Online Supplementary Appendix

Unawareness without AU Introspection

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November 27, 2020

The Online Appendix is structured as follows. Appendix B supplements Section 3 (of the main text). It provides two supplementary results (Propositions S.1 and S.2) on generalized-state-space possibility-correspondence models mentioned in Section 3.2. It also supplements the examples in Section 3. Appendix C provides two supplementary results (Propositions S.3 and S.4) on the conjunctive property of (un)awareness. This is supplementary to Propositions 3 and 4 in Section 4.3. Appendix D supplements Section 5. It provides a characterization of knowledgeability by ignorance and unawareness (Proposition S.5). Appendix E introduces the notion of common knowledge. Proofs are relegated to Appendix F.

B Section 3

B.1 Possibility-Correspondence Models (Section 3.2.1)

I provide the following four basic properties on a possibility correspondence $\overline{\Pi}_i^{\uparrow}$ under which it induces a knowledge operator. To that end, I introduce two notations. First, for any state $\omega \in \Omega$, denote by $S(\omega) \in \mathcal{S}$ the subspace to which ω belongs. Second, for any $(\omega, S, S') \in \Omega \times \mathcal{S} \times \mathcal{S}$ with $\omega \in S'$ (i.e., $S(\omega) = S'$) and $S' \succeq S$, denote by $\omega_S := r_S^{S'}(\omega)$ the projection of ω onto S. Then, the four basic properties are as follows.

1. Regularity: $\{\omega \in S \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant (E, S)\} \in \mathcal{D}$ for each $(E, S) \in \mathcal{E}$.
2. Projections Preserve Ignorance (PPI): If $S \preceq S(\omega)$, then $\overline{\Pi}_i^{\uparrow}(\omega) \leqslant \overline{\Pi}_i^{\uparrow}(\omega_S)$.
3. Projections Preserve Knowledge (PPK): Let $S \preceq S(\omega)$. If $\overline{\Pi}_i^{\uparrow}(\omega) \leqslant (E, S)$ for some $E \in \mathcal{P}(\Omega)$, then $\overline{\Pi}_i^{\uparrow}(\omega_S) \leqslant (E, S)$.
4. Confinedness: $\sigma_i(\omega) \preceq S(\omega)$ for all $\omega \in \Omega$.

While these four basic properties are slightly different from the corresponding basic assumptions of Heifetz, Meier, and Schipper (2006), Proposition S.1 shows that the four basic properties hold iff the possibility correspondence $\overline{\Pi}_i^{\uparrow}$ induces an operator.

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Regularity states that, for each complete algebra $(S, \mathcal{D})$, the set of states in S at which agent i "knows" an event (E, S) is a well-defined element of $\mathcal{D}$. If $\mathcal{D}$ is the power set $\mathcal{P}(S)$ as in Heifetz, Meier, and Schipper (2006), then Regularity is trivially satisfied. PPI states that knowledge in a lower subspace is preserved in an upper subspace: let $S \preceq S(\omega)$. For any event which agent i "knows" at ω_S, she knows it at ω. In contrast, PPK states that knowledge in an upper subspace is preserved in a lower subspace. Let $S \preceq S(\omega)$. If agent i "knows" an event (E, S) at ω, then she knows it at ω_S. One way to interpret Confinedness is that agent i "knows" the event $\overline{\Pi}_i^{\uparrow}(\omega)$ at ω.[1]

Proposition S.1. *The following are equivalent.*

1. $\overline{\Pi}_i^{\uparrow}$ *satisfies the basic properties: Regularity, PPI, PPK, and Confinedness.*
2. $\overline{K}_i(B^{\uparrow}, S) = (\{\omega \in S \mid \Pi_i(\omega) \subseteq B \text{ and } \sigma_i(\omega) = S\}^{\uparrow}, S) \in \mathcal{E}$ *for any* $(B^{\uparrow}, S) \in \mathcal{E}$.

The framework of this paper nests any possibility-correspondence model on a generalized state space which satisfies Generalized Reflexivity (i.e., Truth Axiom) and Generalized Transitivity (i.e., Positive Introspection). The stationary possibility-correspondence model of Heifetz, Meier, and Schipper (2006) is identified as a possibility-correspondence model satisfying Generalized Reflexivity, Generalized Transitivity, and Generalized Euclideanness. As discussed in the main text, the following proposition shows that Generalized Euclideanness characterizes Weak Necessitation, under which unawareness as the lack of knowledge and one as the lack of conception coincide.

Proposition S.2. *The following are equivalent.*

1. $\overline{\Pi}_i^{\uparrow}$ *satisfies Generalized Euclideanness.*
2. *Generalized Negative Introspection:* $(\neg\overline{K}_i)^2(\overline{E}) \wedge \overline{K}_i(\overline{S}^{\uparrow}) \leqslant \overline{K}_i(\overline{E})$ *for any* $\overline{E} = (E, S) \in \mathcal{E}$.
3. *Weak Necessitation of* $\overline{A}_i^{(2)}$: $\overline{A}_i^{(2)}(\overline{E}) = \overline{K}_i(\overline{S}^{\uparrow})$ *for all* $\overline{E} = (E, S) \in \mathcal{E}$.

B.2 Embedding Example 1 into a Generalized State Space (Sections 3.1.2 and 3.2.2)

As mentioned in footnote 11 of the main text, I embed agents' knowledge and unawareness in Example 1 into generalized-state-space possibility-correspondence models.

Example S.1. Let $\mathcal{S} = \{S, S'\}$, $(S, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(S))$, and $(S', \mathcal{D}') = (\{\omega_4\}, \mathcal{P}(S'))$. The projections satisfy $r^S_{S'}(\cdot) = \omega_4$ (r^S_S and $r^{S'}_{S'}$ are an identity). I define the knowledge operators of agents $\{j_1, j_2, j_3, j_4\}$ in a way such that they correspond to those of $\{i_1, i_2, i_3, i_4\}$ in Example 1 as follows.

[1] Board, Chung, and Schipper (2011) show that a possibility correspondence induces an operator if it satisfies PPI, PPK, and Confinedness (of their formulations). See also Grant et al. (2015).

First, recall that the knowledge operators of agents i_1 and i_3 in Example 1 are induced by the non-partitional possibility correspondences Π_{i_1} and Π_{i_3}. Thus, I define the corresponding non-stationary possibility correspondences $\overline{\Pi}^{\uparrow}_{j_1}$ and $\overline{\Pi}^{\uparrow}_{j_3}$ on the generalized state space. Let $\ell \in \{1,3\}$. Define $\overline{\Pi}^{\uparrow}_{j_\ell}(\omega) := (\Pi_{i_\ell}(\omega), S)$ for each $\omega \in \{\omega_1, \omega_2, \omega_3\}$, and $\overline{\Pi}^{\uparrow}_{j_\ell}(\omega_4) := (\{\omega_4\}^{\uparrow}, S')$. Letting $K_{i_\ell}(E)$ be the knowledge operator of i_ℓ in Table 1 of Example 1, the knowledge operator $\overline{K}_{j_\ell}$ satisfies $\overline{K}_{j_\ell}(E,S) = (K_{i_\ell}(E), S)$ for each $E \in \mathcal{D}$, and $\overline{K}_{j_\ell}(E', S') = (E', S')$ for each $E' \in \mathcal{D}'$. Thus, agent j_ℓ knows an event (E,S) at state $\omega \in S$ iff agent i_ℓ knows E at ω. This generalized-state-space model nests not only knowledge but also unawareness: $\overline{U}^{(n)}_{j_\ell}(E,S) = (U^{(n)}_{i_\ell}(E), S)$ for each $E \in \mathcal{D}$, and $\overline{U}^{(n)}_{j_\ell}(E', S') = (\emptyset, S')$ for each $E' \in \mathcal{D}'$. The possibility correspondence $\overline{\Pi}^{\uparrow}_{j_\ell}$ fails to satisfy Generalized Euclideanness (or Stationarity).

Second, in Example 1, the knowledge operators of agents i_2 and i_4 are not induced by any possibility correspondence, as Necessitation fails. However, the knowledge operator of i_2 would be induced by a partitional possibility correspondence if the entire state space is restricted to $\{\omega_1, \omega_3\}$. Likewise, the knowledge operator of i_4 would be induced by a partitional possibility correspondence if the entire state space is restricted to $\{\omega_1\}$. With these special features in mind, I define stationary possibility correspondences $\overline{\Pi}^{\uparrow}_{j_2}$ and $\overline{\Pi}^{\uparrow}_{j_4}$ in a way such that the induced knowledge operators $\overline{K}_{j_2}$ and $\overline{K}_{j_4}$ nest K_{i_2} and K_{i_4}.[2] For j_2, define $\overline{\Pi}^{\uparrow}_{j_2}(\omega) = (\{\omega\}, S)$ for each $\omega \in \{\omega_1, \omega_3\}$, and $\overline{\Pi}^{\uparrow}_{j_2}(\omega) = (\{\omega_4\}^{\uparrow}, S')$ for each $\omega \in \{\omega_2, \omega_4\}$. For j_4, define $\overline{\Pi}^{\uparrow}_{j_4}(\omega_1) = (\{\omega_1\}, S)$, and $\overline{\Pi}^{\uparrow}_{j_4}(\omega) = (\{\omega_4\}^{\uparrow}, S')$ for each $\omega \in \{\omega_2, \omega_3, \omega_4\}$. For each $\ell \in \{2,4\}$, $\overline{K}_{j_\ell}(E,S) = (K_{i_\ell}(E), S)$ for each $E \in \mathcal{D}$, and $\overline{K}_{j_\ell}(E', S') = (E', S')$ for each $E' \in \mathcal{D}'$. Hence, agent j_ℓ knows an event (E,S) at state $\omega \in S$ iff agent i_ℓ knows E at ω. The generalized-state-space model nests not only knowledge but also unawareness: $\overline{U}^{(n)}_{j_\ell}(E,S) = (U^{(n)}_{i_\ell}(E), S)$ for each $E \in \mathcal{D}$, and $\overline{U}^{(n)}_{j_\ell}(E', S') = (\emptyset, S')$ for each $E' \in \mathcal{D}'$. The possibility correspondence $\overline{\Pi}^{\uparrow}_{j_\ell}$ satisfies Generalized Euclideanness (or Stationarity).

B.3 Example 2 (Section 3.2.2)

Table 2 depicts the knowledge and unawareness operators of the caregiver (i) in the standard-state-space model of Example 2. The first column depicts each event $E \in \mathcal{P}(\Omega)$. The second column represents $K_i(E)$ for each event E in the first column. The six and seventh columns depict $U^{(2)}_i(E)$ and $U^{(\infty)}_i(E)$, respectively, for each event E in the first column.

[2]Generally, let $\overline{K}_j : \mathcal{E} \to \mathcal{E}$ be a knowledge operator induced by a possibility correspondence $\overline{\Pi}^{\uparrow}_j$ on a generalized state space. Denote by $\overline{K}_j(E,S) = (K_j(E), S)$ for each $(E,S) \in \mathcal{E}$. Fix a subspace $S \in \mathcal{S}$, and define a knowledge operator $K^S_i : \mathcal{D} \to \mathcal{D}$ (where $\mathcal{D}$ is the complete algebra on S) as $K^S_i(E) := K_j(E)$ for each $E \in \mathcal{D}$. Then, K^S_i satisfies Monotonicity, Truth Axiom, and Positive Introspection. While it also satisfies Non-empty Conjunction, it may not satisfy $K^S_i(S) = S$.

E	$K_i(E)$	$(\neg K_i)(E)$	$(\neg K_i)^2(E)$	$(\neg K_i)^3(E)$	$U_i^{(2)}(E)$	$U_i^{(\infty)}(E)$
$\emptyset$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_2,\omega_4\}$	$\{\omega_2,\omega_4\}$
$\{\omega_2\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$
$\{\omega_3\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_2,\omega_4\}$	$\{\omega_2,\omega_4\}$
$\{\omega_4\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$
$\{\omega_1,\omega_2\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_2,\omega_4\}$	$\{\omega_2,\omega_4\}$
$\{\omega_1,\omega_3\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_4\}$	Ω	$\emptyset$	$\{\omega_2,\omega_4\}$	$\emptyset$
$\{\omega_1,\omega_4\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_2,\omega_4\}$	$\{\omega_2,\omega_4\}$
$\{\omega_2,\omega_3\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_2,\omega_4\}$	$\{\omega_2,\omega_4\}$
$\{\omega_2,\omega_4\}$	$\emptyset$	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$
$\{\omega_3,\omega_4\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_2,\omega_4\}$	$\{\omega_2,\omega_4\}$
$\{\omega_1,\omega_2,\omega_3\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_4\}$	Ω	$\emptyset$	$\{\omega_2,\omega_4\}$	$\emptyset$
$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_2,\omega_4\}$	$\{\omega_2,\omega_4\}$
$\{\omega_1,\omega_3,\omega_4\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_4\}$	Ω	$\emptyset$	$\{\omega_2,\omega_4\}$	$\emptyset$
$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_2,\omega_3,\omega_4\}$	$\{\omega_1,\omega_2,\omega_4\}$	$\{\omega_2,\omega_4\}$	$\{\omega_2,\omega_4\}$
Ω	Ω	$\emptyset$	Ω	$\emptyset$	$\emptyset$	$\emptyset$

Table 2: The Caregiver's Knowledge and Unawareness Operators. For ease of notation, the states ω_{pt}, $\omega_{p\neg t}$, $\omega_{\neg pt}$, and $\omega_{\neg p\neg t}$ are replaced by ω_1, ω_2, ω_3, and ω_4, respectively.

C Properties of Unawareness (Section 4)

Here I provide two supplementary results on the properties of unawareness when knowledge satisfies Finite Conjunction: $\overline{K}_i(\overline{E}) \wedge \overline{K}_i(\overline{F}) \leqslant \overline{K}_i(\overline{E} \wedge \overline{F})$.

C.1 Section 4.1

Proposition A.1 has shown that if agent i is k^2-unaware of $\overline{E}$ *and* its negation $\neg\overline{E}$ then she is k^∞-unaware of $\overline{E}$ (and $\neg\overline{E}$). The following proposition shows that, under Finite Conjunction of the knowledge operator, agent i is k^2-unaware of an event *or* its negation iff she is k^2-unaware of knowing whether the event holds.

Proposition S.3. *For any $\overline{E} \in \mathcal{E}$, $\overline{U}_i^{(2)}(\overline{E}) \vee \overline{U}_i^{(2)}(\neg\overline{E}) \leqslant \overline{U}_i^{(2)}\overline{J}_i(\overline{E})$. The expression holds with equality if $\overline{K}_i$ satisfies Finite Conjunction.*

C.2 Propositions 3 and 4 (Section 4.3)

The following proposition supplements Propositions 3 and 4 in Section 4.3.

Proposition S.4. *Let $\mathcal{M}^{(n)}$ be a model such that $\overline{K}_i$ satisfies Finite Conjunction.*

1. *For any $\overline{E}, \overline{F} \in \mathcal{E}$ with $S(E) = S(F)$, $\overline{A}_i^{(n)}(\overline{E}) \wedge \overline{A}_i^{(n)}(\overline{F}) \leqslant \overline{A}_i^{(n)}(\overline{E} \wedge \overline{F})$.*
2. *$\overline{A}_i^{(n)}$ satisfies Weak Necessitation (i.e., $\overline{A}^{(n)}(E, S) = \overline{K}_i(\overline{S}^{\uparrow})$ for all $(E, S) \in \mathcal{E}$) iff $\overline{A}_i^{(n)}(\overline{E}) \wedge \overline{A}_i^{(n)}(\overline{F}) = \overline{A}_i^{(n)}(\overline{E} \wedge \overline{F})$ for any $\overline{E}, \overline{F} \in \mathcal{E}$.*

Proposition S.4 (1) states that the agent's awareness operator inherits the conjunctive property from her knowledge operator with respect to the events that reside in the same subspace. Namely, suppose that if the agent knows events $(E, S(E))$ and $(F, S(F))$ then she knows the conjunction $(E, S(E)) \wedge (F, S(F))$. Then, whenever the agent is aware of events $(E, S(E))$ and $(F, S(F))$, she is aware of the conjunction $(E, S(E)) \wedge (F, S(F))$, provided that $(E, S(E))$ and $(F, S(F))$ reside in the same subspace: $S(E) = S(F)$. Proposition S.4 (2) states that the agent's (k^n-)awareness operator satisfies Weak Necessitation iff the following holds: the agent is aware of $(E, S(E))$ and $(F, S(F))$ iff she is aware of its conjunction $(E, S(E)) \wedge (F, S(F))$. Assuming that the agent's knowledge operator satisfies Finite Conjunction, Proposition S.4 (1) is a property that is satisfied in any model $\mathcal{M}^{(n)}$ as in Proposition 3, while Proposition S.4 (2) characterizes Weak Necessitation as in Proposition 4.

Proposition S.4 clarifies the differences in the conjunctive properties of awareness between standard and generalized state-spaces models. First, Part (1) implies that, in a standard state space, the agent's awareness operator inherits the conjunctive property from her knowledge operator. Suppose that if the agent knows events E and F then she knows the conjunction $E \cap F$. Then, whenever the agent is aware of events E and F, she is aware of the conjunction $E \cap F$. Second, Example S.2 below demonstrates that, in a generalized state space, it may not be the case that the agent is aware of the conjunction $(E, S) \wedge (F, S')$ when she is aware of events (E, S) and (F, S'), if the base spaces of these events are different. Third, if the agent's (k^n-)awareness operator satisfies Weak Necessitation (in addition to Finite Conjunction of the knowledge operator), then irrespective of whether the underlying state space is a standard or a generalized one, awareness satisfies the stronger form of Finite Conjunction: the agent is aware of $(E, S(E))$ and $(F, S(F))$ iff she is aware of its conjunction $(E, S(E)) \wedge (F, S(F))$.[3]

Now, the following counterexample to Proposition S.4 (1) demonstrates that the assumption that the events $\overline{E}$ and $\overline{F}$ have the same base space cannot be dropped.

Example S.2. First, I define the generalized state space by $\mathcal{S} = \{S, S'\}$, where $S = \{\omega_1, \omega_2, \omega_3\}$, $S' = \{\omega_4, \omega_5\}$, and $S \succeq S'$. Each subspace is endowed with its power set: $(S, \mathcal{D}) = (S, \mathcal{P}(S))$ and $(S', \mathcal{D}') = (S', \mathcal{P}(S'))$. Define the projection $r^S_{S'}$ as $r^S_{S'}(\omega_1) = \omega_4$ and $r^S_{S'}(\omega_2) = r^S_{S'}(\omega_3) = \omega_5$ (the other projections r^S_S and $r^{S'}_{S'}$ are the identity maps).

Next, I define the knowledge and consequently k^n-awareness operators of the single agent i. For any event (E, S) which belongs to the subspace S (i.e., any subset E of S), define $\overline{K}_i(E, S) := (K_{i_1}(E), S)$ where K_{i_1} is the knowledge operator of agent i_1 in Example 1 (see also Table 1). For any event (E, S') which belongs to the subspace S' (i.e., any subset E of S'), define $\overline{K}_i(E, S') := (E, S')$. Then, $\overline{A}_i^{(n)}(E, S') = \overline{S'}^{\uparrow}$.

Now, take $\overline{E} = (S, S)$ and $F = (\{\omega_4\}^{\uparrow}, S')$. Then, $\overline{A}_i^{(n)}(\overline{E}) = (S, S)$ and $\overline{A}_i^{(n)}(\overline{F}) = (S'^{\uparrow}, S')$. Also, $\overline{E} \wedge \overline{F} = (\{\omega_1\}, S)$ and $\overline{A}_i^{(n)}(\overline{E} \wedge \overline{F}) = (\{\omega_1, \omega_2\}, S)$. Hence, Proposition S.4 (1) does not hold when $S(E) \neq S(F)$: $\overline{A}_i^{(n)}(\overline{E}) \wedge \overline{A}_i^{(n)}(\overline{F}) = (S, S) \not\leq (\{\omega_1, \omega_2\}, S) =$

[3] One can also show that, under the assumption that the agent's knowledge operator satisfies Non-empty Conjunction, $\overline{A}_i^{(n)}$ satisfies Weak Necessitation iff $\bigwedge_{\lambda \in \Lambda} \overline{A}_i^{(n)}(\overline{E}_\lambda) = \overline{A}_i^{(n)}(\bigwedge_{\lambda \in \Lambda} \overline{E}_\lambda)$.

$\overline{A}_i^{(n)}(\overline{E} \wedge \overline{F})$.

I also remark that the "$\geqslant$" direction of Proposition S.4 (1) and (2) does not necessarily hold:

$$\overline{A}_i^{(n)}(\overline{E}) \wedge \overline{A}_i^{(n)}(\overline{F}) \geqslant \overline{A}_i^{(n)}(\overline{E} \wedge \overline{F}).^{4} \tag{S.1}$$

For example, in Example 1, $A_{i_1}^{(n)}(\{\omega_1, \omega_3\}) \cap A_{i_1}^{(n)}(\{\omega_2, \omega_3\}) = \{\omega_1, \omega_2\} \subsetneq \Omega = A_{i_1}^{(n)}(\{\omega_3\})$. For Proposition S.4 (1), it means that Expression (S.1) does not hold even on a standard state space. For Proposition S.4 (2), this counterexample means that the awareness operator of agent i_1 in Example 1 violates Weak Necessitation.

D Characterizing Knowledgeability by Ignorance and Unawareness (Section 5.2.1)

While awareness may not necessarily be monotonic in knowledgeability, ignorance (and, by definition, knowledge) are always monotonic in knowledgeability. Thus, for example, if agent i is at least as knowledgeable as j, then KU Introspection of $(\overline{K}_i, \overline{U}_i^{(n)})$ yields $\overline{K}_j \overline{U}_i^{(n)}(\overline{E}) = (\emptyset, S(E))$. Ignorance is "decreasing" in knowledge because an agent is ignorant of an event $\overline{E}$ iff she does not know $\overline{E}$ and she does not know its negation $\neg\overline{E}$. Thus, for any event $\overline{E}$, if i is ignorant of $\overline{E}$ then so is j. Here, I characterize knowledgeability (recall Figure 2) in terms of unawareness and ignorance.

Proposition S.5. *Fix a model $\mathcal{M}^{(n)}$ with $n \in \{2, \infty\}$.*

1. $\overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$ *iff* $\overline{\partial}_i \overline{K}_j(\cdot) \leqslant \overline{U}_j^{(2)}(\cdot)$, *equivalently,* $\overline{A}_j^{(2)}(\cdot) \leqslant \overline{J}_i \overline{K}_j(\cdot)$.
2. *Suppose agent i is at least as knowledgeable as agent j. Then:*
   (a) $\overline{U}_i^{(n)} \overline{U}_j^{(n)}(\cdot) \leqslant \overline{U}_j^{(n)}(\cdot)$.
   (b) $\overline{A}_j^{(n)}(\cdot) \leqslant \overline{A}_i^{(n)} \overline{A}_j^{(n)}(\cdot)$.
   (c) $\overline{U}_i^{(n)}(\cdot) \leqslant \overline{\partial}_j \overline{U}_i^{(n)}(\cdot)$. *Equivalently,* $\overline{J}_j \overline{A}_i^{(n)}(\cdot) \leqslant \overline{A}_i^{(n)}(\cdot)$.

Part (1) states that agent i is at least as knowledgeable as agent j iff agent j is k^2-unaware of an event whenever agent i is ignorant of the event that agent j knows the event. Equivalently, in terms of awareness, agent i is at least as knowledgeable as agent j iff, whenever agent j is k^2-aware of an event, agent i knows whether agent j knows it. While agent's knowledge is not identified from her (un)awareness alone, one can characterize knowledgeability by unawareness and ignorance.

Part (2) provides possible forms of monotonicity of unawareness when agent i is at least as knowledgeable as agent j. First, Part (2a) states that if agent i is k^n-unaware that agent j is k^n-unaware of an event, then agent j is k^n-unaware of the

[4] Fagin and Halpern (1987), Halpern (2001) and Modica and Rustichini (1999) study this property.

event. Second, Part (2b) states instead that if agent j is k^n-aware of an event then agent i is k^n-aware that agent j is k^n-aware of the event. Third, Part (2c) means that if agent i is k^n-unaware of an event then agent j is ignorant of the event that agent i is k^n-unaware of the event. In terms of awareness, noting that agent j knows whether agent i is k^n-aware of an event iff agent j knows whether agent i is k^n-unaware of the event, Part (2c) equivalently states that if agent j knows whether agent i is k^n-aware of an event then agent i is k^n-aware of the event.

Mathematically, Part (2) holds when $i = j$. Part (2a) reduces to Reverse AU Introspection (i.e., Proposition 3 (2)). Part (2b) reduces to Weak AA Self-Reflection (i.e., Proposition 3 (7)). Part (2c) reduces to JU Introspection (i.e., Proposition 3 (3)).

E Common Knowledge

Here, I introduce common knowledge (e.g., Aumann, 1976, 1999). To that end, I first introduce three definitions. First, agent i's knowledge operator $\overline{K}_i$ satisfies *Countable Conjunction* if $\bigwedge_{n\in\mathbb{N}} \overline{K}_i(\overline{E}_n) \leqslant \overline{K}_i(\bigwedge_{n\in\mathbb{N}} \overline{E}_n)$. Second, an event $\overline{E}$ is *self-evident* to agent i if $\overline{E} \leqslant \overline{K}_i(\overline{E})$, i.e., agent i knows $\overline{E}$ whenever $\overline{E}$ obtains. Denote by $\mathcal{J}_i := \{\overline{E} \in \mathcal{E} \mid \overline{E} \leqslant \overline{K}_i(\overline{E})\}$ the collection of events self-evident to i. Third, an event $\overline{E}$ is *publicly evident* among agents I if it is self-evident to every $i \in I$ (Milgrom, 1981). The collection of publicly-evident events is $\bigcap_{i\in I} \mathcal{J}_i$.

Since the framework does not necessarily presuppose that agents' knowledge satisfies Countable Conjunction or is induced by a possibility correspondence, I introduce common knowledge as the "infimum" of the agents' individual knowledge.[5] Especially, the common knowledge operator satisfies Truth Axiom, Monotonicity, and Positive Introspection as individual knowledge operators do. In the formulation, common knowledge is associated with the knowledge of some hypothetical agent with the following two properties: (i) the hypothetical agent is at least as less knowledgeable as the agents I (in that anything that the hypothetical individual knows is known by any individual agent); and (ii) the hypothetical agent is most knowledgeable among those agents who are at least as less knowledgeable as the agents I. The formalization of common knowledge here extends, for example, those of Galanis (2013) in terms of publicly-evident events and Heifetz, Meier, and Schipper (2006) on their generalized state-space models while requiring weaker assumptions on agents' knowledge. The formalization of common knowledge also nests Monderer and Samet (1989) on a standard state space.

Define the *common knowledge operator* $\overline{C}_I : \mathcal{E} \to \mathcal{E}$ by

$$\overline{C}_I(\overline{E}) = \sup\{\overline{F} \in \mathcal{E} \mid \overline{F} \in \bigcap_{i\in I} \mathcal{J}_i \text{ and } \overline{F} \leqslant \overline{E}\} \text{ for each } \overline{E} \in \mathcal{E}.$$

Since $(\emptyset, S(E)) \leqslant \overline{C}_I(\overline{E}) \leqslant \overline{E}$, it follows that $S(\overline{C}_I(\overline{E})) = S(E)$. Letting $\overline{C}_I(\overline{E}) := (C_I(E), S(E))$, an event $\overline{E} \in \mathcal{E}$ is *commonly known* (or *common knowledge*) among

[5] For example, Aumann (1976, 1999) defines common knowledge from the finest partition which is coarser than every agent's partition in a partitional standard-state-space model.

I at a state ω if $\omega \in C_I(E)$. Since $\overline{C}_I(\overline{E}) \in \bigcap_{i \in I} \mathcal{J}_i$ can be shown, an event $\overline{E}$ is common knowledge among I at ω iff there is $\overline{F} \in \bigcap_{i \in I} \mathcal{J}_i$ with $\omega \in F$ and $\overline{F} \leqslant \overline{E}$. By construction, $\overline{C}_I$ satisfies Truth Axiom, Positive Introspection, and Monotonicity. It can be seen that $\overline{C}_I$ inherits each of Necessitation, conjunction properties, and Negative Introspection from the agents' knowledge operators.

Common knowledge implies mutual knowledge: $\overline{C}_I(\cdot) \leqslant \overline{K}_I(\cdot) := \bigwedge_{i \in I} \overline{K}_i(\cdot)$, where $\overline{K}_I(\overline{E})$ is the event that everyone in I knows $\overline{E}$. Since the common knowledge of an event is publicly-evident, if $\overline{E}$ is commonly known among I at ω, then everyone in I knows $\overline{E}$ at ω, everyone knows that everyone knows $\overline{E}$ at ω, and so forth *ad infinitum*. Conversely, if every agent's knowledge satisfies Countable Conjunction, then it can be seen that $\overline{C}_I(\cdot) = \bigwedge_{n \in \mathbb{N}} \overline{K}_I^n(\cdot)$.

To conclude the discussions on common knowledge, two remarks are in order.

Remark S.1. In Example 1, the common knowledge operator among $\{i_1, i_2, i_3, i_4\}$ is equal to the knowledge operator of agent i_4. In Example S.1, the common knowledge operator among $\{j_1, j_2, j_3, j_4\}$ is equal to the knowledge operator of agent j_4.

Remark S.2 (Section 5.2.1)**.** Since the results of this paper (on knowledge) apply to common knowledge, I remark on the non-monotonicity of awareness in knowledgeability studied in Section 5.2.1. It is possible that if some event is not commonly known then it is commonly known that the event is not common knowledge. When each agent receives some events, on the contrary, it may be the case that it is not common knowledge that the event is not common knowledge.

As a degenerate case, suppose that any agent in I has the knowledge operator K_{i_3} of Example 1. By construction, the common knowledge operator is also given by K_{i_3}. At ω_3, it is common knowledge that $E = \{\omega_2\}$ is not common knowledge. Suppose that each agent learns new information so that her knowledge operator becomes K_{i_1}. At ω_3, it is now not common knowledge that $E = \{\omega_2\}$ is not common knowledge.

F Proofs

F.1 Appendix B

Proof of Proposition S.1. I start with showing that, under Confinedness,

$$\{\omega \in S \mid \Pi_i(\omega) \subseteq B \text{ and } \sigma_i(\omega) = S\} = \{\omega \in S \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant (B^{\uparrow}, S)\}. \tag{S.2}$$

The "$\subseteq$" part follows from the definition of the operation $\leqslant$ (without assuming Confinedness). For the "$\supseteq$" part, assume $\omega \in S$ and $\overline{\Pi}_i^{\uparrow}(\omega) \leqslant (B^{\uparrow}, S)$. Then, $\Pi_i(\omega) \subseteq B^{\uparrow}$, $S \preceq \sigma_i(\omega)$, and $S = S(\omega)$. By Confinedness, $\sigma_i(\omega) = S$, and thus $\Pi_i(\omega) \subseteq B$.

Now, assume (1). Fix $(B^{\uparrow}, S) \in \mathcal{E}$. By Expression (S.2) and Regularity,

$$\{\omega \in S \mid \Pi_i(\omega) \subseteq B \text{ and } \sigma_i(\omega) = S\} = \{\omega \in S \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant (B^{\uparrow}, S)\} \in \mathcal{D}.$$

Hence, it suffices to show

$$K_i(B^{\uparrow}) = \{\omega \in S \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant (B^{\uparrow}, S)\}^{\uparrow}. \tag{S.3}$$

Suppose $\omega \in K_i(B^{\uparrow})$. Since $S(\omega) \succeq S$, it follows from PPK that $\overline{\Pi}_i^{\uparrow}(\omega_S) \leqslant (B^{\uparrow}, S)$, where ω_S is the projection of ω onto S. Also, $\sigma(\omega_S) = S$, where $\sigma(\omega_S) \succeq S$ follows from $\overline{\Pi}_i^{\uparrow}(\omega_S) \leqslant (B^{\uparrow}, S)$, and $\sigma(\omega_S) \preceq S$ from Confinedness. Thus, $\omega_S \in \{\omega' \in S \mid \overline{\Pi}_i^{\uparrow}(\omega') \leqslant (B^{\uparrow}, S)\}$, and hence $\omega \in \{\omega \in S \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant (B^{\uparrow}, S)\}^{\uparrow}$, establishing the "$\subseteq$" part of Expression (S.3).

Next, if $\omega \in \{\omega' \in S \mid \overline{\Pi}_i^{\uparrow}(\omega') \leqslant (B^{\uparrow}, S)\}^{\uparrow}$, then $S(\omega) \succeq S$ and $\overline{\Pi}_i^{\uparrow}(\omega_S) \leqslant (B^{\uparrow}, S)$. By PPI, $\overline{\Pi}_i^{\uparrow}(\omega) \leqslant (B^{\uparrow}, S)$, and thus $\omega \in K_i(B^{\uparrow})$, establishing the "$\supseteq$" part of Expression (S.3).

Conversely, assume (2). First, I show Confinedness. Since $\omega \in K_i(\Pi_i^{\uparrow}(\omega))$ and $S(K_i(\Pi_i^{\uparrow}(\omega)) = \sigma_i(\omega)$, it follows that $S(\omega) \succeq \sigma_i(\omega)$.

Second, I prove Regularity. It follows from (2) that $\{\omega \in S \mid \Pi_i(\omega) \subseteq B \text{ and } \sigma_i(\omega) = S\} \in \mathcal{D}$. Then, Regularity follows from Expression (S.2):

$$\{\omega \in S \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant (B^{\uparrow}, S)\} = \{\omega \in S \mid \Pi_i(\omega) \subseteq B \text{ and } \sigma_i(\omega) = S\} \in \mathcal{D}.$$

Third, I show PPI. Let $S \preceq S(\omega)$. Since $\overline{\Pi}_i^{\uparrow}(\omega_S) \leqslant \overline{\Pi}_i^{\uparrow}(\omega_S)$, $\omega_S \in K_i(\Pi_i^{\uparrow}(w_S))$. Since $\overline{K}_i(\overline{\Pi}_i^{\uparrow}(w_S)) \in \mathcal{E}$ by (2), $\omega \in K_i(\Pi_i^{\uparrow}(w_S))$, i.e., $\overline{\Pi}_i^{\uparrow}(\omega) \leqslant \overline{\Pi}_i^{\uparrow}(w_S)$.

Fourth, I show PPK. Assume $S \preceq S(\omega)$ and $\overline{\Pi}_i^{\uparrow}(\omega) \leqslant (B^{\uparrow}, S)$. Then, $\omega_S \in K_i(B^{\uparrow})$. It follows from (2) that $\Pi_i(\omega_S) \subseteq B$ and $\sigma_i(\omega_S) = S$. Then, $\overline{\Pi}_i^{\uparrow}(\omega_S) \leqslant (B^{\uparrow}, S)$. □

Proof of Proposition S.2. First, I establish the equivalence between (1) and (2). To that end, I start with showing that, for any $(E, S) \in \mathcal{E}$,

$$\omega \in (\neg K_i)^2(E) \cap K_i(S^{\uparrow}) \text{ iff } \Pi_i(\omega) \cap K_i(E) \neq \emptyset.$$

Assume $\omega \in (\neg K_i)^2(E) \cap K_i(S^{\uparrow})$. Since $\omega \in K_i(S^{\uparrow})$, $\sigma_i(\omega) \succeq S$. Since $\omega \in (\neg K_i)^2(E)$, $\Pi_i^{\uparrow}(\omega) \not\subseteq (\neg K_i)(E)$. Then, there is $\omega' \in \Pi_i^{\uparrow}(\omega) \cap K_i(E)$. By taking the projection of ω' onto the subspace $\sigma_i(\omega)$ to which $\overline{\Pi}_i^{\uparrow}(\omega)$ belongs, $\omega'_{\sigma_i(\omega)} \in \Pi_i(\omega) \cap K_i(E) \neq \emptyset$. Conversely, since there exists $\omega' \in \Pi_i(\omega) \cap K_i(E)$, $\sigma_i(\omega) = S(\omega') \succeq S$. Thus, $\sigma_i(\omega) \succeq S$, and hence $\omega \in K_i(S^{\uparrow})$. If $\omega \in S^{\uparrow}$ satisfies $\omega \in K_i(\neg K_i)(E)$, then $\omega' \in \Pi_i^{\uparrow}(\omega) \subseteq (\neg K_i)(E)$. Thus, it has to be the case that $\omega \in (\neg K_i)^2(E)$.

Now, assume (1). If $\omega \in (\neg K_i)^2(E) \cap K_i(S^{\uparrow})$ then there is $\omega' \in \Pi_i(\omega) \cap K_i(E)$. By (1), $\Pi_i(\omega) \subseteq \Pi_i^{\uparrow}(\omega') \subseteq E$. Since $\sigma_i(\omega) \succeq S$, it follows $\omega \in K_i(E)$. Conversely, assume (2). Let $\omega' \in \Pi_i(\omega)$. Since $\omega' \in \Pi_i(\omega) \cap K_i(\Pi_i^{\uparrow}(\omega'))$, $\omega \in (\neg K_i)^2(\Pi_i^{\uparrow}(\omega')) \cap K_i(\sigma_i^{\uparrow}(\omega')) \subseteq K_i(\Pi_i^{\uparrow}(\omega'))$, where the set inclusion follows from (2). Then, $\Pi_i^{\uparrow}(\omega) \subseteq \Pi_i^{\uparrow}(\omega')$.

Second, I prove that (2) and (3) are equivalent. Assume (2). Fix $\overline{E} = (E, S)$. Since $\overline{A}_i^{(n)}(\overline{E}) \leqslant \overline{K}_i(\overline{S}^{\uparrow})$ by Remark A.1 (8), it suffices to establish $\overline{K}_i(\overline{S}^{\uparrow}) \leqslant \overline{A}_i^{(n)}(\overline{E})$.

Indeed, I show:

$$\begin{aligned}\overline{K}_i(\overline{S}^{\uparrow}) &= (\overline{K}_i(\neg\overline{K}_i)(\overline{E}) \wedge \overline{K}_i(\overline{S}^{\uparrow})) \vee ((\neg\overline{K}_i)(\neg\overline{K}_i)(\overline{E}) \wedge \overline{K}_i(\overline{S}^{\uparrow}))\\ &\leqslant \overline{K}_i(\neg\overline{K}_i)(\overline{E}) \vee \overline{K}_i(\overline{E}) = \overline{A}_i^{(2)}(\overline{E}) \leqslant \overline{A}_i^{(n)}(\overline{E}).\end{aligned}$$

The first equality follows from the distributive law on the generalized state space. For the first inequality, Monotonicity of $\overline{K}_i$ implies $\overline{K}_i(\neg\overline{K}_i)(\overline{E}) \wedge \overline{K}_i(\overline{S}^{\uparrow}) = \overline{K}_i(\neg\overline{K}_i)(\overline{E})$. Also, (2) implies $(\neg\overline{K}_i)(\neg\overline{K}_i)(\overline{E}) \wedge \overline{K}_i(\overline{S}^{\uparrow}) \leqslant \overline{K}_i(\overline{E})$. Then, the inequality holds because each term of both sides of the inequality has the same base space.

Conversely, assume (3). Then:

$$\begin{aligned}(\neg\overline{K}_i)^2(\overline{E}) \wedge \overline{K}_i(\overline{S}^{\uparrow}) &= (\neg\overline{K}_i)^2(\overline{E}) \wedge \overline{A}_i^{(2)}(\overline{E}) = (\neg\overline{K}_i)^2(\overline{E}) \wedge (\overline{K}_i(\overline{E}) \vee K_i(\neg\overline{K}_i)(\overline{E}))\\ &= ((\neg\overline{K}_i)^2(\overline{E}) \wedge \overline{K}_i(\overline{E})) \vee (\emptyset, S) = (\neg\overline{K}_i)^2(\overline{E}) \wedge \overline{K}_i(\overline{E}) \leqslant \overline{K}_i(\overline{E}),\end{aligned}$$

where the first equality follows from (3), the second equality from the definition of $\overline{A}^{(2)}$, and the rest from the operations on events. Now, (2) obtains. □

F.2 Appendix C

Proof of Proposition S.3. First, I establish:

$$\begin{aligned}\overline{U}_i^{(2)}\overline{J}_i(\overline{E}) &= (\neg\overline{K}_i)\overline{J}_i(\overline{E}) \wedge (\neg\overline{K}_i)^2\overline{J}_i(\overline{E}) = \overline{\partial}_i(\overline{E}) \wedge (\neg\overline{K}_i)\overline{\partial}_i(\overline{E})\\ &= \overline{\partial}_i(\overline{E}) \wedge (\neg\overline{K}_i)((\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)(\neg\overline{E}))\\ &\geqslant \overline{\partial}_i(\overline{E}) \wedge \neg(\overline{K}_i(\neg\overline{K}_i)(\overline{E}) \wedge \overline{K}_i(\neg\overline{K}_i)(\neg\overline{E})) \qquad \text{(S.4)}\\ &= (\overline{\partial}_i(\overline{E}) \wedge (\neg\overline{K}_i)^2(\overline{E})) \vee (\overline{\partial}_i(\overline{E}) \wedge (\neg\overline{K}_i)^2(\neg\overline{E}))\\ &= (\overline{\partial}_i(\overline{E}) \wedge (\neg\overline{K}_i)^2(\overline{E})) \vee (\overline{\partial}_i(\neg\overline{E}) \wedge (\neg\overline{K}_i)^2(\neg\overline{E})).\end{aligned}$$

The first equality follows from the definition of $\overline{U}_i^{(2)}$. The second equality follows from $(\neg\overline{K}_i)\overline{J}_i = \overline{\partial}_i$ (i.e., Remark A.1 (3)). The third equality follows from the definition of $\overline{\partial}_i$. The first inequality (i.e., Expression (S.4)) follows from taking the negation on both sides of

$$\overline{K}_i((\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)(\neg\overline{E})) \leqslant \overline{K}_i(\neg\overline{K}_i)(\overline{E}) \wedge \overline{K}_i(\neg\overline{K}_i)(\neg\overline{E}), \qquad \text{(S.5)}$$

which follows from Monotonicity of $\overline{K}_i$. The fourth equality follows from the operations of events (i.e., De Morgan and distributive laws). The fifth equality follows from $\overline{\partial}_i(\overline{E}) = \overline{\partial}_i(\neg\overline{E})$ (i.e., Remark A.1 (5)).

Next, since Lemma A.1 (1) implies $(\neg\overline{K}_i)^2(\overline{F}) \leqslant (\neg\overline{K}_i)(\neg\overline{F})$, for each $\overline{F} \in \{\overline{E}, \neg\overline{E}\}$,

$$\overline{\partial}_i(\overline{F}) \wedge (\neg\overline{K}_i)^2(\overline{F}) = (\neg\overline{K}_i)(\overline{F}) \wedge (\neg\overline{K}_i)(\neg\overline{F}) \wedge (\neg\overline{K}_i)^2(\overline{F}) = \overline{U}_i^{(2)}(\overline{F}).$$

Hence, $\overline{U}_i^{(2)}\overline{J}_i(\overline{E}) \geqslant \overline{U}_i^{(2)}(\overline{E}) \vee \overline{U}_i^{(2)}(\neg\overline{E})$, as desired. The equality holds when $\overline{K}_i$ satisfies Finite Conjunction, as Expression (S.5) and consequently Expression (S.4) hold with equality. □

To prove Proposition S.4, I provide the following lemma. Note that, in addition, Lemma A.2 in Appendix A.5 will also be used.

Lemma S.1. *Let $\mathcal{M}^{(n)}$ be a model such that $\overline{K}_i$ satisfies Finite Conjunction. Let $\mathcal{F}$ be a non-empty finite subset of $\mathcal{E}$ such that $S(E) = S(F)$ for any $\overline{E}, \overline{F} \in \mathcal{F}$. Then,*

$$(\neg\overline{K}_i)^2(\bigwedge \mathcal{F}) \leqslant \bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K}_i)^2(\overline{F}).$$

Proof of Lemma S.1. Fix $\overline{F} \in \mathcal{F}$. Since $\bigwedge \mathcal{F} \leqslant \overline{F}$ and since both sides have the same base space, it follows from Remark A.2 (2) that $(\neg\overline{K}_i)^2(\bigwedge \mathcal{F}) \leqslant (\neg\overline{K}_i)^2(\overline{F})$. Since $\overline{F} \in \mathcal{F}$ is arbitrary, take the conjunction to obtain the desired inequality. □

Proof of Proposition S.4. Part (1). Let $\mathcal{F}$ be a non-empty finite subset of $\mathcal{E}$ such that $S(E) = S(F)$ for any $\overline{E}, \overline{F} \in \mathcal{F}$. Since $\overline{K}_i$ satisfies Finite Conjunction and Monotonicity, $\overline{K}_i(\bigwedge \mathcal{F}) = \bigwedge_{\overline{E}\in\mathcal{F}} \overline{K}_i(\overline{E})$. Taking the negation,

$$(\neg\overline{K}_i)(\bigwedge \mathcal{F}) = \neg \bigwedge_{\overline{E}\in\mathcal{F}} \overline{K}_i(\overline{E}) = \bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K}_i)(\overline{E}). \tag{S.6}$$

For each $n \in \{2, \infty\}$, it suffices to show

$$\overline{U}_i^{(n)}(\bigwedge \mathcal{F}) \leqslant \bigvee_{\overline{E}\in\mathcal{F}} \overline{U}_i^{(n)}(\overline{E}), \tag{S.7}$$

as taking the negation on both sides yields the desired inequality.

Let $n = 2$. I have

$$\begin{aligned}\overline{U}_i^{(2)}(\bigwedge \mathcal{F}) &= \bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)^2(\bigwedge \mathcal{F}) \leqslant \bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K}_i)(\overline{E}) \wedge \bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K}_i)^2(\overline{F}) \\ &= \bigvee_{\overline{E}\in\mathcal{F}} \left((\neg\overline{K}_i)(\overline{E}) \wedge \bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K}_i)^2(\overline{F}) \right),\end{aligned}$$

where the first equality follows from Expression (S.6), the inequality from Lemma S.1, and the second equality from the distributive law. Then, since $(\neg\overline{K}_i)(\overline{E}) \wedge \bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K}_i)^2(\overline{F}) \leqslant (\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)^2(\overline{E}) = \overline{U}_i^{(2)}(\overline{E})$ for each $\overline{E} \in \mathcal{F}$ and since both sides of the inequality have the same base space, taking the disjunction yields

$$\bigvee_{\overline{E}\in\mathcal{F}} \left((\neg\overline{K}_i)(\overline{E}) \wedge \bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K}_i)^2(\overline{F}) \right) \leqslant \bigvee_{\overline{E}\in\mathcal{F}} \overline{U}_i^{(2)}(\overline{E}),$$

which establishes Expression (S.7). Note that the same proof works, for example, for any non-empty subset $\mathcal{F}$ of $\mathcal{E}$ if $\overline{K}_i$ satisfies Non-empty Conjunction.

Next, consider $n = \infty$. I have

$$\overline{U}_i^{(\infty)}(\bigwedge \mathcal{F}) = (\neg\overline{K}_i)^2(\bigwedge \mathcal{F}) \wedge (\neg\overline{K}_i)^3(\bigwedge \mathcal{F}) \leqslant \bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K}_i)^2(\overline{F}) \wedge (\neg\overline{K}_i)^2(\bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K}_i)(\overline{E})).$$

The equality follows because $\overline{U}_i^{(\infty)}(\cdot) = (\neg\overline{K}_i)^2(\cdot) \wedge (\neg\overline{K}_i)^3(\cdot)$ by Lemma A.1. The inequality follows from Lemma S.1 (the first term) and $(\neg\overline{K}_i)^3(\bigwedge \mathcal{F}) = (\neg\overline{K}_i)^2(\bigvee_{\overline{E}\in\mathcal{F}}(\neg\overline{K}_i)(\overline{E}))$, which follows from operating $(\neg\overline{K}_i)^2$ on Expression (S.6) (the second term).

To establish Expression (S.7) for $n = \infty$, I show

$$(\neg\overline{K}_i)^2(\bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K}_i)(\overline{E})) \leqslant \bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K}_i)^3(\overline{E}). \tag{S.8}$$

Once Expression (S.8) is established,

$$\begin{aligned} \bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K}_i)^2(\overline{F}) \wedge \bigvee_{\overline{E}\in\mathcal{F}} (\neg\overline{K}_i)^3(\overline{E}) &= \bigvee_{\overline{E}\in\mathcal{F}} \left(\bigwedge_{\overline{F}\in\mathcal{F}} (\neg\overline{K}_i)^2(\overline{F}) \wedge (\neg\overline{K}_i)^3(\overline{E}) \right) \\ &\leqslant \bigvee_{\overline{E}\in\mathcal{F}} ((\neg\overline{K}_i)^2(\overline{E}) \wedge (\neg\overline{K}_i)^3(\overline{E})) = \bigvee_{\overline{E}\in\mathcal{F}} \overline{U}_i^{(\infty)}(\overline{E}), \end{aligned}$$

where the first equality follows from the distributive law, and the inequality follows because $\bigwedge_{\overline{F}\in\mathcal{F}}(\neg\overline{K}_i)^2(\overline{F}) \wedge (\neg\overline{K}_i)^3(\overline{E}) \leqslant (\neg\overline{K}_i)^2(\overline{E}) \wedge (\neg\overline{K}_i)^3(\overline{E})$ for each $\overline{E} \in \mathcal{F}$ and both sides of this inequality always reside in the same subspace.

Hence, to prove Expression (S.7), it suffices to establish Expression (S.8). Moreover, one can assume $\mathcal{F} = \{\overline{E}_1, \overline{E}_2\}$. Also, for each $\lambda \in \{1, 2\}$, let $\overline{F}_\lambda := (\neg\overline{K}_i)(\overline{E}_\lambda)$. With this notation, Expression (S.8) is rewritten as

$$(\neg\overline{K}_i)^2(\overline{F}_1 \vee \overline{F}_2) \leqslant (\neg\overline{K}_i)^2(\overline{F}_1) \vee (\neg\overline{K}_i)^2(\overline{F}_2). \tag{S.9}$$

The proof of Expression (S.9) consists of four steps. The first step provides the following preliminary result: for each $\lambda \in \{1, 2\}$,

$$(\neg\overline{K}_i)(\neg\overline{F}_\lambda) = \overline{F}_\lambda. \tag{S.10}$$

By definition, $(\neg\overline{K}_i)(\neg\overline{F}_\lambda) = (\neg\overline{K}_i)(\neg\neg)\overline{K}_i(\overline{E}_\lambda) = (\neg\overline{K}_i)\overline{K}_i(\overline{E}_\lambda)$. By Remark A.1 (1), $(\neg\overline{K}_i)\overline{K}_i(\overline{E}_\lambda) = (\neg\overline{K}_i)(\overline{E}_\lambda)$. Then, by definition, $(\neg\overline{K}_i)(\overline{E}_\lambda) = \overline{F}_\lambda$.

In the second step, I show, in the following two sub-steps,

$$(\neg\overline{K}_i)^2(\overline{F}_1 \vee \overline{F}_2) \leqslant (\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2. \tag{S.11}$$

The first sub-step is to establish

$$(\neg\overline{K}_i)^2(\overline{F}_1 \vee \overline{F}_2) \leqslant (\neg\overline{K}_i)(\neg((\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2)) \tag{S.12}$$

by operating $(\neg\overline{K}_i\neg)$ on both sides of $\overline{K}_i(\overline{F}_1 \vee \overline{F}_2) \leqslant (\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2$ by Remark A.2 (3). Indeed, the latter inequality follows because

$$\overline{K}_i(\overline{F}_1 \vee \overline{F}_2) \leqslant \overline{K}_i(\overline{F}_1 \vee \overline{F}_2) \vee (\neg\overline{K}_i)(\neg\overline{F}_2) \leqslant \overline{K}_i(\overline{F}_1) \vee (\neg\overline{K}_i)(\neg\overline{F}_2) \leqslant (\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2.$$

The first inequality follows because $\overline{K}_i(\overline{F}_1 \vee \overline{F}_2)$ and $(\neg\overline{K}_i)(\neg\overline{F}_2)$ reside in the same subspace (recall Remark 2). The second inequality follows from Lemma A.2. For the third inequality, $\overline{K}_i(\overline{F}_1) \leqslant (\neg\overline{K}_i)^2(\overline{F}_1)$ follows from Remark A.1 (7), and $(\neg\overline{K}_i)(\neg\overline{F}_2) = \overline{F}_2$ from Expression (S.10).

The second sub-step is to show

$$\begin{aligned}
(\neg\overline{K}_i)(\neg((\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2)) &= (\neg\overline{K}_i)(\overline{K}_i(\neg\overline{K}_i)(F_1) \wedge (\neg\overline{F}_2)) \\
&= \neg(\overline{K}_i\overline{K}_i(\neg\overline{K}_i)(F_1) \wedge \overline{K}_i(\neg\overline{F}_2)) \\
&= (\neg\overline{K}_i)\overline{K}_i(\neg\overline{K}_i)(F_1) \vee (\neg\overline{K}_i)(\neg\overline{F}_2) \\
&= (\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2. \qquad \text{(S.13)}
\end{aligned}$$

First, by the definition of the negation, $\neg((\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2) = \overline{K}_i(\neg\overline{K}_i)(F_1) \wedge (\neg\overline{F}_2)$. Thus, by operating $(\neg\overline{K}_i)$ on both sides establishes the first equality. Second, since $\overline{K}_i$ satisfies Monotonicity and Finite Conjunction, $\overline{K}_i(\overline{K}_i(\neg\overline{K}_i)(F_1) \wedge (\neg\overline{F}_2)) = \overline{K}_i\overline{K}_i(\neg\overline{K}_i)(F_1) \wedge \overline{K}_i(\neg\overline{F}_2)$. Taking the negation establishes the second equality. Third, the third equality follows from the definition of the negation. Fourth, by Remark A.1 (2), $(\neg\overline{K}_i)\overline{K}_i(\neg\overline{K}_i)(\overline{F}_1) = (\neg\overline{K}_i)^2(\overline{F}_1)$. Also, by Expression (S.10), $(\neg\overline{K}_i)(\neg\overline{F}_2) = \overline{F}_2$.

Now, Expression (S.11) follows from Expressions (S.12) and (S.13), completing the second step of the proof of Expression (S.9).

The third step is to show, in the following two sub-steps,

$$(\neg\overline{K}_i)^2((\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2) \leqslant (\neg\overline{K}_i)^2(\overline{F}_1) \vee (\neg\overline{K}_i)^2(\overline{F}_2). \qquad \text{(S.14)}$$

The first sub-step is to establish

$$\overline{K}_i((\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2) \leqslant (\neg\overline{K}_i)^2(\overline{F}_1) \vee (\neg\overline{K}_i)^2(\overline{F}_2). \qquad \text{(S.15)}$$

By Monotonicity of $\overline{K}_i$, operate $\overline{K}_i$ on both sides of Expression (S.11) to obtain $\overline{K}_i(\neg\overline{K}_i)^2(\overline{F}_1 \vee \overline{F}_2) \leqslant \overline{K}_i((\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2)$. Substituting $\overline{E} = \overline{F}_2$ and $\overline{F} = (\neg\overline{K}_i)^2(\overline{F}_1)$ into Lemma A.2 (precisely, $\overline{K}_i(\overline{F} \vee \overline{E}) \leqslant \overline{K}_i(\overline{E}) \vee (\neg\overline{K}_i)(\neg\overline{F})$) yields

$$\overline{K}_i((\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2) \leqslant \overline{K}_i(\overline{F}_2) \vee (\neg\overline{K}_i\neg)(\neg\overline{K}_i)(\neg\overline{K}_i)(\overline{F}_1) = \overline{K}_i(\overline{F}_2) \vee (\neg\overline{K}_i)^2(\overline{F}_1),$$

where $(\neg\overline{K}_i\neg)(\neg\overline{K}_i)(\neg\overline{K}_i) = \neg\overline{K}_i\overline{K}_i(\neg\overline{K}_i) = (\neg\overline{K}_i)^2$ follows from Remark A.1 (2). By Remark A.1 (7), $\overline{K}_i(\overline{F}_2) \leqslant (\neg\overline{K}_i)^2(\overline{F}_2)$. Hence, Expression (S.15) obtains.

The second sub-step is to prove Expression (S.14) by showing

$$\begin{aligned}
&(\neg\overline{K}_i)^2((\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2) \leqslant (\neg\overline{K}_i)(\neg((\neg\overline{K}_i)^2(\overline{F}_1) \vee (\neg\overline{K}_i)^2(\overline{F}_2))) \\
=&(\neg\overline{K}_i)(\overline{K}_i(\neg\overline{K}_i)(\overline{F}_1) \wedge \overline{K}_i(\neg\overline{K}_i)(\overline{F}_2)) = \neg(\overline{K}_i\overline{K}_i(\neg\overline{K}_i)(\overline{F}_1) \wedge \overline{K}_i\overline{K}_i(\neg\overline{K}_i)(\overline{F}_2)) \\
=&\neg(\overline{K}_i(\neg\overline{K}_i)(\overline{F}_1) \wedge \overline{K}_i(\neg\overline{K}_i)(\overline{F}_2)) = (\neg\overline{K}_i)^2(\overline{F}_1) \vee (\neg\overline{K}_i)^2(\overline{F}_2).
\end{aligned}$$

The (first) inequality follows from operating $(\neg\overline{K}_i\neg)$ on both sides of Expression (S.15) by Remark A.2 (3) as both sides of the expression reside in the same subspace. The first equality follows from the definition of the negation. The second equality follows because $\overline{K}_i$ satisfies Monotonicity and Finite Conjunction. The third equality follows from Remark A.1 (1). The fourth equality follows from the definition of the negation.

Finally, the fourth step establishes Expression (S.9) by proving

$$(\neg\overline{K}_i)^2(\overline{F}_1 \vee \overline{F}_2) = (\neg\overline{K}_i)^4(\overline{F}_1 \vee \overline{F}_2) \leqslant (\neg\overline{K}_i)^2((\neg\overline{K}_i)^2(\overline{F}_1) \vee \overline{F}_2)$$
$$\leqslant (\neg\overline{K}_i)^2(\overline{F}_1) \vee (\neg\overline{K}_i)^2(\overline{F}_2).$$

First, the first equality follows from Lemma A.1 (1) (i.e., $(\neg\overline{K}_i)^2 = (\neg\overline{K}_i)^4$). Second, since both sides of Expression (S.11) reside in the same subspace, by Remark A.2 (2), operating $(\neg\overline{K}_i)^2$ yields the first inequality. Third, the second inequality follows from Expression (S.14).

Part (2). For the "if" part, first take $\overline{E} = (E, S)$ and $\overline{F} = (\emptyset, S)$. Since $\overline{E} \wedge \overline{F} = \overline{F}$, it follows from the supposition that $\overline{A}_i^{(n)}(\emptyset, S) = \overline{A}_i^{(n)}(\overline{E}) \wedge \overline{A}_i^{(n)}(\emptyset, S) \leqslant \overline{A}_i^{(n)}(\overline{E})$. Second, observe that, by Truth Axiom, $\overline{K}_i(\emptyset, S) = (\emptyset, S)$. Then, it follows from the definition of $\overline{A}_i^{(2)}$ that $\overline{A}_i^{(2)}(\emptyset, S) = \overline{K}_i(\emptyset, S) \vee \overline{K}_i(\neg\overline{K}_i)(\emptyset, S) = \overline{K}_i(\overline{S}^{\uparrow})$. Third, $\overline{K}_i(\overline{S}^{\uparrow}) = \overline{A}_i^{(2)}(\emptyset, S) \leqslant \overline{A}_i^{(n)}(\overline{E}) \leqslant \overline{K}_i(\overline{S}^{\uparrow})$, where the last inequality follows from Remark A.1 (8). Thus, $\overline{A}_i^{(n)}$ satisfies Weak Necessitation: $\overline{A}_i^{(n)}(\overline{E}) = \overline{K}_i(\overline{S}^{\uparrow})$.

For the "only if" part, let $\overline{A}^{(n)}$ satisfy Weak Necessitation. Let $S := \sup(S(E), S(F))$. Then, observe $S(\overline{E} \wedge \overline{F}) = S$ and $\overline{S(E)}^{\uparrow} \wedge \overline{S(F)}^{\uparrow} = \overline{S}^{\uparrow}$. Also, since $\overline{K}_i$ satisfies Monotonicity and Finite Conjunction, $\overline{K}_i(\overline{S(E)}^{\uparrow}) \wedge \overline{K}_i(\overline{S(F)}^{\uparrow}) = \overline{K}_i(\overline{S}^{\uparrow})$. Then,

$$\overline{A}_i^{(n)}(\overline{E}) \wedge \overline{A}_i^{(n)}(\overline{F}) = \overline{K}_i(\overline{S(E)}^{\uparrow}) \wedge \overline{K}_i(\overline{S(F)}^{\uparrow}) = \overline{K}_i(\overline{S}^{\uparrow}) = \overline{A}_i^{(n)}(\overline{E} \wedge \overline{F}),$$

where the first and third equalities follow from Weak Necessitation. □

F.3 Appendix D

Proposition S.5 (especially, Part (1)) hinges on the following two observations. First, agent i is at least as knowledgeable as agent j (i.e., $\overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$) iff $\overline{\partial}_i(\cdot) \leqslant \overline{\partial}_j(\cdot)$. Second, as Proposition 1 (2) shows, k^2-unawareness can be expressed as a particular form of ignorance: $\overline{U}_i^{(2)} = \overline{\partial}_i \overline{K}_i$. The following remark proves the first observation.

Remark S.3. In any model $\mathcal{M}$, $\overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$ iff $\overline{\partial}_i(\cdot) \leqslant \overline{\partial}_j(\cdot)$.

Proof of Remark S.3. Observe that $\overline{\partial}_i(\cdot) \leqslant \overline{\partial}_j(\cdot)$ iff $\overline{J}_j(\cdot) \leqslant \overline{J}_i(\cdot)$: recalling that $\overline{J}_i(\overline{E})$, $\overline{J}_j(\overline{E})$, $\overline{\partial}_i(\overline{E})$, and $\overline{\partial}_j(\overline{E})$ have the same base space $S(E)$, taking the negation on both sides of $\overline{\partial}_i(\cdot) \leqslant \overline{\partial}_j(\cdot)$ yields $\overline{J}_j(\cdot) \leqslant \overline{J}_i(\cdot)$, and vice versa.

Hence, I show $\overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$ iff $\overline{J}_j(\cdot) \leqslant \overline{J}_i(\cdot)$. For the "if" part, since $\overline{K}_i(\overline{E}) = \overline{J}_j(\overline{E}) \wedge \overline{E}$ by Remark A.1 (10), it follows from the supposition that $\overline{K}_j(\overline{E}) = \overline{J}_j(\overline{E}) \wedge \overline{E} \leqslant \overline{J}_i(\overline{E}) \wedge \overline{E} = \overline{K}_i(\overline{E})$ for all $\overline{E} \in \mathcal{E}$.

For the "only if" part, fix $\overline{E} = (E, S) \in \mathcal{E}$. It follows from the supposition that $(K_j(E), S) \leqslant (K_i(E), S)$ and $(K_j(\neg E), S) \leqslant (K_i(\neg E), S)$. Then, $\overline{J}_j(\overline{E}) = (K_j(E), S) \vee (K_j(\neg E), S) \leqslant (K_i(E), S) \vee (K_i(\neg E), S) = \overline{J}_i(\overline{E})$. □

Proof of Proposition S.5. Part (1). Observe that, by Proposition 1 (2), $\overline{U}_j^{(2)}(\cdot) = \overline{\partial}_j \overline{K}_j(\cdot)$. Thus, it suffices to show: $\overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$ iff $\overline{\partial}_i \overline{K}_j(\cdot) \leqslant \overline{\partial}_j \overline{K}_j(\cdot)$.

For the "only if" part, as shown in Remark S.3, $\overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$ implies $\overline{\partial}_i(\overline{F}) \leqslant \overline{\partial}_j(\overline{F})$ for all $\overline{F} \in \mathcal{E}$. Substitute $\overline{F} = \overline{K}_j(\cdot)$ to obtain $\overline{\partial}_i \overline{K}_j(\cdot) \leqslant \overline{\partial}_j \overline{K}_j(\cdot)$.

For the "if" part, take the negation on both sides of $\overline{\partial}_i \overline{K}_j(\cdot) \leqslant \overline{\partial}_j \overline{K}_j(\cdot)$ to get $\overline{J}_j \overline{K}_j(\cdot) \leqslant \overline{J}_i \overline{K}_j(\cdot)$. Since $\overline{K}_\ell(\overline{F}) = \overline{J}_\ell(\overline{F}) \wedge \overline{F}$ for each $\ell \in \{i, j\}$ by Remark A.1 (10), substitute $\overline{F} = \overline{K}_j(\cdot)$ to get $\overline{K}_j \overline{K}_j(\cdot) \leqslant \overline{K}_i \overline{K}_j(\cdot)$. As $\overline{K}_j$ satisfies Positive Introspection, $\overline{K}_j(\cdot) \leqslant \overline{K}_j \overline{K}_j(\cdot)$. Since $\overline{K}_j$ satisfies Truth Axiom and since $\overline{K}_i$ satisfies Monotonicity, $\overline{K}_i \overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$. Hence, $\overline{K}_j(\cdot) \leqslant \overline{K}_j \overline{K}_j(\cdot) \leqslant \overline{K}_i \overline{K}_j(\cdot) \leqslant \overline{K}_i(\cdot)$, as desired.

Part (2). Let agent i be at least as knowledgeable as agent j:

$$\overline{K}_j(\overline{F}) \leqslant \overline{K}_i(\overline{F}) \text{ for all } \overline{F} \in \mathcal{E}. \tag{S.16}$$

Part (2a). I start with showing $\overline{K}_j = \overline{K}_i \overline{K}_j$. Indeed, it follows because $\overline{K}_j(\cdot) \leqslant \overline{K}_j \overline{K}_j(\cdot) \leqslant \overline{K}_i \overline{K}_j(\cdot) \leqslant \overline{K}_j(\cdot)$, where the first inequality follows from Positive Introspection of $\overline{K}_j$, the second inequality from Expression (S.16) with $\overline{F} = \overline{K}_j(\cdot)$, and the third inequality from Truth Axiom of $\overline{K}_i$.

The rest of the proof of this part establishes $\overline{U}_i^{(n)} \overline{U}_j^{(n)}(\cdot) \leqslant (\neg \overline{K}_i) \overline{A}_j^{(n)}(\cdot) = \overline{U}_j^{(n)}(\cdot)$ in two steps. The first step shows $\overline{U}_i^{(n)} \overline{U}_j^{(n)}(\cdot) \leqslant (\neg \overline{K}_i) \overline{A}_j^{(n)}(\cdot)$. Since $\overline{U}_i^{(n)}(\overline{F}) \leqslant \overline{\partial}_i(\overline{F})$ holds by Proposition 1 (2), substituting $\overline{F} = \overline{U}_j^{(n)}(\cdot)$ yields $\overline{U}_i^{(n)} \overline{U}_j^{(n)}(\cdot) \leqslant \overline{\partial}_i \overline{U}_j^{(n)}(\cdot)$. Next, $\overline{\partial}_i \overline{U}_j^{(n)}(\cdot) = (\neg \overline{K}_i) \overline{U}_j^{(n)}(\cdot) \wedge (\neg \overline{K}_i) \overline{A}_j^{(n)}(\cdot) \leqslant (\neg \overline{K}_i) \overline{A}_j^{(n)}(\cdot)$, where the equality follows from the definitions of $\overline{\partial}_i$ and $\overline{A}_i^{(n)} := \neg \overline{U}_i^{(n)}$.

The second step shows $(\neg \overline{K}_i) \overline{A}_j^{(n)} = \overline{U}_j^{(n)}$. Indeed, one has $\overline{K}_i \overline{A}_j^{(n)} = \overline{K}_i \overline{K}_j \overline{A}_j^{(n)} = \overline{K}_j \overline{A}_j^{(n)} = \overline{A}_j^{(n)}$, where the first and third equalities follow from Proposition 3 (6) (i.e., $\overline{A}_j^{(n)} = \overline{K}_j \overline{A}_j^{(n)}$), and the second equality from the preliminary argument of this proof, i.e., $\overline{K}_i \overline{K}_j = \overline{K}_j$. Then, by taking the negation, $(\neg \overline{K}_i) \overline{A}_j^{(n)} = \overline{U}_j^{(n)}$.

Part (2b). I show $\overline{A}_j^{(n)}(\cdot) = \overline{K}_j \overline{A}_j^{(n)}(\cdot) \leqslant \overline{K}_i \overline{A}_j^{(n)}(\cdot) \leqslant \overline{A}_i^{(n)} \overline{A}_j^{(n)}(\cdot)$. The equality follows from Proposition 3 (6). The first inequality follows from the assumption that agent i is at least as knowledgeable as j (specifically, substitute $\overline{F} = \overline{A}_j^{(n)}(\cdot)$ into Expression (S.16)). The second inequality follows because $\overline{K}_i(\overline{F}) \leqslant \overline{A}_i^{(n)}(\overline{F})$ holds by the definition of $\overline{A}_i^{(n)}$ (specifically, substitute $\overline{F} = \overline{A}_j^{(n)}(\cdot)$).

Part (2c). I show

$$\overline{\partial}_j \overline{U}_i^{(n)}(\cdot) \geqslant (\neg \overline{K}_i)\overline{U}_i^{(n)}(\cdot) \wedge (\neg \overline{K}_j)\overline{A}_i^{(n)}(\cdot) \geqslant \overline{U}_i^{(n)}(\cdot). \tag{S.17}$$

Once Expression (S.17) is established, observe that, since $\overline{J}_j(\overline{F}) = \overline{J}_j(\neg \overline{F})$ for any $\overline{F} \in \mathcal{E}$ by Remark A.1 (4), $\overline{U}_i^{(n)}(\cdot) \leqslant \overline{\partial}_j \overline{U}_i^{(n)}(\cdot)$ iff $\overline{J}_j \overline{A}_i^{(n)}(\cdot) = \overline{J}_j \overline{U}_i^{(n)}(\cdot) \leqslant \overline{A}_i^{(n)}(\cdot)$.

Now, I establish the first inequality of Expression (S.17) by showing

$$\overline{\partial}_j \overline{U}_i^{(n)}(\cdot) = (\neg \overline{K}_j)\overline{U}_i^{(n)}(\cdot) \wedge (\neg \overline{K}_j)\overline{A}_i^{(n)}(\cdot) \geqslant (\neg \overline{K}_i)\overline{U}_i^{(n)}(\cdot) \wedge (\neg \overline{K}_j)\overline{A}_i^{(n)}(\cdot).$$

The (first) equality follows from the definitions of $\overline{\partial}_i$ and $\overline{A}_i^{(n)} := \neg \overline{U}_i^{(n)}$. The (first) inequality follows from $(\neg \overline{K}_j)\overline{U}_i^{(n)}(\cdot) \geqslant (\neg \overline{K}_i)\overline{U}_i^{(n)}(\cdot)$, which, in turn, follows from the assumption that i is at least as knowledgeable as j (specifically, substitute $\overline{F} = \overline{U}_i^{(n)}(\cdot)$ into Expression (S.16) and then take the negation).

Next, I show the second inequality of (S.17). By Remark A.1 (11), $(\neg \overline{K}_i)\overline{U}_i^{(n)}(\cdot) \wedge (\neg \overline{K}_j)\overline{A}_i^{(n)}(\cdot) = \overline{S}^{\uparrow} \wedge (\neg \overline{K}_j)\overline{A}_i^{(n)}(\cdot) = (\neg \overline{K}_j)\overline{A}_i^{(n)}(\cdot)$. Since $\overline{K}_j \overline{A}_i^{(n)}(\cdot) \leqslant \overline{A}_i^{(n)}(\cdot)$ by Truth Axiom of $\overline{K}_j$, taking the negation yields $(\neg \overline{K}_j)\overline{A}_i^{(n)}(\cdot) \geqslant \overline{U}_i^{(n)}(\cdot)$. □

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