}} \in \mathcal{D}$ is the event that player i plays a_i. For each $(a'_i, a_i) \in A_i^2$, define $[a'_i \succ_i a_i] := \{a_{-i} \in A_{-i} \mid (a'_i, a_{-i}) \succ_i (a_i, a_{-i})\}$ and $[\![a'_i \succ_i a_i]\!]_{\overrightarrow{\Omega}} := (\Theta_{-i})^{-1}([a'_i \succ_i a_i]) \in \mathcal{D}$. The set $[\![a'_i \succ_i a_i]\!]_{\overrightarrow{\Omega}}$ is the event that player i strictly prefers a'_i to a_i given the other players' strategies. Define the set RAT_i (or $\mathrm{RAT}_i^{\overrightarrow{\Omega}}$) of states at which player i is rational: $\mathrm{RAT}_i := \{\omega \in \Omega \mid \omega \in B_i[\![a'_i \succ_i \Theta_i(\omega)]\!]_{\overrightarrow{\Omega}} \text{ for no } a'_i \in A_i\}$ (e.g., Bonanno (2008)). Player i is rational at ω if, for no action a'_i, she believes playing a'_i is strictly better than $\Theta_i(\omega)$ given the opponents' strategies. It can be seen:

$$\mathrm{RAT}_i = \bigcap_{a_i \in A_i} \left((\Theta_i^{-1}(\{a_i\}))^c \cup \bigcap_{a_i' \in A_i} (\neg B_i)([\![a_i' \succ_i a_i]\!]_{\overrightarrow{\Omega}}) \right) \in \mathcal{D}.$$

Let $\mathrm{RAT}_I := \bigcap_{i \in I} \mathrm{RAT}_i \in \mathcal{D}$. Hence, in the terminal space $\overrightarrow{\Omega^*}$ (indeed, in any belief space), the players' rationality is well defined.

If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism, then $\varphi^{-1}([\![a_i' \succ_i a_i]\!]_{\overrightarrow{\Omega'}}) = [\![a_i' \succ_i a_i]\!]_{\overrightarrow{\Omega}}$ and $\varphi^{-1}(\mathrm{RAT}_i^{\overrightarrow{\Omega'}}) = \mathrm{RAT}_i^{\overrightarrow{\Omega}}$. Especially, $D^{-1}(C^*(\mathrm{RAT}_I^{\overrightarrow{\Omega^*}})) = C(\mathrm{RAT}_I^{\overrightarrow{\Omega}})$: if the players' rationality is common belief at ω in a given belief space $\overrightarrow{\Omega}$, then so is it at $\omega^* = D(\omega)$ in the terminal space.

5.5 Futher Possible Extensions

Terminal Knowledge-Unawareness Space. A *knowledge-unawareness space* is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (K_i, U_i)_{i \in I}, \Theta \rangle$ where $K_i : \mathcal{D} \to \mathcal{D}$ is player i's knowledge operator and $U_i : \mathcal{D} \to \mathcal{D}$ is i's unawareness operator. By Theorem 1, a terminal knowledge-unawareness space exists under various assumptions on knowledge and unawareness.

Call the terminal knowledge-unawareness space $\overrightarrow{\Omega^*} := \langle (\Omega^*, \mathcal{D}^*), (K_i^*, U_i^*)_{i \in I}, \Theta^* \rangle$ *non-trivial* if $U_i^*([e]) \neq \emptyset$ for some $(i, e) \in I \times \mathcal{L}$. This is equivalent to stating that there is a knowledge-unawareness space $\overrightarrow{\Omega}$ (within the category of knowledge-unawareness spaces at hand) such that $U_i([\![e]\!]_{\overrightarrow{\Omega}}) \neq \emptyset$ for some $(i, e) \in I \times \mathcal{L}$.

It would be an interesting research avenue for generalizing the framework of this paper to a generalized state space consisting of multiple state spaces as in Heifetz, Meier, and Schipper (2006, 2008) due to some limitation of standard state space models to describe richer notions of unawareness (e.g., Chen, Ely, and Luo (2012), Dekel, Lipman, and Rustichini (1998), Fukuda (2018), and Modica and Rustichini (1994)). While the framework of this paper requires the domain (the collection of events) $\mathcal{D}$ to be a κ-algebra of sets, the domain in the generalized state space has a more general lattice structure. I conjecture that the idea of this paper can be applied when players' knowledge and unawareness operators are defined on a κ-complete lattice.

Terminal Preference Space. Consider a preference-based type space where players reason about their interactive preferences instead of beliefs (e.g., Di Tillio (2008), Epstein and Wang (1996), and Ganguli, Hiefetz, and Lee (2016)). Each player i's preference-type mapping associates, with each state of the world $\omega \in \Omega$, her preference relation over the set of acts (i.e., bounded measurable functions) on Ω, where Ω is endowed with a κ-algebra. Call a preference-type space a κ-preference-type space if it is defined on a κ-algebra.

The natural conjecture of this paper is that a terminal κ-preference-based type space would exist, irrespective of such nature of preferences as "continuity," if one considers hierarchies of interactive preferences up to the ordinality of κ. For example, Di Tillio (2008) constructs a terminal $\aleph_0$-preference-type space consisting of finite preference hierarchies in the category of $\aleph_0$-preference-type spaces.[27] Epstein and Wang (1996) and Ganguli, Hiefetz, and Lee (2016) construct a terminal $\aleph_1$-preference-type space consisting of finite hierarchies of preferences in the category of $\aleph_1$-preference-type spaces by employing some regularity properties of preferences or preference representations (so that finite preference hierarchies extend to countable ones).[28]

6 Comparison with the Previous Negative Results

So far, I have established the existence of a terminal belief space irrespective of whether beliefs are qualitative or probabilistic. This section compares the existence of a terminal knowledge space with the previous non-existence results (e.g., Fagin et al. (1999), Heifetz and Samet (1998a), and Meier (2005)). The framework of this paper admits the category of κ-knowledge spaces $\overrightarrow{\Omega}$ of I on $(S, \mathcal{S})$ (in which $(\Omega, \mathcal{D})$ is a κ-algebra and) in which each B_i is induced by a partitional possibility correspondence $b_{B_i} : \Omega \to \mathcal{P}(\Omega)$ (satisfying all the properties of Definition 2).[29] Theorem 1 implies

[27] In the context of κ-knowledge spaces, an earlier version of this paper (Fukuda, 2017, Sections 5 and 6) provides a type-space reformulation of a knowledge space (where each type takes either 0 or 1 instead of probability $p \in [0, 1]$), and constructs a terminal space consisting of hierarchies of "knowledge types" up to $\overline{\kappa}$.

[28] In Epstein and Wang (1996), preferences satisfy some continuity properties (their P3 and P4). In Ganguli, Hiefetz, and Lee (2016), preferences are represented by a countable collection of continuous real-valued functionals over acts.

[29] Recall discussions following Definition 2. Especially, Truth Axiom, Negative Introspection, and the Kripke property suffice. Also, in the category of κ-belief spaces $\overrightarrow{\Omega}$ of I on $(S, \mathcal{S})$ in which each B_i is induced by a serial, transitive, and Euclidean possibility correspondence (i.e., B_i satisfies Consistency, Positive Introspection, and Negative Introspection in addition to the Kripke property), a terminal κ-belief space exists.

that there is a terminal κ-knowledge space $\overrightarrow{\Omega^*}$ of this category (i.e., $(\Omega^*, \mathcal{D}^*)$ is a κ-algebra and each B_i^* is induced by a partitional possibility correspondence $b^*_{B_i^*} : \Omega^* \to \mathcal{P}(\Omega^*)$) such that, for any given κ-knowledge space $\overrightarrow{\Omega}$ in this category, there is a unique morphism $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. For the rest of this section, I assert and examine the claim that, for a fixed infinite regular cardinal κ, the terminal space $\overrightarrow{\Omega^*}$ (of a given category of κ-belief spaces) describes all possible belief hierarchies of depth up to $\overline{\kappa}$ but only belief hierarchies of depth up to $\overline{\kappa}$. Throughout this section, unless otherwise stated, fix an infinite regular cardinal κ.

6.1 Belief Hierarchies of Depth $\overline{\kappa}$

The terminal κ-belief space $\overrightarrow{\Omega^*}$ consists of belief hierarchies up to $\overline{\kappa}$ that are generated by some state of some κ-belief space. Hence, as long as belief hierarchies up to $\overline{\kappa}$ are concerned, the terminal κ-belief space $\overrightarrow{\Omega^*}$ contains all possible belief hierarchies of depth $\overline{\kappa}$ that can obtain within the category of κ-belief spaces.

I define the κ-rank of a κ-belief space, extending the rank of a standard partitional knowledge space (where the domain is the power set of underlying states) studied by Heifetz and Samet (1998a).[30] The κ-rank represents the maximal ordinality of the non-trivial belief hierarchies up to $\overline{\kappa}$ that a given κ-belief space can generate.

On the one hand, Heifetz and Samet (1998a) demonstrate that there is no terminal standard partitional knowledge space on the following two grounds. First, a morphism preserves the ranks. Second, there is a standard partitional knowledge space with arbitrarily high rank. Hence, for any candidate terminal standard partitional knowledge space, there exists a standard partitional knowledge space with a higher rank, and thus the candidate space must not be terminal. On the other hand, I show: (i) a morphism between κ-belief spaces preserves the κ-ranks; and (ii) the κ-rank of any κ-belief space (indeed, any λ-belief space with $\lambda \geq \kappa$) is at most $\overline{\kappa}$. Thus, while one can construct a non-trivial belief hierarchy of an arbitrary length in some belief space, such belief space may not contain all possible ways in which the players reason about their interactive beliefs up to $\overline{\kappa}$. In contrast, for a fixed infinite regular cardinal κ, the terminal κ-belief space Ω^* contains all possible belief hierarchies of depth $\overline{\kappa}$ attained in some state of some κ-belief space.

[30] Fagin (1994) considers a closely related concept, "distinguishing ordinals," in his logical system.

Definition 13 (κ-Rank)**.** *The κ-rank of a κ-belief space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{S})$ is the least ordinal α with $\mathcal{C}_\alpha = \mathcal{C}_{\alpha+1}$, where the sequence $(\mathcal{C}_\alpha)_\alpha$ is defined as follows:*

$$\mathcal{C}_\alpha := \begin{cases} \mathcal{A}_\kappa(\{\Theta^{-1}(E) \in \mathcal{D} \mid E \in \mathcal{S}\}) = \Theta^{-1}(\mathcal{A}_\kappa(\mathcal{S})) & \text{if } \alpha = 0 \\ \mathcal{A}_\kappa\left(\left(\bigcup_{\beta<\alpha} \mathcal{C}_\beta\right) \cup \bigcup_{i \in I}\{B_i(E) \in \mathcal{D} \mid E \in \bigcup_{\beta<\alpha} \mathcal{C}_\beta\}\right) & \text{if } \alpha > 0 \end{cases}.$$

Proposition 5 (κ-Rank of a Terminal κ-Belief Space)**.**

1. *If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism between κ-belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$, then the κ-rank of $\overrightarrow{\Omega'}$ is at least as high as that of $\overrightarrow{\Omega}$.*

2. *Let λ be an infinite regular cardinal with $\lambda \geq \kappa$ or the symbol $\lambda = \infty$. Then, the κ-rank of any λ-belief space $\overrightarrow{\Omega}$ is at most $\overline{\kappa}$. Especially, the κ-rank of any κ-belief space $\overrightarrow{\Omega}$ is at most $\overline{\kappa}$.*

By Proposition 5 (1), any two isomorphic κ-belief spaces have the same κ-rank. Especially, the κ-rank of a terminal κ-belief space is unique. Part (2) hinges on the fact that the expressions $\mathcal{L} = \mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S}))$ involve the player's interactive beliefs up to $\overline{\kappa}$ (Remark 1). Specifically, define $\mathcal{D}_\alpha := \mathcal{A}_\kappa(\{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}_\alpha\})$ for each ordinal $\alpha \leq \overline{\kappa}$, where $(\mathcal{L}_\alpha)_{\alpha=0}^{\overline{\kappa}}$ is defined as in Remark 1 so that $\mathcal{L} = \mathcal{L}_{\overline{\kappa}}$. Note that $\mathcal{D}_{\overline{\kappa}} = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$. I show in the proof that $\mathcal{D}_\alpha = \mathcal{C}_\alpha$ for each ordinal $\alpha \leq \overline{\kappa}$. Then, $\mathcal{D}_{\overline{\kappa}} = \mathcal{C}_{\overline{\kappa}} = \mathcal{C}_{\overline{\kappa}+1}$, i.e., the κ-rank of $\overrightarrow{\Omega}$ is at most $\overline{\kappa}$. Also, since $\{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\} = \mathcal{D}_{\overline{\kappa}} = \mathcal{C}_{\overline{\kappa}}$, one can check whether a given κ-belief space $\overrightarrow{\Omega}$ is non-redundant through its primitives alone (i.e., $\mathcal{C}_{\overline{\kappa}}$).

Two remarks on Proposition 5 (2) are in order. First, using the terminologies of Definition 13, Heifetz and Samet (1998a) study ∞-ranks of (particular) ∞-belief spaces of I on $(S, \mathcal{P}(S))$ with $|S| \geq 2$ and $|I| \geq 2$.[31] Their construction implies that the κ-rank of the terminal partitional κ-knowledge space $\overrightarrow{\Omega^*}$ is generically $\overline{\kappa}$. The particular standard partitional (κ-)knowledge spaces constructed by Heifetz and Samet (1998a) also attain κ-rank $\overline{\kappa}$, but the κ-rank of any such space never exceeds $\overline{\kappa}$ (for the fixed infinite regular cardinal κ). Also, such a particular space does not contain all possible belief hierarchies of depth $\overline{\kappa}$. Thus, for the infinite regular cardinal κ, Heifetz and Samet (1998a)'s non-existence argument does not apply to a given class

[31] Definition 13 is also well defined for $\kappa = \infty$. Also, if $\mathcal{S}$ satisfies the separative property (see Footnote 13) then $\mathcal{A}_\infty(\mathcal{S}) = \mathcal{P}(S)$.

of κ-belief spaces.[32] Note that the same argument works for a category of ∞-belief spaces which includes, as a subclass, the category of partitional ∞-knowledge spaces.

Second, one needs to take care of all the $\overline{\kappa}$ levels of interactive beliefs in order to incorporate possible "discontinuity" of qualitative beliefs (knowledge).[33] The existence of a terminal qualitative belief (knowledge) space does not hinge on the continuity of beliefs (knowledge) itself by keeping track of all possible transfinite belief hierarchies (see also Zhou (2010) in the context of finitely-additive beliefs).

In contrast, one only needs to take care of all possible finite belief hierarchies for countably-additive probabilistic beliefs. As shown in Section 5, one can accommodate probabilistic beliefs by considering p-belief operators $(B_i^p)_{(i,p)\in I\times[0,1]}$. Countably-additive beliefs are continuous with respect to a monotone sequence of events (see Definition 10 (2g) and (2h)).[34] Proposition 4 shows that, within the class of probabilistic belief spaces, the terminal probabilistic belief space has the $\aleph_1$-rank $\aleph_0$.[35]

By fixing the language that the players are allowed to use in reasoning about their interactive beliefs (within the ordinality of κ), or by making the domain of a belief space explicit, a morphism (a description map) preserves interactive beliefs in a given κ-belief space to the terminal κ-belief space. At the same time, such preservation concerns only to the extent that belief hierarchies of depth $\overline{\kappa}$ are preserved.

6.2 Comparison of Terminal κ-Belief and λ-Belief Spaces

The previous subsection has examined the sense in which the terminal κ-belief space $\overrightarrow{\Omega^*}$ contains all possible belief hierarchies up to $\overline{\kappa}$. This subsection shows that the terminal κ-belief space does not accommodate any further belief hierarchy. If κ and λ are infinite regular cardinals with $\kappa \leq \lambda$ then the description map $D_{\overrightarrow{\Omega^*_\lambda}}$ from the

[32]On the contrary, if a terminal partitional ∞-knowledge space existed (put differently, if an infinite regular cardinal κ is not fixed), then, for any (infinite regular) cardinal κ, its ∞-rank is at least $\overline{\kappa}$, a contradiction because the ∞-rank of the terminal space is a fixed ordinal.

[33]Heifetz and Samet (1998a,b) attribute the non-existence of a terminal standard partitional knowledge space to the "lack of continuity" of knowledge with respect to an increasing sequence of events. Fagin, Halpern, and Vardi (1991), Fagin (1994), Fagin et al. (1999), and Heifetz and Samet (1999) attribute the non-existence of the space of all coherent hierarchies of knowledge to the lack of "continuity" property of knowledge structures, as opposed to that of σ-additive probability measures. See also Meier (2006, 2008) for the discussion of the use of infinitary expressions.

[34]Unlike qualitative or probability-one belief alone, the continuity of beliefs with respect to an increasing sequence of events (Definition 10 (2h)) requires degrees of p-beliefs.

[35]In the terminal probabilistic belief space, a chain of finite belief hierarchies uniquely extends to the limit countable belief hierarchy (the $\aleph_1$-rank is $\aleph_0 < \aleph_1$), but does not uniquely extend to an uncountable belief hierarchy ($\kappa = \aleph_1$).

terminal λ-space $\overrightarrow{\Omega^*_\lambda}$ into the terminal κ-space $\overrightarrow{\Omega^*_\kappa}$ is surjective (note that both spaces reside in the category of κ-belief spaces at hand). Thus, the cardinality of the terminal κ-space $\overrightarrow{\Omega^*_\kappa}$ is at least as small as that of $\overrightarrow{\Omega^*_\lambda}$.

Proposition 6 (Cardinality of a Terminal Space)**.** *Fix I, $(S, \mathcal{S})$, and some properties in Definition 2 for the players' beliefs. Take infinite regular cardinals κ and λ with $\kappa \leq \lambda$. Denote by Ω^*_κ and Ω^*_λ the terminal κ- and λ-belief spaces constructed in Section 3, respectively.*

1. *$D_{\overrightarrow{\Omega^*_\lambda}} : \overrightarrow{\Omega^*_\lambda} \to \overrightarrow{\Omega^*_\kappa}$ is a surjective morphism so that $|\Omega^*_\kappa| \leq |\Omega^*_\lambda|$.*
2. *Let $|I| \geq 2$ and $|S| \geq 2$, and suppose $\mathcal{S}$ contains E with $\emptyset \subsetneq E \subsetneq S$. Then, $\max(2^{\aleph_0}, \kappa) \leq |\Omega^*_\kappa|$.*

Proposition 6 sheds light on the difference between the existence and non-existence results: while a terminal κ-belief space exists for the fixed infinite regular cardinal κ (observe κ can be fixed arbitrarily), the cardinality of the ∞-belief space (i.e., $\lambda = \infty$) blows up as κ arbitrarily increases. Also, for κ and λ with $\kappa < \lambda$, the terminal λ-belief space $\overrightarrow{\Omega^*_\lambda}$ is at least as rich as $\overrightarrow{\Omega^*_\kappa}$ in cardinality: $|\Omega^*_\kappa| \leq |\Omega^*_\lambda|$. When it comes to belief hierarchies up to $\overline{\kappa}$, however, the space $\overrightarrow{\Omega^*_\lambda}$ would be redundant in that there would be two states which induce the same belief hierarchy up to $\overline{\kappa}$.

Proposition 6 sheds light on the non-existence results. Somewhat informally, define the class $\mathcal{L}_\infty$ of expressions as in Definition 5. The class $\mathcal{L}_\infty$ is too big to be set in the standard set theory (e.g., for any ordinal α, there is $e_\alpha \in \mathcal{L}_\infty$ such that if $\alpha > 0$ then $e_\alpha \neq e_\beta$ for all $\beta < \alpha$, a contradiction). Define the class Ω^*_∞ as in Equation (1), and let $\overrightarrow{\Omega^*_\infty}$ be the terminal space. Since $\overrightarrow{\Omega^*_\infty}$ would be non-redundant, $\mathcal{D}^*_\infty$ is a complete algebra that separates any two states, i.e., the power class of Ω^*_∞. Proposition 6 suggests that $\kappa \leq |\Omega^*_\infty|$ for any (infinite regular) cardinal, meaning that Ω^*_∞ is too big to be a set.

Or, consider the terminal κ-belief (knowledge) space $\overrightarrow{\Omega^*_\kappa}$ of the category of κ-belief (knowledge) spaces that satisfy all the properties in Definition 2. One can introduce the players' beliefs about any subset of $\overrightarrow{\Omega^*_\kappa}$ (see Remark A.2 in Appendix A for a formal statement). In the extended space $\langle(\Omega^*_\kappa, \mathcal{P}(\Omega^*_\kappa)), (\overline{B}^*_i)_{i \in I}, \Theta^*\rangle$, every state ω^*_κ induces a belief hierarchy of an arbitrary length uniquely extended from the original hierarchy (of depth $\overline{\kappa}$). If the terminal ∞-knowledge space $\overrightarrow{\Omega^*_\infty}$ existed and contained any such belief hierarchy, then there would be an injection from Ω^*_κ into Ω^*_∞, asserting again that $\kappa \leq |\Omega^*_\infty|$ for any κ.

6.3 Informational Content of the Domain

The terminal κ-belief space constructed in Section 3 has a distinct feature that the domain $\mathcal{D}^*$ is generically not the power set of the underlying states Ω^*.[36] Does this feature have some limitation on the representation of players' interactive beliefs?

On the one hand, the syntactic formulas $\mathcal{L} = \mathcal{L}_{\overline{\kappa}}$ express nature $(S, \mathcal{S})$ and interactive beliefs up to $\overline{\kappa}$. On the other hand, an arbitrary κ-belief space $\overrightarrow{\Omega}$ represents how each nature state s, each nature event E, and players' interactive beliefs about nature events are described. Formally, the semantic interpretation function $[\![\cdot]\!]_{\overrightarrow{\Omega}}$ associates, with each expression $e \in \mathcal{L}$ that describes nature states and interactive beliefs, the corresponding event $[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D}$ in the space $\overrightarrow{\Omega}$. Hence, as long as exogenously given nature states $(S, \mathcal{S})$ and interactive beliefs of depth $\overline{\kappa}$ are concerned, $\mathcal{D}_{\overline{\kappa}} = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$ contains sufficient information about how much $\overrightarrow{\Omega}$ can describe nature states and interactive beliefs determined by $\mathcal{L}$; and any event $E \in \mathcal{D} \setminus \mathcal{D}_{\overline{\kappa}}$, if there is any, cannot be captured by the given language $\mathcal{L}$.[37]

Since $\mathcal{D}_{\overline{\kappa}} = D_{\overrightarrow{\Omega}}^{-1}(\mathcal{D}^*)$ (recall the discussion in Section 3), the sufficient information that the given belief space $\overrightarrow{\Omega}$ can capture takes a form of a κ-algebra. Formally:

Remark 4. If $\overrightarrow{\Omega}$ is a κ-belief space in a given category, then $\overrightarrow{\Omega_\kappa} := \langle (\Omega, \mathcal{D}_{\overline{\kappa}}), (B_i|_{\mathcal{D}_{\overline{\kappa}}})_{i \in I}, \Theta \rangle$ is a κ-belief space in the same category with the following properties: (i) the identity map $\mathrm{id}_\Omega : \overrightarrow{\Omega} \to \overrightarrow{\Omega_\kappa}$ is a morphism (an isomorphism iff $\overrightarrow{\Omega}$ is minimal); and (ii) $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega_\kappa}} \circ \mathrm{id}_\Omega$. Moreover, by construction, $\overrightarrow{\Omega_\kappa}$ is minimal: $\mathcal{D}_{\overline{\kappa}} = D_{\overrightarrow{\Omega_\kappa}}^{-1}(\mathcal{D}^*)$.

Remark 4 states that belief hierarchies of depth $\overline{\kappa}$ generated by some state of some κ-belief space $\overrightarrow{\Omega}$ are generated by restricting attention to the minimal κ-belief spaces the domains of which are always κ-algebras. For any $\omega^* \in \Omega^*$, there are a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $\omega^* = D_{\overrightarrow{\Omega}}(\omega)$. However, one can always take the space $\overrightarrow{\Omega_\kappa}$ and $\omega \in \Omega$ with $\omega^* = D_{\overrightarrow{\Omega_\kappa}}(\omega)$. Thus, as long as the nature state $s \in_0 \omega^*$ and the players' interactive beliefs $\{e \in \mathcal{L} \mid e \in_1 \omega^*\}$ are concerned, nothing is lost by restricting attention to the space $\overrightarrow{\Omega_\kappa}$. Hence, as long as belief hierarchies up to $\overline{\kappa}$ are concerned, $\overrightarrow{\Omega_\kappa}$ is sufficient in that any belief hierarchy up to $\overline{\kappa}$ generated by $\overrightarrow{\Omega}$ is also generated by $\overrightarrow{\Omega_\kappa}$.

[36] While the domain of a standard possibility correspondence model in the previous literature is often assumed to be the power set, the domain of an arbitrary (probabilistic) type space is implicitly assumed to be a σ-algebra. If one accommodates both probabilistic beliefs and knowledge, then one needs to define them on a σ-algebra.

[37] Such $E \in \mathcal{D} \setminus \mathcal{D}_{\overline{\kappa}}$ might be captured by a richer language, i.e., $E \in \mathcal{D}_{\overline{\lambda}}$ with $\overline{\lambda} > \overline{\kappa}$.

7 Concluding Remarks

The main result of this paper (Theorem 1) constructs the terminal belief space $\overrightarrow{\Omega^*}$ for varieties of assumptions on beliefs. Within a given class of κ-belief spaces, the players can reason about their interactive beliefs up to depth $\overline{\kappa}$ (for a fixed infinite regular cardinal κ). The space Ω^* exhausts all possible interactive beliefs up to $\overline{\kappa}$ that can realize at some state of some belief space. Thus, the space Ω^* encodes interactive beliefs within itself (Proposition 1) and exhausts any statement regarding interactive beliefs that holds at some state of some belief space (Proposition 2). In Ω^*, only the explicit assumptions on beliefs made by the outside analysts are imposed. That is, Ω^* is free from implicit assumptions imposed by how the model is represented. Each state in Ω^* coherently and completely describes the corresponding nature state and interactive beliefs (Proposition 3 and Corollary 1). Explicitly, Theorem 2 shows that $\overrightarrow{\Omega^*}$ is the largest set of coherent descriptions that reflects assumptions on beliefs.

Section 5 constructs a terminal belief space irrespective of nature of beliefs (Corollaries 2 to 6). Thus, one can understand the existence of a terminal belief space under the common framework regardless of whether beliefs are qualitative or probabilistic. Proposition 4 shows, under the framework of this paper, finite belief hierarchies uniquely extend to countable ones for countably-additive probabilistic beliefs.

This paper circumvents the previous non-existence of a terminal knowledge space by restricting attention to all possible knowledge (belief) hierarchies of depth $\overline{\kappa}$. The κ-algebra $\mathcal{D}^*$ accommodates all possible interactive beliefs of depth $\overline{\kappa}$ (Proposition 5 and Remark 4). While an infinite regular cardinal κ can be taken arbitrarily for given nature states $(S, \mathcal{S})$, the terminal κ-belief space $\overrightarrow{\Omega^*}$ may not contain belief hierarchies of depth $\overline{\lambda}$ with $\overline{\lambda} > \overline{\kappa}$ (Proposition 6).

A Appendix

Proof of Remark 1. First, $\mathcal{L}_{\overline{\lambda}} \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$ follows because (i) $\mathcal{L}_0 \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$; and (ii) if $\mathcal{L}_\beta \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$ for all $\beta < \alpha (\leq \overline{\lambda})$ then $\mathcal{L}_\alpha \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$. Conversely, it can be seen that if $e \in \mathcal{L}_{\overline{\lambda}}$ then $e \in \mathcal{L}_\alpha$ for some $\alpha < \overline{\lambda}$. I show $\mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S})) \subseteq \mathcal{L}_{\overline{\lambda}}$. First, $\mathcal{A}_\kappa(\mathcal{S}) \subseteq \mathcal{L}_{\overline{\lambda}}$. Second, if $e \in \mathcal{L}_{\overline{\lambda}}$ then $e \in \mathcal{L}_\alpha$ for some $\alpha < \overline{\lambda}$ and thus $(\neg e) \in \mathcal{L}_{\alpha+1} \subseteq \mathcal{L}_{\overline{\lambda}}$. Third, take $\mathcal{F} \subseteq \mathcal{L}_{\overline{\lambda}}$ with $0 < |\mathcal{F}| < \lambda$. For each $f \in \mathcal{F}$, there is $\alpha_f < \overline{\lambda}$ with $f \in \mathcal{L}_{\alpha_f}$. Since $|\mathcal{F}| < \overline{\lambda}$, the definition of an infinite regular cardinal

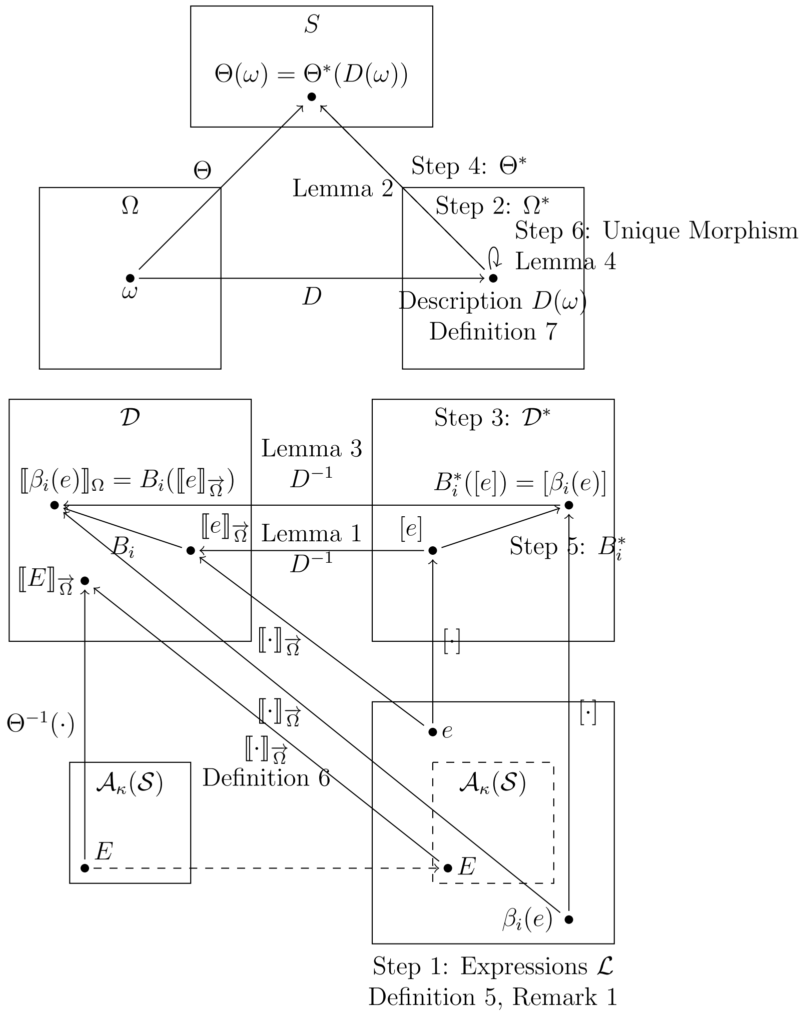

Figure 1: Interrelations among the Definitions and Lemmas for Theorem 1.

yields $\gamma := \sup_{f \in \mathcal{F}} \alpha_f < \overline{\lambda}$. Since $\mathcal{F} \subseteq \mathcal{L}_\gamma \subseteq \mathcal{L}_{\overline{\lambda}}$, it follows $\bigwedge \mathcal{F} \in \mathcal{L}_{\overline{\lambda}}$. Hence, $\mathcal{L}_\lambda^I(\mathcal{A}_\kappa(\mathcal{S})) \subseteq \mathcal{L}_{\overline{\lambda}}$. $\square$

Proof Sketch of Remark 2. The proof is similar to Heifetz and Samet (1998b, Proposition 4.1) and Meier (2006, Proposition 2). For nature events, since φ is a morphism, $[\![\cdot]\!]_{\overrightarrow{\Omega}} = \Theta^{-1}(\cdot) = \varphi^{-1}(\Theta')^{-1}(\cdot) = \varphi^{-1}([\![\cdot]\!]_{\overrightarrow{\Omega'}})$. Then, use the property that φ^{-1} commutes with set-algebraic operations and belief operators. $\square$

Remark A.1 (Behavioral Equivalence)**.** Two states, possibly residing in different belief spaces, are identified when the descriptions are identical. While this notion of equivalence is closely related to Fagin (1994, Section 4) and Mertens and Zamir (1985), the notion is extensively studied as one notion of bisimulations called "behavioral equivalence" (Kurz, 2000) in theoretical computer science. For notions of bisimulations ("observational equivalence"), see, for instance, Jacobs and Rutten (2012), Kurz (2000), and Rutten (2000).

Let $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ be belief spaces in a given category. States $(\omega, \omega') \in \Omega \times \Omega'$ are *behaviorally equivalent* if there are a belief space $\overrightarrow{\Omega''}$ and morphisms $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega''}$ and $\varphi' : \overrightarrow{\Omega'} \to \overrightarrow{\Omega''}$ with $\varphi(\omega) = \varphi'(\omega')$.

I show $(\omega, \omega') \in \Omega \times \Omega'$ are behaviorally equivalent iff $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$. This means that, in order for two states to be identical in terms of belief hierarchies, it suffices to show they are behaviorally equivalent. The proof goes as follows. If $(\omega, \omega') \in \Omega \times \Omega'$ are behaviorally equivalent, then $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega''}}(\varphi(\omega)) = D_{\overrightarrow{\Omega''}}(\varphi'(\omega')) = D_{\overrightarrow{\Omega'}}(\omega')$. The converse holds once the description map is shown to be a morphism.

If $\overrightarrow{\Omega}$ is non-redundant, then the "co-induction proof principle" (e.g., Jacobs and Rutten (2012), Kurz (2000), and Rutten (2000)) holds: in order for two states ω and ω' in Ω to be identical, it is enough to show they are behaviorally equivalent.

Proof of Lemma 1. The proof consists of three steps. The first step establishes the following correspondence between syntactic and semantic operations.

1. $[(\neg e)] = \neg[e](= [e]^c)$ for any $e \in \mathcal{L}$.

2. $[S] = \Omega^*$ (and $[\emptyset] = \emptyset$). In other words, $[\bigwedge \emptyset] = \Omega^*$ (and $[\bigvee \emptyset] = \emptyset$).

3. $[\bigwedge \mathcal{E}] = \bigcap_{e \in \mathcal{E}} [e]$ (and $[\bigvee \mathcal{E}] = \bigcup_{e \in \mathcal{E}} [e]$) for any $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$.

To prove (1), fix $e \in \mathcal{L}$. Then, $\omega^* \in [(\neg e)]$ iff $(\neg e) \in_1 \omega^* = D(\omega)$ iff $\omega \in [\![\neg e]\!]_{\overrightarrow{\Omega}} = \neg[\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega \notin [\![e]\!]_{\overrightarrow{\Omega}}$ iff $e \notin_1 D(\omega) = \omega^*$ iff $\omega^* \notin [e]$ iff $\omega^* \in (\neg[e])$, where a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. Thus, $[(\neg e)] = \neg[e]$.

To prove (2), if $\omega^* \in \Omega^*$ then $\omega^* = D(\omega)$ for some belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$. Since $\omega \in \Omega = \Theta^{-1}(S) = [\![S]\!]_{\overrightarrow{\Omega}}$, I get $S \in_1 D(\omega) = \omega^*$ and thus $\omega^* \in [S]$.

For (3), $\omega^* \in [\bigwedge \mathcal{E}]$ iff $\bigwedge \mathcal{E} \in_1 \omega^* = D(\omega)$ iff $\omega \in [\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}} = \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega^* \in \bigcap_{e \in \mathcal{E}} [e]$, where a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$.

The second step establishes that $\mathcal{D}^*$ is a κ-algebra on Ω^*. If $[e] \in \mathcal{D}^*$, then it follows from the first step and $(\neg e) \in \mathcal{L}$ that $\neg[e] = [(\neg e)] \in \mathcal{D}^*$. Next, take $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. It follows from the first step and $\bigwedge \mathcal{E} \in \mathcal{L}$ that $\bigcap_{e \in \mathcal{E}} [e] = [\bigwedge \mathcal{E}] \in \mathcal{D}^*$. For the case that $\mathcal{E} = \emptyset$, observe $\Omega^* = [S] \in \mathcal{D}^*$.

The third step establishes $D^{-1}([\cdot]) = [\![\cdot]\!]_{\overrightarrow{\Omega}}$ for any belief space $\overrightarrow{\Omega}$. For any $e \in \mathcal{L}$, $\omega \in D^{-1}([e])$ iff $D(\omega) \in [e]$ iff $e \in_1 D(\omega)$ iff $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$. $\square$

Proof of Lemma 2. For any $\omega^* \in \Omega^*$, choose a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = D(\omega)$ and define $\Theta^*(\omega^*) := \Theta(\omega)$, where $\Theta(\omega) \in_0 D(\omega)$. I show $\Theta^*(\omega^*)$ does not depend on a particular choice of $\overrightarrow{\Omega}$ and ω (i.e., $\Theta^* : \Omega^* \to S$ is well defined). If $\omega^* = D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$ for some $\omega \in \Omega$ and $\omega' \in \Omega'$, then $(0, \Theta(\omega)) = (0, \Theta'(\omega'))$.

Next, for each $E \in \mathcal{A}_\kappa(\mathcal{S})$, $\omega^* \in (\Theta^*)^{-1}(E)$ iff $\Theta^*(\omega^*) = \Theta(\omega) \in E$ iff $\omega \in \Theta^{-1}(E) = [\![E]\!]_{\overrightarrow{\Omega}}$ iff $E \in_1 D(\omega) = \omega^*$ iff $\omega^* \in [E]$, where $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. $\square$

To establish Lemma 3, I provide Lemma A.1 below. Suppose that a certain property of beliefs is represented by operators $f_{\overrightarrow{\Omega}} : \mathcal{D} \to \mathcal{D}$ in each belief space $\overrightarrow{\Omega}$. Operators would be generated by composing belief operators $(B_i)_{i \in I}$ and set-algebraic as well as constant and identity operations. For example, let $f_{\overrightarrow{\Omega}}(\cdot) = B_i(\cdot)$ and $g_{\overrightarrow{\Omega}}(\cdot) = B_i B_i(\cdot)$. Positive Introspection is characterized by $f_{\overrightarrow{\Omega}}(\cdot) \subseteq g_{\overrightarrow{\Omega}}(\cdot)$. Truth Axiom is characterized by $f_{\overrightarrow{\Omega}}(\cdot) \subseteq \mathrm{id}_{\mathcal{D}}(\cdot)$. Monotonicity is expressed as $f_{\overrightarrow{\Omega}}$ being *monotone*: $f_{\overrightarrow{\Omega}}(E) \subseteq f_{\overrightarrow{\Omega}}(F)$ for all $E, F \in \mathcal{D}$ with $E \subseteq F$. Likewise, Non-empty λ-Conjunction is expressed as $f_{\overrightarrow{\Omega}}$ satisfying *non-empty λ-conjunction*: $\bigcap_{E \in \mathcal{E}} f_{\overrightarrow{\Omega}}(E) \subseteq f_{\overrightarrow{\Omega}}(\bigcap \mathcal{E})$ for all $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$. Abusing the notation, denote by $f_{\overrightarrow{\Omega^*}}$ the corresponding operation in $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (B_i^*)_{i \in I}, \Theta^* \rangle$ (note that B_i^* is shown to be well defined on $(\Omega^*, \mathcal{D}^*)$ irrespective of Lemma A.1 below).

Lemma A.1 (Preservation of Properties of Beliefs)**.** *Suppose that $f_{\overrightarrow{\Omega}} : \mathcal{D} \to \mathcal{D}$ and $g_{\overrightarrow{\Omega}} : \mathcal{D} \to \mathcal{D}$ are defined in each κ-belief space $\overrightarrow{\Omega}$. Suppose further that if $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ is measurable then $\varphi^{-1} f_{\overrightarrow{\Omega'}}(\cdot) = f_{\overrightarrow{\Omega}} \varphi^{-1}(\cdot)$ and $\varphi^{-1} g_{\overrightarrow{\Omega'}}(\cdot) = g_{\overrightarrow{\Omega}} \varphi^{-1}(\cdot)$.*

1. *(a) If $f_{\overrightarrow{\Omega}}(\cdot) \subseteq g_{\overrightarrow{\Omega}}(\cdot)$ holds for every belief space $\overrightarrow{\Omega}$, then $f_{\overrightarrow{\Omega^*}}(\cdot) \subseteq g_{\overrightarrow{\Omega^*}}(\cdot)$.*

   (b) If $f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq g_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}})$ for some belief space $\overrightarrow{\Omega}$ and some $e \in \mathcal{L}$, then $f_{\overrightarrow{\Omega^*}}([e]) \not\subseteq g_{\overrightarrow{\Omega^*}}([e])$.

   (c) If there exits a surjective measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$, then $f_{\overrightarrow{\Omega}}(\cdot) \subseteq g_{\overrightarrow{\Omega}}(\cdot)$ implies $f_{\overrightarrow{\Omega'}}(\cdot) \subseteq g_{\overrightarrow{\Omega'}}(\cdot)$.

2. *(a) If $f_{\overrightarrow{\Omega}}$ is monotone for every belief space $\overrightarrow{\Omega}$, then so is $f_{\overrightarrow{\Omega^*}}$.*

   (b) Suppose $f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq f_{\overrightarrow{\Omega}}([\![f]\!]_{\overrightarrow{\Omega}})$ for some belief space $\overrightarrow{\Omega}$ and some $e, f \in \mathcal{L}$ with $[\![e]\!]_{\overrightarrow{\Omega}} \subseteq [\![f]\!]_{\overrightarrow{\Omega}}$. Then, $f_{\overrightarrow{\Omega^*}}([e]) \not\subseteq f_{\overrightarrow{\Omega^*}}([f])$.

(c) *Suppose there exits a surjective measurable map* $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$. *If* $f_{\overrightarrow{\Omega}}$ *is monotone, then so is* $f_{\overrightarrow{\Omega'}}$.

3. (a) *If* $f_{\overrightarrow{\Omega}}$ *satisfies non-empty* λ*-conjunction for every belief space* $\overrightarrow{\Omega}$*, then so does* $f_{\overrightarrow{\Omega^*}}$.

   (b) *Suppose* $\bigcap_{e\in\mathcal{E}} f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq f_{\overrightarrow{\Omega}}(\bigcap_{e\in\mathcal{E}}[\![e]\!]_{\overrightarrow{\Omega}})$ *for some belief space* $\overrightarrow{\Omega}$ *and some* $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ *with* $|\mathcal{E}| < \lambda$. *Then,* $\bigcap_{e\in\mathcal{E}} f_{\overrightarrow{\Omega^*}}([e]) \not\subseteq f_{\overrightarrow{\Omega^*}}(\bigcap_{e\in\mathcal{E}}[e])$.

   (c) *Suppose there exits a surjective measurable map* $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$. *If* $f_{\overrightarrow{\Omega}}$ *satisfies non-empty* λ*-conjunction, then so does* $f_{\overrightarrow{\Omega'}}$.

Proof of Lemma A.1. 1. Let $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ be measurable. If $\omega \in g_{\overrightarrow{\Omega}}(E)$ for all $E \in \mathcal{D}$ with $\omega \in f_{\overrightarrow{\Omega}}(E)$ then, for any $E' \in \mathcal{D}'$, $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(E')$ (i.e., $\omega \in f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'))$) implies $\varphi(\omega) \in g_{\overrightarrow{\Omega'}}(E')$.

   (a) If $\omega^* \in f_{\overrightarrow{\Omega^*}}([e])$ then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = D(\omega)$. Now, $\omega^* = D(\omega) \in g_{\overrightarrow{\Omega^*}}([e])$.

   (b) By hypothesis, there is $\omega \in \Omega$ such that $\omega \in f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1} f_{\overrightarrow{\Omega^*}}([e])$ and $\omega \notin g_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1} g_{\overrightarrow{\Omega^*}}([e])$.

   (c) If $\omega' \in f_{\overrightarrow{\Omega'}}(E')$ then $\omega' = \varphi(\omega)$ for some $\omega \in \Omega$. Now, $\omega' = \varphi(\omega) \in g_{\overrightarrow{\Omega'}}(E')$.

2. Let $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ be measurable, and let $f_{\overrightarrow{\Omega}}$ be monotone. For any $E', F' \in \mathcal{D}'$ with $E' \subseteq F'$, if $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(E')$ (i.e., $\omega \in f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'))$) then $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(F')$.

   (a) Let $[e] \subseteq [f]$. If $\omega^* \in f_{\overrightarrow{\Omega^*}}([e])$ then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = D(\omega)$. Now, $\omega^* = D(\omega) \in f_{\overrightarrow{\Omega^*}}([f])$.

   (b) By hypothesis, there is $\omega \in \Omega$ such that $\omega \in f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1} f_{\overrightarrow{\Omega^*}}([e])$ and $\omega \notin f_{\overrightarrow{\Omega}}([\![f]\!]_{\overrightarrow{\Omega}}) = D^{-1} f_{\overrightarrow{\Omega^*}}([f])$.

   (c) Let $E' \subseteq F'$. If $\omega' \in f_{\overrightarrow{\Omega'}}(E')$ then $\omega' = \varphi(\omega)$ for some $\omega \in \Omega$. Now, $\omega' = \varphi(\omega) \in f_{\overrightarrow{\Omega'}}(F')$.

3. Let $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ be measurable. If $f_{\overrightarrow{\Omega}}$ satisfies non-empty λ-conjunction, then $\varphi(\omega) \in \bigcap_{E'\in\mathcal{E}'} f_{\overrightarrow{\Omega'}}(E')$ implies $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(\bigcap \mathcal{E}')$ for all $\mathcal{E}' \in \mathcal{P}(\mathcal{D}') \setminus \{\emptyset\}$ with $|\mathcal{E}'| < \lambda$.

(a) Fix $\mathcal{E}^* \in \mathcal{P}(\mathcal{D}^*)\setminus\{\emptyset\}$ with $|\mathcal{E}^*| < \lambda$. If $\omega^* \in \bigcap_{[e]\in\mathcal{E}^*} f_{\overrightarrow{\Omega^*}}([e])$ then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = D(\omega)$. Now, $\omega^* = D(\omega) \in f_{\overrightarrow{\Omega^*}}(\bigcap \mathcal{E}^*)$.

(b) By hypothesis, there is $\omega \in \Omega$ with $\omega \in \bigcap_{e\in\mathcal{E}} f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1}(\bigcap_{e\in\mathcal{E}} f_{\overrightarrow{\Omega^*}}([e]))$ and $\omega \notin f_{\overrightarrow{\Omega}}(\bigcap_{e\in\mathcal{E}}[\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1}(f_{\overrightarrow{\Omega^*}}(\bigcap_{e\in\mathcal{E}}[e]))$.

(c) Fix $\mathcal{E}' \in \mathcal{P}(\mathcal{D}') \setminus \{\emptyset\}$ with $|\mathcal{E}'| < \lambda$. If $\omega' \in \bigcap_{E'\in\mathcal{E}'} f_{\overrightarrow{\Omega'}}(E')$ then there is $\omega \in \Omega$ with $\omega' = \varphi(\omega)$. Now, $\omega' = \varphi(\omega) \in f_{\overrightarrow{\Omega'}}(\bigcap \mathcal{E}')$.

□

Two remarks on Lemma A.1 are in order. First, B_i^* violates some property of beliefs if there exists a belief space $\overrightarrow{\Omega}$ such that B_i violates the corresponding property with respect to $\mathcal{D}_{\overline{\kappa}} = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$. Second, for belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$, if there is a surjective measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ with $B_i\varphi^{-1}(\cdot) = \varphi^{-1}B_i'(\cdot)$, then B_i' inherits the properties of B_i. Now, I prove Lemma 3.

Proof of Lemma 3. Fix $i \in I$. I show that B_i^* is well defined and inherits all the properties imposed in the given category. Then, for any $e \in \mathcal{L}$, $B_i(D^{-1}([e])) = B_i([\![e]\!]_{\overrightarrow{\Omega}}) = [\![\beta_i(e)]\!]_{\overrightarrow{\Omega}} = D^{-1}([\beta_i(e)]) = D^{-1}(B_i^*([e]))$.

To show B_i^* is well defined, take $e, f \in \mathcal{D}$ with $[e] = [f]$. If $\omega^* \in B_i^*([e]) = [\beta_i(e)]$, then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $D(\omega) \in [\beta_i(e)]$, i.e., $\beta_i(e) \in_1 D(\omega)$. Thus, $\omega \in [\![\beta_i(e)]\!]_{\overrightarrow{\Omega}} = B_i([\![e]\!]_{\overrightarrow{\Omega}}) = B_i(D^{-1}([e]))$. Since $[e] = [f]$, $\omega \in [\![\beta_i(f)]\!]_{\overrightarrow{\Omega}}$, i.e., $\omega^* = D(\omega) \in [\beta_i(f)] = B_i^*([f])$. By changing the role of e and f, $B_i^*([e]) = B_i^*([f])$.

Next, I show B_i^* inherits properties specified in Definition 2. For Monotonicity, apply Lemma A.1 (2a) by taking $f_{\overrightarrow{\Omega}} = B_i$. For Non-empty λ-Conjunction, apply Lemma A.1 (3a) by taking $f_{\overrightarrow{\Omega}} = B_i$. Next, apply Lemma A.1 (1a) to the following. For Necessitation, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (\Omega, B_i(\Omega))$. For Consistency, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i(\cdot) \cap (\neg B_i)(\cdot), \emptyset)$. For Truth Axiom, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i, \mathrm{id}_{\mathcal{D}})$. For Positive Introspection, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i, B_iB_i)$. For Negative Introspection, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i, B_i(\neg B_i))$.

Finally, consider the Kripke property. If $b_{B_i^*}^*(\omega^*) \subseteq [e]$ then $b_{B_i}(\omega) = \bigcap_{F\in\mathcal{D}:\omega\in B_i(F)} F \subseteq D^{-1}\bigcap_{[f]\in\mathcal{D}^*:\omega^*\in B_i^*[f]}[f] \subseteq D^{-1}[e]$, where a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. Since B_i satisfies the Kripke property, $\omega \in B_iD^{-1}[e] = D^{-1}B_i^*[e]$, as desired. □

Proof of Lemma 4. First, if $s \in_0 \omega^*$ and $s' \in_0 D(\omega^*)$, then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $s = \Theta(\omega) = \Theta^*(D(\omega)) = \Theta^*(\omega^*) = s'$. Note that the argument does not depend on a particular choice of belief spaces. Second, in a similar way to Heifetz and Samet (1998b, Lemma 4.6) and Meier (2006, Lemma 6), I show by induction that $[\![\cdot]\!]_{\overrightarrow{\Omega^*}} = [\cdot]$. Then, $\omega^* = \{s\} \sqcup \{e \in \mathcal{L} \mid e \in_1 \omega^*\} = \{s\} \sqcup \{e \in \mathcal{L} \mid \omega^* \in [e]\} = \{s\} \sqcup \{e \in \mathcal{L} \mid \omega^* \in [\![e]\!]_{\overrightarrow{\Omega^*}}\} = D(\omega^*)$ for any $\omega^* \in \Omega^*$.

To establish $[\![\cdot]\!]_{\overrightarrow{\Omega^*}} = [\cdot]$, start from $E \in \mathcal{A}_\kappa(\mathcal{S})$. Then, $\omega^* \in [\![E]\!]_{\overrightarrow{\Omega^*}} = (\Theta^*)^{-1}(E)$ iff $\Theta^*(\omega^*) = \Theta^*(D(\omega)) = \Theta(\omega) \in E$ iff $\omega \in \Theta^{-1}(E) = [\![E]\!]_{\overrightarrow{\Omega}}$ iff $E \in_1 D(\omega)$ iff $\omega^* = D(\omega) \in [E]$, where $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$.

Next, let $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. Assume the induction hypothesis that $[\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$ for all $e \in \mathcal{E}$. Then, $[\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega^*}} = \bigcap_{e\in\mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega^*}} = \bigcap_{e\in\mathcal{E}}[e] = [\bigwedge \mathcal{E}]$.

Next, assume the induction hypothesis that $[\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$. By definition, $[\beta_i(e)] = B_i^*([e]) = B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = [\![\beta_i(e)]\!]_{\overrightarrow{\Omega^*}}$. Also, $[(\neg e)] = \neg[e] = \neg[\![e]\!]_{\overrightarrow{\Omega^*}} = [\![\neg e]\!]_{\overrightarrow{\Omega^*}}$. □

Proof of Theorem 1. I have already shown that $\overrightarrow{\Omega^*} = \langle(\Omega^*, \mathcal{D}^*), (B_i^*)_{i\in I}, \Theta^*\rangle$ is a belief space of I on $(S, \mathcal{S})$ of the given category such that, for any belief space $\overrightarrow{\Omega}$, the description map $D_{\overrightarrow{\Omega}}$ is a morphism. Thus, I need only to show that $D_{\overrightarrow{\Omega}}$ is a unique morphism. If $\varphi : \Omega \to \Omega^*$ is a morphism, then Remark 3 and Lemma 4 imply $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega^*}} \circ \varphi = \varphi$. □

Proof of Proposition 1. First, the mapping defined in the proposition is surjective, because it follows from Lemmas 2, 3, and 4 that

$$\begin{aligned}(\Theta(\omega), (\{e \in \mathcal{L} \mid \omega \in B_i[\![e]\!]_{\overrightarrow{\Omega}}\})_{i\in I}) &= (\Theta^*(D(\omega)), (\{e \in \mathcal{L} \mid D(\omega) \in B_i^*[\![e]\!]_{\overrightarrow{\Omega^*}}\})_{i\in I})\\ &= (\Theta^*(D(\omega)), (\{e \in \mathcal{L} \mid D(\omega) \in B_i^*[e]\})_{i\in I}).\end{aligned}$$

Second, I show that the mapping defined in the proposition is injective. Suppose that $\Theta^*(\omega^*) = \Theta^*(\tilde{\omega}^*) = s$ and $\{e \in \mathcal{L} \mid \omega^* \in B_i^*[e]\} = \{e \in \mathcal{L} \mid \tilde{\omega}^* \in B_i^*[e]\}$ for each $i \in I$ (recall $[\![\cdot]\!]_{\overrightarrow{\Omega^*}} = [\cdot]$). Since $\Theta^*(\omega^*) \in_0 \omega^*$ and $\Theta^*(\tilde{\omega}^*) \in_0 \tilde{\omega}^*$, states ω^* and $\tilde{\omega}^*$ contain the same unique nature state s. Now, I show by induction that ω^* and $\tilde{\omega}^*$ contain the same set of expressions. First, for any $E \in \mathcal{A}_\kappa(\mathcal{S})$, if $E \in_1 \omega^*$ (i.e., $\omega^* \in [E] = (\Theta^*)^{-1}(E)$), then $\Theta^*(\tilde{\omega}^*) = \Theta^*(\omega^*) \in E$ and thus $E \in_1 \tilde{\omega}^*$. The converse also holds. Second, $(\neg e) \in_1 \omega^*$ iff $e \notin_1 \omega^*$ iff $e \notin_1 \tilde{\omega}^*$ iff $(\neg e) \in_1 \tilde{\omega}^*$. Third, let $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. Then, $\bigwedge \mathcal{E} \in_1 \omega^*$ iff $e \in_1 \omega^*$ for all $e \in \mathcal{E}$ iff $e \in_1 \tilde{\omega}^*$ for all $e \in \mathcal{E}$ iff $\bigwedge \mathcal{E} \in_1 \tilde{\omega}^*$. Fourth, fix $i \in I$. Then, $\beta_i(e) \in_1 \omega^*$ iff $\omega^* \in B_i^*[e]$ iff $\tilde{\omega}^* \in B_i^*[e]$ iff $\beta_i(e) \in_1 \tilde{\omega}^*$. The induction is complete, and $\omega^* = \tilde{\omega}^*$. □

Proof of Proposition 2. Recall $D_{\overrightarrow{\Omega}}^{-1}([\cdot]) = [\![\cdot]\!]_{\overrightarrow{\Omega}}$ and $[\cdot] = [\![\cdot]\!]_{\overrightarrow{\Omega^*}}$.

1. It suffices to show that if Φ is satisfiable then it is satisfiable in $\overrightarrow{\Omega^*}$. If there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega \in [\![f]\!]_{\overrightarrow{\Omega}} = D^{-1}([f])$ for all $f \in \Phi$, then $D(\omega) \in [f] = [\![f]\!]_{\overrightarrow{\Omega^*}}$ for all $f \in \Phi$.

2. For the first assertion, it is enough to show that $\Phi \models_{\overrightarrow{\Omega^*}} e$ implies $\Phi \models e$. Let $\overrightarrow{\Omega}$ be a belief space. If $\omega \in [\![f]\!]_{\overrightarrow{\Omega}} = D^{-1}([f])$ for all $f \in \Phi$, then $D(\omega) \in [f] = [\![f]\!]_{\overrightarrow{\Omega^*}}$ for all $f \in \Phi$. By assumption, $D(\omega) \in [\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$, i.e., $\omega \in D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}}$. Thus, $\Phi \models e$. The second assertion can be seen as a special case of the first. Or, for any belief space $\overrightarrow{\Omega}$, $[\![e]\!]_{\overrightarrow{\Omega}} = D^{-1}([e]) = D^{-1}([\![e]\!]_{\overrightarrow{\Omega^*}}) = D^{-1}(\Omega^*) = \Omega$.

□

Proof of Proposition 3. First, $e \notin_1 \omega^*$ iff $\omega^* \notin [e]$ iff $\omega^* \in \neg[e] = [\neg e]$ iff $(\neg e) \in_1 \omega^*$. Second, suppose $e \in_1 \omega^*$ and $(e \to f) \in_1 \omega^*$. Then, $\omega^* \in [e]$ and $\omega^* \in [e \to f] = [(\neg e) \vee f] = \neg[e] \cup [f]$. Thus, $\omega^* \in [f]$, i.e., $f \in_1 \omega^*$. Third, $\bigwedge \mathcal{E} \in_1 \omega^*$ iff $\omega^* \in [\bigwedge \mathcal{E}] = \bigcap_{e \in \mathcal{E}}[e]$ iff $\omega^* \in [e]$ for all $e \in \mathcal{E}$ iff $e \in_1 \omega^*$ for all $e \in \mathcal{E}$. □

Proof of Corollary 1. The first part follows from Proposition 3. The third part follows from Proposition 3 and the fact that $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg\beta_i)(e) \in_1 \omega^*$ iff $\omega^* \in (\neg B_i^*)([e]) \cap (\neg B_i^*)(\neg B_i^*)([e])$. Thus, I prove the second part.

First, if $\beta_i(e) \notin_1 \omega^*$ and $\beta_i(\neg e) \notin_1 \omega^*$, then by Proposition 3, $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e) \in_1 \omega^*$. Second, assume Consistency on i's beliefs. Since $B_i^*([e]) \cap B_i^*(\neg[e]) = \emptyset$, $\beta_i(e) \in_1 \omega^*$ and $\beta_i(\neg e) \in_1 \omega^*$ do not hold simultaneously. If $\beta_i(e) \in_1 \omega^*$ then $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e) \notin_1 \omega^*$. If $\beta_i(\neg e) \in_1 \omega^*$ then $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e) \notin_1 \omega^*$. Also, $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e) \in_1 \omega^*$ implies $\beta_i(e) \notin_1 \omega^*$ and $\beta_i(\neg e) \notin_1 \omega^*$. Conversely, suppose exactly one of the three conditions holds. If $\omega^* \in B_i^*([e])$ then $\beta_i(e) \in_1 \omega^*$. Then, $\beta_i(\neg e) \notin_1 \omega^*$, i.e., $\omega^* \in (\neg B_i^*)([\neg e]) = (\neg B_i^*)([e]^c)$, establishing Consistency. □

Proof of Theorem 2. Step 1. The proof consists of two steps. The first step shows that Ω^* is a coherent set of descriptions. For (1a), take $\omega^* \in \Omega^*$. There is a unique nature state $s = \Theta^*(\omega^*)$ with $s \in_0 \omega^*$. For any $E \in \mathcal{A}_\kappa(\mathcal{S})$ with $s \in E$, $\omega^* \in (\Theta^*)^{-1}(E) = [E]$, i.e., $E \in_1 \omega^*$. Conditions (1b) to (1e) follow from Proposition 3.

Next, consider (2a). If $(e \leftrightarrow f)$ is valid in $\overrightarrow{\Omega^*}$, then $[\![e]\!]_{\overrightarrow{\Omega^*}} = [\![f]\!]_{\overrightarrow{\Omega^*}}$ and $B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = B_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})$, i.e., $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega^*}} = [\![\beta_i(f)]\!]_{\overrightarrow{\Omega^*}}$. It follows that $[\![\beta_i(e) \leftrightarrow \beta_i(f)]\!]_{\overrightarrow{\Omega}} = \Omega^*$, i.e., $(\beta_i(e) \leftrightarrow \beta_i(f))$ is valid in $\overrightarrow{\Omega^*}$.

For (2b), similarly to the above argument, if $(e \to f)$ is valid in $\overrightarrow{\Omega^*}$, then $[\![e]\!]_{\overrightarrow{\Omega^*}} \subseteq [\![f]\!]_{\overrightarrow{\Omega^*}}$ and thus $B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) \subseteq B_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})$, i.e., $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega^*}} \subseteq [\![\beta_i(f)]\!]_{\overrightarrow{\Omega^*}}$. Then, $[\![\beta_i(e) \to \beta_i(f)]\!]_{\overrightarrow{\Omega^*}} = \Omega^*$, i.e., $(\beta_i(e) \to \beta_i(f))$ is valid in $\overrightarrow{\Omega^*}$.

For (2c), by supposition, $\bigcap\{[\![f]\!]_{\overrightarrow{\Omega^*}} \in \mathcal{D}^* \mid \omega^* \in B_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})\} \subseteq [\![e]\!]_{\overrightarrow{\Omega^*}}$. By the Kripke property of $\overrightarrow{\Omega^*}$, $\omega^* \in B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = [\beta_i(e)]$, i.e., $\beta_i(e) \in_1 \omega^*$.

Next, I show (3). Fix $\omega^* \in \Omega^*$. It is enough to show that each of the following expressions is valid in $\overrightarrow{\Omega^*}$. For Necessitation, consider $[\![\beta_i(S)]\!]_{\overrightarrow{\Omega^*}} = B_i^*([\![S]\!]_{\overrightarrow{\Omega^*}}) = B_i^*(\Omega^*) = \Omega^*$. For Non-empty λ-Conjunction, consider:

$$\left[\!\!\left[(\bigwedge_{e\in\mathcal{E}} \beta_i(e)) \to \beta_i(\bigwedge \mathcal{E})\right]\!\!\right]_{\overrightarrow{\Omega^*}} = \left(\neg \bigcap_{e\in\mathcal{E}} B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}})\right) \cup B_i^*(\bigcap_{e\in\mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*.$$

For Consistency, consider $[\![\beta_i(e) \to (\neg\beta_i)(\neg e)]\!]_{\overrightarrow{\Omega^*}} = (\neg B_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup (\neg B_i^*)(\neg[\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*$. For Truth Axiom, consider $[\![\beta_i(e) \to e]\!]_{\overrightarrow{\Omega^*}} = (\neg B_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup [\![e]\!]_{\overrightarrow{\Omega^*}} = \Omega^*$. For Positive Introspection, consider $[\![\beta_i(e) \to \beta_i\beta_i(e)]\!]_{\overrightarrow{\Omega^*}} = (\neg B_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup B_i^* B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*$. For Negative Introspection, consider $[\![(\neg\beta_i)(e) \to \beta_i(\neg\beta_i)(e)]\!]_{\overrightarrow{\Omega^*}} = B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup B_i^*(\neg B_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*$.

Step 2. The second step shows that $\Omega \subseteq \Omega^*$ for any set Ω of coherent descriptions. To that end, I introduce a belief structure on Ω, and show that the description map $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map.

Step 2.1. By slightly abusing the notation, let $[e]_{\overrightarrow{\Omega}} := \{\omega \in \Omega \mid e \in_1 \omega\}$ for each $e \in \mathcal{L}$. Let $\mathcal{D} := \{[e]_{\overrightarrow{\Omega}} \in \mathcal{P}(\Omega) \mid e \in \mathcal{L}\}$. Note that $[\cdot]_{\overrightarrow{\Omega}} \in \mathcal{D}$ is different from $[\cdot] \in \mathcal{D}^*$. I show $(\Omega, \mathcal{D})$ is a κ-algebra. First, I show $[\emptyset]_{\overrightarrow{\Omega}} = \emptyset$. Suppose to the contrary that $\omega \in [\emptyset]_{\overrightarrow{\Omega}}$. By definition, $\emptyset \in_1 \omega$, which is impossible. Hence, $\emptyset = [\emptyset]_{\overrightarrow{\Omega}} \in \mathcal{D}$. Second, $\Omega = [S]_{\overrightarrow{\Omega}} \in \mathcal{D}$. Third, I show $[\neg e]_{\overrightarrow{\Omega}} = \neg[e]_{\overrightarrow{\Omega}}$. Indeed, $\omega \in [\neg e]_{\overrightarrow{\Omega}}$ iff $(\neg e) \in_1 \omega$ iff $e \notin_1 \omega$ iff $\omega \in \neg[e]_{\overrightarrow{\Omega}}$. Hence, $\mathcal{D}$ is closed under complementation. Fourth, I show $[\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}} = \bigcap_{e\in\mathcal{E}}[e]_{\overrightarrow{\Omega}}$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. Indeed, $\omega \in \bigcap_{e\in\mathcal{E}}[e]_{\overrightarrow{\Omega}}$ iff $e \in_1 \omega$ for all $e \in \mathcal{E}$ iff $\bigwedge \mathcal{E} \in_1 \omega$ iff $\omega \in [\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}}$.

Step 2.2. Define $\Theta : \Omega \to S$ which associates, with each $\omega \in \Omega$, the unique $s \in S$ with $s \in_0 \omega$. The map Θ is a well-defined measurable map such that $(\Theta)^{-1}(E) = [E]_{\overrightarrow{\Omega}}$ for each $E \in \mathcal{A}_\kappa(\mathcal{S})$. If $\omega \in [E]_{\overrightarrow{\Omega}}$, then $E \in_1 \omega$. Hence, $\Theta(\omega) \in E$, i.e., $\omega \in \Theta^{-1}(E)$.

Conversely, if $\omega \in \Theta^{-1}(E)$ then $\Theta(\omega) \in E$, and thus $E \in_1 \omega$. Hence, $\omega \in [E]_{\overrightarrow{\Omega}}$.

Step 2.3. Fix $i \in I$, and define i's belief operator $B_i : \mathcal{D} \to \mathcal{D}$ as follows: $B_i([e]_{\overrightarrow{\Omega}}) := [\beta_i(e)]_{\overrightarrow{\Omega}}$ for each $[e] \in \mathcal{D}$. I show B_i is well defined. If $[e]_{\overrightarrow{\Omega}} = [f]_{\overrightarrow{\Omega}}$, then $[(e \leftrightarrow f)]_{\overrightarrow{\Omega}} = \Omega$. This implies $[(\beta_i(e) \leftrightarrow \beta_i(f))]_{\overrightarrow{\Omega}} = \Omega$. Thus, $[\beta_i(e)] = [\beta_i(f)]$.

Next, I show that B_i reflects assumptions on beliefs. For Necessitation, $\Omega = [\beta_i(S)]_{\overrightarrow{\Omega}} = B_i([S]_{\overrightarrow{\Omega}}) = B_i(\Omega)$. For Monotonicity, take $[e]_{\overrightarrow{\Omega}}, [f]_{\overrightarrow{\Omega}} \in \mathcal{D}$ with $[e]_{\overrightarrow{\Omega}} \subseteq [f]_{\overrightarrow{\Omega}}$. Then, $[e \to f]_{\overrightarrow{\Omega}} = \Omega$. It follows $[\beta_i(e) \to \beta_i(f)]_{\overrightarrow{\Omega}} = \Omega$, i.e., $[\beta_i(e)]_{\overrightarrow{\Omega}} \subseteq [\beta_i(f)]_{\overrightarrow{\Omega}}$. Thus, $B_i([e]_{\overrightarrow{\Omega}}) \subseteq B_i([f]_{\overrightarrow{\Omega}})$.

For Non-empty λ-Conjunction, take $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$. If $\omega \in \bigcap_{e \in \mathcal{E}} B_i([e]_{\overrightarrow{\Omega}}) = [\bigwedge_{e \in \mathcal{E}} \beta_i(e)]_{\overrightarrow{\Omega}}$ then $\bigwedge_{e \in \mathcal{E}} \beta_i(e) \in_1 \omega$. Since $(\bigwedge_{e \in \mathcal{E}} \beta_i(e) \to \beta_i(\bigwedge \mathcal{E})) \in_1 \omega$, it follows $\beta_i(\bigwedge \mathcal{E}) \in_1 \omega$, i.e., $\omega \in [\beta_i(\bigwedge \mathcal{E})]_{\overrightarrow{\Omega}} = B_i(\bigcap_{e \in \mathcal{E}} [e]_{\overrightarrow{\Omega}})$. For the Kripke property, $\omega \in B_i([e]_{\overrightarrow{\Omega}})$ for any $(\omega, [e]_{\overrightarrow{\Omega}}) \in \Omega \times \mathcal{D}$ with $\bigcap \{[f]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid \omega \in B_i([f]_{\overrightarrow{\Omega}})\} \subseteq [e]_{\overrightarrow{\Omega}}$.

For Consistency, if $\omega \in B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$ then $\beta_i(e) \in_1 \omega$. Since $(\beta_i(e) \to (\neg\beta_i)(\neg e)) \in_1 \omega$, it follows $(\neg\beta_i)(\neg e) \in_1 \omega$, i.e., $\omega \in [(\neg\beta_i)(\neg e)]_{\overrightarrow{\Omega}} = (\neg B_i)(\neg [e]_{\overrightarrow{\Omega}})$. For Truth Axiom, if $\omega \in B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$ then $\beta_i(e) \in_1 \omega$. Since $(\beta_i(e) \to e) \in_1 \omega$, it follows $e \in_1 \omega$, i.e., $\omega \in [e]_{\overrightarrow{\Omega}}$. For Positive Introspection, if $\omega \in B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$ then $\beta_i(e) \in_1 \omega$. Since $(\beta_i(e) \to \beta_i\beta_i(e)) \in_1 \omega$, it follows $\beta_i\beta_i(e) \in_1 \omega$, i.e., $\omega \in [\beta_i\beta_i(e)]_{\overrightarrow{\Omega}} = B_iB_i([e]_{\overrightarrow{\Omega}})$. For Negative Introspection, if $\omega \in (\neg B_i)([e]_{\overrightarrow{\Omega}}) = [(\neg\beta_i)(e)]_{\overrightarrow{\Omega}}$ then $(\neg\beta_i)(e) \in_1 \omega$. Since $((\neg\beta_i)(e) \to \beta_i(\neg\beta_i)(e)) \in_1 \omega$, it follows $\beta_i(\neg\beta_i)(e) \in_1 \omega$, i.e., $\omega \in [\beta_i(\neg\beta_i)(e)]_{\overrightarrow{\Omega}} = B_i(\neg B_i)([e]_{\overrightarrow{\Omega}})$.

Step 2.4. So far, I have established that $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta\rangle$ is a belief space (of the given category). Finally, I show that the description map $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map, and thus $\overrightarrow{\Omega}$ is non-redundant and $\Omega \subseteq \Omega^*$.

I start with showing by induction that $[\cdot]_{\overrightarrow{\Omega}} : \mathcal{L} \to \mathcal{D}$, viewed as a mapping, coincides with the semantic interpretation function $[\![\cdot]\!]_{\overrightarrow{\Omega}}$ (consequently, $\overrightarrow{\Omega}$ is minimal). First, fix $E \in \mathcal{A}_\kappa(\mathcal{S})$. Then, $\omega \in [\![E]\!]_{\overrightarrow{\Omega}} = \Theta^{-1}(E)$ iff $\Theta(\omega) \in E$ iff $\omega \in [E]_{\overrightarrow{\Omega}}$. Second, supposing $[\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$, $\omega \in [\![\neg e]\!]_{\overrightarrow{\Omega}} = \neg[\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega \notin [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega \notin [e]_{\overrightarrow{\Omega}}$ iff $\omega \in [\neg e]_{\overrightarrow{\Omega}}$. Third, suppose $[\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$ for all $e \in \mathcal{E}$ with $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ and $|\mathcal{E}| < \kappa$. Then, $\omega \in [\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}} = \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega \in \bigcap_{e \in \mathcal{E}} [e]_{\overrightarrow{\Omega}} = [\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}}$. Fourth, supposing $[\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$, $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega}} = B_i([\![e]\!]_{\overrightarrow{\Omega}}) = B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$.

I show that $D(\omega) = \omega$ for all $\omega \in \Omega$. First, $e \in_1 D(\omega)$ iff $D(\omega) \in [e]$ iff $\omega \in D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$ iff $e \in_1 \omega$. Second, $s \in_0 \omega$ iff $s = \Theta(\omega) = \Theta^*(D(\omega))$ iff

$s \in_0 D(\omega)$. Hence, D is an inclusion map. $\square$

Proof of Corollary 2. Construct $\overrightarrow{\Omega^*} := \langle(\Omega^*, \mathcal{D}^*), (B_i^{*p})_{(i,p)\in I\times[0,1]}, \Theta^*\rangle$ as in the proof of Theorem 1. The set Ω^* is not empty (consider $\overrightarrow{\{s\}}$). To see that $\overrightarrow{\Omega^*}$ is terminal, it suffices to show that the p-belief operators B_i^{*p} satisfy the properties specified in Definition 10 (2).

First, (2a), (2b), (2d), and (2h) follow from Lemma A.1 (1a). Next, (2c) follows from Lemma A.1 (2a). Next, (2g) follows from Lemma A.1 (2a) and (3a).

Next, (2e) and (2f) follow from the following variant of Lemma A.1 (1a). Let $f_{\overrightarrow{\Omega}} : \mathcal{D}^2 \to \mathcal{D}$ and $g_{\overrightarrow{\Omega}} : \mathcal{D}^2 \to \mathcal{D}$ be defined in each probabilistic belief space $\overrightarrow{\Omega}$ and satisfy, for any measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$, $\varphi^{-1} f_{\overrightarrow{\Omega'}}(E', F') = f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'), \varphi^{-1}(F'))$ and $\varphi^{-1} g_{\overrightarrow{\Omega'}}(E', F') = g_{\overrightarrow{\Omega}}(\varphi^{-1}(E'), \varphi^{-1}(F'))$ for all $E', F' \in \mathcal{D}'$. If $f_{\overrightarrow{\Omega}}(E, F) \subseteq g_{\overrightarrow{\Omega}}(E, F)$ (for all $E, F \in \mathcal{D}$) for every probabilistic belief space $\overrightarrow{\Omega}$, then $f_{\overrightarrow{\Omega^*}}([e], [f]) \subseteq g_{\overrightarrow{\Omega^*}}([e], [f])$ for all $[e], [f] \in \mathcal{D}^*$.

For (2i), if $[t^*_{B_i^*}(\omega^*)] \subseteq [e]$ then $[t_{B_i}(\omega)] \subseteq D^{-1}[e]$ and thus $\omega \in B_i^1(D^{-1}[e]) = D^{-1} B_i^{*1}([e])$, where a probabilistic belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. $\square$

To prove Proposition 4, I show the following preliminary result.

Lemma A.2. *Let $(\Omega, \mathcal{D})$ be an $\aleph_0$-algebra, and let $(B_i^p)_{p\in[0,1]}$ be a collection of player i's p-belief operators $B_i^p : \mathcal{D} \to \mathcal{D}$ satisfying Definition 10 (2). Then, there is a unique collection $(\overline{B}_i^p)_{p\in[0,1]}$ of p-belief operators $\overline{B}_i^p : \mathcal{A}_{\aleph_1}(\mathcal{D}) \to \mathcal{A}_{\aleph_1}(\mathcal{D})$ satisfying Definition 10 (2) and $\overline{B}_i^p|_{\mathcal{D}} = B_i^p$.*

Proof of Lemma A.2. Denote by $t_{B_i} : (\Omega, \mathcal{D}) \to (\Delta(\Omega, \mathcal{D}), \mathcal{A}_{\aleph_0}\{\{\mu \in \Delta(\Omega, \mathcal{D}) \mid \mu(E) \geq p\} \mid (E, p) \in \mathcal{D} \times [0,1]\})$ the measurable type mapping associated with $(B_i^p)_{p\in[0,1]}$, where $\Delta(\Omega, \mathcal{D})$ is the set of countably-additive probability measures on $(\Omega, \mathcal{D})$. Each $t_{B_i}(\omega)$ admits a unique Carathéodory extension $\bar{t}_{B_i}(\omega)$ on $\Delta(\Omega, \sigma(\mathcal{D}))$. For each $(E, p) \in \mathcal{D} \times [0,1]$, it follows from $\{\omega \in \Omega \mid \bar{t}_{B_i}(\omega)(E) \geq p\} = \{\omega \in \Omega \mid t_{B_i}(\omega)(E) \geq p\}$ that $\bar{t}_{B_i}^{-1}(\{\mu \in \Delta(\Omega, \sigma(\mathcal{D})) \mid \mu(E) \geq p\}) \in \mathcal{D}$. Hence, $\bar{t}_{B_i} : (\Omega, \sigma(\mathcal{D})) \to (\Delta(\Omega, \sigma(\mathcal{D})), \Sigma)$ is a measurable map, where $\Sigma := \sigma(\{\{\mu \in \Delta(\Omega, \sigma(\mathcal{D})) \mid \mu(E) \geq p\} \mid (E, p) \in \mathcal{D} \times [0,1]\})$. Now, it follows from Heifetz and Samet (1998b, Lemma 4.5) that $\Sigma = \sigma(\{\{\mu \in \Delta(\Omega, \sigma(\mathcal{D})) \mid \mu(E) \geq p\} \mid (E, p) \in \sigma(\mathcal{D}) \times [0,1]\})$. Denote $\overline{B}_i^p(E) := \{\omega \in \Omega \mid \bar{t}_{B_i}(\omega)(E) \geq p\}$ for each $(E, p) \in \sigma(\mathcal{D}) \times [0,1]$.

To show uniqueness, let $\tilde{B}_i^p$ be an extension. If $\omega \in \tilde{B}_i^p(E)$, then $\bar{t}_{B_i}(\omega)(E) = t_{\tilde{B}_i}(\omega)(E) \geq p$. Thus, $\omega \in \overline{B}_i^p(E)$. The converse also holds. $\square$

Proof of Proposition 4. Denote $\mathcal{L} = \mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S}))$. Let $(\lambda_\alpha)_{\alpha=0}^{\overline{\kappa}}$ be the auxiliary sequence that generates $\mathcal{L}$ as in Remark 1. For ease of notation, denote $\mathcal{L}^I_\lambda = \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$.

Define $[\mathcal{L}^I_\lambda] := \{[e] \in \mathcal{D}^* \mid e \in \mathcal{L}^I_\lambda\}$. As in Lemma 1, $[\mathcal{L}^I_\lambda]$ is an algebra on Ω^*. Since $\mathcal{L}^I_\lambda \subseteq \mathcal{L}$ and since $\mathcal{D}^*$ is a σ-algebra, $\sigma([\mathcal{L}^I_\lambda]) \subseteq \mathcal{D}^*$. To prove the converse set inclusion, I show $[e] \in \sigma([\mathcal{L}^I_\lambda])$ for all $e \in \mathcal{L}$. To that end, I first establish $B^{*p}_i([e]) \in \sigma([\mathcal{L}^I_\lambda])$ for any $[e] \in \sigma([\mathcal{L}^I_\lambda])$, i.e., $B^{*p}_i|_{\sigma([\mathcal{L}^I_\lambda])} : \sigma([\mathcal{L}^I_\lambda]) \to \sigma([\mathcal{L}^I_\lambda])$. Applying Lemma 3 to $B^{*p}_i|_{[\mathcal{L}^I_\lambda]} : [\mathcal{L}^I_\lambda] \to [\mathcal{L}^I_\lambda]$, the operator $B^{*p}_i|_{[\mathcal{L}^I_\lambda]}$ satisfies Definition 10 (2) (with the slight modification that events are restricted to $[\mathcal{L}^I_\lambda]$). It follows from Lemma A.2 that $B^{*p}_i|_{[\mathcal{L}^I_\lambda]}$ uniquely extends to $\sigma([\mathcal{L}^I_\lambda])$, and it coincides with $B^{*p}_i|_{\sigma([\mathcal{L}^I_\lambda])}$. Then, $B^{*p}_i|_{\sigma([\mathcal{L}^I_\lambda])} : \sigma([\mathcal{L}^I_\lambda]) \to \sigma([\mathcal{L}^I_\lambda])$.

Now, I show $[e] \in \sigma([\mathcal{L}^I_\lambda])$ for all $e \in \mathcal{L}$ by induction on the construction of $\mathcal{L}$. For $\alpha = 0$, $[E] \in \sigma([\mathcal{L}^I_\lambda])$ for all $E \in \mathcal{L}_0 = \mathcal{A}_\kappa(\mathcal{S})$. Suppose $[e] \in \sigma([\mathcal{L}^I_\lambda])$ for all $e \in \bigcup_{\beta<\alpha} \mathcal{L}_\beta$. For any $e \in \bigcup_{\beta<\alpha} \mathcal{L}_\beta$, $[\beta^p_i(e)] = B^{*p}_i|_{\sigma([\mathcal{L}^I_\lambda])}([e]) \in \sigma([\mathcal{L}^I_\lambda])$ (note that $\beta^p_i(e)$ is the (syntactic) expression for "i p-believes e"). Thus, $[e] \in \sigma([\mathcal{L}^I_\lambda])$ for all $e \in \mathcal{L}'_\alpha$. Then, $[\neg e] = \neg[e] \in \sigma([\mathcal{L}^I_\lambda])$ for any $e \in \mathcal{L}'_\alpha$. Also, $[\bigwedge \mathcal{F}] = \bigcap_{e\in\mathcal{F}}[e] \in \sigma([\mathcal{L}^I_\lambda])$ for any $\mathcal{F} \subseteq \mathcal{L}'_\alpha$ with $0 < |\mathcal{F}| < \kappa$. $\square$

Proof of Corollary 3. Adding the player "0" to I, for any $\overrightarrow{\Omega}$, define B^p_0 as follows: $B^p_0(E) = \Omega$ if $\mu(E) \geq p$; and $B^p_0(E) = \emptyset$ if $\mu(E) < p$. Put differently, the player 0 has a state-independent type mapping $t_{B_0}(\omega, \cdot) = \mu(\cdot)$ for all $\omega \in \Omega$. For any spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega}'$, $B^p_0\varphi^{-1} = \varphi^{-1}B'^p_0$ is equivalent to $\mu \circ \varphi^{-1} = \mu'$. Hence, a probabilistic belief space with a common prior can be identified as a probabilistic belief space $\langle(\Omega, \mathcal{D}), (B^p_i)_{(i,p)\in(I\cup\{0\})\times[0,1]}, \Theta\rangle$ satisfying Expression (3).

Following the construction of a terminal space in Section 3, the p-belief operators $(B^{*p}_0)_{p\in[0,1]}$ satisfy $D^{-1}B^{*p}_0([e]) = B^p_0 D^{-1}([e])$ for all $[e] \in \mathcal{D}^*$. Define $\mu^* : \mathcal{D}^* \to [0,1]$ by $\mu^*([e]) := \sup\{p \in [0,1] \mid \Omega^* = B^{*p}_0([e])\}$ for each $[e] \in \mathcal{D}^*$. Since the p-belief operators $(B^{*p}_0)_{p\in[0,1]}$ satisfy the properties in Definition 10 (2), μ^* is a countably-additive probability measure. Also, $\mu^* = \mu \circ D^{-1}$ follows from $D^{-1}B^{*p}_0(\cdot) = B^p_0 D^{-1}(\cdot)$. Thus, μ^* satisfies Equation (3) (see Footnote 25), and the proof is complete. $\square$

Proof of Corollary 4. Construct Ω^*, as in the proof of Theorem 1, by viewing the set of players in each conditional belief space as $\overline{I} := I \times [0,1] \times \mathcal{C}_S$. To see that Ω^* is not empty, take a CPS μ on $(S, \mathcal{A}_{\aleph_1}(\mathcal{S}), \mathcal{C}_S)$. Consider $\langle(S, \mathcal{A}_{\aleph_1}(\mathcal{S}), \mathcal{C}_S), (B^p_{i,C})_{(i,p,C)\in\overline{I}}, \mathrm{id}_S\rangle$, where (i) $B^p_{i,C}(E) := \emptyset$ if $\mu(E|C) < p$; and (ii) $B^p_{i,C}(E) := S$ if $\mu(E|C) \geq p$.

Next, as in the proof of Theorem 1, define $\mathcal{D}^*$, Θ^*, and an auxiliary collection of p-belief operators $(B^{*p}_{i,C_S})_{(i,p,C_S)\in\bar{I}}$ as $B^{*p}_{i,C_S}([e]) := [\beta^p_{i,C_S}(e)]$ for each $[e] \in \mathcal{D}^*$. By construction, $D^{-1}(B^{*p}_{i,C_S}([e])) = B^p_{i,C_S}(D^{-1}[e])$. Since $(\Theta^*)^{-1}(C_S) = [C_S] \in \mathcal{D}^*$, let $\mathcal{C}^* := \{[C_S] \in \mathcal{D}^* \mid C_S \in \mathcal{A}_{\aleph_1}(\mathcal{S})\}$. By construction, $\mathcal{C}^* \subseteq \mathcal{D}^*$, $(\Theta^*)^{-1}(\mathcal{C}_S) = \mathcal{C}^*$, and $\emptyset \notin \mathcal{C}^*$ (this is because Θ^* is surjective). Then, $(\Omega^*, \mathcal{D}^*, \mathcal{C}^*)$ is a conditional space, and $(B^{*p}_{i,C_S})_{(i,p,C_S)\in\bar{I}}$ is a well-defined collection of p-belief operators (observe $B^{*p}_i(\cdot|[C_S]) = B^{*p}_{i,C_S}$). As in the proof of Corollary 2, the p-belief operators satisfy the specified properties, i.e., $\overrightarrow{\Omega^*} := \langle(\Omega^*, \mathcal{D}^*, \mathcal{C}^*), (B^{*p}_i(\cdot|[C_S]))_{(i,p,[C_S])\in I\times[0,1]\times\mathcal{C}^*}, \Theta^*\rangle$ is a conditional belief space. By construction, $\overrightarrow{\Omega^*}$ is terminal. □

Proof of Corollary 6. Identify the common belief operator as the belief operator of a hypothetical player who represents common belief in each belief space (with a common belief operator). As in Section 3, construct a candidate terminal κ-belief space $\overrightarrow{\Omega^*} = \langle(\Omega^*, \mathcal{D}^*), (B^*_i)_{i\in I}, C^*, \Theta^*\rangle$. For any belief space $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, C, \Theta\rangle$, $D^{-1}C^*[\cdot] = CD^{-1}[\cdot]$. The candidate space is terminal if C^* is a common belief operator: $C^*[e] = \max\{[f] \in \mathcal{J}^*_I \mid [f] \subseteq B^*_I[e]\}$ for any $[e] \in \mathcal{D}^*$.

Let $\omega^* \in C^*[e]$. There are a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $\omega^* = D(\omega)$. Thus, $\omega \in D^{-1}C^*[e] = CD^{-1}[e]$. Since $CD^{-1}[e] \subseteq B_I D^{-1}[e] = D^{-1}B^*_I[e]$, $\omega^* = D(\omega) \in B^*_I[e]$. Thus, $C^*[e] \subseteq B^*_I[e]$. To show $C^*[e] \in \mathcal{J}^*_I$, take any $[f] \in \mathcal{D}^*$ with $C^*[e] \subseteq [f]$. Take $\omega^* \in C^*[e]$. There are a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $\omega^* = D(\omega)$. Since $CD^{-1}[e] = D^{-1}C^*[e] \subseteq D^{-1}[f]$, it follows $D^{-1}C^*[e] = CD^{-1}[e] \subseteq B_I D^{-1}[f] = D^{-1}B^*_I[f]$. Thus, $\omega^* = D(\omega) \in B^*_I[f]$. It follows that $C^*[e] \subseteq \max\{[f] \in \mathcal{J}^*_I \mid [f] \subseteq B^*_I[e]\}$.

To get the converse set inclusion, take any $[f] \in \mathcal{J}^*_I$ with $[f] \subseteq B^*_I[e]$. If $\omega^* \in [f]$ then there are a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $\omega \in D^{-1}[f] \subseteq D^{-1}B^*_I[e] = B_I D^{-1}[e]$. For the belief space $\overrightarrow{\Omega}$, consider $\overrightarrow{\Omega'} = \langle(\Omega, \mathcal{D}'), (B_i|_{\mathcal{D}'})_{i\in I}, C|_{\mathcal{D}'}, \Theta\rangle$ with $\mathcal{D}' = D^{-1}(\mathcal{D}^*)$. One can show that $\overrightarrow{\Omega'}$ is a belief space and that the identify map $\mathrm{id}_\Omega : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism (see also Remark 4 in Section 6.3). Then, $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \mathrm{id}_\Omega$. Since one can retake $\omega^* = D_{\overrightarrow{\Omega'}}(\omega)$, without loss, assume $\mathcal{D} = D^{-1}(\mathcal{D}^*)$.

To establish $\omega^* \in C^*[e]$, it suffices to show that $D^{-1}[f] \in \mathcal{J}_I$, because it implies $\omega \in D^{-1}[f] \subseteq C(D^{-1}[e]) = D^{-1}C^*[e]$. Take any $E' \in \mathcal{D} = D^{-1}(\mathcal{D}^*)$ with $D^{-1}[f] \subseteq E'$. Without loss, one can assume $E' = D^{-1}[e']$ for some $[e'] \in \mathcal{D}^*$ with $[f] \subseteq [e]'$, because $D^{-1}[e' \vee f] = D^{-1}([e]' \cup [f]) = D^{-1}[e'] \cup D^{-1}[f] = D^{-1}[e'] = E'$. Since $[f] \in \mathcal{J}^*_I$, it follows $[f] \subseteq B^*_I[e']$. Thus, $D^{-1}[f] \subseteq D^{-1}B^*_I[e'] = B_I D^{-1}[e'] = B_I(E')$. □

Proof of Proposition 5. Part 1. Let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ be a morphism. In a similar way to Heifetz and Samet (1998a), I show by induction that $\mathcal{C}_\alpha = \varphi^{-1}(\mathcal{C}'_\alpha)$ for all α. For $\alpha = 0$, $\mathcal{C}_0 = \varphi^{-1}(\mathcal{C}'_0)$ follows because $(\Theta)^{-1}(E) = \varphi^{-1}((\Theta')^{-1}(E))$ for any $E \in \mathcal{A}_\kappa(\mathcal{S})$. Suppose $\mathcal{C}_\beta = \varphi^{-1}(\mathcal{C}'_\beta)$ for all $\beta < \alpha$. Then,

$$\mathcal{C}_\alpha = \mathcal{A}_\kappa\left(\{\varphi^{-1}(E') \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\} \cup \bigcup_{i \in I}\{\varphi^{-1}(B'_i(E')) \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\}\right)$$
$$= \varphi^{-1}\left(\mathcal{A}_\kappa\left(\{E' \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\} \cup \bigcup_{i \in I}\{B'_i(E') \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\}\right)\right) = \varphi^{-1}(\mathcal{C}'_\alpha).$$

Thus, if $\mathcal{C}'_\alpha = \mathcal{C}'_{\alpha+1}$, then $\mathcal{C}_\alpha = \mathcal{C}_{\alpha+1}$. In other words, if the κ-rank of $\overrightarrow{\Omega'}$ is α then that of $\overrightarrow{\Omega}$ is at most α.

Part 2. Fix a λ-belief space $\overrightarrow{\Omega}$ in the category of κ-belief spaces (where $\lambda \geq \kappa$). Define $\mathcal{D}_\alpha := \mathcal{A}_\kappa(\{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}_\alpha\})$ for each $\alpha \leq \overline{\kappa}$, where $(\mathcal{L}_\alpha)_{\alpha=0}^{\overline{\kappa}}$ is defined as in Remark 1 so that $\mathcal{L} = \mathcal{L}_{\overline{\kappa}}$. Especially, $\mathcal{D}_{\overline{\kappa}} = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$. I show $\mathcal{D}_\alpha = \mathcal{C}_\alpha$ for all $\alpha \leq \overline{\kappa}$. For $\alpha = 0$, $\mathcal{D}_0 = \{\Theta^{-1}(E) \in \mathcal{D} \mid E \in \mathcal{A}_\kappa(\mathcal{S})\} = \mathcal{C}_0$. Next, if $\mathcal{D}_\beta = \mathcal{C}_\beta$ for all $\beta < \alpha$, then

$$\mathcal{D}_\alpha = \mathcal{A}_\kappa\left(\left(\bigcup_{\beta<\alpha} \mathcal{D}_\beta\right) \cup \bigcup_{i \in I}\{B_i([\![e]\!]) \in \mathcal{D} \mid [\![e]\!] \in \bigcup_{\beta<\alpha} \mathcal{D}_\beta\}\right) = \mathcal{C}_\alpha.$$

Hence, $\mathcal{C}_{\overline{\kappa}} = \mathcal{D}_{\overline{\kappa}} = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$, implying $\mathcal{C}_{\overline{\kappa}} = \mathcal{C}_{\overline{\kappa}+1}$, i.e., the κ-rank of $\overrightarrow{\Omega}$ is at most $\overline{\kappa}$. □

Proof of Proposition 6. Part 1. Since the λ-belief space belongs to the category of κ-belief spaces, there is a unique morphism $D_{\overrightarrow{\Omega^*_\lambda}} : \overrightarrow{\Omega^*_\lambda} \to \overrightarrow{\Omega^*_\kappa}$, which takes the following form. While each $\omega^* = \{s \in S \mid s \in_0 \omega^*\} \sqcup \{e \in \mathcal{L}^I_\lambda(\mathcal{A}_\lambda(\mathcal{S})) \mid e \in_1 \omega^*\} \in \Omega^*_\lambda$ consists of the unique nature state s and expressions $e \in \mathcal{L}^I_\lambda(\mathcal{A}_\lambda(\mathcal{S}))$ that are true e, $D_{\overrightarrow{\Omega_\lambda}}(\omega^*) = \{s \in S \mid s \in_0 \omega^*\} \sqcup \{e \in \mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S})) \mid e \in_1 \omega^*\}$ consists of the same unique nature state s and expressions $e \in \mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S}))$ (observe $\mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S})) \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_\lambda(\mathcal{S}))$) that obtain. Thus, $D_{\overrightarrow{\Omega_\lambda}}$ is a surjective morphism (that identifies two states ω^* and $\tilde{\omega}^*$ in $\overrightarrow{\Omega^*_\lambda}$ if they induce the same κ-belief hierarchies, i.e., $D_{\overrightarrow{\Omega_\lambda}}(\omega^*) = D_{\overrightarrow{\Omega_\lambda}}(\tilde{\omega}^*)$). Hence, $|\Omega^*_\kappa| \leq |\Omega^*_\lambda|$.

Part 2. To simplify the proof, I make the following assumptions. Since the proof does not depend on the cardinality of I, let $I = \{1, 2\}$. Next, by the (second) remark following Definition 7, assume all the properties of beliefs in Definition 2. Next, assume $(S, \mathcal{S}) = (\{s_0, s_1\}, \mathcal{P}(S))$ (the proof goes through by taking $s_1 \in E$ and $s_0 \in E^c$ for E in the statement of the proposition). First, the knowledge space $\overrightarrow{\Omega}$ constructed by Hart, Heifetz, and Samet (1996) is a non-redundant κ-belief space with $|\Omega| \geq 2^{\aleph_0}$. Since the morphism $D_{\overrightarrow{\Omega}}$ is injective, $2^{\aleph_0} \leq |\Omega| \leq |\Omega^*_\kappa|$. Second, Heifetz and Samet (1998a, Theorem 2.5) constructs a non-redundant κ-belief (knowledge) space $\overrightarrow{\Omega'}$ with $|\Omega'| = \kappa$. Since the morphism $D_{\overrightarrow{\Omega'}}$ is injective, $\kappa = |\Omega'| \leq |\Omega^*_\kappa|$. $\square$

Remark A.2. Let $(\Omega, \mathcal{D})$ be a κ-algebra. Let $B_i : \mathcal{D} \to \mathcal{D}$ satisfy the Kripke property. Define $\overline{B}_i : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ by $\overline{B}_i(E) := \{\omega \in \Omega \mid b_{B_i}(\omega) \subseteq E\}$ for each $E \in \mathcal{P}(\Omega)$. By construction, $\overline{B}_i$ satisfy the Kripke property. Also, $\overline{B}_i$ inherits Consistency, Truth Axiom, Positive Introspection, and Negative Introspection from B_i. Moreover, $B_i = \overline{B}_i|_{\mathcal{D}}$ and $b_{\overline{B}_i} = b_{B_i}$.

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