Unawareness without AU Introspection*

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Abstract

This paper studies unawareness in terms of the lack of knowledge in a model that generalizes both a non-partitional standard-state-space model and a stationary generalized-state-space model. The resulting model may not necessarily satisfy AU Introspection: an agent, who is unaware of an event, is unaware of being unaware of it. Yet, the paper shows that such agent does not know whether she is unaware of it, i.e., she is ignorant of being unaware of it. First, the paper asks when and how the generalized model (in particular, a standard-state-space model) has a non-trivial form of unawareness and sensible properties of unawareness. Second, the paper studies the implications of the violation of AU Introspection. An agent, when facing infinitely many objects of knowledge, may know that there is an event of which she is unaware. Treating new information only at face value can cause an agent to become unaware of some event.

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1 Introduction

Since 2009, November 12th has been declared as World Pneumonia Day to "raise awareness about pneumonia, the world's leading infectious killer of children under the age of 5" (WHO, 2018; see also Greenslade, 2020). If one is unaware of the fact that "pneumonia accounts for 19 per cent of all under-five deaths" (UNICEF and WHO, 2006; Wardlaw et al., 2006), the unawareness would refer to the lack of

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knowledge. One does not know the fact, and one does not know that one does not know the fact. In contrast, when it has been claimed that "only one in five caregivers in the developing world know the two key symptoms of pneumonia—fast and difficult breathing—" (UNICEF and WHO, 2006; Wardlaw et al., 2006), one could interpret it as "four in five caregivers in the developing world are unaware of the symptoms of pneumonia." The unawareness would refer to the lack of conception. Four in five caregivers, presumably familiar with pneumonia, would have limited understanding on some aspects (i.e., the symptoms) of pneumonia.

Two different models of unawareness in the literature embody two different notions of unawareness: one as the lack of knowledge and the other as the lack of conception. In a standard-state-space model in which the state space consists of a single space, unawareness refers to the lack of knowledge as in pioneering papers by Modica and Rustichini (1994, 1999): an agent is unaware of a statement if she does not know it and she does not know that she does not know it.[1] In contrast, in a generalized-state-space model in which the state space consists of multiple subspaces as in Heifetz, Meier, and Schipper (2006, 2008), an agent is unaware of a statement if she does not know the subspace within which the statement is described. Since multiple subspaces differ in how rich their vocabulary is for describing statements, the non-knowledge of the subspace is interpreted as the lack of conception.[2] Unawareness as the lack of conception is stronger than one as the lack of knowledge: if an agent does not know the subspace within which the statement is described, then she does not know the statement and she does not know that she does not know it.

This paper provides a tractable model of unawareness on a generalized state space that nests the above two models. The model enables one to directly compare a "non-partitional" standard-state-space model and a "stationary" generalized-state-space model of Heifetz, Meier, and Schipper (2006) regarding the following questions. What properties of unawareness do they satisfy? When do they represent a non-trivial form of unawareness? How do they differ in possible behavioral implications?

The resulting model defines unawareness as the lack of knowledge, and is understood as a generalized-state-space model without the axiom of AU Introspection: AU Introspection, first postulated by Dekel, Lipman, and Rustichini (1998), states that if an agent is unaware of an event, then she is unaware of being unaware of it. The paper asks when unawareness as the lack of knowledge coincides with one as the lack of conception. If the underlying state space reduces to a single space and if agents' knowledge is induced by a possibility correspondence on the single state space, then

[1] First, Fagin and Halpern (1987) and Pires (1994) are other pioneering papers. Second, previous studies on unawareness and information-processing errors in standard-state-space "non-partitional" possibility-correspondence models include: Bacharach (1985), Brandenburger, Dekel, and Geanakoplos (1992), Dekel and Gul (1997), Geanakoplos (1989), Morris (1996), Samet (1990), and Shin (1993).

[2] Other pioneering attempts to describe unawareness in a richer structure include: Board and Chung (2007, 2008), Board, Chung, and Schipper (2011), Galanis (2011, 2013), Halpern (2001), Halpern and Rêgo (2008, 2009), Heifetz, Meier, and Schipper (2006, 2008, 2013), Heinsalu (2014), and Li (2009). See Schipper (2015) for an overview.

the resulting model is the non-partitional standard-state-space model. If the agents' knowledge is induced by a possibility correspondence on the generalized state space and if the model satisfies AU Introspection, then the model is the stationary generalized-state-space model. In fact, I characterize a property of a possibility correspondence on a generalized state space that yields AU Introspection. The model can also capture agents' knowledge which is not necessarily induced by possibility correspondences.

Roughly, the paper answers the above questions as follows. First, although the model may not necessarily satisfy AU Introspection, whenever an agent is unaware of an event, she is *ignorant* of being unaware of it: she does not know she is unaware of it, and she does not know she is aware of it. I call this property JU Introspection.

Second, an agent is unaware of an event through two channels: (i) whether unawareness satisfies AU Introspection; and (ii) whether the agent knows the subspace to which the event belongs (in a standard state space, whether the agent knows the entire space).[3] The paper shows that AU Introspection is a property that connects two notions of unawareness, one as the lack of conception and the other as the lack of knowledge. In a non-partitional standard-state-space model in which the agent knows the entire space, the model has a non-trivial form of unawareness if and only if (hereafter, iff) it violates AU Introspection. In a stationary generalized-state-space model in which AU Introspection holds, unawareness is determined exclusively by whether the agent knows subspaces.

Third, suppose unawareness may not satisfy AU Introspection. Then, an agent may know that there exists an event of which she is unaware, i.e., she may know her self-unawareness. Also, more knowledge may not necessarily lead to more awareness. In a (standard or generalized) state-space possibility-correspondence model, AU Introspection fails when the agent processes (new) information only at face value. Suppose the agent receives new information (say, E) so that she knows E at a state iff E holds at that state. Since she takes information only at face value, assume that, when new information does not hold, she may not know that new information does not hold (i.e., the negation of E). Note that the former statement implies that, when new information does not obtain, the agent does not know E. These two statements imply that if new information E does not hold then she may not know that she does not know E.[4] Thus, when new information E does not obtain, the agent does not know E and she does not know that she does not know E, i.e., she is unaware of new information E.

The paper is organized as follows. The rest of this section provides a technical

[3] In economic problems, an agent may not fully recognize all possible contingencies she may face. The degree to which the agent may not recognize contingencies is subsumed in this second channel of unawareness. As examples of economic problems, Auster (2013), Filiz-Ozbay (2012), and Zhao (2011) study a principal-agent problem in which the agent is aware of a subset of the full contingencies (of which the principal is fully aware). Modica, Rustichini, and Tallon (1998) study a general-equilibrium model in which agents may not foresee all possible future contingencies. Quiggin (2016) studies a decision problem in which the agent is aware of a subset of the entire state space.

[4] The latter statement implies that if new information E does not hold then she may not know that E does not hold, and the former states that E does not hold iff she does not know E.

overview. Section 2 defines a model. Section 3 demonstrates that the model nests any non-partitional standard-state-space model and any (non-)stationary generalized-state-space model. Section 4 studies properties of unawareness. Section 4.1 restates unawareness in terms of ignorance and possibility. Section 4.2 characterizes non-trivial unawareness. Section 4.3 investigates the existing axioms of unawareness. Section 5 studies knowledge of self-unawareness (Section 5.1) and non-monotonicity of unawareness in knowledgeability (Section 5.2). Section 6 provides concluding remarks. Proofs are relegated to Appendix A. Online Appendix provides supplementary discussions.

1.1 Technical Overview

The model introduced in Section 2 imposes the following conditions on knowledge and unawareness. First, Section 2.3 assumes the following properties of knowledge: (i) an agent can only know what is true (Truth Axiom); (ii) if the agent knows an event then she knows that she knows it (Positive Introspection); and (iii) at each state, the agent knows any logical consequence of what she knows (Monotonicity). As discussed in Section 3, these properties are commonly assumed in previous standard and generalized state-space models. I drop Negative Introspection (if the agent does not know an event then she knows that she does not know it), as such agent is never unaware of any event. I also relax Necessitation (the agent knows a tautology), and instead show that the violation of it is the second channel of unawareness in a standard-state-space model.

Second, Section 2.4 defines unawareness solely in terms of the lack of knowledge, in order to ask when unawareness as the lack of knowledge coincides with one as the lack of conception. Precisely, an agent is k^n-*unaware* of an event if she does not know it, she does not know that she does not know it, and so forth n times, including the case of $n = \infty$. Thus, notions of unawareness are fully determined by underlying properties of knowledge (e.g., Necessitation) and a level n of the lack of knowledge.

Section 4 studies properties of unawareness. Section 4.1 relates the derived notions of unawareness to such notions derived from knowledge as possibility and ignorance. Following Modica and Rustichini (1999), an agent considers an event E possible if she does not know its negation $\neg E$ and if she is aware (not unaware) of E. The agent is ignorant of an event E if she does not know E and she does not know its negation $\neg E$.

Proposition 2 in Section 4.2 fully characterizes when unawareness is non-trivial. In the context of standard-state-space models, Dekel, Lipman, and Rustichini (1998) show that no standard-state-space model can capture a sensible form of unawareness if a logical agent satisfies the following three axioms: Plausibility, KU Introspection, and AU Introspection. Plausibility states that if the agent is unaware of an event, then she does not know it and she does not know that she does not know it. KU Introspection means that the agent does not know that she is unaware of any particular event. Modica and Rustichini (1994) demonstrate another negative result when unawareness is symmetric (i.e., the unawareness of an event entails that of its negation).

I show that unawareness can only take two forms: either two levels of lack of knowledge or infinitely many levels of lack of knowledge. In fact, three levels of lack of knowledge imply any higher level. Proposition 1 in Section 4.1 shows unawareness is a particular form of ignorance: the agent is (k^n-)unaware of an event iff she is ignorant of the possibility that she knows the event. I use this result to show that k^2-unawareness coincides with the lack of conception iff it does with k^∞-unawareness. In fact, these two forms of k^n-unawareness coincide iff k^2-unawareness satisfies AU Introspection (or equivalently, Symmetry). Thus, if these two forms of k^n-unawareness coincide in a standard-state-space model, then unawareness comes only from the violation of Necessitation. In contrast, unawareness is non-trivial in a standard-state-space possibility-correspondence model whenever unawareness violates AU Introspection (equivalently, Negative Introspection), as Necessitation always holds in the possibility-correspondence model. Thus, on a standard state space, any properly non-partitional possibility-correspondence model can capture a non-trivial form of k^2-unawareness.

What properties of unawareness do standard and generalized state-space models commonly satisfy? Proposition 3 in Section 4.3 demonstrates that unawareness satisfies such properties as Plausibility, KU Introspection, the converse of AU Introspection, and JU Introspection. The converse of AU introspection states that an agent, who is unaware of being unaware of an event, is unaware of the event. To restate, if agents are logical and introspective and notions of unawareness are defined in terms of the lack of knowledge, then any model satisfies Plausibility, KU Introspection, and JU Introspection (instead of AU Introspection).

In contrast, Proposition 4 in Section 4.3 clarifies the difference between standard and generalized state-space models. It studies properties such as AU Introspection and Symmetry that connect unawareness as the lack of knowledge and one as the lack of conception. In a standard-state-space possibility-correspondence model, the violation of AU Introspection is the only channel through which unawareness is non-trivial, because Necessitation closes the second channel of unawareness. In contrast, in a generalized-state-space stationary possibility-correspondence model, AU Introspection holds. Thus, unawareness as the lack of knowledge and one as the lack of conception coincide, and unawareness is non-trivial when an agent does not know a subspace.

Finally, since AU Introspection (or any equivalent property in Proposition 4) is the key difference between standard and generalized state-space models, Section 5 studies two implications of the violation of AU Introspection. First, recall that, under KU Introspection, there is no state at which an agent knows that she is unaware of a particular event. Can the agent know that she is unaware of "something" (i.e., the fact that she is unaware of some event)? Can she be aware that she is unaware of something? Knowledge of unawareness or awareness of unwareness is substantially interesting because if the agent is at least aware of her own unawareness, then she would be able to make her action contingent on "something" she is unaware of. In the context of semantic models, the question is also theoretically interesting because semantic models can only represent the knowledge and unawareness of specific events.[5]

[5]First, see also Board and Chung (2007), Halpern and Rêgo (2009), Schipper (2015), and the references therein for representing the event that an agent is unaware of *something* using the first-

Section 5.1 defines an event that captures whether there is an event the agent is unaware of. Proposition 5 shows that if a given model has an infinite number of events and if AU Introspection fails, then it can be the case that she knows there is an event of which she is unaware (while she does not know that she is unaware of any particular event). If the given model has finitely many events or if AU Introspection holds, then the agent does not know that there is an event of which she is unaware. In fact, under AU Introspection, there is even no state at which the agent is unaware of something and is aware that she is unaware of something.

Second, Section 5.2 shows by example that getting more information can cause an agent to become unaware of some event in the absence of AU Introspection. Thus, unawareness as the lack of knowledge may not be monotonic in knowledgeability while unawareness as the lack of conception is.

The non-monotonicity of awareness in knowledgeability is not necessarily related to the negative value of information (e.g., Galanis, 2015; Geanakoplos, 1989) in a generalized-state-space model, as AU Introspection alone does not necessarily guarantee the non-negative value of information.[6] The non-negative value of information is related to how knowledge and awareness interact with probabilistic beliefs.[7] Proposition 6 provides a Blackwell-type result under AU Introspection, when preferences over possibility correspondences are induced from preferences over acts: an agent always prefers one possibility correspondence to another iff the former is more informative.

2 Model

This section presents the framework, which represents agents' knowledge and unawareness by their knowledge and unawareness operators on a generalized state space.[8] Specifically, Section 2.1 defines a generalized state space. On the generalized state

order logic. Second, see Grant and Quiggin (2015), Karni and Vierø (2017), and Kochov (2018) for the decision-theoretic foundation of awareness of unawareness. Third, in the context of contracts, see Tirole (2009) for renegotiation and Zhao (2011) for the "force majeure clause."

[6]Morris and Shin (1997) and Zhao (2008) also show that the value of information (or the value of awareness) may be negative in their respective models. Quiggin (2016) studies an environment in which the value of information and the value of awareness add up to a constant. Galanis (2016) studies the value of information in a risk-sharing environment, in which the disclosure of public information may benefit some market participants at the expense of others. For example, this rationalizes why some more-aware market participants may have an incentive to release their private information and awareness to less-aware participants in order to take advantage of them.

[7]In a standard-state-space model, in contrast, if the value of information is negative or if awareness is not monotonic in knowledgeability, then AU Introspection must be violated. Conversely, Fukuda (2019b) shows that, in a standard-state-space model, dynamic consistency conditions between knowledge and probabilistic beliefs lead to a non-trivial form of unawareness, i.e., AU Introspection.

[8]Such operator-based framework is useful for examining dependencies among axioms of knowledge and unawareness, for capturing higher-order knowledge and unawareness by iterating agents' knowledge and unawareness operators, and for constructing a canonical space that accommodates agents' interactive reasoning (see, for instance, Fukuda, 2020; Meier, 2008).

space, Section 2.2 defines events, which are objects of agents' knowledge and unawareness, and operations on events. Section 2.3 defines a model, which represents agents' knowledge and unawareness by knowledge and unawareness operators. Particularly, Section 2.4 defines unawareness operators derived from the lack of knowledge. Section 2.5 defines additional properties of knowledge and basic properties of unawareness.

2.1 Generalized State Space

Throughout the paper, let I be a non-empty set of *agents*. I introduce a generalized state space of Heifetz, Meier, and Schipper (2006) in a slightly more abstract way.

A *generalized state space* $\langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}, \succeq, r \rangle$ consists of three primitives. First, $(S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}$ is a non-empty collection of complete algebras of sets. Each $S_\alpha \in \mathcal{S} := \{S_\alpha\}_{\alpha \in \mathcal{A}}$ is non-empty and referred to as a *subspace*. The collection $\mathcal{S}$ is assumed to be disjoint. Each S_α is endowed with a collection $\mathcal{D}_\alpha$ of subsets of S_α (i.e., $\mathcal{D}_\alpha$ is a subset of the power set $\mathcal{P}(S_\alpha)$) that is closed under arbitrary union, arbitrary intersection, and complementation. Within a complete algebra $(S_\alpha, \mathcal{D}_\alpha)$, I follow the set-theoretical conventions that $\bigcup \emptyset = \emptyset \in \mathcal{D}_\alpha$ and $\bigcap \emptyset = S_\alpha \in \mathcal{D}_\alpha$. The set of *states of the world* is the entire union $\Omega := \bigcup_{\alpha \in \mathcal{A}} S_\alpha$.

Second, $\langle \mathcal{S}, \succeq \rangle$ is a complete lattice. A subspace S' is intended to be interpreted as being at least as "expressive" as a subspace S if $S' \succeq S$, as in Heifetz, Meier, and Schipper (2006). Thus, the partial order $\succeq$ is intended to be interpreted as ranking subspaces $\mathcal{S}$ by amounts of "concepts" or "expressive power."

Third, $r := (r_S^{S'})_{S' \succeq S}$ is a collection of surjective projections $r_S^{S'} : (S', \mathcal{D}') \to (S, \mathcal{D})$ for each pair $(S, S') \in \mathcal{S}^2$ with $S' \succeq S$. I assume: (i) each projection is measurable: $(r_S^{S'})^{-1}(B) \in \mathcal{D}'$ for all $B \in \mathcal{D}$; (ii) each r_S^S is the identity mapping; and (iii) projections are commutative: $S'' \succeq S' \succeq S$ implies $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$.

The generalized state space is *standard* if $\mathcal{S}$ is a singleton, i.e., $\mathcal{S} = \{\Omega\}$.

2.2 Events

This subsection defines events (objects of agents' knowledge and unawareness) and operations on the collection of events. Namely, an *event* is a pair $(B^\uparrow, S) \in \mathcal{P}(\Omega) \times \mathcal{S}$ with $B^\uparrow := \bigcup \{(r_S^{S'})^{-1}(B) \in \mathcal{P}(\Omega) \mid S' \succeq S \text{ for some } S' \in \mathcal{S}\}$ and $B \in \mathcal{D}$. Define the *domain* $\mathcal{E}$ as the collection of events. Fix an event $(B^\uparrow, S_\alpha)$. By denoting $(\emptyset^S, S) := (\emptyset, S)$, the set $B^\uparrow$ alone determines the subspace S_α to which the event $(B^\uparrow, S_\alpha)$ belongs.

The event $(B^\uparrow, S_\alpha)$ has two components. First, call S_α the *base space* of $(B^\uparrow, S_\alpha)$ (or simply $B^\uparrow$), and denote $S(B^\uparrow, S_\alpha) = S_\alpha$ (or simply $S(B^\uparrow) = S_\alpha$). Second, call B the *basis* of $B^\uparrow$. For any event (E, S), the basis B of E satisfies $E = B^\uparrow$.

I introduce the following four operations on the domain $\mathcal{E}$. The first is a partial order $\leqslant$ on events: for any events $(E, S(E))$ and $(F, S(F))$, define $(E, S(E)) \leqslant (F, S(F))$ as $E \subseteq F$ and $S(E) \succeq S(F)$. The greatest element is $(\Omega, \inf \mathcal{S}) = ((\inf \mathcal{S})^\uparrow, \inf \mathcal{S})$ while the least element is $(\emptyset, \sup \mathcal{S})$.

Second, for any collection of events $(B_\lambda^\uparrow, S_\lambda)_{\lambda\in\Lambda}$, define its *conjunction* as

$$\bigwedge_{\lambda\in\Lambda}(B_\lambda^\uparrow, S_\lambda) := \left(\bigcap_{\lambda\in\Lambda} B_\lambda^\uparrow, \sup_{\lambda\in\Lambda} S_\lambda\right) = \left(\left(\bigcap_{\lambda\in\Lambda}(r_{S_\lambda}^{\sup_{\lambda\in\Lambda} S_\lambda})^{-1}(B_\lambda)\right)^\uparrow, \sup_{\lambda\in\Lambda} S_\lambda\right) \in \mathcal{E}.$$

Since $\bigwedge_{\lambda\in\Lambda}(B_\lambda^\uparrow, S_\lambda)$ is the infimum of events $(B_\lambda^\uparrow, S_\lambda)_{\lambda\in\Lambda}$ in a partially ordered set $\langle\mathcal{E}, \leqslant\rangle$, it forms a complete lattice.

Third, define the *negation* of an event $(B^\uparrow, S)$ by $\neg(B^\uparrow, S) := (\neg B^\uparrow, S) := ((S \setminus B)^\uparrow, S) \in \mathcal{E}$. By definition, $\neg\neg(B^\uparrow, S) = (B^\uparrow, S)$. By letting $\neg\emptyset^S := S^\uparrow$ and $\neg S^\uparrow := \emptyset^S$, I can unambiguously write $\neg\neg B^\uparrow = B^\uparrow$ for any $(B^\uparrow, S) \in \mathcal{E}$. Since the negation of $(B^\uparrow, S)$ is taken within the subspace S, the negation is generally different from the set- or lattice-theoretical complement. As in Heifetz, Meier, and Schipper (2006):

Remark 1. If $S(E) = S(F)$, then $E \subseteq F$ iff $\neg F \subseteq \neg E$. Put differently, if $S(E) = S(F)$, then $(E, S(E)) \leqslant (F, S(F))$ iff $\neg(F, S(F)) \leqslant \neg(E, S(E))$.

Fourth, define the *disjunction* of $(B_\lambda^\uparrow, S_\lambda)_{\lambda\in\Lambda}$ as $\bigvee_{\lambda\in\Lambda}(B_\lambda^\uparrow, S_\lambda) := \neg(\bigwedge_{\lambda\in\Lambda}\neg(B_\lambda^\uparrow, S_\lambda)) \in \mathcal{E}$. The disjunction $\bigvee_{\lambda\in\Lambda}(B_\lambda^\uparrow, S_\lambda)$ is generally different from the supremum of $(B_\lambda^\uparrow, S_\lambda)_{\lambda\in\Lambda}$ in $\langle\mathcal{E}, \leqslant\rangle$. As in Heifetz, Meier, and Schipper (2006), the following can be verified:

Remark 2. If $S = S_\lambda$ for all $\lambda \in \Lambda$, then $\bigvee_{\lambda\in\Lambda}(B_\lambda^\uparrow, S_\lambda) = (\bigcup_{\lambda\in\Lambda} B_\lambda^\uparrow, S)$.

In particular, $(B^\uparrow, S) \vee (\neg B^\uparrow, S) = (S^\uparrow, S)$. It can be seen that the distributive laws hold: $(E, S) \wedge \bigvee_{\lambda\in\Lambda}(E_\lambda, S_\lambda) = \bigvee_{\lambda\in\Lambda}((E, S) \wedge (E_\lambda, S_\lambda))$ and $(E, S) \vee \bigwedge_{\lambda\in\Lambda}(E_\lambda, S_\lambda) = \bigwedge_{\lambda\in\Lambda}((E, S) \vee (E_\lambda, S_\lambda))$.

If the state space is standard, i.e., $\mathcal{S} = \{\Omega\}$, then one can identify a complete lattice $\langle\mathcal{E}, \leqslant\rangle$ with $\langle\mathcal{D}, \subseteq\rangle$. Also, one can identify the operations of conjunction $\bigwedge$, negation $\neg$, and disjunction $\bigvee$ as those of intersection, complementation, and union.

2.3 Model

A model consists of a generalized state space and a profile of agents' knowledge and unawareness operators. Through a knowledge operator, the knowledge of an event $(E, S(E))$ is expressed as another event. As in the previous literature, denote by $K_i(E)$ the set of states at which agent i knows $(E, S(E))$. Then, agent i's knowledge operator associates, with each event $(E, S(E))$, the event $(K_i(E), S(K_i(E)))$ that agent i knows $(E, S(E))$.

Since objects of knowledge and unawareness are events, I simplify the notation of an event $(B^\uparrow, S(B^\uparrow))$. Namely, I add the over-line to the set $B^\uparrow$ to denote the event $\overline{B}^\uparrow := (B^\uparrow, S(B^\uparrow))$, as the set $B^\uparrow$ determines its base space $S(B^\uparrow)$. Thus, I denote any event $(E, S(E))$ by $\overline{E} = (E, S(E))$. Then, for any operator such as i's knowledge operator $(E, S(E)) \mapsto (K_i(E), S(K_i(E)))$, I add the over-line to K_i to denote

$$\overline{K}_i : \mathcal{E} \ni (E, S(E)) = \overline{E} \mapsto \overline{K}_i(\overline{E}) := (K_i(E), S(K_i(E))) \in \mathcal{E}.$$

I add the over-line to K_i because the knowledge $\overline{K}_i(\overline{E})$ of an event $\overline{E}$ is itself an event.

Now, a *model* (of I) is a tuple $\mathcal{M} := \langle\langle(S_\alpha, \mathcal{D}_\alpha)_{\alpha\in\mathcal{A}}, \succeq, r\rangle, (\overline{K}_i, \overline{U}_i)_{i\in I}\rangle$ with the following ingredients. First, $\langle(S_\alpha, \mathcal{D}_\alpha)_{\alpha\in\mathcal{A}}, \succeq, r\rangle$ is a generalized state space. Let $\mathcal{E}$ be the collection of events, i.e., the domain.

Second, $\overline{K}_i : \mathcal{E} \ni (E, S(E)) = \overline{E} \mapsto \overline{K}_i(\overline{E}) = (K_i(E), S(K_i(E))) \in \mathcal{E}$ is agent i's *knowledge operator* satisfying (at least) the following four conditions: (i) $S(K_i(E)) = S(E)$ for any $(E, S(E)) \in \mathcal{E}$; (ii) Truth Axiom: $\overline{K}_i(\overline{E}) \leqslant \overline{E}$ (for all $\overline{E} \in \mathcal{E}$); (iii) Positive Introspection: $\overline{K}_i(\cdot) \leqslant \overline{K}_i\overline{K}_i(\cdot)$ (i.e., $\overline{K}_i(\overline{E}) \leqslant \overline{K}_i\overline{K}_i(\overline{E})$ for all $\overline{E} \in \mathcal{E}$); and (iv) Monotonicity: $\overline{E} \leqslant \overline{F}$ implies $\overline{K}_i(\overline{E}) \leqslant \overline{K}_i(\overline{F})$.

For any event $\overline{E} = (E, S(E))$, $\overline{K}_i(\overline{E}) = (K_i(E), S(K_i(E)))$ is the event that i knows $\overline{E}$. The set $K_i(E)$ constitutes a part of the event $\overline{K}_i(\overline{E})$, and is interpreted as the set of states at which i knows $\overline{E}$. The base space of the event $\overline{K}_i(\overline{E})$ is assumed to be $S(E)$: $S(K_i(E)) = S(E)$. Truth Axiom distinguishes knowledge from beliefs in that knowledge is truthful. Positive Introspection allows the agent to know what she knows. Monotonicity renders the agent a logical inference ability.

Third, $\overline{U}_i : \mathcal{E} \ni (E, S(E)) = \overline{E} \mapsto \overline{U}_i(\overline{E}) = (U_i(E), S(U_i(E))) \in \mathcal{E}$ is i's *unawareness operator* satisfying: $S(U_i(E)) = S(E)$ for all $(E, S(E)) \in \mathcal{E}$. For any event $\overline{E} = (E, S(E))$, $\overline{U}_i(\overline{E}) = (U_i(E), S(U_i(E)))$ is the event that i is unaware of $\overline{E}$. The set $U_i(E)$ constitutes a part of the event $\overline{U}_i(\overline{E})$, and is interpreted as the set of states at which i is unaware of $\overline{E}$. The base space of the event $\overline{U}_i(\overline{E})$ is assumed to be $S(E)$: $S(U_i(E)) = S(E)$. The next subsection defines particular unawareness operators derived from the lack of knowledge.

For ease of notation, I often identify $\mathcal{M} := \langle\langle(S_\alpha, \mathcal{D}_\alpha)_{\alpha\in\mathcal{A}}, \succeq, r\rangle, (\overline{K}_i, \overline{U}_i)_{i\in I}\rangle$ with $\mathcal{M} = \langle\mathcal{E}, (\overline{K}_i, \overline{U}_i)_{i\in I}\rangle$. Define agent i's *awareness operator* by $\overline{A}_i(\cdot) := (\neg\overline{U}_i)(\cdot)$ (Modica and Rustichini, 1994, 1999). That is, $\overline{A}_i(\overline{E})$ is the event that i is aware of $\overline{E}$ in that i is not unaware of $\overline{E}$.

2.4 Unawareness Operators Defined as the Lack of Knowledge

The previous subsection has defined a model. This subsection defines particular unawareness operators defined from a knowledge operator. Fix agent i's knowledge operator $\overline{K}_i : \mathcal{E} \to \mathcal{E}$. Let $n \in \{m \in \mathbb{N} \mid m \geq 2\} \cup \{\infty\}$.

Define the *k^n-unawareness operator* $\overline{U}_i^{(n)}(\cdot) := \bigwedge_{r=1}^{n}(\neg\overline{K}_i)^r(\cdot)$, where $(\neg\overline{K}_i)^r$ is the r-th power of $(\neg\overline{K}_i)$. Denoting $\overline{U}_i^{(n)}(E, S(E)) = (U_i^{(n)}(E), S(E))$, the set $U_i^{(n)}(E) = \bigcap_{r=1}^{n}(\neg K_i)^r(E)$ consists of the states at which agent i is (k^n-)*unaware* of the event $(E, S(E))$. Thus, agent i is (k^n-)unaware of an event $(E, S(E))$ at a state ω if $\omega \in U_i^{(n)}(E) = \bigcap_{r=1}^{n}(\neg K_i)^r(E)$. Modica and Rustichini (1994) define unawareness by $\overline{U}_i^{(2)}(\cdot) = (\neg\overline{K}_i)(\cdot) \wedge (\neg\overline{K}_i)^2(\cdot)$, i.e., agent i does not know an event, and she does not know that she does not know it. Dekel, Lipman, and Rustichini (1998) consider $\overline{U}_i^{(\infty)}(\cdot) = \bigwedge_{r\in\mathbb{N}}(\neg\overline{K}_i)^r(\cdot)$, i.e., agent i does not know an event, she does not know that she does not know it, and so on *ad infinitum*.

Also, define the k^n-*awareness operator* by $\overline{A}_i^{(n)}(\cdot) := (\neg \overline{U}_i^{(n)})(\cdot)$. By definition, $\overline{A}_i^{(n)}(\cdot) = \bigvee_{r=1}^{n} \overline{K}_i(\neg \overline{K}_i)^{r-1}(\cdot)$, where $(\neg \overline{K}_i)^0(E,S) := (E,S)$.

2.5 Additional Properties of Knowledge and Unawareness

Fix a model $\mathcal{M}$. I introduce three additional properties of knowledge. First, $\overline{K}_i$ satisfies *Necessitation* if $\overline{K}_i(\Omega, \inf \mathcal{S}) = (\Omega, \inf \mathcal{S})$: the agent knows the entire space $(\Omega, \inf \mathcal{S})$. Second, $\overline{K}_i$ satisfies *Non-empty Conjunction* if $\bigwedge_{\lambda \in \Lambda} \overline{K}_i(\overline{E}_\lambda) \leqslant \overline{K}_i(\bigwedge_{\lambda \in \Lambda} \overline{E}_\lambda)$ for any non-empty index set Λ: if the agent knows each of a non-empty collection of events then she knows its conjunction. *Finite Conjunction* refers to the case in which Λ is finite. A knowledge operator in the generalized-state-space model of Heifetz, Meier, and Schipper (2006) satisfies Necessitation and Non-empty Conjunction. Third, $\overline{K}_i$ satisfies *Negative Introspection* if $(\neg \overline{K}_i)(\cdot) \leqslant \overline{K}_i(\neg \overline{K}_i)(\cdot)$: if the agent does not know an event then she knows that she does not know it.

Next, I define three joint properties of knowledge and unawareness proposed in Dekel, Lipman, and Rustichini (1998) (Section 4.3 examines other properties). First, $(\overline{K}_i, \overline{U}_i)$ is *plausible* if $\overline{U}_i(\cdot) \leqslant (\neg \overline{K}_i)(\cdot) \wedge (\neg \overline{K}_i)^2(\cdot)$: if the agent is unaware of an event then she does not know it and she does not know that she does not know it. Call $\mathcal{M}$ *plausible* if every $(\overline{K}_i, \overline{U}_i)$ is plausible. Second, $(\overline{K}_i, \overline{U}_i)$ satisfies *KU Introspection* if $\overline{K}_i \overline{U}_i(E,S) = (\emptyset, S)$: for any event, there is no state at which the agent knows that she is unaware of it. Third, $(\overline{K}_i, \overline{U}_i)$ satisfies *AU Introspection* if $\overline{U}_i(\cdot) \leqslant \overline{U}_i \overline{U}_i(\cdot)$: if the agent is unaware of an event then she is unaware of being unaware of it.

3 Possibility-Correspondence Models

This section demonstrates that the framework defined in Section 2 accommodates any standard-state-space non-partitional possibility-correspondence model (Section 3.1) and any generalized-state-space (non-)stationary possibility-correspondence model (Section 3.2).[9] Also, within the generalized-state-space possibility-correspondence models, Section 3.2 identifies the condition on a possibility correspondence under which unawareness defined as the lack of knowledge coincides with one as the lack of conception.

3.1 Non-partitional Standard-State-Space Models

I demonstrate that the framework nests any standard-state-space possibility-correspondence model. To that end, I remark on the following notational conventions for standard-state-space models. First, as discussed, I identify the underlying complete algebra $\mathcal{D}$ on a standard state space Ω and the collection of events $\mathcal{E} = \{(E, \Omega) \in \mathcal{D} \times \{\Omega\} \mid E \in \mathcal{D}\}$. Second, recalling that the knowledge operator $\overline{K}_i$ is written as $\overline{K}_i(E,S) = (K_i(E), S)$

[9]A particular case is the generalized-state-space stationary possibility-correspondence model of Heifetz, Meier, and Schipper (2006). Their model is also related to Board and Chung (2008) (see also Board, Chung, and Schipper, 2011) and Fagin and Halpern (1987) (see also Halpern and Rêgo, 2008).

for each $(E, S) \in \mathcal{E}$, I identify the operators $\overline{K}_i : \mathcal{E} \to \mathcal{E}$ and $K_i : \mathcal{D} \to \mathcal{D}$. That is, I suppress the over-line on the knowledge operator in any standard-state-space model. For the unawareness operator, I also identify $\overline{U}_i$ and U_i. The same notational convention applies to other operators in any standard-state-space model.

3.1.1 Non-partitional Possibility-Correspondence Model

A *possibility correspondence* $\Pi_i : \Omega \to \mathcal{D}$ *induces* an operator $K_i : \mathcal{D} \to \mathcal{D}$ if $K_i(E) = \{\omega \in \Omega \mid \Pi_i(\omega) \subseteq E\} \in \mathcal{D}$ for every $E \in \mathcal{D}$. Calling each $\Pi_i(\omega)$ the *possibility set* at ω, the condition says that agent i knows an event E at ω if the possibility set at ω implies (i.e., is included in) E. If K_i is induced by Π_i, then it is well-known that K_i satisfies: (i) Monotonicity (i.e., $E \subseteq F$ implies $K_i(E) \subseteq K_i(F)$); (ii) Non-empty Conjunction (i.e., $\bigcap_{\lambda \in \Lambda} K_i(E_\lambda) \subseteq K_i(\bigcap_{\lambda \in \Lambda} E_\lambda)$); and (iii) Necessitation (i.e., $K_i(\Omega) = \Omega$).

Also, if K_i is induced by Π_i, then the following introspective properties of K_i can be characterized in terms of Π_i. First, K_i satisfies Truth Axiom iff Π_i is reflexive (i.e., $\omega \in \Pi_i(\omega)$ for all $\omega \in \Omega$). Second, K_i satisfies Positive Introspection iff Π_i is transitive (i.e., $\omega' \in \Pi_i(\omega)$ implies $\Pi_i(\omega') \subseteq \Pi_i(\omega)$). Third, K_i satisfies Negative Introspection iff Π_i is Euclidean (i.e., $\omega' \in \Pi_i(\omega)$ implies $\Pi_i(\omega) \subseteq \Pi_i(\omega')$).

Hence, the framework nests any non-partitional (i.e., reflexive-and-transitive) possibility-correspondence model on a standard state space. The knowledge operator induced by any such non-partitional possibility correspondence satisfies Necessitation. Note that if the possibility correspondence is partitional (i.e., reflexive, transitive, and Euclidean), then $U_i(\cdot) = \emptyset$, provided that (K_i, U_i) is plausible.

3.1.2 Example

I provide an example of a model on a standard state space. The example will be used to examine (non-)trivial forms and properties of unawareness on a standard state space.

Example 1. Let $I = \{i_1, i_2, i_3, i_4\}$, and consider a standard state space $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Table 1 depicts each agent's knowledge and unawareness operators. Specifically, the table has four blocks corresponding to the four agents $\{i_1, i_2, i_3, i_4\}$. The first column of each block depicts every event $E \in \mathcal{D}$, and the second column of each block illustrates the knowledge of every event $K_{i_\ell}(E)$ (with $\ell \in \{1, 2, 3, 4\}$) where E is the corresponding element in the first column. To compute the unawareness operators $U^{(2)}_{i_\ell}$ and $U^{(n)}_{i_\ell}$ (with $n \geq 3$ or $n = \infty$), the third, fourth, and fifth columns depict the lack of knowledge $(\neg K_{i_\ell})(E)$, $(\neg K_{i_\ell})^2(E)$, and $(\neg K_{i_\ell})^3(E)$, respectively. Then, the sixth column illustrates $U^{(2)}_{i_\ell}(E)$, and the seventh column $U^{(n)}_{i_\ell}(E)$.

In this example, agent i_1's knowledge coincides with Dekel, Lipman, and Rustichini (1998, Example 1). Once the properties of unawareness are studied, Example 3 of Section 4.3.1 revisits this example. Every K_i satisfies Non-empty Conjunction as well as Truth Axiom, Positive Introspection, and Monotonicity. Every pair $(K_i, U^{(n)}_i)$ satisfies Plausibility and KU Introspection.

E	$K_{i_1}(E)$	$(\neg K_{i_1})(E)$	$(\neg K_{i_1})^2(E)$	$(\neg K_{i_1})^3(E)$	$U_{i_1}^{(2)}(E)$	$U_{i_1}^{(n)}(E)$
$\emptyset$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_3\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_3\}$	Ω	$\emptyset$	$\{\omega_3\}$	$\emptyset$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_3\}$	$\{\omega_3\}$
Ω	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$

E	$K_{i_2}(E)$	$(\neg K_{i_2})(E)$	$(\neg K_{i_2})^2(E)$	$(\neg K_{i_2})^3(E)$	$U_{i_2}^{(2)}(E)$	$U_{i_2}^{(n)}(E)$
$\emptyset$	$\emptyset$	Ω	$\{\omega_2\}$	Ω	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_2\}$	$\emptyset$	Ω	$\{\omega_2\}$	Ω	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2\}$	Ω	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
$\{\omega_2, \omega_3\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$
Ω	$\{\omega_1, \omega_3\}$	$\{\omega_2\}$	Ω	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$

E	$K_{i_3}(E)$	$(\neg K_{i_3})(E)$	$(\neg K_{i_3})^2(E)$	$(\neg K_{i_3})^3(E)$	$U_{i_3}^{(2)}(E)$	$U_{i_3}^{(n)}(E)$
$\emptyset$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\emptyset$	$\{\omega_2, \omega_3\}$	$\emptyset$
$\{\omega_2\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$
$\{\omega_3\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\emptyset$	$\{\omega_2, \omega_3\}$	$\emptyset$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\emptyset$	$\{\omega_2, \omega_3\}$	$\emptyset$
$\{\omega_2, \omega_3\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$
Ω	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$

E	$K_{i_4}(E)$	$(\neg K_{i_4})(E)$	$(\neg K_{i_4})^2(E)$	$(\neg K_{i_4})^3(E)$	$U_{i_4}^{(2)}(E)$	$U_{i_4}^{(n)}(E)$
$\emptyset$	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_2\}$	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_3\}$	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_1, \omega_2\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\{\omega_2, \omega_3\}$	$\emptyset$	Ω	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
Ω	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	Ω	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$

Table 1: Agents' Knowledge and Unawareness in Example 1 ($n \geq 3$ or $n = \infty$)

For each agent $j \in \{i_1, i_3\}$, the knowledge operator K_j also satisfies Necessitation. In fact, K_j is induced by a reflexive-and-transitive possibility correspondence $\Pi_j : \Omega \to \mathcal{D}$: $\Pi_{i_1}(\omega_1) = \{\omega_1\}$, $\Pi_{i_1}(\omega_2) = \{\omega_2\}$, and $\Pi_{i_1}(\omega_3) = \Omega$; and $\Pi_{i_3}(\omega_1) = \{\omega_1\}$ and $\Pi_{i_3}(\omega_2) = \Pi_{i_3}(\omega_3) = \Omega$. When $n \geq 3$ or $n = \infty$, while $U_{i_3}^{(n)}$ is degenerate, $U_{i_1}^{(n)}$ is not.

In contrast, for each agent $i \in \{i_2, i_4\}$, while the knowledge operator K_i satisfies Truth Axiom, Positive Introspection, and Monotonicity, K_i violates Necessitation. Thus, no possibility correspondence can induce K_{i_2} or K_{i_4}.

Additional remarks are in order. First, while $(K_i, U_i^{(n)})$ satisfies AU Introspection for each $i \in \{i_2, i_4\}$, the other pairs $(K_j, U_j^{(n)})$ $(j \in \{i_1, i_3\})$ do not. The fact that $(K_j, U_j^{(n)})$ does not satisfy AU Introspection is consistent with Dekel, Lipman, and Rustichini (1998) that there is no non-trivial possibility-correspondence model which satisfies all of Plausibility, KU Introspection, and AU Introspection.

Second, whenever agent i_3 knows some event at some state, agent i_1 knows the event at that state. Yet, there are a state (e.g., ω_3) and an event (e.g., $\{\omega_2\}$) such that, while "more knowledgeable" agent i_1 is unaware of the event at the state, "less knowledgeable" agent i_3 is aware (i.e., not unaware) of the event at that state. In contrast, whenever agent i_4 knows some event at some state, agent i_2 knows the event at that state. Whenever "more knowledgeable" agent i_2 is unaware of some event at some state, "less knowledgeable" agent i_4 is unaware of the event at that state.

Third, agent i_2's (resp. i_4's) knowledge operator can be identified as being defined on $\{\omega_1, \omega_3\}$ (resp. $\{\omega_1\}$). In other words, any state $\omega \in \{\omega_2\}$ (resp. $\omega \in \{\omega_2, \omega_3\}$) is deemed "impossible" by agent i_2 (resp. i_4). In this example, their unawareness is always determined by "impossible" states or the violation of Necessitation.

3.2 (Non-)Stationary Generalized-State-Space Models

Next, I show that my framework accommodates any generalized-state-space possibility-correspondence model (which satisfies Truth Axiom and Positive Introspection).

3.2.1 (Non-)Stationary Possibility-Correspondence Model

Fix a generalized state space $\langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}, \succeq, r \rangle$. Agent i's *possibility correspondence* $\overline{\Pi}_i^{\uparrow} : \Omega \to \mathcal{E}$ *induces* an operator $\overline{K}_i : \mathcal{E} \to \mathcal{E}$ if $\overline{K}_i(E, S) = (\{\omega \in \Omega \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant (E, S)\}, S) \in \mathcal{E}$ for each $(E, S) \in \mathcal{E}$. Denote by $\overline{\Pi}_i^{\uparrow}(\cdot) = (\Pi_i^{\uparrow}(\cdot), \sigma_i(\cdot))$, and call $\sigma_i : \Omega \to \mathcal{S}$ agent i's *awareness function.* The possibility correspondence presupposes the mapping $\overline{\Pi}_i : \Omega \ni \omega \mapsto \overline{\Pi}_i(\omega) = (\Pi_i(\omega), \sigma_i(\omega)) \in \mathcal{P}(\Omega) \times \mathcal{S}$ with $\Pi_i(\cdot) \in \mathcal{D}_{\sigma_i(\cdot)}$. Agent i considers state ω' *possible* at a state ω if $\omega' \in \Pi_i(\omega)$. The mapping $\overline{\Pi}_i^{\uparrow}$ associates, with each state, the event that agent i considers possible at ω. Call each $\Pi_i(\omega)$ the *possibility set* at ω. Proposition S.1 in Online Appendix B.1 provides the four properties on a possibility correspondence under which it induces an operator. The induced operator $\overline{K}_i$ satisfies Monotonicity and $S(K_i(E)) = S(E)$ for all $(E, S(E)) \in \mathcal{E}$. The induced operator $\overline{K}_i$ also satisfies Non-empty Conjunction and Necessitation.

Next, I identify the conditions on the possibility correspondence under which the induced operator $\overline{K}_i$ satisfies Truth Axiom and Positive Introspection. Similarly to standard-state-space models, the following can be shown: $\overline{K}_i$ satisfies Truth Axiom iff $\overline{\Pi}_i^{\uparrow}$ satisfies *Generalized Reflexivity*: $\omega \in \Pi_i^{\uparrow}(\omega)$ for all $\omega \in \Omega$. Also, $\overline{K}_i$ satisfies Positive Introspection iff $\overline{\Pi}_i^{\uparrow}$ satisfies *Generalized Transitivity*: $\omega' \in \Pi_i^{\uparrow}(\omega)$ implies $\overline{\Pi}_i^{\uparrow}(\omega') \leqslant \overline{\Pi}_i^{\uparrow}(\omega)$. Note that Reflexivity of Π_i (i.e., $\omega \in \Pi_i(\omega)$) is sufficient but not necessary for $\overline{K}_i$ to satisfy Truth Axiom.

The crucial difference with the stationary generalized-state-space model of Heifetz, Meier, and Schipper (2006) is that the possibility correspondence $\overline{\Pi}_i^{\uparrow}$ may fail to satisfy *Stationarity*: $\omega' \in \Pi_i(\omega)$ implies $\Pi_i^{\uparrow}(\omega') = \Pi_i^{\uparrow}(\omega)$ (i.e., $\overline{\Pi}_i^{\uparrow}(\omega') = \overline{\Pi}_i^{\uparrow}(\omega)$). In fact, the missing part between Stationarity and Generalized Transitivity is *Generalized Euclideanness*: if $\omega' \in \Pi_i(\omega)$ then $\Pi_i^{\uparrow}(\omega) \subseteq \Pi_i^{\uparrow}(\omega')$. Proposition S.2 in Online Appendix B.1 demonstrates that $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Euclideanness iff $\overline{A}_i^{(2)} = \overline{K}_i \vee \overline{K}_i(\neg\overline{K}_i)$ satisfies *Weak Necessitation*: $\overline{A}_i^{(2)}(E,S) = \overline{K}_i(\overline{S}^{\uparrow})$ for all $(E,S) \in \mathcal{E}$. Hence, as in standard-state-space models, this paper divides Stationarity into two parts, Generalized Transitivity and Generalized Euclideanness.

The framework of this paper can accommodate the possibility-correspondence model of Heifetz, Meier, and Schipper (2006) on a generalized state space in a way such that Weak Necessitation can be relaxed. Weak Necessitation characterizes unawareness as the "lack of conception" in Heifetz, Meier, and Schipper (2006, 2008): the agent is unaware of (E,S) if she does not know $\overline{S}^{\uparrow}$. Unawareness as the lack of conception is stronger than k^n-unawareness:

$$(\neg\overline{K}_i)(\overline{S}^{\uparrow}) \leqslant \overline{U}_i^{(n)}(E,S) \text{ for any } (E,S) \in \mathcal{E}.^{10} \tag{1}$$

Proposition 4 in Section 4.3 asks under what conditions Expression (1) holds with equality, i.e., k^n-unawareness satisfies Weak Necessitation. For instance, it shows that Weak Necessitation and AU Introspection are equivalent. In a standard-state-space model with Necessitation, Weak Necessitation (or AU Introspection) is equivalent to Negative Introspection. In fact, Generalized Euclideanness reduces to Euclideanness, which characterizes Negative Introspection in the standard-state-space model.

Also, the framework can capture such interactive knowledge as common knowledge in addition to agents' individual knowledge and unawareness. Online Appendix E formulates common knowledge in the framework of this paper.

3.2.2 Example

The next example compares a standard-state-space non-partitional possibility-correspondence model and a generalized-state-space stationary possibility-correspondence model.[11]

Example 2. Suppose that a caregiver diagnoses a child with fast and difficult breathing. Suppose that some test (say, an X-ray) is available. For both models, suppose that the most expressive subspace (the entire space for the standard-state-space model) describes whether the test is conducted and whether the child contracts pneumonia.

[10] This follows because, if agent i does not know $\overline{S}^{\uparrow}$, then she does not know any (F,S) by Monotonicity of $\overline{K}_i$. Formally, see the proofs of Remark A.1 (8) and (9) in Appendix A.1.

[11] Example S.1 in Online Appendix B.2 shows that the knowledge operators of agents i_2 and i_4 in Example 1, which cannot be induced by any standard-state-space possibility correspondence, can be embedded into a non-stationary generalized-state-space possibility-correspondence model.

Before formally presenting the models, I informally discuss them. In the standard-state-space model, when the test is not conducted, suppose the caregiver fails to exclude the possibility that the test is conducted. This makes her possibility correspondence non-partitional. Then, when the test is not conducted, the caregiver turns out to be k^n-unaware that the child contracts pneumonia, i.e., she does not know that the child contracts pneumonia, and she does not know that she does not know it (when $n = 2$).

In the generalized-state-space model, at a state at which the test is not conducted, suppose the caregiver is only able to conceive whether the test is conducted or not (i.e., her awareness function associates, with such state, the subspace describing only whether the test is conducted). The possibility correspondence is stationary. However, when the test is not conducted, the caregiver does not know the most expressive subspace describing both test and pneumonia. At such state, the caregiver is unaware that the child contracts pneumonia as the lack of conception. By Weak Necessitation, the caregiver is equivalently k^n-unaware that the child contracts pneumonia.

Now, I formally provide the standard-state-space non-partitional model. The state space Ω consists of four states $\{\omega_{pt}, \omega_{p\neg t}, \omega_{\neg pt}, \omega_{\neg p\neg t}\}$, depending on whether the child contracts pneumonia and the test is conducted. At ω_{pt}, the child contracts pneumonia (p), and the test is conducted (t). At $\omega_{p\neg t}$, the child contracts pneumonia, and the test is not conducted ($\neg t$). At $\omega_{\neg pt}$, the child does not contract pneumonia ($\neg p$), and the test is conducted. At $\omega_{\neg p\neg t}$, the child does not contract pneumonia, and the test is not conducted. Identify the domain with the subsets of the state space $\mathcal{D} = \mathcal{P}(\Omega)$.

Define the possibility correspondence $\Pi_i : \Omega \to \mathcal{D}$ of the caregiver (denoted by i) as follows. At a state $\omega \in \{\omega_{pt}, \omega_{\neg pt}\}$ at which the test is conducted, assume that the test enables the caregiver to diagnose whether the child contracts pneumonia, and consequently she considers ω possible: $\Pi_i(\omega) = \{\omega\}$ for each $\omega \in \{\omega_{pt}, \omega_{\neg pt}\}$. In contrast, at a state $\omega \in \{\omega_{p\neg t}, \omega_{\neg p\neg t}\}$ at which the test is not conducted, assume that she cannot diagnose whether the child contracts pneumonia and that she cannot rule out the possibility that the test is conducted. Consequently, the caregiver considers every state to be possible: $\Pi_i(\omega) = \Omega$ for each $\omega \in \{\omega_{p\neg t}, \omega_{\neg p\neg t}\}$.

Consider the event $E = \{\omega_{pt}, \omega_{p\neg t}\}$ that the child contracts pneumonia. I show that the caregiver is k^2-unaware (in fact, k^n-unaware) of E at a state at which the test is not conducted (i.e., at $\omega_{p\neg t}$ or $\omega_{\neg p\neg t}$). Since the caregiver knows E only at ω_{pt} at which the child contracts pneumonia and the test is conducted, the caregiver does not know E iff ω_{pt} does not hold (either the child does not contract pneumonia or the test is not conducted): $(\neg K_i)(E) = \neg\{\omega_{pt}\}$. The caregiver knows $\neg\{\omega_{pt}\}$ only at $\omega_{\neg pt}$ at which the child does not contract pneumonia and the test is conducted. Hence, the caregiver does not know that she does not know E iff $\omega_{\neg pt}$ does not hold (either the child contracts pneumonia or the test is not conducted): $(\neg K_i)^2(E) = \neg\{\omega_{\neg pt}\}$. Hence, the caregiver is k^2-unaware of E iff the test is not conducted: $U_i^{(2)}(E) = \{\omega_{p\neg t}, \omega_{\neg p\neg t}\}$.

Similarly, it can be seen that the caregiver is k^n-unaware of E iff the test is not conducted. Also, the caregiver is k^n-unaware of the event that the child does not contract pneumonia (i.e., $\neg E = \{\omega_{\neg pt}, \omega_{\neg p\neg t}\}$) iff the test is not conducted. More generally, as shown in Online Appendix B.3, the unawareness operators satisfy:

$$U_i^{(2)}(E) = \begin{cases} \emptyset & \text{if } E \in \{\emptyset, \{\omega_{p\neg t}\}, \{\omega_{\neg p\neg t}\}, \{\omega_{p\neg t}, \omega_{\neg p\neg t}\}, \Omega\} \\ \{\omega_{p\neg t}, \omega_{\neg p\neg t}\} & \text{otherwise} \end{cases}; \text{ and}$$

$$U_i^{(\infty)}(E) = \begin{cases} \emptyset & \text{if } E \in \{\emptyset, \{\omega_{p\neg t}\}, \{\omega_{\neg p\neg t}\}, \{\omega_{pt}, \omega_{\neg pt}\}, \{\omega_{p\neg t}, \omega_{\neg p\neg t}\}, \\ & \quad \{\omega_{pt}, \omega_{p\neg t}, \omega_{\neg pt}\}, \{\omega_{pt}, \omega_{\neg pt}, \omega_{\neg p\neg t}\}, \Omega\} \\ \{\omega_{p\neg t}, \omega_{\neg p\neg t}\} & \text{otherwise} \end{cases}.$$

Next, I move on to the generalized-state-space stationary model. Let the generalized state space consist of four subspaces $\mathcal{S} = \{S, S_p, S_t, S_\varnothing\}$, where $S = \{\omega_{pt}, \omega_{p\neg t}, \omega_{\neg pt}, \omega_{\neg p\neg t}\}$, $S_p = \{\omega_p, \omega_{\neg p}\}$, $S_t = \{\omega_t, \omega_{\neg t}\}$, and $S_\varnothing = \{\omega_\varnothing\}$. The subspace S_p describes only whether the child contracts pneumonia (i.e., ω_p denotes the state at which the child contracts pneumonia, and $\omega_{\neg p}$ the state at which the child does not contract pneumonia). The subspace S_t describes only whether the test is conducted (i.e., ω_t denotes the state at which the test is conducted, and $\omega_{\neg t}$ the state at which the test is not conducted). The least expressible subspace $S_\varnothing$ consists of a single state which does not describe whether the child contracts pneumonia or whether the test is conducted.

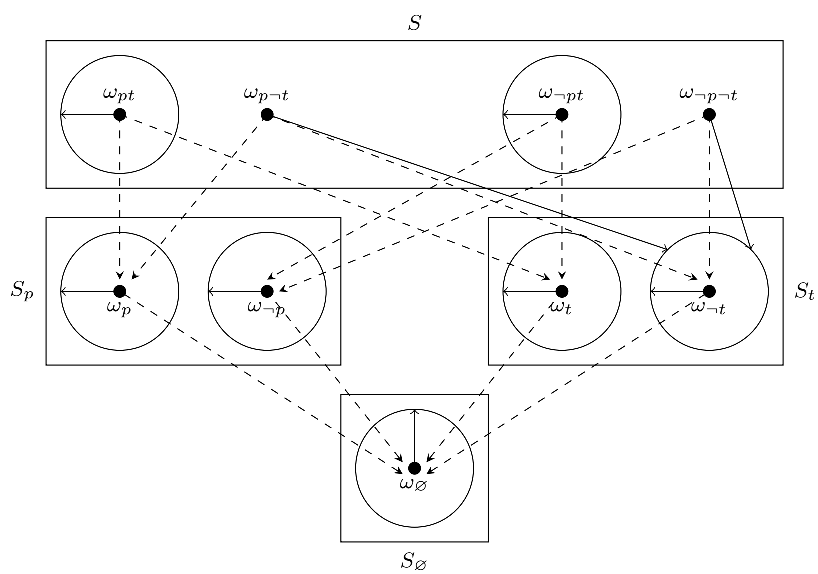

Figure 1: Subspaces, Projections, and Possibility Correspondence $\overline{\Pi}_j^{\uparrow}$ in Example 2

The rectangles in Figure 1 represent the subspaces, and the dashed arrows indicate projections (identity mappings and compositions are omitted). The circles and solid arrows represent the possibility correspondence. In the state space S, at $\omega \in \{\omega_{pt}, \omega_{\neg pt}\}$ at which the test is conducted, the caregiver (for ease of exposition, denote by j) considers ω possible: $\overline{\Pi}_j^{\uparrow}(\omega) = (\{\omega\}, S)$ for each $\omega \in \{\omega_{pt}, \omega_{\neg pt}\}$. At $\omega \in \{\omega_{p\neg t}, \omega_{\neg p\neg t}\}$ at which the test is not conducted, however, the caregiver considers $\omega_{\neg t}$ possible: $\overline{\Pi}_j^{\uparrow}(\omega) =$

$(\{\omega_{\neg t}\}^{\uparrow}, S_t)$ for each $\omega \in \{\omega_{p\neg t}, \omega_{\neg p \neg t}\}$. In any other state space $S' \in \{S_p, S_t, S_\varnothing\}$, at any state $\omega \in S'$, the caregiver considers ω possible: $\overline{\Pi}_j^{\uparrow}(\omega) = (\{\omega\}^{\uparrow}, S')$. It can be seen that the possibility correspondence satisfies Generalized Reflexivity and Stationarity (i.e., Generalized Transitivity and Generalized Euclideanness).

By Weak Necessitation (which follows from Generalized Euclideanness), $\overline{U}_j^{(n)}(\overline{E}) = (\neg \overline{K}_j)(\overline{S(E)}^{\uparrow})$ for any $\overline{E} \in \mathcal{E}$. More specifically, for any subspace $S' \in \{S_p, S_t, S_\varnothing\}$, $\overline{U}_j^{(n)}(E, S') = (\emptyset, S')$ for any $E \in \mathcal{P}(S')$. For the subspace S, $\overline{U}_j^{(n)}(E, S) = (\{\omega_{p\neg t}, \omega_{\neg p \neg t}\}, S)$ for any $E \in \mathcal{P}(S)$. That is, whenever the test is not conducted (i.e., at $\omega_{p\neg t}, \omega_{\neg p \neg t}$) in S, the caregiver is (k^n-)unaware of any event which belongs to the subspace S. This is because, at $\omega_{p\neg t}$ or $\omega_{\neg p \neg t}$ at which the test is not conducted, the caregiver lacks the conception of the most expressive subspace S. Once the properties of unawareness are examined, I revisit this example in Remark 4.

4 Unawareness without AU Introspection

Having defined the basic framework, I now proceed with the main analyses. First, Section 4.1 shows that k^n-unawareness can take only two forms, $n = 2$ or $n = \infty$. Second, Section 4.2 characterizes when unawareness is non-trivial (i.e., there exists an event of which an agent is unaware). Third, Section 4.3 characterizes the properties of unawareness that hold in my framework (and consequently both a non-partitional standard-state-space model and a stationary generalized-state-space model) and the properties of unawareness that may not necessarily hold in my framework (i.e., the properties of unawareness that may distinguish these two models).

Throughout the section, unless otherwise specified, I fix a single agent i, and focus on a single-agent model.

4.1 Equivalent Representations

Throughout the subsection, fix the knowledge operator $\overline{K}_i$. Let $\overline{U}_i^{(n)}$ ($n \geq 2$ or $n = \infty$) be the unawareness operator defined from the knowledge operator $\overline{K}_i$ as in Section 2.4.

The first benchmark result is that, under Truth Axiom, Positive Introspection, and Monotonicity, $(\neg \overline{K}_i)^2 = (\neg \overline{K}_i)^{2n}$ for all $n \in \mathbb{N}$ (documented in Lemma A.1 in Appendix A.2).[12] It implies that $\overline{U}_i^{(\infty)} = \overline{U}_i^{(n)}$ for all $n \geq 3$. In other words, if the chain of the lack of knowledge holds repeatedly three times, then the chain continues without an end. Hence, as long as the notions of unawareness are derived from the lack of knowledge, I can restrict attention to $\overline{U}_i^{(n)}$ with $n \in \{2, \infty\}$, and I can replace $\overline{U}_i^{(\infty)}$ with $\overline{U}_i^{(3)}$. Note

[12]Mathematically, this property is related to the notion of regularly open/closed sets in general topology (e.g., Willard, 2004) in the sense that the assumptions on the knowledge operator are related to a part of the properties of the interior operator on a topological space.

that $\overline{U}_i^{(2)}$ and $\overline{U}_i^{(\infty)}$ may be different (e.g., agents i_1 and i_3 in Example 1). Proposition 4 in Section 4.3 shows that $\overline{U}_i^{(2)} = \overline{U}_i^{(\infty)}$ iff $\overline{U}_i^{(2)}$ satisfies AU Introspection.

As k^n-unawareness takes only two forms, how are k^2- and k^∞-unawareness related to such notions as ignorance and possibility? Dekel, Lipman, and Rustichini (1998), based on Hart (1995), give the following motivating example of unawareness (and unforeseen contingencies): "a person contracting to have a home built fails to foresee the possibility that city regulations preclude locating [her] driveway where [s]he wants it." Although the possibility in this quotation may be rewritten as the event, I restate unawareness so that the person contracting to have a home built is ignorant of knowing the possibility that city regulations preclude locating her driveway where she wants it.

To that end, I define the following three operators on $\mathcal{E}$: the possibility, knowing-whether, and ignorance operators. Fix $\overline{E} \in \mathcal{E}$, and let $n \geq 2$ or $n = \infty$. First, define the *possibility* operator by $\overline{M}_i^{(n)}(\overline{E}) := (\neg\overline{K}_i)(\neg\overline{E}) \wedge \overline{A}_i^{(n)}(\overline{E})$ (Modica and Rustichini, 1999). Thus, $\overline{M}_i^{(n)}(\overline{E})$ is the event that i considers $\overline{E}$ possible in that i does not know its negation $\neg\overline{E}$ and i is (k^n-)aware of $\overline{E}$. Unlike the standard notion of possibility stating that i considers an event possible when she does not know its negation, for agent i to consider an event possible, she has to be aware of the event.

Second, define the *ignorance* operator by $\overline{\partial}_i(\overline{E}) := (\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)(\neg\overline{E})$ (e.g., Lehrer and Samet, 2011). Thus, $\overline{\partial}_i(\overline{E})$ is the event that i is ignorant of $\overline{E}$ in that i does not know $\overline{E}$ nor $\neg\overline{E}$.[13]

Third, define the *knowing-whether* operator by $\overline{J}_i(\overline{E}) := (\neg\overline{\partial}_i)(\overline{E}) = \overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg\overline{E})$ (e.g., Hart, Heifetz, and Samet, 1996). Thus, $\overline{J}_i(\overline{E})$ is the event that i knows whether $\overline{E}$ obtains (or not) in that either i knows $\overline{E}$ or she knows its negation $\neg\overline{E}$. By definition, the negation of the ignorance operator is the knowing-whether operator, and vice versa: $\overline{J}_i = \neg\overline{\partial}_i$ and $\overline{\partial}_i = \neg\overline{J}_i$.

With these definitions, the following proposition answers the above questions. First, Proposition 1 (1) shows that agent i is k^∞-unaware of an event iff she is k^2-unaware of not knowing the event. Thus, not only does unawareness reduce to two forms, but also k^∞-unawareness is reduced to a special form of k^2-unawareness.

Second, the first statement of Proposition 1 (2) (resp. (3)) shows that k^2- (resp. k^∞-) unawareness is a special form of ignorance: agent i is k^2-unaware of an event iff she is ignorant of knowing the event (Part (2)); and agent i is k^∞-unaware of an event iff she is ignorant of knowing that she does not know the event (Part (3)).

Third, the second statement of Proposition 1 (2) (resp. (3)) shows that k^2- (resp. k^∞-) unawareness can also be expressed as ignorance and possibility. Recall the person contracting to have a home built fails to foresee the possibility that city regulations preclude locating her driveway where she wants it (Dekel, Lipman, and Rustichini, 1998; Hart, 1995). Proposition 1 (2) and (3) show that agent i is k^n-unaware of an event iff she is ignorant of the possibility that she knows the event (for $n = 2$ in Part (2) and $n = \infty$ in Part (3)). Thus, formally, the person is unaware of the city regulations

[13] I use the symbol "∂" of the boundary operator on a topological space in the sense that the knowledge operator satisfies a part of the properties of the interior operator.

iff she is ignorant of the possibility that she knows the city regulations.

Proposition 1. *Fix $\overline{E} \in \mathcal{E}$.*

1. $\overline{U}_i^{(\infty)}(\overline{E}) = \overline{U}_i^{(2)}(\neg\overline{K}_i)(\overline{E})$. *Equivalently,* $\overline{A}_i^{(\infty)}(\overline{E}) = \overline{A}_i^{(2)}(\neg\overline{K}_i)(\overline{E})$.
2. $\overline{U}_i^{(2)}(\overline{E}) = \overline{\partial}_i\overline{K}_i(\overline{E}) \leqslant \overline{\partial}_i(\overline{E})$. *Also,* $\overline{U}_i^{(2)}(\overline{E}) = \overline{\partial}_i\overline{M}_i^{(2)}\overline{K}_i(\overline{E})$.
3. $\overline{U}_i^{(\infty)}(\overline{E}) = \overline{\partial}_i\overline{K}_i(\neg\overline{K}_i)(\overline{E})$. *Also,* $\overline{U}_i^{(\infty)}(\overline{E}) = \overline{\partial}_i\overline{M}_i^{(\infty)}\overline{K}_i(\overline{E})$.

In Proposition 1 (2), $\overline{U}_i^{(2)}(\overline{E}) \leqslant \overline{\partial}_i(\overline{E})$ states that k^n-unawareness is stronger than ignorance: if the agent is k^n-unaware of $\overline{E}$, then she is ignorant of $\overline{E}$. As an implication, k^2-unawareness can be restated as ignorance and non-knowledge of non-knowledge, that is, $\overline{U}_i^{(2)}(\cdot) = \overline{\partial}_i(\cdot) \wedge (\neg\overline{K}_i)^2(\cdot)$: the agent is k^2-unaware of an event iff she is ignorant of it and she does not know that she does not know it.

4.2 Characterization of Non-triviality

Agent i's unawareness operator $\overline{U}_i$ on a model $\mathcal{M} = \langle \mathcal{E}, (\overline{K}_i, \overline{U}_i)_{i\in I}\rangle$ represents a *non-trivial form of unawareness* (or $\overline{U}_i$ is *non-trivial*) if $\overline{U}_i(\overline{E}) \neq (\emptyset, S(E))$ (i.e., $U_i(E) \neq \emptyset$) for some $\overline{E} \in \mathcal{E}$. Agent i's unawareness operator $\overline{U}_i$ is *trivial* otherwise. Dekel, Lipman, and Rustichini (1998) show an unawareness operator on a standard state space is trivial under Plausibility, KU Introspection, and AU Introspection. Modica and Rustichini (1994) show it is trivial under Symmetry ($\overline{U}_i(\overline{E}) = \overline{U}_i(\neg\overline{E})$ for all $\overline{E} \in \mathcal{E}$).

Section 4.2.1 characterizes non-triviality. It also shows that one's underlying knowledge is not necessarily identified from her (un)awareness. Section 4.2.2 shows that there are two channels through which an agent is unaware of events.

To simplify the analysis, as mentioned, this section focuses on a single-agent model. Thus, a model $\mathcal{M} = \langle \mathcal{E}, (\overline{K}_i, \overline{U}_i)\rangle$ represents a *non-trivial form of unawareness* (or $\mathcal{M}$ is *non-trivial*) if the unawareness operator $\overline{U}_i$ is non-trivial. The model $\mathcal{M}$ is *trivial* otherwise. Also, denote by $\mathcal{M}^{(n)} := \langle \mathcal{E}, (\overline{K}_i, \overline{U}_i^{(n)})\rangle$ (with $n \in \{2, \infty\}$) the model in which the unawareness operator $\overline{U}_i^{(n)}$ is derived from the knowledge operator.

4.2.1 Characterization

To characterize when a model $\mathcal{M}^{(n)} = \langle \mathcal{E}, (\overline{K}_i, \overline{U}_i^{(n)})\rangle$ (with $n \in \{2, \infty\}$) is non-trivial, I define a collection of events that represents one's knowledge. Call an event $\overline{E}$ *self-evident* to agent i if $\overline{E} \leqslant \overline{K}_i(\overline{E})$ (i.e., $E \subseteq K_i(E)$). Define $\mathcal{J}_i := \{\overline{E} \in \mathcal{E} \mid \overline{E} \leqslant \overline{K}_i(\overline{E})\}$, and call $\mathcal{J}_i$ agent i's *self-evident collection.*[14] To examine the qualitative feature of

[14] The self-evident collection summarizes or recovers $\overline{K}_i$ in that $\overline{K}_i(\overline{E}) = \sup\{\overline{F} \in \mathcal{E} \mid \overline{F} \in \mathcal{J}_i \text{ and } \overline{F} \leqslant \overline{E}\}$, where the supremum is taken on $\langle \mathcal{E}, \leqslant\rangle$. Fukuda (2019a) studies self-evident collections on a standard state space to represent agents' knowledge and common knowledge.

knowledge, that is, whether the self-evident collection is closed under negation, define $\mathcal{J}_i^* := \{\overline{E} \in \mathcal{E} \mid \neg \overline{E} \in \mathcal{J}_i\}$.[15]

The following proposition characterizes non-triviality by whether $\mathcal{J}_i$ is closed under negation, i.e., whether there is a self-evident event $\overline{E}$ whose negation is not self-evident: $\overline{E} \in \mathcal{J}_i \setminus \mathcal{J}_i^*$. The characterization implies that $\mathcal{M}^{(2)}$ on a standard state space is generically non-trivial even when the knowledge operator is induced from a reflexive-and-transitive possibility correspondence (e.g., agents i_1 and i_3 in Example 1).

Proposition 2. *1. (a) For any $\overline{E} \in \mathcal{E}$, $\overline{U}_i^{(2)}(\overline{E}) \neq (\emptyset, S(E))$ iff $\overline{K}_i(\overline{E}) \in \mathcal{J}_i \setminus \mathcal{J}_i^*$.*

(b) $\mathcal{M}^{(2)}$ is non-trivial iff $\mathcal{J}_i \setminus \mathcal{J}_i^* \neq \emptyset$.

2. (a) For any $\overline{E} \in \mathcal{E}$, $\overline{U}_i^{(\infty)}(\overline{E}) \neq (\emptyset, S(E))$ iff $\overline{K}_i(\neg\overline{K}_i)(\overline{E}) \in \mathcal{J}_i \setminus \mathcal{J}_i^*$.

(b) $\mathcal{M}^{(\infty)}$ is non-trivial iff $\{\overline{F} \in \mathcal{J}_i \setminus \mathcal{J}_i^* \mid \overline{K}_i(\neg\overline{F}) \in \mathcal{J}_i \setminus \mathcal{J}_i^*\} \neq \emptyset$.

Proposition 2 (1b) and (2b) characterize when models $\mathcal{M}^{(2)}$ and $\mathcal{M}^{(\infty)}$ are non-trivial, respectively. To that end, Parts (1a) and (2a) characterize when the k^2- and k^∞-unawareness of an event are not empty, respectively.

Take the model $\mathcal{M}^{(2)}$. Part (1a) states that the unawareness $\overline{U}_i^{(2)}(\overline{E})$ of an event $\overline{E}$ is not empty iff the non-knowledge $(\neg\overline{K}_i)(\overline{E})$ is not self-evident (the knowledge $\overline{K}_i(\overline{E})$ is self-evident by Positive Introspection). Part (1b), in words, states that the model $\mathcal{M}^{(2)}$ is non-trivial iff $\mathcal{J}_i$ is *not* closed under the operation of negation.

For the model $\mathcal{M}^{(\infty)}$, recall Proposition 1 (1): k^∞-unawareness $\overline{U}_i^{(\infty)}$ is equivalent to k^2-unawareness $\overline{U}_i^{(2)}(\neg\overline{K}_i)$ of non-knowledge. Thus, Part (2a) states that the unawareness $\overline{U}_i^{(\infty)}(\overline{E})$ of an event $\overline{E}$ is not empty iff the non-knowledge of the non-knowledge $(\neg\overline{K}_i)^2(\overline{E})$ is not self-evident (the knowledge of non-knowledge $\overline{K}_i(\neg\overline{K}_i)(\overline{E})$ is self-evident by Positive Introspection). This condition is stronger than (1a): while (1a) holds as long as the non-knowledge $(\neg\overline{K}_i)(\overline{F})$ is not self-evident for some event $\overline{F}$, (1b) holds when the event $\overline{F}$ has to be $\overline{F} = (\neg\overline{K}_i)(\overline{E})$ for some $\overline{E}$.

Part (2b) is also stronger than (1b), which states that $\mathcal{M}^{(2)}$ is non-trivial iff there is $\overline{F} \in \mathcal{J}_i \setminus \mathcal{J}_i^*$. (2b) states that the model $\mathcal{M}^{(\infty)}$ is non-trivial iff there is an event $\overline{F} \in \mathcal{J}_i \setminus \mathcal{J}_i^*$ such that the non-knowledge of its negation $(\neg\overline{K}_i)(\neg\overline{F})$ is not self-evident.

As an example, consider agent i_3 in Example 1. Since $\mathcal{J}_{i_3} \setminus \mathcal{J}_{i_3}^* = \{\{\omega_1\}\}$, $U_{i_3}^{(2)}$ is non-trivial. However, $U_{i_3}^{(\infty)}$ is trivial because $K_{i_3}(\neg\{\omega_1\}) = \emptyset \in \mathcal{J}_{i_3} \cap \mathcal{J}_{i_3}^*$.

Proposition 2 states that whether a model is non-trivial does not hinge on knowledgeability but on the qualitative feature of the self-evident collection $\mathcal{J}_i$. On a related point, two different knowledge operators may induce the same unawareness operator (especially when the induced unawareness operator is trivial).

Remark 3. While the underlying knowledge operator $\overline{K}_i$ is identified from the ignorance operator $\overline{\partial}_i$ through $\overline{K}_i(\overline{E}) = (\neg\overline{\partial}_i)(\overline{E}) \wedge \overline{E}$ for each $\overline{E} \in \mathcal{E}$ by Truth Axiom of $\overline{K}_i$ (see Remark A.1 (10) in Appendix A.1 for the proof), the underlying knowledge

[15] It can be seen that $\overline{K}_i$ satisfies Negative Introspection iff $\mathcal{J}_i = \mathcal{J}_i^*$.

operator is not necessarily identifiable from the k^n-unawareness operator. I provide an example (of a standard-state-space model) in which two different knowledge operators induce the same unawareness operator.

Define agent i's knowledge operator K_i as $K_i = K_{i_2}$ in Example 1. Define agent j's knowledge operator K_j as follows: $K_j(E) = \{\omega_1, \omega_3\}$ if $E \in \{\{\omega_1, \omega_3\}, \Omega\}$; and $K_j(E) = \emptyset$ otherwise. On the one hand, $K_j(\cdot) \subseteq K_i(\cdot)$ and $K_j \neq K_i$. Agent i is at least as knowledgeable as agent j in that whenever agent j knows some event at some state, agent i knows the event at that state. On the other hand, their unawareness operators are identical: $U_i^{(n)}(\cdot) = U_j^{(n)}(\cdot) = \{\omega_2\}$ for each $n \in \{2, \infty\}$.

4.2.2 Two Channels of Unawareness

Next, I argue that there are two channels through which the agent is unaware of events. By Monotonicity of the knowledge operator $\overline{K}_i$, for any event (E, S),

$$\overline{A}_i^{(n)}(E, S) \leqslant \overline{K}_i(\overline{S}^{\uparrow}) \leqslant \overline{S}^{\uparrow} \text{ or equivalently } (\emptyset, S) \leqslant (\neg\overline{K}_i)(\overline{S}^{\uparrow}) \leqslant \overline{U}_i^{(n)}(E, S).$$

Thus, there are two channels through which $\overline{U}_i^{(n)}(\overline{E}) \neq (\emptyset, S)$ (i.e., $\overline{A}_i^{(n)}(\overline{E}) \neq \overline{S}^{\uparrow}$). The first is the failure of Weak Necessitation: $\overline{A}_i^{(n)}(\overline{E}) \neq \overline{K}_i(\overline{S}^{\uparrow})$, which turns out to be equivalent to AU Introspection. The second is whether $\overline{K}_i(\overline{S}^{\uparrow}) = \overline{S}^{\uparrow}$ in the subspace S to which the event belongs. In a standard-state-space model, the second channel reduces to the violation of Necessitation.

I discuss (i) the role of the second channel in a generalized-state-space model, (ii) the role of the first channel (i.e., the violation of AU Introspection) in a standard-state-space model, and (iii) the role of the second channel (i.e., the violation of Necessitation) in the standard-state-space model.

First, for a stationary possibility correspondence on a generalized state space as in Heifetz, Meier, and Schipper (2006), Generalized Euclideanness guarantees that the model satisfies AU Introspection (recall Section 3.2). Hence, the only channel through which the agent is unaware of events is through the second channel. Specifically, since $K_i(S^{\uparrow}) = \{\omega \in S \mid \sigma_i(\omega) = S\}^{\uparrow}$, the second channel hinges on whether the awareness function σ_i assigns each state $\omega \in S$ to the state space S in which ω resides.

Second, in contrast, in a standard-state-space possibility-correspondence model in which Necessitation holds, the only channel through which the agent is unaware of events is through the violation of AU Introspection, equivalently, Negative Introspection. In fact, Proposition 2 implies that $\mathcal{M}^{(2)}$ on a standard state space is non-trivial iff $\mathcal{J}_i$ is not closed under complementation, i.e., Negative Introspection fails. Put differently, a partitional standard-state-space model represents a trivial form of unawareness because $\mathcal{J}_i = \mathcal{J}_i^*$, which is equivalent to Negative Introspection. It follows that a standard-state-space possibility-correspondence model is non-trivial iff it is not partitional (recall Section 3.1). An underlying intuition is simple: $\mathcal{M}^{(2)}$ is non-trivial iff Negative Introspection (equivalently, AU Introspection) fails. When Negative Introspection fails, unawareness is non-trivial even with Necessitation. The failure of AU Introspection also leads to non-trivial unawareness in a generalized-state-space model.

Third, to discuss the role of the second channel (i.e., Necessitation) in a standard-state-space model, consider agent i_1 in Example 1. Her knowledge operator K_{i_1} satisfies Necessitation, and her unawareness operator $U_{i_1}^{(n)}$ is non-trivial. In fact, $\{\omega_2\} \in \mathcal{J}_{i_1} \setminus \mathcal{J}_{i_1}^*$ satisfies $K_{i_1}(\neg\{\omega_2\}) = \{\omega_1\} \in \mathcal{J}_{i_1} \setminus \mathcal{J}_{i_1}^*$. The difference with the second channel is that, in order for agent i_1 with Necessitation to be unaware of an event E at ω, she does not know E at ω but she *knows* E at another state ω'. This is because, under Necessitation, $K_{i_1}(E) = \emptyset$ implies $\emptyset = (\neg K_{i_1})^2(E) \supseteq U_{i_1}^{(n)}(E)$. Put differently, if the agent does not know E at any state, then she knows that she does not know E at any state (by Necessitation), and thus she is not unaware of E at any state.

Suppose that, in addition to AU Introspection, Necessitation also fails. At any state ω at which agent i does not know a tautology, she does not know any event, and thus she is unaware of every event: $\emptyset \neq (\neg K_i)(\Omega) \subseteq U_i^{(\infty)}(E) \subseteq U_i^{(2)}(E)$ for all E. As an example, consider agent j whose knowledge operator K_j satisfies $K_j(E) = K_{i_1}(E)$ if $E \neq \Omega$; and $K_j(\Omega) = \{\omega_1, \omega_2\}$ in Example 1. Then, $U_j^{(n)}(\cdot) = \{\omega_3\}$. At ω_3, agent j is unaware of $E \in \{\emptyset, \{\omega_3\}, \Omega\}$ of which agent i_1 is aware.

4.3 Properties of Unawareness

Next, Section 4.3.1 studies the properties of unawareness that hold in a model, i.e., the common properties of unawareness between a non-partitional standard-state-space model and a stationary generalized-state-space model. Section 4.3.2, in contrast, characterizes properties that distinguish a non-partitional standard-state-space model and a stationary generalized-state-space model. For example, it establishes the equivalence among Weak Necessitation, AU Introspection, and Symmetry. Throughout the subsection, consider a single-agent model $\mathcal{M}^{(n)} = \langle \mathcal{E}, (\overline{K}_i, \overline{U}_i^{(n)}) \rangle$ with $n \in \{2, \infty\}$.

4.3.1 Common Properties of Unawareness

I examine properties of k^n-unawareness. First, by definition, any model $\mathcal{M}^{(n)}$ satisfies Plausibility, and any $\mathcal{M}^{(\infty)}$ satisfies Strong Plausibility: $\overline{U}_i^{(\infty)}(\cdot) = \bigwedge_{r\in\mathbb{N}}(\neg\overline{K}_i)^r(\cdot)$ (Heifetz, Meier, and Schipper, 2006, 2008).[16]

Second, as Expression (1) shows, unawareness as the lack of conception implies one as the lack of knowledge: $(\neg\overline{K}_i)(\overline{S}^{\uparrow}) \leqslant \overline{U}_i^{(n)}(E, S)$. Equivalently, if the agent is k^n-aware of an event (E, S) then she knows its subspace $\overline{S}^{\uparrow}$: $\overline{A}_i^{(n)}(E, S) \leqslant \overline{K}_i(\overline{S}^{\uparrow})$.

Third, any model $\mathcal{M}^{(n)}$ satisfies KU Introspection as Dekel, Lipman, and Rustichini (1998) note that Truth Axiom, Monotonicity, and Plausibility yield KU Introspection (these three properties imply KU Introspection on a generalized state space as well). The following proposition summarizes other properties of $\mathcal{M}^{(n)}$.

[16] I also use other terminologies coined by Heifetz, Meier, and Schipper (2006, 2008) and Schipper (2015) for some properties in Proposition 3.

Proposition 3. *Any model $\mathcal{M}^{(n)}$ satisfies the following. Let $\overline{E} = (E, S) \in \mathcal{E}$.*

1. *Second-order-k^n-Unawareness: $(\neg \overline{K}_i)(\overline{S}^{\uparrow}) = \overline{U}_i^{(n)} \overline{U}_i^{(n)}(E, S)$.*
2. *Reverse AU Introspection: $\overline{U}_i^{(n)} \overline{U}_i^{(n)}(\overline{E}) \leqslant \overline{U}_i^{(n)}(\overline{E})$. Also, $\overline{U}_i^{(n)} \overline{U}_i^{(n)} \overline{U}_i^{(n)}(\overline{E}) = \overline{U}_i^{(n)} \overline{U}_i^{(n)}(\overline{E})$.*
3. *JU Introspection: $\overline{U}_i^{(n)}(\overline{E}) \leqslant \overline{\partial}_i \overline{U}_i^{(n)}(\overline{E})$ with equality. Equivalently, $\overline{J}_i \overline{A}_i^{(n)}(\overline{E}) \leqslant \overline{A}_i^{(n)}(\overline{E})$ with equality.*
4. *Weak A-Negative Introspection of $\overline{A}_i^{(2)}$: $(\neg \overline{K}_i)(\overline{E}) \wedge \overline{A}_i^{(2)}(\overline{E}) = \overline{K}_i(\neg \overline{K}_i)(\overline{E})$.*
5. *AK Self-Reflection: $\overline{A}_i^{(n)}(\overline{E}) = \overline{A}_i^{(n)} \overline{K}_i(\overline{E})$. Equivalently, $\overline{U}_i^{(n)}(\overline{E}) = \overline{U}_i^{(n)} \overline{K}_i(\overline{E})$.*
6. *A-Introspection: $\overline{A}_i^{(n)}(\overline{E}) = \overline{K}_i \overline{A}_i^{(n)}(\overline{E})$. Equivalently, $\overline{U}_i^{(n)}(\overline{E}) = (\neg \overline{K}_i) \overline{A}_i^{(n)}(\overline{E})$.*
7. *Weak AA Self-Reflection: $\overline{A}_i^{(n)}(\overline{E}) \leqslant \overline{A}_i^{(n)} \overline{A}_i^{(n)}(\overline{E})$ with equality if $n = 2$. Also, $\overline{A}_i^{(n)} \overline{A}_i^{(n)}(\overline{E}) = \overline{A}_i^{(n)} \overline{A}_i^{(n)} \overline{A}_i^{(n)}(\overline{E})$.*
8. *Possibility-of-Awareness: $\overline{M}_i^{(n)} \overline{A}_i^{(n)}(\overline{E}) = \overline{A}_i^{(n)} \overline{A}_i^{(n)}(\overline{E})$.*

Property (1) states that unawareness as the lack of conception (the left-hand side) is equivalent to k^n-unawareness of k^n-unawareness. This property follows from KU Introspection and Monotonicity of $\overline{K}_i$. Technically, the event $\overline{U}_i^{(n)} \overline{U}_i^{(n)}(E, S)$ (agent i is (k^n-)unaware of being (k^n-)unaware of an event (E, S)) is a complicated object according to the definition of the k^n-unawareness operator $\overline{U}_i^{(n)}$.[17] Again, Property (1) provides an explicit characterization $\overline{U}_i^{(n)} \overline{U}_i^{(n)}(E, S) = (\neg \overline{K}_i)(\overline{S}^{\uparrow})$.

Property (2) is the "converse" of AU Introspection: if the agent is unaware of being unaware of an event, then she is unaware of the event. Reverse AU Introspection follows from $\overline{U}_i^{(n)} \overline{U}_i^{(n)}(E, S) = (\neg \overline{K}_i)(\overline{S}^{\uparrow})$ (i.e., Property (1)) and the part of Weak Necessitation $(\neg \overline{K}_i)(\overline{S}^{\uparrow}) \leqslant \overline{U}_i^{(n)}(E, S)$. Since Reverse AU Introspection always holds, I will show that k^n-unawareness, defined directly from the lack of knowledge, coincides with unawareness as the lack of conception when the k^n-unawareness operator $\overline{U}_i^{(n)}$ satisfies AU Introspection. I remark that the ignorance operator satisfies $\overline{\partial}_i \overline{\partial}_i(\cdot) \leqslant \overline{\partial}_i(\cdot)$: if the agent is ignorant of being ignorant of an event, then she is ignorant of the event.

Property (3), JU Introspection, states that if the agent is unaware of an event, then she is *ignorant* of being unaware of it. Indeed, JU Introspection holds with equality. Reverse AU Introspection is also seen as a consequence of JU Introspection (with equality). Namely, since Proposition 1 (2) states that unawareness implies ignorance: $\overline{U}_i^{(n)}(\cdot) \leqslant \overline{\partial}_i(\cdot)$, it follows that $\overline{U}_i^{(n)} \overline{U}_i^{(n)}(\cdot) \leqslant \overline{\partial}_i \overline{U}_i^{(n)}(\cdot) = \overline{U}_i^{(n)}(\cdot)$.

[17] Technically, $\overline{U}_i^{(n)} \overline{U}_i^{(n)}(\overline{E}) = \bigwedge_{r=1}^{n} (\neg \overline{K}_i)^r (\overline{U}_i^{(n)}(\overline{E})) = \bigwedge_{r=1}^{n} (\neg \overline{K}_i)^r (\bigwedge_{s=1}^{n} (\neg \overline{K}_i)^s (\overline{E}))$.

Property (4), Weak A-Negative Introspection, is proposed by Li (2009): if the agent does not know an event and she is aware of the event, then she knows that she does not know it. For an arbitrary unawareness operator $\overline{U}_i$ such that the model $\langle \mathcal{E}, (\overline{K}_i, \overline{U}_i) \rangle$ satisfies Plausibility, the model satisfies Weak A-Negative Introspection iff $\overline{A}_i = \overline{A}_i^{(2)}$.

Weak A-Negative Introspection for $n = \infty$, however, may not hold (i.e., the "$\leqslant$" part may fail). For example, consider Example 1: $(\neg K_{i_1})(\{\omega_1, \omega_2\}) \cap A_{i_1}^{(\infty)}(\{\omega_1, \omega_2\}) = \{\omega_3\} \not\subseteq \emptyset = K_{i_1}(\neg K_{i_1})(\{\omega_1, \omega_2\})$. Proposition 4 shows that Weak A-Negative Introspection for $n = \infty$ is equivalent to Weak Necessitation for $n = 2$. Weak A-Negative Introspection for $n = \infty$ is also equivalent to Weak Negative Introspection for $n = 2$: $(\neg \overline{K}_i)(\overline{E}) \wedge \overline{A}_i^{(2)}(\neg \overline{K}_i)(\overline{E}) = \overline{K}_i(\neg \overline{K}_i)(\overline{E})$ (Fagin and Halpern, 1987; Halpern, 2001), as Proposition 1 (1) implies $\overline{A}_i^{(\infty)} = \overline{A}_i^{(2)}(\neg \overline{K}_i)$.

Properties (5) and (6) are proposed by Modica and Rustichini (1994, 1999). Property (5), AK Self-Reflection, is equivalent to $\overline{U}_i^{(n)} = \overline{U}_i^{(n)} \overline{K}_i$: the agent is unaware of an event iff she is unaware of knowing it. Property (6), A-Introspection, is equivalent to $\overline{U}_i^{(n)} = (\neg \overline{K}_i) \overline{A}_i^{(n)}$: the agent is unaware iff she does not know that she is aware.

Property (7), Weak AA Self-Reflection, is a part of AA Self-Reflection (Modica and Rustichini, 1994, 1999): $\overline{A}_i^{(n)} = \overline{A}_i^{(n)} \overline{A}_i^{(n)}$. AA Self-Reflection states that the agent is aware of an event iff she is aware of being aware of it. While AA Self-Reflection holds when $n = 2$, only the "$\leqslant$" direction (i.e., Weak AA Self-Reflection) holds when $n = \infty$. For instance, in Example 1, $A_{i_1}^{(\infty)}(\{\omega_1\}) = \{\omega_1, \omega_2\} \neq \Omega = A_{i_1}^{(\infty)} A_{i_1}^{(\infty)}(\{\omega_1\})$. Although the first-order and second-order (un)awareness may differ for $n = \infty$, the second-order and any higher-order (un)awareness coincide for any n, e.g., $\overline{A}_i^{(n)} \overline{A}_i^{(n)} = \overline{A}_i^{(n)} \overline{A}_i^{(n)} \overline{A}_i^{(n)}$ and $\overline{U}_i^{(n)} \overline{U}_i^{(n)} = \overline{U}_i^{(n)} \overline{U}_i^{(n)} \overline{U}_i^{(n)}$ (i.e., Property (2)).

Property (8) states that the agent is aware of being aware of an event iff she considers it possible that she is aware of the event. Properties (7) and (8) imply that if the agent is aware of an event then she considers it possible that she is aware of the event.

I discuss the question raised by Dekel, Lipman, and Rustichini (1998) and subsequently studied by Chen, Ely, and Luo (2012): which of the three axioms of Plausibility, KU Introspection, and AU Introspection is to be retained in a standard-state-space model so as to capture a non-trivial form of unawareness. Section 4.2.2 has characterized two channels through which the agent is unaware of events. In the context of a standard-state-space model, the first channel is the violation of AU Introspection, and the second channel that of Necessitation. Now, Proposition 3 implies that any model satisfies Plausibility (by definition), KU Introspection, Reverse AU Introspection, and JU Introspection (instead of AU Introspection). Hence, a standard-state-space possibility-correspondence model can represent a non-trivial form of unawareness when the model violates AU Introspection. Chen, Ely, and Luo (2012) also show that, in a standard-state-space model, Negative Introspection and AU Introspection are equivalent, and hence AU Introspection clashes with the other two properties. Yet, Proposition 3 shows that the model satisfies Reverse AU Introspection and JU Introspection, instead of AU Introspection. The following example discusses how agent i_1

in Example 1 satisfies Reverse AU Introspection and JU Introspection. Section 4.3.2 characterizes AU Introspection (equivalently, Weak Necessitation). Section 5 studies the implications of the violation of AU Introspection.

Example 3. To discuss non-trivial unawareness on a standard state space, consider the well-known example of the "curious incident" based on Dekel, Lipman, and Rustichini (1998) and Geanakoplos (1989). Suppose that Watson and Sherlock Homes were reasoning about whether there had been a human or dog intruder on the previous night. Let the state space be $\Omega = \{\omega_1, \omega_2, \omega_3\}$, where: ω_1 is the state at which there is a human intruder so that the dog barks, ω_2 the state at which there is a dog intruder so that the cat howls, and ω_3 the state at which there is no intruder.

When Watson reported that the dog did not bark and the cat did not howl, Holmes replied "that is the curious incident." The difference between the reasoning of Homes and Watson is that Watson took information only at face value. To see this, define Watson's (agent i_1's) possibility correspondence. At ω_1, the dog barks, and thus Watson considers ω_1 possible: $\Pi_{i_1}(\omega_1) = \{\omega_1\}$. At ω_2, the cat howls, and thus Watson considers ω_2 possible: $\Pi_{i_1}(\omega_2) = \{\omega_2\}$. At ω_3, however, Watson does not receive any signal (the dog did not bark and the cat did not howl), and he considers *any* state to be possible: $\Pi_{i_1}(\omega_3) = \{\omega_1, \omega_2, \omega_3\}$.

For Homes (denote by agent j), at ω_1, the dog barks, and Homes considers ω_1 possible: $\Pi_j(\omega_1) = \{\omega_1\}$. At ω_2, the cat howls, and Homes considers ω_2 possible: $\Pi_j(\omega_2) = \{\omega_2\}$. At ω_3, however, Holmes inferred that neither man nor dog had intruded on the premises: $\Pi_j(\omega_3) = \{\omega_3\}$. This is because, if a man had intruded, then the dog would have barked; and if a dog had intruded, then the cat would have howled. While Watson took the absence of two signals only at face value so that Watson was not able to make any inference from the absence of signals at ω_3, Homes was able to infer that there was no human or canine intruder from the absence of two signals.

Since the possibility correspondence of Homes is partitional, at any state, Homes is aware of every event. In contrast, since the possibility correspondence of Watson is not partitional, the unawareness of Watson is non-trivial. Watson's unawareness operator satisfies Reverse AU Introspection and JU Introspection (in addition to Plausibility and KU Introspection). At ω_3 at which there is no human or canine intruder, Watson is k^n-unaware of a human intruder ($\{\omega_1\}$) or a dog intruder ($\{\omega_2\}$). For the event that there was a human intruder or the one that there was a dog intruder, Watson never knows that he is unaware of the event at any state (KU Introspection). However, Watson knows a tautology, a statement that holds at every state. Thus, for each of these two events, Watson knows that he never knows that he is unaware of the event at any state. Consequently, at ω_3, while Watson is unaware of a human intruder or a dog intruder, Watson is not unaware that he is unaware of a human intruder or a dog intruder (i.e., while AU Introspection fails, Reverse AU Introspection holds).[18]

[18] Recall that, in a standard-state-space possibility-correspondence model, the unique channel for non-trivial unawareness is through the failure of AU Introspection, as Necessitation holds. Hence, AU Introspection must violate in order to represent a non-trivial form of unawareness.

Instead of AU Introspection, Watson is ignorant of being unaware of each event (JU Introspection): he does not know that he is unaware of the event, and he does not know that he is aware of the event: $\omega_3 \in \partial_{i_1} U_{i_1}^{(n)}(E) = U_{i_1}^{(n)}(E)$ for each $E \in \{\{\omega_1\}, \{\omega_2\}\}$.

4.3.2 Lack of Knowledge and Lack of Conception

I examine properties of unawareness under which unawareness as the lack of knowledge coincides with one as the lack of conception. The following proposition characterizes Weak Necessitation of $\overline{U}_i^{(n)}$ (the first part for $n = 2$, and the second part for $n = \infty$).

Proposition 4. *Let $\mathcal{M}^{(n)}$ be a model.*

1. *Fix $n = 2$. Weak Necessitation of $\overline{U}_i^{(2)}$ (i.e., $\overline{U}_i^{(2)}(E, S) = (\neg\overline{K}_i)(\overline{S}^{\uparrow})$ for all $(E, S) \in \mathcal{E}$) is equivalent to any of (a)-(h) where $n = 2$ is substituted.*
2. *Fix $n = \infty$. Weak Necessitation of $\overline{U}_i^{(\infty)}$ (i.e., $\overline{U}_i^{(\infty)}(E, S) = (\neg\overline{K}_i)(\overline{S}^{\uparrow})$ for all $(E, S) \in \mathcal{E}$) is equivalent to any of (f)-(i) where $n = \infty$ is substituted.*

(a) Subjective Negative Introspection: $\overline{K}_i(\overline{S}^{\uparrow}) \wedge (\neg\overline{K}_i)(E, S) \leqslant \overline{K}_i(\overline{K}_i(\overline{S}^{\uparrow}) \wedge (\neg\overline{K}_i)(E, S))$.

(b) Negative Non-Introspection: $\overline{U}_i^{(2)}(\cdot) \leqslant (\neg\overline{K}_i)^3(\cdot)$.

(c) Strong Plausibility of $\overline{U}_i^{(2)}$: $\overline{U}_i^{(2)} = \overline{U}_i^{(\infty)}$.

(d) Weak A-Negative Introspection of $\overline{A}_i^{(\infty)}$: $(\neg\overline{K}_i)(\cdot) \wedge \overline{A}_i^{(\infty)}(\cdot) = \overline{K}_i(\neg\overline{K}_i)(\cdot)$.

(e) Symmetry of $\overline{U}_i^{(2)}$: $\overline{U}_i^{(2)}(\overline{E}) = \overline{U}_i^{(2)}(\neg\overline{E})$.

(f) AU Introspection of $\overline{U}_i^{(n)}$: $\overline{U}_i^{(n)}(\cdot) \leqslant \overline{U}_i^{(n)}\overline{U}_i^{(n)}(\cdot)$.

(g) MU Introspection of $\overline{U}_i^{(n)}$: $\overline{M}_i^{(n)}\overline{U}_i^{(n)}(E, S) = (\emptyset, S)$.

(h) Monotonicity of $\overline{A}_i^{(n)}$: $\overline{E} \leqslant \overline{F}$ implies $\overline{A}_i^{(n)}(\overline{E}) \leqslant \overline{A}_i^{(n)}(\overline{F})$.

(i) AA Self Reflection of $\overline{A}_i^{(\infty)}$: $\overline{A}_i^{(\infty)} = \overline{A}_i^{(\infty)}\overline{A}_i^{(\infty)}$.

In Proposition 4, the meaning of each of (f), (g), and (h) depends on whether $n = 2$ or $n = \infty$. In Part (1), for example, Weak Necessitation of $\overline{U}_i^{(2)}$ is equivalent to AU Introspection of $\overline{U}_i^{(2)}$ (i.e., (f) for $n = 2$). In Part (2), Weak Necessitation of $\overline{U}_i^{(\infty)}$ is equivalent to AU Introspection of $\overline{U}_i^{(\infty)}$ (i.e., (f) for $n = \infty$). By (c), any of (a)-(h) with $n = 2$ implies any of (f)-(i) with $n = \infty$.

Proposition 4 sheds light on the difference between standard and generalized state-space models with Necessitation. When a standard-state-space model satisfies Necessitation, the only channel for non-trivial k^n-unawareness is whether AU Introspection holds. Thus, under each property in Proposition 4, the k^n-unawareness operator is trivial. In contrast, if the k^n-unawareness operator in a generalized-state-space possibility-correspondence model satisfies any property in Proposition 4, then the only channel of unawareness is whether the agent knows the subspace to which a given event belongs.

Five remarks on Proposition 4 are in order. First, Proposition 4 sheds light on Chen, Ely, and Luo (2012), which show Negative Introspection and AU Introspection are equivalent in a standard state space under Necessitation. Proposition 4 shows that Subjective Negative Introspection (i.e., (a)) and AU Introspection (i.e., (f)) are equivalent for k^2-unawareness. Subjective Negative Introspection reduces to Negative Introspection in a standard state space under Necessitation.

Second, recall that Proposition 2 characterizes non-trivial unawareness by using the self-evident collection. Now, Subjective Negative Introspection (i.e., (a)) implies that Weak Necessitation of $\overline{U}_i^{(2)}$ can be characterized in terms of the following closure property of the self-evident collection $\mathcal{J}_i$: if $(E, S) \in \mathcal{J}_i$ then $\overline{K}_i(\overline{S}^{\uparrow}) \wedge (\neg(E, S)) \in \mathcal{J}_i$.

Third, while Proposition 3 shows that Weak A-Negative Introspection of $\overline{A}_i^{(2)}$ always holds, Proposition 4 states that Weak A-Negative Introspection of $\overline{A}_i^{(\infty)}$ (i.e., (d)) holds if $\overline{U}_i^{(2)}$ satisfies Weak Necessitation. Likewise, while Proposition 3 shows that AA Self Reflection of $\overline{A}_i^{(2)}$ always holds, Proposition 4 states that AA Self Reflection of $\overline{A}_i^{(\infty)}$ (i.e., (i)) holds if $\overline{U}_i^{(2)}$ satisfies Weak Necessitation. The converse, however, does not necessarily hold. For agent i_3 in Example 1, while AA Self Reflection of $\overline{A}_{i_3}^{(\infty)}$ holds, Weak Necessitation of $\overline{U}_{i_3}^{(2)}$ fails.

Forth, Modica and Rustichini (1994) show k^2-unawareness is trivial under Symmetry (i.e., (e)) in their framework. While Symmetry of $\overline{U}_i^{(2)}$ is equivalent to Weak Necessitation of $\overline{U}_i^{(2)}$, Symmetry of $\overline{U}_i^{(\infty)}$ does not necessarily imply Symmetry of $\overline{U}_i^{(2)}$ (e.g., agent i_1 in Example 1). Similarly, while Weak Necessitation of $\overline{U}_i^{(2)}$ implies that of $\overline{U}_i^{(\infty)}$, the converse does not necessarily hold (e.g., agent i_3 in Example 1).

Fifth, MU Introspection states that there is no state at which the agent considers it possible that she is unaware of any particular event. By A-Introspection (Proposition 3 (6)), $\overline{M}_i^{(n)}\overline{U}_i^{(n)}(\cdot) = \overline{U}_i^{(n)}(\cdot) \wedge \overline{A}_i^{(n)}\overline{U}_i^{(n)}(\cdot)$: the agent considers it possible that she is unaware of an event iff she is unaware of the event and is aware of being unaware of it. Thus, MU Introspection and AU Introspection are equivalent.

Remark 4. To conclude this section, I compare how standard-state-space non-partitional and generalized-state-space stationary possibility-correspondence models represent a non-trivial form of unawareness in light of Example 2 in Section 3.2.2. To see that the standard-state-space model violates any property in Proposition 4, consider the event $\{\omega_{pt}, \omega_{\neg pt}\}$ that the test is conducted. When the test is not conducted (i.e., at $\omega_{p\neg t}$ or $\omega_{\neg p \neg t}$), the caregiver is unaware of the event that the test is conducted. However, observe that there is no state at which the caregiver knows that the test is not conducted. Thus, at any state, the caregiver knows that she does not know that the test is not conducted. Hence, at any state, the caregiver is aware that the test is not conducted. This implies that the model violates Symmetry for $\{\omega_{pt}, \omega_{\neg p \neg t}\}$.

In contrast, the generalized-state-space model satisfies any property in Proposition 4. When the test is not conducted (i.e., at $\omega_{p\neg t}$ or $\omega_{\neg p \neg t}$), the caregiver's awareness function assigns the subspace S_t, which describes only whether the test is conducted. Hence, the caregiver is unaware of any event which belongs to the most expressive space S. Especially, she is unaware of the event $(\{\omega_{pt}, \omega_{\neg pt}\}, S)$ that the test is conducted and its negation $(\{\omega_{p\neg t}, \omega_{\neg p \neg t}\}, S)$ that the test is not conducted.

5 Implications of the Violation of AU Introspection

This section studies the implications of the violation of AU Introspection. First, Section 5.1 shows that an agent may know that she is unaware of *something*. In contrast, under AU Introspection, there is no state at which the agent is unaware of something and is aware that she is unaware of something. Section 5.2 studies the non-monotonicity of unawareness with respect to knowledge. When an agent receives new information, she may become unaware of some event if she treats new information only at face value.

5.1 Knowledge of Self-awareness

Although KU Introspection requires that an agent do not know that she is unaware of any *particular* event, does there exist a state in which the agent knows that she is unaware of *something*? Here, I study the knowledge of self-unawareness. Agents, who know or are aware of their unawareness, may be able to partially prepare for unforeseen contingencies by making their actions contingent on "something unexpected."

Fix a model $\mathcal{M}^{(n)}$ with $n \in \{2, \infty\}$. I define the event $\overline{\mathbb{U}}_i^{(n)} := (\mathbb{U}_i^{(n)}, \sup \mathcal{S})$ that the agent is unaware of something by

$$\overline{\mathbb{U}}_i^{(n)} := \bigvee_{\overline{E} \in \mathcal{E}} \overline{U}_i^{(n)}(\overline{E}) = \left(\bigcup_{\overline{E} \in \mathcal{E}} (r_{S(E)}^{\sup \mathcal{S}})^{-1}(B(U_i^{(n)}(E))), \sup \mathcal{S} \right) \in \mathcal{E}, \tag{2}$$

where $B(U_i^{(n)}(E))$ is the basis of $U_i^{(n)}(E)$, i.e., $U_i^{(n)}(E) = (B(U_i^{(n)}(E)))^{\uparrow}$. The set $\mathbb{U}_i^{(n)}$ consists of states at which the agent is unaware of some event: $\mathbb{U}_i^{(n)} = \{\omega \in \sup \mathcal{S} \mid$ there is $(E, S(E)) \in \mathcal{E}$ with $\omega \in U_i^{(n)}(E)\}$. Also, $\mathcal{M}^{(n)}$ is non-trivial iff $\mathbb{U}_i^{(n)} \neq \emptyset$.

The following proposition examines self-awareness. First, under AU Introspection, the agent can never know that she is unaware of something, and there is no state at which the agent is unaware of something and is aware that she is unaware of something. Second, when the collection of events is finite, the agent can still never know that she is unaware of something. However, when AU Introspection fails, there may exist a state at which the agent is unaware of something and is aware that she is unaware of something. Third, if the collection of events is infinite and if AU Introspection fails, then the agent may know that she is unaware of something.

Proposition 5. *1. Let the model $\mathcal{M}^{(n)}$ satisfy AU Introspection.*

(a) $\overline{\mathbb{U}}_i^{(n)} = \overline{U}_i^{(n)}(\overline{\sup \mathcal{S}})$.

(b) $\overline{K}_i(\overline{\mathbb{U}}_i^{(n)}) = (\emptyset, \sup \mathcal{S})$ *and* $\overline{A}_i^{(n)}(\overline{\mathbb{U}}_i^{(n)}) = \overline{K}_i(\overline{\sup \mathcal{S}})$.

(c) $\overline{\mathbb{U}}_i^{(n)} \wedge \overline{A}_i^{(n)}(\overline{\mathbb{U}}_i^{(n)}) = (\emptyset, \sup \mathcal{S})$.

2. Let $\overline{K}_i$ satisfy Finite Conjunction, and let the domain $\mathcal{E}$ be finite.

(a) $\overline{K}_i(\overline{\mathbb{U}}_i^{(n)}) = (\emptyset, \sup \mathcal{S})$ *and* $\overline{A}_i^{(n)}(\overline{\mathbb{U}}_i^{(n)}) = \overline{K}_i(\overline{\sup \mathcal{S}})$.

(b) When AU Introspection fails, it is possible that $\overline{\mathbb{U}}_i^{(n)} \wedge \overline{A}_i^{(n)}(\overline{\mathbb{U}}_i^{(n)}) \neq (\emptyset, \sup \mathcal{S})$.

3. If $\mathcal{E}$ is infinite and if AU Introspection fails, then it is possible that $(\emptyset, \sup \mathcal{S}) \neq \overline{K}_i(\overline{\mathbb{U}}_i^{(n)}) \leqslant \overline{\mathbb{U}}_i^{(n)} \wedge \overline{A}_i^{(n)}(\overline{\mathbb{U}}_i^{(n)})$.

Under AU Introspection, Part (1a) demonstrates that the event $\overline{\mathbb{U}}_i^{(n)}$ that the agent is unaware of something reduces to the event $\overline{U}_i^{(n)}(\overline{\sup \mathcal{S}})$ that she is unaware of the *particular* event $\overline{\sup \mathcal{S}}$. By KU Introspection, the agent does not know that she is unaware of any particular event. Thus, as shown in Part (1b), the agent does not know that she is unaware of something: $\overline{K}_i(\overline{\mathbb{U}}_i^{(n)}) = (\emptyset, \sup \mathcal{S})$. Then, she is aware that she is unaware of something at any state at which she knows some event in the subspace $\sup \mathcal{S}$. However, Part (1c) means that there is no state at which she is unaware of something and is aware that she is unaware of something.

Parts (2) and (3), in contrast, do not necessarily impose AU Introspection. Part (2a) states that, under Finite Conjunction, if the domain is finite then the agent never knows that she is unaware of something. Yet, she is aware that she is unaware of something at any state at which she knows some event in the subspace $\sup \mathcal{S}$. When AU Introspection fails, Part (2b) shows that it is possible that the agent is unaware of something and is aware that she is unaware of something.

To contrast Parts (1c) and (2b), take Example 1. Agents i_1 and i_3 do not know that they are unaware of something. However, at ω_3, agent i_1 is (k^n-)unaware of something and is aware that she is unaware of something. Likewise, at ω_2 or ω_3, agent i_3 is (k^n-)unaware of something and is aware that she is unaware of something. In fact, in any standard-state-space model with Finite Conjunction and Necessitation, $A_i^{(n)}(\mathbb{U}_i^{(n)}) = K_i(\Omega) = \Omega$, where the first equality follows from Part (2a). Under AU Introspection, as Part (1a) implies, $\mathbb{U}_i^{(n)} = U_i^{(n)}(\Omega) = \emptyset$ (in fact, unawareness is trivial). If AU Introspection fails, in contrast, then $\emptyset \neq \mathbb{U}_i^{(n)}$, i.e., the agent is unaware of some event at some state. Then, $\emptyset \neq \mathbb{U}_i^{(n)} = \mathbb{U}_i^{(n)} \cap A_i^{(n)}(\mathbb{U}_i^{(n)})$: at some state, agent i is unaware of something and is aware that she is unaware of something.

Part (3) says: if a given domain is infinite and if AU Introspection fails, then it is possible (even in a standard state space) that the agent knows that she is unaware of something, even though she never knows that she is unaware of any particular event.

In this case, at a state at which the agent knows that there is something she is unaware of, she is unaware of something and is aware that she is unaware of something.

I remark that one can also examine whether agent i knows or is aware of the event that she is unaware of something in some collection of events $\mathcal{F}$ by restricting attention to events $\overline{E} \in \mathcal{F}$ in Expression (2). For example, one can analyze whether the agent is unaware of something in a given subspace $S \in \mathcal{S}$ by taking $\mathcal{F} = \{\overline{E} \in \mathcal{E} \mid S(E) = S\}$. The result similar to Proposition 5 holds.

5.2 (Non-)monotonicity of Unawareness in Knowledge

This subsection studies the non-monotonicity of unawareness in knowledgeability. Agent i is *at least as knowledgeable as* agent j if $\overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$: whenever agent j knows some event at some state, agent i knows it at that state. Section 5.2.1 shows that, while more knowledge leads to more awareness under AU Introspection, if AU Introspection fails then an agent may become unaware of some event even when she becomes more knowledgeable. Section 5.2.2 briefly compares the non-monotonicity of unawareness in knowledgeability with the negative value of information under unawareness.

5.2.1 Non-Monotonicity of Unawareness in One's Knowledge

Under Weak Necessitation (equivalently, AU Introspection), awareness is monotonic in knowledgeability. Namely, if agent i is at least as knowledgeable as agent j, then whenever agent i is unaware of an event, so is agent j:

$$\overline{U}_i^{(n)}(\overline{E}) = (\neg \overline{K}_i)(\overline{S(E)}^{\uparrow}) \leqslant (\neg \overline{K}_j)(\overline{S(E)}^{\uparrow}) = \overline{U}_j^{(n)}(\overline{E}) \text{ for every } \overline{E} \in \mathcal{E}. \tag{3}$$

In fact, the inequality $\overline{U}_i^{(n)}(\overline{E}) \leqslant \overline{U}_j^{(n)}(\overline{E})$ holds as long as agent i's unawareness operator satisfies Weak Necessitation (by Expression (1), if agent j does not know the subspace $\overline{S}^{\uparrow}$ then she is k^n-unaware of (E, S)).[19]

I remark that the converse does not necessarily hold: "more awareness" does not necessarily imply knowledgeability. In Remark 3, while one agent is at least as knowledgeable as another and their knowledge operators are different, two agents have the identical (un)awareness operator.

More generally when AU Introspection is not assumed, awareness may not necessarily be monotonic in knowledgeability. That is, the fact that $\overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$ (i.e., agent i is at least as knowledgeable as agent j) does not necessarily imply $\overline{A}_j^{(n)}(\cdot) \leqslant \overline{A}_i^{(n)}(\cdot)$.

For one example, take Example 1. While agents $(i, j) = (i_2, i_4)$ satisfy this relation, agents $(i, j) = (i_1, i_3)$ do not. For another example, recall Proposition 2: it shows that

[19] Alternatively, under Weak Necessitation, agent i is at least as aware as agent j in that $\overline{A}_j^{(n)}(\cdot) \leqslant \overline{A}_i^{(n)}(\cdot)$ iff $\sigma_i(\cdot) \succeq \sigma_j(\cdot)$. In fact, in his possibility-correspondence model of unawareness, Galanis (2016) defines the relation, agent i is at least as aware as agent j, by $\sigma_i(\cdot) \succeq \sigma_j(\cdot)$. If agent i is at least as knowledgeable as agent j, then $\overline{\Pi}_i^{\uparrow}(\cdot) \leqslant \overline{\Pi}_j^{\uparrow}(\cdot)$, which implies $\sigma_i(\cdot) \succeq \sigma_j(\cdot)$. Hence, agent i is at least as aware as agent j.

non-trivial unawareness hinges on the qualitative feature of knowledge (e.g., whether the lack of knowledge is self-evident). Thus, introduce the knowledge of agent j on a standard state space as: $K_j(E) = \emptyset$ for all $E \in \mathcal{D} \setminus \{\Omega\}$ and $K_j(\Omega) = \Omega$. Her self-evident collection is $\mathcal{J}_j = \{\emptyset, \Omega\}$. Then, she is always aware of every event. This is because she always knows that she does not know any non-tautological event. The non-monotonicity of awareness holds for any agent i who is at least as knowledgeable as agent j and who violates AU Introspection.

I provide two ways to intuitively understand the non-monotonicity of awareness. One way to look at the non-monotonicity is that, while increase in knowledge enhances awareness through knowledge itself, decrease in knowledge also enhances awareness through the knowledge of the lack of knowledge. To compare it with the case with Weak Necessitation, recall that the k^n-unawareness operators can be written as $\overline{U}_i^{(2)}(\cdot) = (\neg\overline{K}_i)(\cdot) \wedge (\neg\overline{K}_i)^2(\cdot)$ and $\overline{U}_i^{(\infty)}(\cdot) = (\neg\overline{K}_i)^3(\cdot) \wedge (\neg\overline{K}_i)^2(\cdot)$. While the first term of each unawareness operator, $(\neg\overline{K}_i)(\cdot)$ or $(\neg\overline{K}_i)^3(\cdot)$, decreases in knowledgeability, the second term $(\neg\overline{K}_i)^2(\cdot)$ may increase in knowledgeability. When agent i is at least as knowledgeable as agent j, $(\neg\overline{K}_j)^2(\cdot) \leqslant (\neg\overline{K}_i)^2(\cdot)$ while $(\neg\overline{K}_i)(\cdot) \leqslant (\neg\overline{K}_j)(\cdot)$. The overall effect on unawareness is not determinate.

Another way to look at the non-monotonicity is that an agent takes information only at face value when her possibility correspondence fails Generalized Euclideanness. If agent i fully "understands" her possibility sets $\Pi_i(\cdot)$, then she would be able to distinguish states ω and ω' by reasoning about whether the possibility sets $\Pi_i(\omega)$ and $\Pi_i(\omega')$ are identical or not. Given Generalized Transitivity, Generalized Euclideanness states that, at each state ω, if agent i considers ω' possible (i.e., $\omega' \in \Pi_i(\omega)$), then the possibility sets coincide: $\Pi_i(\omega) = \Pi_i(\omega')$. In fact, if $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Euclideanness in addition to Generalized Reflexivity and Generalized Transitivity, then $\Pi_i(\omega) = \{\omega' \in \sigma_i(\omega) \mid \Pi_i(\omega') = \Pi_i(\omega)\}$ (this expression is also a restatement of what Grant et al. (2015) call "Sub-model Information Consistency").

To see this point through an example, take agents i_1 and i_3 in Example 1. Agent i_1 is at least as knowledgeable as agent i_3. More specifically, the knowledge of i_1 is such that, in addition to what agent i_3 knows, agent i_1 knows $\{\omega_2\}$ whenever state ω_2 obtains (precisely, $K_{i_1}(E) = K_{i_3}(E) \cup \{\omega_2\}$ if $\omega_2 \in E$). However, when ω_2 does not hold, she is not necessarily informed that ω_2 does not obtain (technically, recalling $\Pi_{i_1}(\omega_3) = \Omega$, agent i_1 does not exclude ω_2 from the set of states that she considers possible at ω_3). Thus, when i_1 does not know $\{\omega_2\}$, specifically at ω_3, she does not know that she does not know $\{\omega_2\}$. Recalling $\{\omega_2\}$ is new information, agent i_1 is k^n-unaware of her new information $\{\omega_2\}$ at ω_3. Hence, the non-monotonicity of awareness in knowledgeability comes from the fact that a more knowledgeable agent takes new information only at face value (if the more knowledgeable agent's possibility correspondence satisfies Stationarity, i.e., her knowledge satisfies Weak Necessitation, then she is at least as aware as the less knowledgeable agent, as shown in Expression (3)).

Two additional remarks are in order. First, one can compare agents' knowledge and unawareness as one agent's knowledge and unawareness over time. Let a state space be given by $\Omega = \{\omega_1, \omega_2, \omega_3\}$ as in Example 1. Denote agent i's knowledge at time t by $i(t)$. Specifically, let $i(0) = i_4$, $i(1) = i_1$, and $i(2) = j$ (where $\mathcal{J}_j = \{\emptyset, \Omega\}$ as in the previous discussion). At time 1, getting more information causes agent i to get aware of some event at each realized state. At time 2, on the other hand, she "forgets" some events, and this may make her aware of some events at some states. Second, while knowledgeability and more awareness are not in a one-to-one relation, Online Appendix D shows that knowledgeability is characterized jointly by awareness and knowing-whether. Specifically, it shows: $\overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$ (i.e., agent i is at least as knowledgeable as agent j) iff $\overline{A}_j^{(2)}(\cdot) \leqslant \overline{J}_i \overline{K}_j(\cdot)$ (i.e., if agent j is k^2-aware of an event then agent i knows whether agent j knows the event).

5.2.2 Qualitative Blackwell's Theorem

Section 5.2.1 has demonstrated that new information may lead to more unawareness. Does an agent have an incentive to disclose her private information to another agent to take advantage of her (more) unawareness? I examine the relation between the non-monotonicity of awareness in knowledgeability and the negative value of information under unawareness.

In a standard-state-space non-partitional possibility-correspondence model, both the non-monotonicity of awareness in knowledgeability and the negative value of information must imply the violation of AU Introspection.[20] AU Introspection holds iff the possibility correspondence is partitional. Among partitional possibility-correspondence models, more information leads to more awareness, and more information never hurts.

In a generalized-state-space stationary possibility-correspondence model, Galanis (2015, 2016) shows that the value of information may be negative. Hence, the value of information may be negative even with AU Introspection. In Galanis (2015, 2016), a sufficient condition for the value of information to be non-negative is how knowledge and awareness interact with probabilistic beliefs (the "Conditional Independence" condition). In contrast, awareness may not be monotonic in knowledgeability without introducing probabilistic beliefs. Also, AU Introspection (precisely, Generalized Euclideanness) is a property that hinges only on Π_i, not involving the awareness function σ_i. Thus, the non-monotonicity of awareness in knowledgeability may not necessarily be comparable to the value of information.

I demonstrate the role of Weak Necessitation on the relation between preferences and knowledgeability of possibility correspondences in a decision problem that does not involve probabilistic beliefs. Agents' knowledge is induced from a possibility correspondence. Agents' preferences over acts induce preferences over possibility correspondences. I establish the Blackwell theorem studied by Dubra and Echenique (2004) and Hérves-Beloso and Monteiro (2013): roughly, when knowledge satisfies Weak Necessi-

[20] Let agent i violate AU Introspection. Let another agent j, whose knowledge is induced from a partitional possibility correspondence, be at least as less knowledgeable as agent i. If information is not valuable, then agent i is at least as unaware as agent j in that $U_j^{(2)}(\cdot) \subseteq U_i^{(2)}(\cdot)$ and $U_j^{(2)} \neq U_i^{(2)}$.

tation, a possibility correspondence is always preferred to another iff the first is finer (i.e., more knowledgeable).

Fix a generalized state space $\langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}, \succeq, r \rangle$. Suppose agents' knowledge is induced by possibility correspondences satisfying Generalized Reflexivity and Generalized Transitivity. For possibility correspondences $\overline{\Pi}_i^{\uparrow}$ and $\overline{\Pi}_j^{\uparrow}$ of agents i and j, $\overline{\Pi}_i^{\uparrow}$ is *at least as knowledgeable as* $\overline{\Pi}_j^{\uparrow}$ if $\overline{\Pi}_i^{\uparrow}(\cdot) \leqslant \overline{\Pi}_j^{\uparrow}(\cdot)$ (note that this is equivalent to $\overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$).

Let Z be a set of *consequences* with $|Z| \geq 2$. An *act* is a function $f : \Omega \to Z$. The act f is $\overline{\Pi}_i^{\uparrow}$*-feasible* if $\overline{\Pi}_i^{\uparrow}(\omega) = \overline{\Pi}_i^{\uparrow}(\omega')$ implies $f(\omega) = f(\omega')$.

A *preference relation* $\succsim$ is a complete-and-transitive binary relation on acts. The preference relation $\succsim$ *(weakly) prefers* a possibility correspondence $\overline{\Pi}_i^{\uparrow}$ to $\overline{\Pi}_j^{\uparrow}$ if, for any $\overline{\Pi}_j^{\uparrow}$-feasible act f, there exists a $\overline{\Pi}_i^{\uparrow}$-feasible act g such that $g \succsim f$. Now:

Proposition 6. *If a possibility correspondence $\overline{\Pi}_i^{\uparrow}$ is at least as knowledgeable as a possibility correspondence $\overline{\Pi}_j^{\uparrow}$, then every preference relation prefers $\overline{\Pi}_i^{\uparrow}$ to $\overline{\Pi}_j^{\uparrow}$. Conversely, let $\overline{\Pi}_i^{\uparrow}$ satisfy Generalized Euclideanness. If every preference relation prefers $\overline{\Pi}_i^{\uparrow}$ to $\overline{\Pi}_j^{\uparrow}$, then $\overline{\Pi}_i^{\uparrow}$ is at least as knowledgeable as $\overline{\Pi}_j^{\uparrow}$.*

In Proposition 6, Generalized Reflexivity and Generalized Transitivity play an important role in proving that any $\overline{\Pi}_j^{\uparrow}$-feasible act is also $\overline{\Pi}_i^{\uparrow}$-feasible when $\overline{\Pi}_i^{\uparrow}$ is at least as knowledgeable as $\overline{\Pi}_j^{\uparrow}$ (see Remark A.3 in Appendix A.6). Then, any preference relation prefers $\overline{\Pi}_i^{\uparrow}$ to $\overline{\Pi}_j^{\uparrow}$. Since the "if" part of Proposition 6 ("non-negative value of information") follows from Generalized Reflexivity and Generalized Transitivity, every preference relation prefers $\overline{\Pi}_i^{\uparrow}$ to $\overline{\Pi}_j^{\uparrow}$ irrespective of whether i is at least as aware as j.

The fact that the "if" part of Proposition 6 holds irrespective of unawareness comes from the definition of preferences over possibility correspondences in this paper. Thus, I compare Proposition 6 with the related results in the value-of-information literature such as Galanis (2015, 2016) and Geanakoplos (1989).

Recall that, in this paper, a possibility correspondence $\overline{\Pi}_i^{\uparrow}$ is preferred to another $\overline{\Pi}_j^{\uparrow}$ if, for any $\overline{\Pi}_j^{\uparrow}$-*feasible* act f, there exists a $\overline{\Pi}_i^{\uparrow}$-*feasible* act g with $g \succsim f$. In contrast, the value-of-information literature compares possibility correspondences $\overline{\Pi}_i^{\uparrow}$ and $\overline{\Pi}_j^{\uparrow}$ by comparing the expected utility of an "optimal" act f under $\overline{\Pi}_j^{\uparrow}$ (i.e., under the posterior beliefs obtained from a (common) prior conditioned on $\overline{\Pi}_j^{\uparrow}$) and that of an "optimal" act g under $\overline{\Pi}_i^{\uparrow}$.[21] Negative value of information under unawareness states that, an

[21]See Galanis (2015) and Geanakoplos (1989) for their definitions of optimality in the contexts of standard and generalized state-space possibility-correspondence models, respectively. Technically, since the properties of unawareness studied in this paper in terms of the lack of knowledge do not depend on agents' probabilistic beliefs, this paper considers ordinal preferences over acts as a primitive.

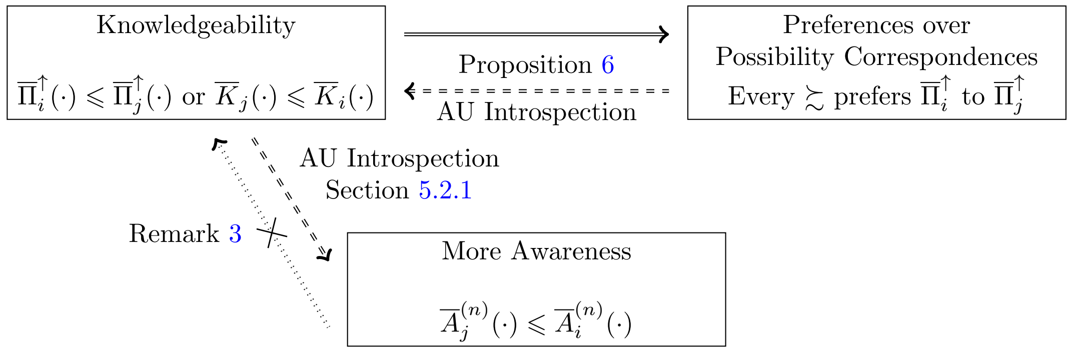

Figure 2: Relation among Knowledgeability, Preferences over Possibility Correspondences, and More Awareness

"optimal" act g under $\overline{\Pi}_i^{\uparrow}$ may yield less expected utility than an "optimal" act f under $\overline{\Pi}_j^{\uparrow}$ does, even if $\overline{\Pi}_i^{\uparrow}$ is at least as knowledgeable as $\overline{\Pi}_j^{\uparrow}$.

In the context of standard-state-space non-partitional possibility-correspondence models, Geanakoplos (1989) identifies an additional condition weaker than Euclideanness (called "nestedness") under which the value of information is non-negative. However, if the value of information under a non-partitional possibility correspondence Π_i is less than that under a less-knowledgeable but partitional possibility correspondence Π_j, then $U_i^{(2)}$ is non-trivial and violates AU Introspection.[22]

In the context of generalized-state-space possibility-correspondence models, Galanis (2015, 2016) shows that, in a decision problem with unawareness, the negative value of information is related to the fact that an agent treats her awareness asymmetrically. Roughly, in his model, the value of information is non-negative if the agent's probabilistic beliefs incorporate the information $\{\omega' \in S \mid \sigma_i(\omega) = \sigma_i(\omega')\}$ that would be retrieved from the awareness function σ_i at each state $\omega \in S \in \mathcal{S}$.[23] The feasibility of an act also incorporates the information derived from the awareness function because $\overline{\Pi}_i^{\uparrow}(\omega) = \overline{\Pi}_i^{\uparrow}(\omega')$ holds iff $\sigma_i(\omega) = \sigma_i(\omega')$ and $\Pi_i(\omega) = \Pi_i(\omega')$.

To compare Sections 5.2.1 and 5.2.2, Figure 2 depicts the relations among knowledgeability, preferences over possibility correspondences, and more awareness. First, consider the relation between knowledgeability and preferences over possibility correspondences. The solid arrow depicts the first part of Proposition 6: if agent i's possibility correspondence $\overline{\Pi}_i^{\uparrow}$ is at least as knowledgeable as agent j's possibility correspondence $\overline{\Pi}_j^{\uparrow}$, then every preference relation prefers $\overline{\Pi}_i^{\uparrow}$ to $\overline{\Pi}_j^{\uparrow}$. Conversely, the dashed arrow depicts the second part of Proposition 6: under AU Introspection, if every pref-

[22] Since K_i is induced by a possibility correspondence, it satisfies Necessitation (recall Section 3.1). Hence, $U_i^{(2)}$ is non-trivial iff it violates AU Introspection iff Π_i fails Euclideanness.

[23] Thus, in a sense, while the non-monotonicity of awareness in knowledge comes from the failure to fully reason about the possibility sets $\Pi_i(\cdot)$, the negative value of information comes from the failure to fully reason about the awareness function σ_i.

erence relation prefers $\overline{\Pi}_i^{\uparrow}$ to $\overline{\Pi}_j^{\uparrow}$, then $\overline{\Pi}_i^{\uparrow}$ is at least as knowledgeable as $\overline{\Pi}_j^{\uparrow}$.

Next, consider the relation between knowledgeability and more awareness. The dashed arrow shows that, as discussed in Section 5.2.1, knowledgeability implies more awareness under AU Introspection. In contrast, the dotted arrow shows that the converse does not necessarily hold, i.e., more awareness does not necessarily imply knowledgeability. This is because if more awareness yields knowledgeability, then $\overline{A}_i^{(n)} = \overline{A}_j^{(n)}$ has to imply $\overline{K}_i = \overline{K}_j$. However, Remark 3 demonstrates that unawareness, unlike ignorance, does not necessarily recover underlying knowledge.

6 Conclusion

This paper studied unawareness from the lack of knowledge in a generalized-state-space model which nests both a non-partitional standard-state-space model and a stationary generalized-state-space model. The resulting model does not necessarily satisfy AU Introspection or equivalently Weak Necessitation.

The paper showed that unawareness can only take two forms: an agent is ignorant of knowing that she does not know an event; or the agent is ignorant of knowing an event. For either form, an agent is unaware of an event iff she is unaware of the possibility that she knows the event. Next, the paper characterized when unawareness is non-trivial by using a self-evident collection. The paper clarified two channels through which unawareness is non-trivial: the failure of AU Introspection and the non-knowledge of a subspace. This paper also examined the properties of unawareness that come from and that do not come from the basic properties of knowledge. Instead of AU Introspection, a weaker form of Introspection called JU Introspection holds: the agent is unaware of an event iff she is ignorant of being unaware of it.

This paper demonstrated two implications of the violation of AU Introspection. First, the paper showed that an agent may know that she is unaware of something if the objects of knowledge are infinite and if the agent's unawareness fails AU Introspection. Second, when unawareness is determined by the lack of knowledge, getting new information may cause the agent to become unaware of some event when the agent treats new information only at face value.

One interesting avenue for future research would be to incorporate agents' probabilistic beliefs in the framework of this paper to capture knowledge, probabilistic beliefs, and unawareness in a single coherent framework.

A Proofs

A.1 Technical Preliminaries

I provide two technical results that are used throughout this appendix. First, the following remark lists some technical properties of operators. For each property, the key assumptions are appended inside the parentheses.

Remark A.1. Let $\overline{K}_i : \mathcal{E} \to \mathcal{E}$ be agent i's knowledge operator.

1. $\overline{K}_i = \overline{K}_i \overline{K}_i$ (Truth Axiom and Positive Introspection of $\overline{K}_i$).
2. $(\neg \overline{K}_i) = (\neg \overline{K}_i)\overline{K}_i$ (Part (1)).
3. $\overline{\partial}_i = (\neg \overline{K}_i)\overline{J}_i$ (Truth Axiom, Monotonicity, and Positive Introspection of $\overline{K}_i$).
4. $\overline{J}_i(\overline{E}) = \overline{J}_i(\neg \overline{E})$ (Definition of $\overline{J}_i$).
5. $\overline{\partial}_i(\overline{E}) = \overline{\partial}_i(\neg \overline{E})$ (Definition of $\overline{\partial}_i$).
6. $\overline{E} \leqslant (\neg \overline{K}_i)(\neg \overline{E})$ (Truth Axiom of $\overline{K}_i$).
7. $\overline{K}_i(\overline{E}) \leqslant (\neg \overline{K}_i)^2(\overline{E})$ (Part (6)).
8. $\overline{A}_i^{(n)}(E, S) \leqslant \overline{K}_i(\overline{S}^{\uparrow})$ (Monotonicity of $\overline{K}_i$).
9. $(\neg \overline{K}_i)(\overline{S}^{\uparrow}) \leqslant \overline{U}_i^{(n)}(E, S)$ (Part (8)).
10. $\overline{K}_i(\overline{E}) = \overline{J}_i(\overline{E}) \wedge \overline{E}$ (Truth Axiom of $\overline{K}_i$).
11. $(\neg \overline{K}_i)\overline{U}_i^{(n)}(E, S) = \overline{S}^{\uparrow}$ (KU Introspection).

Proof of Remark A.1. Part (1). By Positive Introspection, $\overline{K}_i(\cdot) \leqslant \overline{K}_i\overline{K}_i(\cdot)$. By Truth Axiom, $\overline{K}_i(\overline{F}) \leqslant \overline{F}$. Substituting $\overline{F} = \overline{K}_i(\cdot)$ yields $\overline{K}_i\overline{K}_i(\cdot) \leqslant \overline{K}_i(\cdot)$.

Part (2). The property follows from taking the negation on both sides of Part (1).

Part (3). I show $\overline{J}_i = \overline{K}_i\overline{J}_i$. Then, take the negation on both sides to get $\overline{\partial}_i = (\neg \overline{K}_i)\overline{J}_i$. First, $\overline{K}_i\overline{J}_i(\cdot) \leqslant \overline{J}_i(\cdot)$ follows from Truth Axiom of $\overline{K}_i$. Second,

$$\overline{K}_i\overline{J}_i(\overline{E}) = \overline{K}_i(\overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg \overline{E})) \geqslant \overline{K}_i\overline{K}_i(\overline{E}) \vee \overline{K}_i\overline{K}_i(\neg \overline{E}) \geqslant \overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg \overline{E}) = J_i(\overline{E}).$$

The equalities follow from the definition of $\overline{J}_i$. For the first inequality, since $\overline{K}_i(\overline{E})$ and $\overline{K}_i(\neg \overline{E})$ have the same base space, $\overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg \overline{E}) \geqslant \overline{K}_i(\overline{E})$ and $\overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg \overline{E}) \geqslant \overline{K}_i(\neg \overline{E})$ (recall Remark 2). Since $\overline{K}_i$ satisfies Monotonicity, $\overline{K}_i(\overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg \overline{E})) \geqslant \overline{K}_i\overline{K}_i(\overline{E})$ and $\overline{K}_i(\overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg \overline{E})) \geqslant \overline{K}_i\overline{K}_i(\neg \overline{E})$. The second inequality follows from Positive Introspection of $\overline{K}_i$.

Part (4). By definition, $\overline{J}_i(\overline{E}) = \overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg \overline{E}) = \overline{K}_i(\neg \overline{E}) \vee \overline{K}_i(\overline{E}) = \overline{J}_i(\neg \overline{E})$.

Part (5). By definition, $\overline{\partial}_i(\overline{E}) = (\neg \overline{K}_i)(\overline{E}) \wedge (\neg \overline{K}_i)(\neg \overline{E}) = (\neg \overline{K}_i)(\neg \overline{E}) \wedge (\neg \overline{K}_i)(\overline{E}) = \overline{\partial}_i(\neg \overline{E})$. Or take the negation on both sides of Part (4).

Part (6). By Truth Axiom, $\overline{K}_i(\neg\overline{E}) \leqslant (\neg\overline{E})$. By taking the negation on both sides, $\overline{E} = \neg\neg\overline{E} \leqslant (\neg\overline{K}_i)(\neg\overline{E})$.

Part (7). Substitute $\overline{F} = \overline{K}_i(\overline{E})$ into $\overline{F} \leqslant (\neg\overline{K}_i)(\neg\overline{F})$ (i.e., Part (6)).

Part (8). Since $\overline{K}_i$ satisfies Monotonicity, for any $(E, S) \in \mathcal{E}$, $\overline{K}_i(\neg\overline{K}_i)^{r-1}(E, S) \leqslant \overline{K}_i(\overline{S}^{\uparrow})$ for all $r \in \mathbb{N}$. Then, $\overline{A}_i^{(n)}(E, S) = \bigvee_{r=1}^{n} \overline{K}_i(\neg\overline{K}_i)^{r-1}(E, S) \leqslant \overline{K}_i(\overline{S}^{\uparrow})$.

Part (9). Take the negation on both sides of Part (8).

Part (10). For any $\overline{E} \in \mathcal{E}$, by definition,

$$\overline{J}_i(\overline{E}) \wedge \overline{E} = (\overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg\overline{E})) \wedge \overline{E} = (\overline{K}_i(\overline{E}) \wedge \overline{E}) \vee (\overline{K}_i(\neg\overline{E}) \wedge \overline{E}).$$

It follows from Truth Axiom of $\overline{K}_i$ that $\overline{K}_i(\overline{E}) \wedge \overline{E} = \overline{K}_i(\overline{E})$ and $\overline{K}_i(\neg\overline{E}) \wedge \overline{E} = (\emptyset, S(E))$. Thus, $\overline{J}_i(\overline{E}) \wedge \overline{E} = \overline{K}_i(\overline{E}) \vee (\emptyset, S(E)) = \overline{K}_i(\overline{E})$.

Part (11). As discussed in Section 4.3.1, Truth Axiom, Monotonicity, and Plausibility yield KU Introspection: $\overline{K}_i\overline{U}_i^{(n)}(E, S) = (\emptyset, S)$. Take the negation on both sides to obtain $(\neg\overline{K}_i)\overline{U}_i^{(n)}(E, S) = \overline{S}^{\uparrow}$. □

Second, by Remark 1, for any events (E, S) and (F, S) that have the same base space, $(E, S) \leqslant (F, S)$ iff $\neg(F, S) \leqslant \neg(E, S)$. Applying Monotonicity of $\overline{K}_i$, one has:

Remark A.2. Take $(E, S), (F, S) \in \mathcal{E}$ with $(E, S) \leqslant (F, S)$.

1. $(\neg\overline{K}_i)(F, S) \leqslant (\neg\overline{K}_i)(E, S)$.
2. $(\neg\overline{K}_i)^2(E, S) \leqslant (\neg\overline{K}_i)^2(F, S)$.
3. $(\neg\overline{K}_i)(\neg(E, S)) \leqslant (\neg\overline{K}_i)(\neg(F, S))$.

A.2 Section 4.1

In order to prove Proposition 1, I establish the following lemma on the property of higher-order non-knowledge as discussed in the main text: $(\neg\overline{K}_i)^2 = (\neg\overline{K}_i)^{2n}$.

Lemma A.1. *Fix $\overline{E} \in \mathcal{E}$.*

1. $\overline{K}_i(\overline{E}) \leqslant (\neg\overline{K}_i)^2(\overline{E}) = (\neg\overline{K}_i)^{2n}(\overline{E}) \leqslant (\neg\overline{K}_i)(\neg\overline{E})$.
2. $(\neg\overline{K}_i)^2(\neg\overline{E}) \leqslant (\neg\overline{K}_i)^{2n+1}(\overline{E}) = (\neg\overline{K}_i)^3(\overline{E}) \leqslant (\neg\overline{K}_i)(\overline{E})$.

Proof of Lemma A.1. I start with showing

$$\overline{K}_i(\overline{E}) \leqslant \overline{K}_i\overline{K}_i(\overline{E}) \leqslant \overline{K}_i(\neg\overline{K}_i)^2(\overline{E}). \tag{A.1}$$

The first inequality follows from Positive Introspection of $\overline{K}_i$. The second inequality follows from operating $\overline{K}_i$ on both sides of $\overline{K}_i(\overline{E}) \leqslant (\neg\overline{K}_i)^2(\overline{E})$ (i.e., Remark A.1 (7)) through Monotonicity of $\overline{K}_i$.

Part (1). First, by Remark A.1 (7), $\overline{K}_i(\overline{E}) \leqslant (\neg\overline{K}_i)^2(\overline{E})$. Second, I show $(\neg\overline{K}_i)^2(\overline{E}) \leqslant (\neg\overline{K}_i)(\neg\overline{E})$. It follows from Remark A.2 (3) that operating $(\neg\overline{K}_i\neg)$ on both sides of $\overline{K}_i(\overline{E}) \leqslant \overline{E}$ (Truth Axiom) yields $(\neg\overline{K}_i)^2(\overline{E}) \leqslant (\neg\overline{K}_i)(\neg\overline{E})$.

Third, it suffices to show $(\neg\overline{K}_i)^2 = (\neg\overline{K}_i)^4$. Since both sides of $\overline{K}_i(\overline{E}) \leqslant \overline{K}_i(\neg\overline{K}_i)^2(\overline{E})$ in Expression (A.1) have the same base space, it follows from Remark A.2 (3) that operating $(\neg\overline{K}_i\neg)$ yields $(\neg\overline{K}_i)^2(\overline{E}) \leqslant (\neg\overline{K}_i)^4(\overline{E})$.

Conversely, by Truth Axiom, $\overline{K}_i(\neg\overline{K}_i)^2(\overline{E}) \leqslant (\neg\overline{K}_i)^2(\overline{E})$. Since both sides of the inequality have the same base space, by Remark A.2 (3), operating $(\neg\overline{K}_i\neg)$ on both sides gives $(\neg\overline{K}_i)^4(\overline{E}) \leqslant (\neg\overline{K}_i)\overline{K}_i(\neg\overline{K}_i)(\overline{E})$. Since $(\neg\overline{K}_i)\overline{K}_i = (\neg\overline{K}_i)$ by Remark A.1 (2), $(\neg\overline{K}_i)\overline{K}_i(\neg\overline{K}_i) = (\neg\overline{K}_i)^2$. Thus, $(\neg\overline{K}_i)^4(\overline{E}) \leqslant (\neg\overline{K}_i)^2(\overline{E})$.

Part (2). First, by Part (1), $(\neg\overline{K}_i)^2(\overline{F}) \leqslant (\neg\overline{K}_i)(\neg\overline{F})$. Substituting $\overline{F} = \neg\overline{E}$ yields $(\neg\overline{K}_i)^2(\neg\overline{E}) \leqslant (\neg\overline{K}_i)(\overline{E})$. Since both sides of the inequality have the same base space, by Remark A.2 (2), operating $(\neg\overline{K}_i)^2$ on both sides yields $(\neg\overline{K}_i)^4(\neg\overline{E}) \leqslant (\neg\overline{K}_i)^3(\overline{E})$. Since $(\neg\overline{K}_i)^2 = (\neg\overline{K}_i)^4$ by Part (1), $(\neg\overline{K}_i)^2(\neg\overline{E}) \leqslant (\neg\overline{K}_i)^3(\overline{E})$.

Second, by Part (1), $(\neg\overline{K}_i)^2(\overline{E}) = (\neg\overline{K}_i)^{2n}(\overline{E})$. Operating $(\neg\overline{K}_i)$ on both sides yields $(\neg\overline{K}_i)^3(\overline{E}) = (\neg\overline{K}_i)^{2n+1}(\overline{E})$.

Third, since both sides of $\overline{K}_i(\overline{E}) \leqslant \overline{K}_i(\neg\overline{K}_i)^2(\overline{E})$ in Expression (A.1) have the same base space, taking the negation on both sides yields $(\neg\overline{K}_i)^3(\overline{E}) \leqslant (\neg\overline{K}_i)(\overline{E})$. □

Proof of Proposition 1. Part (1). By Lemma A.1, $\overline{U}_i^{(\infty)}(\cdot)$ reduces to $(\neg\overline{K}_i)^2(\cdot) \wedge (\neg\overline{K}_i)^3(\cdot) = \overline{U}_i^{(2)}(\neg\overline{K}_i)(\cdot)$.

Part (2). First, $\overline{U}_i^{(2)}(\overline{E}) = (\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)^2(\overline{E}) = (\neg\overline{K}_i)\overline{K}_i(\overline{E}) \wedge (\neg\overline{K}_i)(\neg\overline{K}_i)(\overline{E}) = \overline{\partial}_i\overline{K}_i(\overline{E})$, where the first equality follows from the definition of $\overline{U}_i^{(2)}$, the second equality from Remark A.1 (2), and the third equality from the definition of $\overline{\partial}_i$.

Second, $\overline{U}_i^{(2)}(\overline{E}) = (\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)^2(\overline{E}) \leqslant (\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)(\neg\overline{E}) = \overline{\partial}_i(\overline{E})$, where $(\neg\overline{K}_i)^2(\overline{E}) \leqslant (\neg\overline{K}_i)(\neg\overline{E})$ follows from Lemma A.1 (1).

Third, to obtain $\overline{\partial}_i\overline{M}_i^{(2)}\overline{K}_i = \overline{U}_i^{(2)}(= \overline{\partial}_i\overline{K}_i)$, I show $\overline{M}_i^{(2)}\overline{K}_i = \overline{K}_i$. Indeed,

$$\begin{aligned}\overline{M}_i^{(2)}\overline{K}_i(\overline{E}) &= (\neg\overline{K}_i)^2(\overline{E}) \wedge \overline{A}_i^{(2)}\overline{K}_i(\overline{E}) = (\neg\overline{K}_i)^2(\overline{E}) \wedge \left(\overline{K}_i\overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg\overline{K}_i)\overline{K}_i(\overline{E})\right)\\ &=(\neg\overline{K}_i)^2(\overline{E}) \wedge \left(\overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg\overline{K}_i)(\overline{E})\right) = ((\neg\overline{K}_i)^2(\overline{E}) \wedge \overline{K}_i(\overline{E})) \vee (\emptyset, S(E)) = \overline{K}_i(\overline{E}),\end{aligned}$$

where the third equality follows from $\overline{K}_i = \overline{K}_i\overline{K}_i$ (i.e., Remark A.1 (1)) and $(\neg\overline{K}_i) = (\neg\overline{K}_i)\overline{K}_i$ (i.e., Remark A.1 (2)), and the fifth equality from $\overline{K}_i(\overline{E}) \leqslant (\neg\overline{K}_i)^2(\overline{E})$ (i.e., Remark A.1 (7)).

Part (3). For the first claim, $\overline{\partial}_i\overline{K}_i(\neg\overline{K}_i) = \overline{U}_i^{(2)}(\neg\overline{K}_i) = \overline{U}_i^{(\infty)}$, where the first equality follows from Part (2) (i.e., $\overline{\partial}_i\overline{K}_i = \overline{U}_i^{(2)}$), and the second equality from Part (1).

For the second claim, I show $\overline{M}_i^{(\infty)}\overline{K}_i = \overline{K}_i(\neg\overline{K}_i)^2$. Then, operating $\overline{\partial}_i$ on both sides of $\overline{M}_i^{(\infty)}\overline{K}_i = \overline{K}_i(\neg\overline{K}_i)^2$ yields

$$\begin{aligned}\overline{\partial}_i\overline{M}_i^{(\infty)}\overline{K}_i(\cdot) &= \overline{\partial}_i\overline{K}_i(\neg\overline{K}_i)^2(\cdot) = (\neg\overline{K}_i)\overline{K}_i(\neg\overline{K}_i)^2(\cdot) \wedge (\neg\overline{K}_i)^4(\cdot) = (\neg\overline{K}_i)^3(\cdot) \wedge (\neg\overline{K}_i)^4(\cdot)\\ &= (\neg\overline{K}_i)^2(\cdot) \wedge (\neg\overline{K}_i)^3(\cdot) = \overline{U}_i^{(\infty)}(\cdot),\end{aligned}$$

where the second equality follows from the definition of $\overline{\partial}_i$, the third equality from $(\neg\overline{K}_i)\overline{K}_i = (\neg\overline{K}_i)$ (i.e., Remark A.1 (2)), the fourth equality from $(\neg\overline{K}_i)^4 = (\neg\overline{K}_i)^2$ (i.e., Lemma A.1 (1)), and the fifth equality from Part (1).

Hence, I establish $\overline{M}_i^{(\infty)}\overline{K}_i = \overline{K}_i(\neg\overline{K}_i)^2$ in three steps. In the first step, by the definition of $\overline{M}_i^{(\infty)}$, $\overline{M}_i^{(\infty)}\overline{K}_i(\cdot) = (\neg\overline{K}_i)^2(\cdot) \wedge \overline{A}_i^{(\infty)}\overline{K}_i(\cdot)$. In the second step, $\overline{A}_i^{(\infty)}\overline{K}_i = \overline{A}_i^{(2)}(\neg\overline{K}_i)\overline{K}_i = \overline{A}_i^{(2)}(\neg\overline{K}_i)$, where the first equality follows from taking the negation on both sides of Part (1) (i.e., $\overline{U}_i^{(\infty)} = \overline{U}_i^{(2)}(\neg\overline{K}_i)$), and the second equality from $(\neg\overline{K}_i)\overline{K}_i = (\neg\overline{K}_i)$ (i.e., Remark A.1 (2)). In the third step,

$$\begin{aligned}\overline{M}_i^{(\infty)}\overline{K}_i(\cdot) &= (\neg\overline{K}_i)^2(\cdot) \wedge \overline{A}_i^{(2)}(\neg\overline{K}_i)(\cdot) = (\neg\overline{K}_i)^2(\cdot) \wedge \left(\overline{K}_i(\neg\overline{K}_i)(\cdot) \vee \overline{K}_i(\neg\overline{K}_i)^2(\cdot)\right)\\ &= (\emptyset, S(\cdot)) \vee ((\neg\overline{K}_i)^2(\cdot) \wedge \overline{K}_i(\neg\overline{K}_i)^2(\cdot)) = \overline{K}_i(\neg\overline{K}_i)^2(\cdot),\end{aligned}$$

where the first equality follows from the previous two steps, and the last equality from Truth Axiom of $\overline{K}_i$. □

The following proposition studies k^2-unawareness especially when it satisfies Symmetry ($\overline{U}_i^{(2)}(\overline{E}) = \overline{U}_i^{(2)}(\neg\overline{E})$): if agent i is k^2-unaware of an event and its negation, then she is k^∞-unaware of the event (and its negation). The statement will be used in establishing Proposition 4 in Section 4.3.

Proposition A.1. *For any $\overline{E} \in \mathcal{E}$, $\overline{U}_i^{(2)}(\overline{E}) \wedge \overline{U}_i^{(2)}(\neg\overline{E}) \leqslant \overline{U}_i^{(\infty)}(\overline{E})$.*

Proof of Proposition A.1. It follows from the definition of $\overline{U}_i^{(2)}$ that

$$\overline{U}_i^{(2)}(\overline{E}) \wedge \overline{U}_i^{(2)}(\neg\overline{E}) = (\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)^2(\overline{E}) \wedge (\neg\overline{K}_i)(\neg\overline{E}) \wedge (\neg\overline{K}_i)^2(\neg\overline{E}).$$

Next, Lemma A.1 (1) yields $(\neg\overline{K}_i)^2(\overline{E}) \leqslant (\neg\overline{K}_i)(\neg\overline{E})$. Also, Lemma A.1 (2) implies $(\neg\overline{K}_i)^2(\neg\overline{E}) \leqslant (\neg\overline{K}_i)^3(\overline{E})$. Then,

$$\overline{U}_i^{(2)}(\overline{E}) \wedge \overline{U}_i^{(2)}(\neg\overline{E}) \leqslant (\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)^2(\overline{E}) \wedge (\neg\overline{K}_i)^3(\overline{E}) = \overline{U}_i^{(3)}(\overline{E}) = \overline{U}_i^{(\infty)}(\overline{E}),$$

where the last equality follows because $\overline{U}_i^{(3)} = \overline{U}_i^{(\infty)}$ by Lemma A.1. □

Proposition S.3 in Online Appendix C.1 shows that, under Finite Conjunction of $\overline{K}_i$, agent i is k^2-unaware of an event or its negation iff she is k^2-unaware of knowing whether the event holds.

A.3 Section 4.2

Proof of Proposition 2. Part (1a). Observe that $\overline{K}_i(\overline{E}) \in \mathcal{J}_i \setminus \mathcal{J}_i^*$ iff $(\neg\overline{K}_i)(\overline{E}) \neq \overline{K}_i(\neg\overline{K}_i)(\overline{E})$. Also, if $(\neg\overline{K}_i)(\overline{E}) = \overline{K}_i(\neg\overline{K}_i)(\overline{E})$ then $\overline{U}_i^{(2)}(\overline{E}) = (\emptyset, S(E))$.

Thus, if $(\emptyset, S(E)) \neq \overline{U}_i^{(2)}(\overline{E})$ for some $\overline{E} \in \mathcal{E}$, then $(\neg\overline{K}_i)(\overline{E}) \neq \overline{K}_i(\neg\overline{K}_i)(\overline{E})$, i.e., $\overline{K}_i(\overline{E}) \in \mathcal{J}_i \setminus \mathcal{J}_i^*$. Conversely, if $\overline{K}_i(\overline{E}) \in \mathcal{J}_i \setminus \mathcal{J}_i^*$ for some $\overline{E} \in \mathcal{E}$, then $K_i(\neg K_i)(E) \subsetneq (\neg K_i)(E)$. Then, $(\emptyset, S(E)) \neq (\neg\overline{K}_i)^2(\overline{E}) \wedge (\neg\overline{K}_i)(\overline{E}) = \overline{U}_i^{(2)}(\overline{E})$.

Part (1b). If $\mathcal{M}^{(2)}$ is non-trivial, then $\overline{U}_i^{(2)}(\overline{E}) \neq (\emptyset, S(E))$ for some $\overline{E} \in \mathcal{E}$. By Part (1a), $\overline{K}_i(\overline{E}) \in \mathcal{J}_i \setminus \mathcal{J}_i^*$, i.e., $\mathcal{J}_i \setminus \mathcal{J}_i^* \neq \emptyset$. Conversely, if $\mathcal{J}_i \setminus \mathcal{J}_i^* \neq \emptyset$, then there is $\overline{K}_i(\overline{E}) = \overline{E} \in \mathcal{J}_i \setminus \mathcal{J}_i^*$ (the equality follows because $\overline{E} \in \mathcal{J}_i$). Thus, by Part (1a), $\overline{U}_i^{(2)}(\overline{E}) \neq (\emptyset, S(E))$, i.e., $\mathcal{M}^{(2)}$ is non-trivial.

Part (2). Part (2a) follows from Part (1a) and $\overline{U}_i^{(\infty)} = \overline{U}_i^{(2)}(\neg\overline{K}_i)$ (i.e., Proposition 1 (1)). Thus, I prove (2b). Observe that, by Part (1a), $\mathcal{M}^{(\infty)}$ is non-trivial iff $\overline{K}_i(\neg\overline{K}_i)(\overline{E}) \in \mathcal{J}_i \setminus \mathcal{J}_i^*$ for some $\overline{E} \in \mathcal{E}$. Let $\mathcal{M}^{(\infty)}$ be non-trivial, and let $\overline{F} := \overline{K}_i(\neg\overline{K}_i)(\overline{E}) \in \mathcal{J}_i \setminus \mathcal{J}_i^*$. I prove $\overline{K}_i(\neg\overline{F}) = \overline{K}_i(\neg\overline{K}_i)^2(\overline{E}) \in \mathcal{J}_i \setminus \mathcal{J}_i^*$ by contraposition. If $\overline{K}_i(\neg\overline{K}_i)^2(\overline{E}) \in \mathcal{J}_i^*$, then $\overline{K}_i(\neg\overline{K}_i)^3(\overline{E}) = (\neg\overline{K}_i)^3(\overline{E})$. Thus, $(\neg\overline{K}_i)^2(\overline{E}) = (\neg\overline{K}_i)^4(\overline{E}) = \overline{K}_i(\neg\overline{K}_i)^2(\overline{E}) \in \mathcal{J}_i$, and hence $\overline{F} = \overline{K}_i(\neg\overline{K}_i)(\overline{E}) \in \mathcal{J}_i^*$. Conversely, suppose $\overline{K}_i(\neg\overline{F}) \in \mathcal{J}_i \setminus \mathcal{J}_i^*$ for some $\overline{F} \in \mathcal{J}_i \setminus \mathcal{J}_i^*$. Since $\overline{K}_i(\overline{F}) = \overline{F}$ (by $\overline{F} \in \mathcal{J}_i$), it follows that $\overline{K}_i(\neg\overline{K}_i)(\overline{F}) \in \mathcal{J}_i \setminus \mathcal{J}_i^*$. Thus, $\mathcal{M}^{(\infty)}$ is non-trivial. □

A.4 Section 4.3

Proof of Proposition 3. Property (6): A-Introspection. Since $\overline{K}_i$ satisfies Truth Axiom, $\overline{K}_i\overline{A}_i^{(n)}(\cdot) \leqslant \overline{A}_i^{(n)}(\cdot)$. Conversely, $\overline{K}_i(\neg\overline{K}_i)^{r-1}(\cdot) \leqslant \overline{K}_i\overline{K}_i(\neg\overline{K}_i)^{r-1}(\cdot) \leqslant \overline{K}_i\overline{A}_i^{(n)}(\cdot)$, where the first inequality follows from Positive Introspection of $\overline{K}_i$, and the second inequality from Monotonicity of $\overline{K}_i$ as $\overline{K}_i(\neg\overline{K}_i)^{r-1}(\cdot) \leqslant \overline{A}_i^{(n)}(\cdot)$. Then, $\overline{A}_i^{(n)}(\cdot) = \bigvee_{r=1}^{n} \overline{K}_i(\neg\overline{K}_i)^{r-1}(\cdot) \leqslant \overline{K}_i\overline{A}_i^{(n)}(\cdot)$.

Property (1): Second-order-k^n-Unawareness. It suffices to show $\overline{K}_i(\overline{S}^{\uparrow}) = \overline{A}_i^{(n)}\overline{U}_i^{(n)}(E, S)$:

$$\overline{K}_i(\overline{S}^{\uparrow}) \geqslant \overline{A}_i^{(n)}\overline{U}_i^{(n)}(E, S) \geqslant \overline{K}_i(\neg\overline{K}_i)\overline{U}_i^{(n)}(E, S) = \overline{K}_i(\overline{S}^{\uparrow}),$$

where the first inequality follows from Remark A.1 (8), the second inequality from the definition of $\overline{A}_i^{(n)}$, and the equality from Remark A.1 (11).

Property (2): Reverse AU Introspection. Property (1) (i.e., $\overline{U}_i^{(n)}\overline{U}_i^{(n)}(\overline{E}) = (\neg\overline{K}_i)(\overline{S}^{\uparrow})$) and Remark A.1 (9) (i.e., $(\neg\overline{K}_i)(\overline{S}^{\uparrow}) \leqslant \overline{U}_i^{(n)}(\overline{E})$) yield $\overline{U}_i^{(n)}\overline{U}_i^{(n)}(\overline{E}) \leqslant \overline{U}_i^{(n)}(\overline{E})$. Also, Property (1) yields $\overline{U}_i^{(n)}\overline{U}_i^{(n)}\overline{U}_i^{(n)}(\overline{E}) = (\neg\overline{K}_i)(\overline{S}^{\uparrow}) = \overline{U}_i^{(n)}\overline{U}_i^{(n)}(\overline{E})$.

Property (3): JU Introspection. I have

$$\overline{\partial}_i \overline{U}_i^{(n)}(E,S) = (\neg\overline{K}_i)\overline{U}_i^{(n)}(E,S) \wedge (\neg\overline{K}_i)\overline{A}_i^{(n)}(E,S) = \overline{S}^{\uparrow} \wedge \overline{U}_i^{(n)}(E,S) = \overline{U}_i^{(n)}(E,S),$$

where the first equality follows from the definition of $\overline{\partial}_i$, the second equality from Remark A.1 (11) and Property (6) (i.e., $\overline{K}_i\overline{A}_i^{(n)} = \overline{A}_i^{(n)}$). Next, $\overline{U}_i^{(n)}(\overline{E}) = \overline{\partial}_i\overline{U}_i^{(n)}(\overline{E})$ is equivalent to $\overline{J}_i\overline{A}_i^{(n)}(\overline{E}) = \overline{A}_i^{(n)}(\overline{E})$ as $\overline{J}_i\overline{A}_i^{(n)}(\overline{E}) = \overline{J}_i\overline{U}_i^{(n)}(\overline{E})$ by Remark A.1 (4).

Property (4): Weak A-Negative Introspection. Weak A-Negative Introspection follows from the definition of $\overline{A}_i^{(2)}$ and Truth Axiom of $\overline{K}_i$.

Property (5): AK Self-Reflection. AK Self-Reflection follows from Remark A.1 (1) (i.e., $\overline{K}_i = \overline{K}_i\overline{K}_i$).

Property (7): Weak AA Self-Reflection. Let $n = 2$. I have

$$\begin{aligned}\overline{U}_i^{(2)}\overline{A}_i^{(2)}(E,S) &= (\neg\overline{K}_i)\overline{A}_i^{(2)}(E,S) \wedge (\neg\overline{K}_i)^2\overline{A}_i^{(2)}(E,S) = \overline{U}_i^{(2)}(E,S) \wedge (\neg\overline{K}_i)\overline{U}_i^{(2)}(E,S)\\ &= \overline{U}_i^{(2)}(E,S) \wedge \overline{S}^{\uparrow} = \overline{U}_i^{(2)}(E,S),\end{aligned}$$

where the first equality follows from the definition of $\overline{U}_i^{(2)}$, the second equality from Property (6) (i.e., $\overline{K}_i\overline{A}_i^{(2)} = \overline{A}_i^{(2)}$), and the third equality from Remark A.1 (11). Taking the negation yields $\overline{A}_i^{(2)}(\overline{E}) = \overline{A}_i^{(2)}\overline{A}_i^{(2)}(\overline{E})$.

Let $n = \infty$. I have $\overline{U}_i^{(\infty)}\overline{A}_i^{(\infty)}(E,S) \leqslant (\neg\overline{K}_i)\overline{A}_i^{(\infty)}(E,S) = \overline{U}_i^{(\infty)}(E,S)$, where the inequality follows from the definition of $\overline{U}_i^{(\infty)}$, and the equality from Property (6). Taking the negation yields $\overline{A}_i^{(\infty)}(E,S) \leqslant \overline{A}_i^{(\infty)}\overline{A}_i^{(\infty)}(E,S)$.

Next, $\overline{K}_i(\overline{S}^{\uparrow}) \geqslant \overline{A}_i^{(\infty)}\overline{A}_i^{(\infty)}(E,S) \geqslant \overline{K}_i(\neg\overline{K}_i)^2\overline{A}_i^{(\infty)}(E,S) = \overline{K}_i(\neg\overline{K}_i)\overline{U}_i^{(\infty)}(E,S) = \overline{K}_i(\overline{S}^{\uparrow})$, where the first inequality follows from Remark A.1 (8), the second inequality from the definition of $\overline{A}_i^{(\infty)}$, the first equality from Property (6) (more precisely, $(\neg\overline{K}_i)\overline{A}_i^{(\infty)} = \overline{U}_i^{(\infty)}$), and the second equality from Remark A.1 (11). Thus,

$$\overline{A}_i^{(\infty)}\overline{A}_i^{(\infty)}\overline{A}_i^{(\infty)}(E,S) = \overline{K}_i(\overline{S}^{\uparrow}) = \overline{A}_i^{(\infty)}\overline{A}_i^{(\infty)}(E,S). \tag{A.2}$$

Property (8): Possibility-of-Awareness. I have $\overline{M}_i^{(n)}\overline{A}_i^{(n)}(E,S) = (\neg\overline{K}_i)\overline{U}_i^{(n)}(E,S) \wedge \overline{A}_i^{(n)}\overline{A}_i^{(n)}(E,S) = \overline{S}^{\uparrow} \wedge \overline{A}_i^{(n)}\overline{A}_i^{(n)}(E,S) = \overline{A}_i^{(n)}\overline{A}_i^{(n)}(E,S)$, where the first equality follows from the definition of $\overline{M}_i^{(n)}$, and the second equality from Remark A.1 (11). □

Proof of Proposition 4. In the proof, I write $\overline{E}$ or $(E, S(E))$ for a given event (E,S). First, I show that (a) is equivalent to Weak Necessitation of $\overline{U}_i^{(2)}$: $(\neg\overline{K}_i)(\overline{S}^{\uparrow}) = \overline{U}_i^{(2)}(\overline{E})$. Suppose (a). I show Weak Necessitation by proving:

$$\begin{aligned}\overline{U}_i^{(2)}(\overline{E}) &= (\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)^2(\overline{E}) \leqslant (\neg\overline{K}_i)(\overline{E}) \wedge ((\neg\overline{K}_i)(\overline{S}^{\uparrow}) \vee \overline{K}_i(\overline{E}))\\ &= ((\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)(\overline{S}^{\uparrow})) \vee (\emptyset, S) = (\neg\overline{K}_i)(\overline{S}^{\uparrow}) \leqslant \overline{U}_i^{(2)}(\overline{E}).\end{aligned}$$

The first equality follows from the definition of $\overline{U}_i^{(2)}$. I prove the first inequality by showing $(\neg\overline{K}_i)^2(\overline{E}) \leqslant (\neg\overline{K}_i)(\overline{S}^{\uparrow}) \vee \overline{K}_i(\overline{E})$. By Remark A.2 (1), operate $(\neg\overline{K}_i)$ on both sides of $(\neg\overline{K}_i)(\overline{E}) \geqslant \overline{K}_i(\overline{S}^{\uparrow}) \wedge (\neg\overline{K}_i)(\overline{E})$ to get $(\neg\overline{K}_i)^2(\overline{E}) \leqslant (\neg\overline{K}_i)(\overline{K}_i(\overline{S}^{\uparrow}) \wedge (\neg\overline{K}_i)(\overline{E}))$. By taking the negation to (a), $(\neg\overline{K}_i)(\overline{K}_i(\overline{S}^{\uparrow}) \wedge (\neg\overline{K}_i)(\overline{E})) \leqslant \neg(\overline{K}_i(\overline{S}^{\uparrow}) \wedge (\neg\overline{K}_i)(\overline{E})) = (\neg\overline{K}_i)(\overline{S}^{\uparrow}) \vee \overline{K}_i(\overline{E})$. Next, the second equality follows by definition. The third equality follows because $(\neg\overline{K}_i)(\overline{S}^{\uparrow}) \leqslant (\neg\overline{K}_i)(\overline{E})$ holds by Monotonicity of $\overline{K}_i$. The second inequality follows from Remark A.1 (9).

Conversely, $\overline{K}_i(\overline{S}^{\uparrow}) = \overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg\overline{K}_i)(\overline{E})$ by Weak Necessitation of $\overline{U}_i^{(2)}$. Then,

$$\begin{aligned}\overline{K}_i(\overline{S}^{\uparrow}) \wedge (\neg\overline{K}_i)(\overline{E}) &= (\overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg\overline{K}_i)(\overline{E})) \wedge (\neg\overline{K}_i)(\overline{E}) \\ &= (\emptyset, S) \vee (\overline{K}_i(\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)(\overline{E})) = \overline{K}_i(\neg\overline{K}_i)(\overline{E}), \end{aligned} \tag{A.3}$$

where the last equality follows from Truth Axiom of $\overline{K}_i$. Then, (a) holds because

$$\overline{K}_i(\overline{S}^{\uparrow}) \wedge (\neg\overline{K}_i)(\overline{E}) = \overline{K}_i(\neg\overline{K}_i)(\overline{E}) \leqslant \overline{K}_i(\overline{K}_i(\neg\overline{K}_i)(\overline{E})) = \overline{K}_i(\overline{K}_i(\overline{S}^{\uparrow}) \wedge (\neg\overline{K}_i)(\overline{E})),$$

where the equalities follow from Expression (A.3), and the inequality from Positive Introspection of $\overline{K}_i$.

Second, (b) follows from Weak Necessitation of $\overline{U}_i^{(2)}$ because $\overline{U}_i^{(2)}(\overline{E}) = (\neg\overline{K}_i)(\overline{S}^{\uparrow}) \leqslant (\neg\overline{K}_i)^3(\overline{E})$, where the inequality follows from Monotonicity of $\overline{K}_i$ (i.e., $(\neg\overline{K}_i)(\overline{S}^{\uparrow}) \leqslant (\neg\overline{K}_i)(F, S)$ for any (F, S) including $(F, S) = (\neg\overline{K}_i)^2(\overline{E})$). Next, I show that (b) implies (c). Recall that, by Lemma A.1, $\overline{U}_i^{(\infty)}(\cdot) = \bigwedge_{r=1}^{3}(\neg\overline{K}_i)^r(\cdot)$. Now, (b) implies that $\overline{U}_i^{(2)}(\cdot) \leqslant \overline{U}_i^{(\infty)}(\cdot)$. In contrast, by definition, $\overline{U}_i^{(\infty)}(\cdot) \leqslant \overline{U}_i^{(2)}(\cdot)$. Next, I show that (c) implies Weak Necessitation of $\overline{U}_i^{(2)}$:

$$\overline{A}_i^{(2)}(\overline{E}) = \overline{A}_i^{(2)}\overline{A}_i^{(2)}(\overline{E}) = \overline{A}_i^{(\infty)}\overline{A}_i^{(2)}(\overline{E}) = \overline{A}_i^{(2)}(\neg\overline{K}_i)\overline{A}_i^{(2)}(\overline{E}) = \overline{A}_i^{(2)}\overline{U}_i^{(2)}(\overline{E}) = \overline{K}_i(\overline{S}^{\uparrow}).$$

The first equality follows from Proposition 3 (7), the second equality from (c), the third equality from Proposition 1 (1) (i.e., $\overline{A}_i^{(\infty)} = \overline{A}_i^{(2)}(\neg\overline{K}_i)$), the fourth equality from taking the negation on Proposition 3 (6) (i.e., $(\neg\overline{K}_i)\overline{A}_i^{(2)} = \overline{U}_i^{(2)}$), and the fifth equality from taking the negation on Proposition 3 (1) (i.e., $\overline{A}_i^{(2)}\overline{U}_i^{(2)}(E, S) = \overline{K}_i(\overline{S}^{\uparrow})$).

Third, (d) follows from (c) and Proposition 3 (4). Conversely, I show (d) implies (c): $\overline{U}_i^{(2)}(\cdot) = (\neg\overline{K}_i)(\cdot) \wedge (\neg\overline{K}_i)^2(\cdot) = (\neg\overline{K}_i)(\cdot) \wedge (\overline{K}_i(\cdot) \vee \overline{U}_i^{(\infty)}(\cdot)) = (\emptyset, S(\cdot)) \vee ((\neg\overline{K}_i)(\cdot) \wedge \overline{U}_i^{(\infty)}(\cdot)) = \overline{U}_i^{(\infty)}(\cdot)$, where the second equality follows from (d).

Fourth, (e) follows from Weak Necessitation of $\overline{U}_i^{(2)}$: $\overline{U}_i^{(2)}(E, S) = (\neg\overline{K}_i)(\overline{S}^{\uparrow}) = \overline{U}_i^{(2)}(\neg(E, S))$. Conversely, by Proposition A.1, (e) implies (c): $\overline{U}_i^{(2)}(\overline{E}) = \overline{U}_i^{(2)}(\overline{E}) \wedge \overline{U}_i^{(2)}(\neg\overline{E}) \leqslant \overline{U}_i^{(\infty)}(\overline{E}) \leqslant \overline{U}_i^{(2)}(\overline{E})$.

Fifth, (f) follows (with equality) from Weak Necessitation of $\overline{U}_i^{(n)}$: $\overline{U}_i^{(n)}(E, S) = (\neg\overline{K}_i)(\overline{S}^{\uparrow}) = \overline{U}_i^{(n)}\overline{U}_i^{(n)}(E, S)$, where the last equality follows from Proposition 3 (1).

Conversely, suppose (f). Then, $\overline{U}_i^{(n)}(E,S) = \overline{U}_i^{(n)}\overline{U}_i^{(n)}(E,S) = (\neg\overline{K}_i)(\overline{S}^{\uparrow})$, where the first equality follows from (f) and Reverse AU Introspection (i.e., Proposition 3 (2)), and the second equality from Proposition 3 (1).

Sixth, it follows from A-Introspection (i.e., Proposition 3 (6)) that (f) and (g) are equivalent: $\overline{M}_i^{(n)}\overline{U}_i^{(n)}(\overline{E}) = (\neg\overline{K}_i)(\overline{A}_i^{(n)})(\overline{E}) \wedge \overline{A}_i^{(n)}\overline{U}_i^{(n)}(\overline{E}) = \overline{U}_i^{(n)}(\overline{E}) \wedge \overline{A}_i^{(n)}\overline{U}_i^{(n)}(\overline{E})$.

Seventh, I show that (h) follows from Weak Necessitation of $\overline{U}_i^{(n)}$. If $(E,S(E)) \leqslant (F,S(F))$, then $\overline{A}_i^{(n)}(E,S(E)) = \overline{K}_i(\overline{S(E)}^{\uparrow}) \leqslant \overline{K}_i(\overline{S(F)}^{\uparrow}) = \overline{A}_i^{(n)}(F,S(F))$. Conversely, I show that (h) implies Weak Necessitation: $\overline{K}_i(\overline{S}^{\uparrow}) = \overline{K}_i(\neg\overline{K}_i)(\emptyset,S) \leqslant \overline{A}_i^{(n)}(\emptyset,S) \leqslant \overline{A}_i^{(n)}(E,S) \leqslant \overline{K}_i(\overline{S}^{\uparrow})$. The equality follows from $\overline{K}_i(\emptyset,S) = (\emptyset,S)$. The first inequality follows from the definition of $\overline{A}_i^{(n)}$, the second inequality from (h), and the third inequality from Remark A.1 (8).

Eighth, (i) (i.e., $\overline{A}_i^{(\infty)}(E,S) = \overline{A}_i^{(\infty)}\overline{A}_i^{(\infty)}(E,S)$) and Weak Necessitation for $n=\infty$ (i.e., $\overline{A}_i^{(\infty)}(E,S) = \overline{K}_i(\overline{S}^{\uparrow})$) are equivalent because $\overline{A}_i^{(\infty)}\overline{A}_i^{(\infty)}(E,S) = \overline{K}_i(\overline{S}^{\uparrow})$ holds by Expression (A.2) in the proof of Proposition 3 (7). □

A.5 Section 5.1

The following technical lemma is used in the proofs of Proposition 5 (2a) (and Proposition S.4 in Online Appendix C.2).

Lemma A.2. *Let $\overline{K}_i : \mathcal{E} \to \mathcal{E}$ be an operator satisfying Finite Conjunction and Monotonicity. For any $\overline{E},\overline{F} \in \mathcal{E}$, $\overline{K}_i(\overline{E} \vee \overline{F}) \vee (\neg\overline{K}_i)(\neg\overline{F}) \leqslant \overline{K}_i(\overline{E}) \vee (\neg\overline{K}_i)(\neg\overline{F})$.*

Proof of Lemma A.2. First, $\overline{K}_i(\overline{E} \vee \overline{F}) \vee (\neg\overline{K}_i)(\neg\overline{F}) = (\overline{K}_i(\overline{E} \vee \overline{F}) \wedge \overline{K}_i(\neg\overline{F})) \vee (\neg\overline{K}_i)(\neg\overline{F})$. Second, $\overline{K}_i(\overline{E} \vee \overline{F}) \wedge \overline{K}_i(\neg\overline{F}) = \overline{K}_i((\overline{E} \vee \overline{F}) \wedge (\neg\overline{F})) = \overline{K}_i(\overline{E} \wedge (\neg\overline{F}))$, where the first equality follows from Finite Conjunction and Monotonicity. Third,

$$\overline{K}_i(\overline{E} \vee \overline{F}) \vee (\neg\overline{K}_i)(\neg\overline{F}) = \overline{K}_i(\overline{E} \wedge (\neg\overline{F})) \vee (\neg\overline{K}_i)(\neg\overline{F}) \leqslant \overline{K}_i(\overline{E}) \vee (\neg\overline{K}_i)(\neg\overline{F}),$$

where the equality follows from the previous two parts, and the inequality from Monotonicity. □

Proof of Proposition 5. I start with showing:

$$\overline{K}_i(\overline{\mathbb{U}}_i^{(n)}) = (\emptyset, \sup\mathcal{S}) \text{ implies } \overline{A}_i^{(n)}(\overline{\mathbb{U}}_i^{(n)}) = \overline{K}_i(\overline{\sup\mathcal{S}}). \tag{A.4}$$

Let $\overline{K}_i(\overline{\mathbb{U}}_i^{(n)}) = (\emptyset, \sup\mathcal{S})$. By taking the negation, $(\neg\overline{K}_i)(\overline{\mathbb{U}}_i^{(n)}) = \overline{\sup\mathcal{S}}$, and thus $\overline{A}_i^{(2)}(\overline{\mathbb{U}}_i^{(n)}) = \overline{K}_i(\overline{\sup\mathcal{S}})$. Then, $\overline{K}_i(\overline{\sup\mathcal{S}}) = \overline{A}_i^{(2)}(\overline{\mathbb{U}}_i^{(n)}) \leqslant \overline{A}_i^{(n)}(\overline{\mathbb{U}}_i^{(n)}) \leqslant \overline{K}_i(\overline{\sup\mathcal{S}})$, where the last inequality follows from Remark A.1 (8). Hence, $\overline{A}_i^{(n)}(\overline{\mathbb{U}}_i^{(n)}) = \overline{K}_i(\overline{\sup\mathcal{S}})$.

Part (1). Assume AU Introspection. For Part (1a), recall that, by Proposition 4, AU Introspection is equivalent to Weak Necessitation: $\overline{A}_i^{(n)}(\overline{E}) = \overline{K}_i(\overline{S(E)}^{\uparrow})$ or equivalently $\overline{U}_i^{(n)}(\overline{E}) = (\neg\overline{K}_i)(\overline{S(E)}^{\uparrow})$ for any $\overline{E} \in \mathcal{E}$. Hence,

$$\overline{\mathbb{U}}_i^{(n)} = \neg \bigwedge_{\overline{E}\in\mathcal{E}} \overline{A}_i^{(n)}(\overline{E}) = \neg \bigwedge_{\overline{E}\in\mathcal{E}} \overline{K}_i(\overline{S(E)}^{\uparrow}) = \neg\overline{K}_i(\overline{\sup\mathcal{S}}) = \overline{U}_i^{(n)}(\overline{\sup\mathcal{S}}),$$

where the first equality follows from the definition of $\overline{\mathbb{U}}_i^{(n)}$, the second and forth equalities from AU Introspection (precisely, Weak Necessitation), and the third equality from Monotonicity of $\overline{K}_i$.

For Part (1b), by KU Introspection, operating $\overline{K}_i$ to Part (1a) gives $\overline{K}_i(\overline{\mathbb{U}}_i^{(n)}) = \overline{K}_i\overline{U}_i^{(n)}(\overline{\sup\mathcal{S}}) = (\emptyset, \sup\mathcal{S})$. Then, Expression (A.4) yields $\overline{A}_i^{(n)}(\overline{\mathbb{U}}_i^{(n)}) = \overline{K}_i(\overline{\sup\mathcal{S}})$.

For Part (1c), Parts (1a) and (1b) imply $\overline{\mathbb{U}}_i^{(n)} \wedge \overline{A}_i^{(n)}(\overline{\mathbb{U}}_i^{(n)}) = \overline{U}_i^{(n)}(\overline{\sup\mathcal{S}}) \wedge \overline{K}_i(\overline{\sup\mathcal{S}}) = (\emptyset, \sup\mathcal{S})$, as $\overline{U}_i^{(n)}(\overline{\sup\mathcal{S}}) \leqslant (\neg\overline{K}_i)(\overline{\sup\mathcal{S}})$.

Part (2a). I start with showing that, for any $\overline{E}, \overline{F} \in \mathcal{E}$,

$$\overline{K}_i(\overline{E} \vee \overline{F}) \leqslant \overline{K}_i(\overline{K}_i(\overline{E}) \vee (\neg\overline{K}_i)(\neg\overline{F})). \tag{A.5}$$

Lemma A.2 implies $\overline{K}_i(\overline{E} \vee \overline{F}) \leqslant \overline{K}_i(\overline{E} \vee \overline{F}) \vee (\neg\overline{K}_i)(\neg\overline{F}) \leqslant \overline{K}_i(\overline{E}) \vee (\neg\overline{K}_i)(\neg\overline{F})$, where the first inequality follows because the both sides of the inequality have the same base space. By operating $\overline{K}_i$ on the expression, it follows from Monotonicity of $\overline{K}_i$ that $\overline{K}_i\overline{K}_i(\overline{E} \vee \overline{F}) \leqslant \overline{K}_i(\overline{K}_i(\overline{E}) \vee (\neg\overline{K}_i)(\neg\overline{F}))$. By Positive Introspection of $\overline{K}_i$, $\overline{K}_i(\overline{E} \vee \overline{F}) \leqslant \overline{K}_i\overline{K}_i(\overline{E} \vee \overline{F})$, establishing Expression (A.5).

Now, let $\mathcal{E}$ be finite, and re-label it by $\mathcal{E} = \{(E_r, S_r)\}_{r=1}^m$ so that $\overline{\mathbb{U}}_i^{(n)} = \bigvee_{r=1}^m \overline{U}_i^{(n)}(\overline{E}_r)$. Substitute $\overline{E} = \overline{U}_i^{(n)}(\overline{E}_m)$ and $\overline{F} = \bigvee_{r=1}^{m-1} \overline{U}_i^{(n)}(\overline{E}_r)$ into Expression (A.5). Then,

$$(\neg\overline{K}_i\neg)(\overline{F}) = (\neg\overline{K}_i)(\bigwedge_{r=1}^{m-1} \overline{A}_i^{(n)}(\overline{E}_r)) = \neg(\bigwedge_{r=1}^{m-1} \overline{K}_i\overline{A}_i^{(n)}(\overline{E}_r)) = \neg(\bigwedge_{r=1}^{m-1} \overline{A}_i^{(n)}(\overline{E}_r)) = \bigvee_{r=1}^{m-1} \overline{U}_i^{(n)}(\overline{E}_r),$$

where the second equality follows from Finite Conjunction and Monotonicity of $\overline{K}_i$, and the third equality from Proposition 3 (6). Then,

$$\overline{K}_i(\overline{\mathbb{U}}_i^{(n)}) \leqslant \overline{K}_i(\overline{K}_i\overline{U}_i^{(n)}(\overline{E}_m) \vee \bigvee_{r=1}^{m-1} \overline{U}_i^{(n)}(\overline{E}_r)) = \overline{K}_i((\emptyset, S_m) \vee \bigvee_{r=1}^{m-1} \overline{U}_i^{(n)}(\overline{E}_r)) \leqslant \overline{K}_i(\bigvee_{r=1}^{m-1} \overline{U}_i^{(n)}(\overline{E}_r)).$$

By induction, $\overline{K}_i(\overline{\mathbb{U}}_i^{(n)}) \leqslant (\emptyset, S_1)$. Since $K_i(\mathbb{U}_i^{(n)}) = \emptyset$, it follows $\overline{K}_i(\overline{\mathbb{U}}_i^{(n)}) = (\emptyset, \sup\mathcal{S})$. Then, Expression (A.4) yields $\overline{A}_i^{(n)}(\overline{\mathbb{U}}_i^{(n)}) = \overline{K}_i(\overline{\sup\mathcal{S}})$.

Part (2b). For an example in which $\overline{\mathbb{U}}_i^{(n)} \wedge \overline{A}_i^{(n)}(\overline{\mathbb{U}}_i^{(n)}) \neq (\emptyset, \sup\mathcal{S})$, as in the discussions in the main text, take agent i_1 or i_3 in Example 1.

Part (3). I provide a counterexample when $\mathcal{E}$ is not finite in the context of a standard state space (one could embed the standard-state-space example into a generalized state space). Let $(\Omega, \mathcal{E}) = (\mathbb{R}, \mathcal{P}(\Omega))$.[24] Let the agent's knowledge operator K_i be the interior operator on the usual Euclidean topology. Her knowledge satisfies Finite Conjunction and Necessitation as well as Truth Axiom, Monotonicity, and Positive Introspection. The unawareness operator $U_i^{(n)}$ satisfies Plausibility and KU Introspection, and violates AU Introspection. Now, for any $\omega \in \Omega$, let $E_\omega = (\omega, +\infty)$. Then, $U_i^{(2)}(E_\omega) = \partial_i K_i E_\omega = \{\omega\}$ and $U_i^{(\infty)}(E_\omega) = \partial_i (\neg K_i)^2 E_\omega = \{\omega\}$. Thus, $K_i(\bigcup_{\omega\in\Omega} U_i^{(n)}(E_\omega)) = K_i(\Omega) = \Omega$. This implies that $\Omega = K_i(\mathbb{U}_i^{(n)}) = \mathbb{U}_i^{(n)} \cap A_i^{(n)}(\mathbb{U}_i^{(n)})$. In words, the agent knows, at any state, that she is unaware of something. Also, at any state, there is an event of which she is unaware, and she is aware that there is an event of which she is unaware. □

A.6 Section 5.2

The "if" part of Proposition 6 follows from the following assertion.

Remark A.3. If $\overline{\Pi}_i^{\uparrow}$ is at least as knowledgeable as $\overline{\Pi}_j^{\uparrow}$, then any $\overline{\Pi}_j^{\uparrow}$-feasible act f is $\overline{\Pi}_i^{\uparrow}$-feasible.

Proof of Remark A.3. Let $\overline{\Pi}_i^{\uparrow}$ be at least as knowledgeable as $\overline{\Pi}_j^{\uparrow}$. Take any $\overline{\Pi}_j^{\uparrow}$-feasible act f. To show that f is $\overline{\Pi}_i^{\uparrow}$-feasible, assume $\overline{\Pi}_i^{\uparrow}(\omega) = \overline{\Pi}_i^{\uparrow}(\omega')$. Since $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Reflexivity and since $\overline{\Pi}_i^{\uparrow}$ is at least as knowledgeable as $\overline{\Pi}_j^{\uparrow}$, it follows that $\omega \in \Pi_i^{\uparrow}(\omega) = \Pi_i^{\uparrow}(\omega') \subseteq \Pi_j^{\uparrow}(\omega')$ and $\omega' \in \Pi_i^{\uparrow}(\omega') = \Pi_i^{\uparrow}(\omega) \subseteq \Pi_j^{\uparrow}(\omega)$. Since $\overline{\Pi}_j^{\uparrow}$ satisfies Generalized Transitivity, $\overline{\Pi}_j^{\uparrow}(\omega) = \overline{\Pi}_j^{\uparrow}(\omega')$ and thus $f(\omega) = f(\omega')$. □

To prove Proposition 6, for any $(\omega, S, S') \in \Omega \times \mathcal{S} \times \mathcal{S}$ with $\omega \in S'$ and $S' \succeq S$, denote by $\omega_S := r_S^{S'}(\omega)$ the projection of ω onto S.

Proof of Proposition 6. The "if" part follows from Remark A.3. For the converse statement, let $\overline{\Pi}_i^{\uparrow}$ satisfy Generalized Euclideanness. I prove the contrapositive in four steps: if $\overline{\Pi}_i^{\uparrow}$ is *not* at least as knowledgeable as $\overline{\Pi}_j^{\uparrow}$, then there exists a preference relation $\succsim$ that does *not* prefer $\overline{\Pi}_i^{\uparrow}$ to $\overline{\Pi}_j^{\uparrow}$. Thus, suppose that there are $\omega, \omega' \in \Omega$ such that $\omega' \in \Pi_i^{\uparrow}(\omega)$ and $\omega' \notin \Pi_j^{\uparrow}(\omega)$ (i.e., $\overline{\Pi}_i^{\uparrow}$ is not at least as knowledgeable as $\overline{\Pi}_j^{\uparrow}$).

First, for agent i, I show $\overline{\Pi}_i^{\uparrow}(\omega) = \overline{\Pi}_i^{\uparrow}(\omega'_{\sigma_i(\omega)})$, where $\omega'_{\sigma_i(\omega)}$ is the projection of ω' onto the subspace $\sigma_i(\omega)$ to which the event $\overline{\Pi}_i^{\uparrow}(\omega)$ belongs. Since $\omega' \in \Pi_i^{\uparrow}(\omega)$, $\omega'_{\sigma_i(\omega)} \in \Pi_i(\omega)$. By Generalized Euclideanness and Generalized Transitivity (i.e., Stationarity), $\Pi_i(\omega'_{\sigma_i(\omega)}) = \Pi_i(\omega)$. Thus, $\overline{\Pi}_i^{\uparrow}(\omega) = \overline{\Pi}_i^{\uparrow}(\omega'_{\sigma_i(\omega)})$.

[24] Two remarks are in order. First, one can provide a similar example on the set $\mathbb{Q}$ of rational numbers, which is a countable set. Second, any infinite complete algebra of sets is uncountable, as it is well known that any infinite σ-algebra is already uncountable.

Second, for agent j, I show $\overline{\Pi}_j^{\uparrow}(\omega) \neq \overline{\Pi}_j^{\uparrow}(\omega'_{\sigma_i(\omega)})$. By Generalized Reflexivity, $\omega'_{\sigma_i(\omega)} \in \Pi_j^{\uparrow}(\omega'_{\sigma_i(\omega)})$. In contrast, since $\omega' \notin \Pi_j^{\uparrow}(\omega)$ by assumption, it follows that $\omega'_{\sigma_i(\omega)} \notin \Pi_j^{\uparrow}(\omega)$.

Third, take $z, z' \in Z$ with $z \neq z'$, and define an act f as:

$$f(\omega'') = \begin{cases} z & \text{if } \Pi_j^{\uparrow}(\omega'') = \Pi_j^{\uparrow}(\omega) \\ z' & \text{otherwise} \end{cases}.$$

By construction, f is $\overline{\Pi}_j^{\uparrow}$-feasible. Also, f is not $\overline{\Pi}_i^{\uparrow}$-feasible: while $\Pi_i^{\uparrow}(\omega) = \Pi_i^{\uparrow}(\omega'_{\sigma_i(\omega)})$ by the first step, the second step implies $f(\omega) = z \neq z' = f(\omega'_{\sigma_i(\omega)})$.

Fourth, define a preference relation $\succsim$ such that $f \succ g$ for any other act g. Then, there exists no $\overline{\Pi}_i^{\uparrow}$-feasible act g such that $g \succsim f$. Hence, the preference relation $\succsim$ does not prefer $\overline{\Pi}_i^{\uparrow}$ to $\overline{\Pi}_j^{\uparrow}$. □

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