The Existence of Universal Qualitative Belief Spaces*

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Abstract

This paper establishes the existence of a canonical representation of players' interactive beliefs, irrespective of nature of beliefs: especially, qualitative, truthful (i.e., knowledge), or probabilistic (e.g., countably-additive, finitely-additive, or non-additive) beliefs. The canonical model is the "largest" interactive belief model to which any particular model can be mapped in a unique belief-preserving way. For any statement regarding players' beliefs and rationality that holds at some state of some particular model, the statement holds at the corresponding state of the canonical model. The key insight for the existence is the need to specify possible depth of players' reasoning, depending on the nature of beliefs. The specification of possible depth of reasoning also has game-theoretic implications in characterizations of some solution concepts of games using the canonical space. The paper also shows the following properties of the canonical space. The canonical space has minimum possible assumptions on how players' beliefs are modeled. Each state of the canonical model encodes players' interactive beliefs at that state within itself in a complete and coherent manner.

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1 Introduction

Consider a group of players who reason interactively about unknown external values, *states of nature* S, such as strategies in a game. Players reason about states of nature—their strategies. Players also reason about their beliefs about states of nature—their beliefs about each other's strategies and consequently their rationality. And so on. This paper constructs the first formal framework general enough to represent any conceivable form of interactive beliefs irrespective of nature of beliefs. In particular, beliefs can be probabilistic or qualitative including knowledge. The key insight behind the construction of the canonical structure and its game-theoretic implications are the need to specify possible depth of players' reasoning.

An arbitrary model of beliefs will capture some possible aspects of players' interactive reasoning but will generally exclude others. Specifically, a model of beliefs (a belief space) consists of the following three ingredients. The first ingredient is a set Ω. Each element $\omega \in \Omega$ is a list of possible specifications of the prevailing nature state $s \in S$ and players' interactive beliefs regarding nature states S (i.e., their beliefs about nature states S, their beliefs about their beliefs about S, and so on). Call each specification ω a state (of the world).

The second ingredient is the set of statements about which the players can reason. These statements, specified as subsets of states of the world Ω, are referred to as events. A belief space requires a description of the language available to the players, modeled as a collection of subsets of Ω which I call the domain.

The third ingredient is players' belief operators defined on the domain. For each event E, player i's belief operator assigns the set of states at which she believes E, i.e., the event that i believes E. One can represent various notions of qualitative or probabilistic beliefs by imposing properties of those beliefs on belief operators. Thus, the framework can accommodate a wide variety of properties of beliefs such as qualitative, truthful (i.e., knowledge), or probabilistic (e.g., countably-additive, finitely-additive, or non-additive) beliefs. Iterative applications of the belief operator generate higher-order interactive reasoning. As will be discussed in Section 1.1, certain logical (i.e., set-theoretic) assumptions on the domain determine the possible depth of players' interactive reasoning through their belief operators. For example, if the domain is closed under countable conjunction (i.e., set-theoretic countable intersection), then Alice and Bob, who are reasoning about their actions in a strategic game, may have common belief in rationality: they believe that they are rational, they believe that they believe that they are rational, and so on, *ad infinitum*.

In applications, typically a specific model of beliefs is assumed a priori. This leaves open the possibility that some relevant aspects of reasoning are excluded. The main result of the paper (Theorem 1 in Section 3) demonstrates the existence of a universal (precisely, terminal) belief space into which any belief space is embedded in a unique manner that maintains players' interactive beliefs in that smaller space.

I formally show the extent to which any form of reasoning in the smaller space can be retrieved in the universal space: for any logical statement generated by nature states and players' interactive beliefs about them, the statement is true at some state of some belief space if and only if it is true at some state of the universal space; and the statement is true at all states of the universal space if and only if it is true at all states of any belief space (Proposition 2). One of game-theoretic applications is that players' rationality and their beliefs about each other's rationality expressed in a particular belief space are preserved in the universal space (Section 6.1 and Appendix D.1). Hence, the universal space enables analysis of formal questions regarding beliefs without restrictions on the nature of reasoning.

Moreover, I characterize the universal space in a way so that each state of the universal space specifies players' interactive beliefs at that state within itself (Corollary 1), which solves the issue of self-reference: a state is supposed to fully describe players' beliefs while their beliefs are defined on states. The space is (belief-)complete in that it includes all possible forms of reasoning (Proposition 1). Thus, the universal space leaves no relevant aspect of players' interactive beliefs unspecified as long as nature states and players' interactive beliefs about them are concerned.

I establish the existence of a universal belief space as long as players' beliefs are represented by belief operators, irrespective of assumptions on players' logical and introspective abilities. The universal belief space satisfies the exact assumptions that are imposed by the outside analysts. The key idea is that the existence of a universal belief space hinges on specifying depth of players' reasoning. My result is theoretically interesting in that the existence of a universal belief space is unrelated to assumptions on players' beliefs. For example, my paper reconciles the previous existence results on canonical probabilistic belief structures and the previous non-existence results on canonical knowledge (or more general qualitative-belief) structures. At the same time, it is substantively interesting because I establish the canonical representation of beliefs even when players are less than "perfectly rational" in terms of their logical or introspective abilities.

My framework nests partitional (Aumann, 1976) and non-partitional possibility correspondence models of knowledge and qualitative beliefs by identifying the conditions on players' belief operators under which their beliefs are induced from information sets on the underlying states of the world. Each player's information set associated with a state represents the set of states she considers possible at that state. While a player in a partitional model is logically omniscient and is fully introspective about what she knows and what she does not know, a player in a non-partitional model may, for example, fail Negative Introspection—she does not know a certain event, and she does not know that she does not know it.[1] My framework also nests

[1]Non-partitional models are motivated in part by notions of unawareness (e.g., Fagin and Halpern, 1987; Modica and Rustichini, 1994, 1999; Schipper, 2015). The study of non-partitional models ranges from implications of common knowledge and common belief (e.g., Agreement theorems (Aumann, 1976)) to solution concepts in game theory. See, for example, Bacharach (1985), Brandenburger, Dekel, and Geanakoplos (1992), Dekel and Gul (1997), Geanakoplos (1989), Morris (1996),

other forms of possibility correspondence models of qualitative beliefs which may fail to be truthful.[2] I can further relax players' logical reasoning abilities inherent in possibility correspondence models. For example, players may fail to believe logical consequences of their beliefs.

As will be discussed in Section 1.1, qualitative beliefs may play an important role in characterizing solution concepts such as common belief in rationality in a game (e.g., Aumann, 1987; Brandenburger and Dekel, 1987; Stalnaker, 1994; Tan and Werlang, 1988) or especially in a game with ordinal payoff structures that do not admit probabilistic beliefs (Bonanno, 2008; Bonanno and Tsakas, 2018). One can also introduce qualitative beliefs as in Bjorndahl, Halpern, and Pass (2013) in psychological games (Battigalli and Dufwenberg, 2009; Geanakoplos, Pearce, and Stacchetti, 1989) in which players' interactive beliefs themselves enter into their preferences.[3]

I construct a universal belief space in a way such that beliefs can be probabilistic as in type spaces (Harsanyi, 1967-68): each player has a type mapping that associates, with each state, a probability distribution on the underlying states.[4] As Samet (2000) demonstrates the correspondence between a type mapping and a collection of p-belief operators (Monderer and Samet, 1989), the main result of this paper also asserts the existence of a universal probabilistic (e.g., countably-/finitely-/non-additive) belief space (Corollary 2). Also, this paper establishes such universal belief space with or without a common prior. Probabilistic beliefs can reduce to whether a player believes an event with probability at least p or not. Technically, my construction of a universal qualitative-belief space follows the topology-free construction of a universal (countably-additive) type space by Heifetz and Samet (1998b). Another game-theoretic application is that a correlated equilibrium (Aumann, 1974) can be embedded into a subset of a universal belief space that consists of players' belief hierarchies about play, whenever different states in the correlated equilibrium yield different belief hierarchies (Appendix D.1).

My framework can endow players with both knowledge and belief on a general domain.[5] Indeed, the consideration of the domain of knowledge has often been ne-

Samet (1990), and Shin (1993).

[2]In the literature, knowledge is distinguished from belief in that a player can only know what is true while she can believe something false.

[3]As an example of the use of qualitative belief in psychological games, consider the example of Geanakoplos, Pearce, and Stacchetti (1989) in which Alice can send Bob either flowers or chocolates. Alice enjoys surprising Bob. So, if Alice believes that Bob expects (i.e., believes that he receives) flowers, she sends chocolates, and vice versa.

[4]The existence of a universal type/belief space is pioneered by Armbruster and Böge (1979), Böge and Eisele (1979), and Mertens and Zamir (1985). Their topological constructions are extended by, for example, Brandenburger and Dekel (1993) and Pintér (2005).

[5]Such consideration would be needed for analyzing: (i) players' knowledge about their own strategy and their beliefs about opponents' strategies (e.g., Dekel and Gul, 1997) or (ii) players' knowledge about their past-observed moves and their beliefs about past-unobserved and future moves in an extensive-form game (e.g., Battigalli and Bonanno, 1997). Knowledge and probability-one belief would differ in a continuous model: a player may believe with probability one that a

glected. Standard possibility correspondence models allow any subset of states Ω to be an object of knowledge. Under such power-set specification, the knowledge of an event is not in the domain of the probability space, which hampers the epistemic analyses of players' knowledge and beliefs. Not only is my framework capable of capturing both knowledge and belief, but also the framework admits a universal space.

In sum, I construct a universal belief space within the class of belief spaces that satisfy given assumptions on beliefs. Beliefs can be probabilistic (e.g., countably-/finitely-/non-additive) or qualitative including (fully-/not-necessarily-introspective) knowledge. Special cases include such previous belief models as possibility correspondences and type mappings. The existence of a universal space hinges on the specification of depth of reasoning rather than on properties of beliefs themselves.

The paper is organized as follows. The rest of this section discusses the role that the key insight, depth of players' reasoning, plays on the construction of a universal belief space and its game-theoretic applications. Section 2 defines a belief space, properties of beliefs, and a universal belief space. Section 3 constructs a universal space (Theorem 1). Section 4 studies the properties of the universal space. Especially, Section 4.4 characterizes the universal space as the "largest" set describing players' interactive beliefs in a complete and coherent manner (Theorem 2). Section 5 discusses further theoretical applications such as various forms of probabilistic beliefs. Section 6 provides game-theoretic applications. Section 7 compares the existence result of a universal knowledge space with the previous non-existence results. Section 8 provides concluding remarks. Proofs are relegated to Appendix A. Supplementary Appendix (Appendices B to F), available online, provides supplementary discussions on the extensions of the existence result to richer forms of beliefs and on further game-theoretic applications.

1.1 Specification of Depth of Players' Reasoning

A *standard* belief space is a model in which any subset of underlying states Ω is an object of beliefs. Heifetz and Samet (1998a) demonstrate that a universal standard partitional knowledge space generically does not exist. They show that, unlike σ-additive probabilistic beliefs, a non-trivial sequence of interactive knowledge can develop beyond any given ordinal.[6] Moreover, Meier (2005) shows that there is no universal standard qualitative-belief (including knowledge) space represented by a more general non-partitional possibility correspondence (i.e., a general Kripke frame). If there were a universal space in a class of such general qualitative-belief spaces, then one could construct a universal standard partitional knowledge space from the given class, which is impossible.

How do my positive results reconcile with the negative results? What plays a

random draw from the interval $[0, 1]$ is irrational while she does not know it.

[6] The negative results are also obtained by Fagin (1994), Fagin et al. (1999), Fagin, Halpern, and Vardi (1991), and Heifetz and Samet (1999).

crucial role in establishing a universal knowledge (i.e., truthful-belief) space is to specify a set algebra as objects of players' knowledge, i.e., a specification of the language that the players are allowed to use in their reasoning.

To see this point, let κ be an infinite cardinal number. Call a collection of subsets of underlying states Ω a κ-*algebra* (a shorthand for a κ-complete algebra) if it is closed under complementation and under union (and consequently intersection) of any subcollection with cardinality less than κ. The power set of Ω is always a κ-algebra. For example, a κ-algebra subsumes an algebra of sets if κ is the least infinite cardinal $\aleph_0$. A κ-algebra subsumes a σ-algebra if κ is the least uncountable cardinal $\aleph_1$. Call a knowledge space (a belief space with players' beliefs truthful) a κ-knowledge space if its domain is a κ-algebra.

Specifying the domain of a knowledge space by a κ-algebra amounts to determining the language available to the players in reasoning about their interactive knowledge up to the ordinal depth of κ. Any κ-knowledge space can capture interactive knowledge of a form, Alice knows that Bob knows that ..., up to the ordinal level of κ. For example, any $\aleph_0$-knowledge space can capture any finite-level interactive knowledge, because $\kappa = \aleph_0$ is the least infinite cardinal and a knowledge hierarchy up to the ordinality of $\aleph_0 = |\{0, 1, 2, \dots\}|$ consists of all finite levels of interactive knowledge. Likewise, any $\aleph_1$-knowledge space can capture any countable level of interactive knowledge of the form, Alice knows that Bob knows that Alice knows that ..., which would naturally emerge when one considers common knowledge among players.

In fact, transfinite levels of reasoning would be necessary if one considers implications of common knowledge (or common belief) of rationality in a general infinite game. Consider the following two-player strategic game $\langle (A_i)_{i\in I}, (u_i)_{i\in I}\rangle$. Each player i announces an element from an ordered set $A_i := \{0, 1, 2, 3, 4, \dots\} \cup \{a, b\}$, where $0 < 1 < 2 < 3 < 4 < \cdots\cdots < a < b$. Announcing b always yields a payoff of 1 irrespective of the opponent's announcement. For any other announcements, if i's announcement is (strictly) higher than the opponent's, she obtains a payoff of 2; if not, she obtains a payoff of 0. Table 1 depicts player i's payoff $u_i(a_i, a_{-i})$ as a function of a_i (Row) and a_{-i} (Column).

Consider a process of iterated elimination of strictly dominated actions (IESDA). At each round of elimination, the minimal element is always strictly dominated: first, $a_i = 0$ is a unique strictly dominated action in A_i; next, $a_i = 1$ is a unique strictly dominated action in $A_i \setminus \{0\}$; and so on. Once $\{0, 1, 2, 3, 4, \dots\}$ have been deleted from each player's action space, in the subgame in which each player's action set is $\{a, b\}$, action a is strictly dominated by b. Thus, the action profile $(a_1, a_2) = (b, b)$ is the unique prediction under IESDA after one more elimination (namely, a) after eliminating $0, 1, 2, 3, 4, \dots$. Hence, in order to reach the unique prediction under IESDA, the players need to engage in a transfinite process of IESDA.[7]

[7] A pioneering work pointing out the need for a transfinite process of iterated elimination of never-best-replies is Lipman (1994). In his model, the players are expected-utility maximizes with respect to their countably-additive beliefs. Thus, a belief space would refer to an $\aleph_1$-belief space in which

	0	1	2	3	4	⋯⋯⋯	a	b
0	0	0	0	0	0	⋯⋯⋯	0	0
1	2	0	0	0	0	⋯⋯⋯	0	0
2	2	2	0	0	0	⋯⋯⋯	0	0
3	2	2	2	0	0	⋯⋯⋯	0	0
4	2	2	2	2	0	⋯⋯⋯	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋯⋯⋯	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋯⋯⋯	⋮	⋮
a	2	2	2	2	2	⋯⋯⋯	0	0
b	1	1	1	1	1	⋯⋯⋯	1	1

Table 1: Player i's payoff $u_i(a_i, a_{-i})$ as a function of a_i (Row) and a_{-i} (Column).

One can extend this game (see Section 6.1.3) so that the action profile $(a_1, a_2) = (b, b)$ is the unique prediction under IESDA after one more elimination (namely, a) after an arbitrary ordinal level of elimination by replacing the actions $\{0, 1, 2, 3, 4, \dots\}$ with the given ordinal (note that $\{0, 1, 2, 3, 4, \dots\}$ is indeed the least infinite ordinal).[8] In such an extended game, the unique prediction based on IESDA requires interactive reasoning of an arbitrary fixed ordinal level: say, Alice and Bob are rational, they believe that they are rational, they believe that they believe that they are rational, and so forth, up to an arbitrarily fixed ordinal level.

The main result (Theorem 1 in Section 3) establishes that, for each fixed cardinal κ (e.g., take the least uncountable cardinal for the game in Table 1), there is a universal κ-belief space in each class of κ-belief spaces that respect some given assumptions on players' beliefs by taking care of exactly all the levels of interactive beliefs up to the ordinal level of κ (e.g., all countable levels of beliefs for the least uncountable cardinal). In particular, a universal κ-knowledge space exists within a class of truthful κ-belief spaces. In light of the extended game of Table 1, it would be natural for the outside analysts to consider a class of belief spaces in which the players can engage in interactive reasoning of an arbitrary but predetermined ordinal level of κ.

The construction circumvents the previous non-existence results by explicitly specifying a domain of qualitative beliefs (or knowledge) as a κ-algebra. On the one hand,

players' beliefs are countably-additive probabilistic beliefs. His example suggests that the players need to engage in a transfinite (yet countable) level of reasoning. Such transfinite levels of interactive reasoning are used not only in rationazability solution concepts but also in robust mechanism design (see, for instance, Bergemann and Morris, 2005, 2011).

[8]This example is based on Chen, Long, and Luo (2007), in which each player's action set A_i is $[0, 1]$. Fukuda (2020) also shows that, for an arbitrary fixed ordinal, there exist a strategic game and a particular belief model of the game in which the players' mutual beliefs ("everybody believes") would not converge until the fixed ordinal level. This paper, in contrast, shows that when a strategic game in question is fixed, the analysts can fix large enough depth of the players' reasoning and there exists a universal (terminal) belief space which incorporates all possible ways in which the players can interactively reason about their actions up to that depth.

the previous negative results imply that a sequence of interactive knowledge can generally develop beyond any depth of reasoning in a discontinuous way if any subset of states of the world is an object of knowledge. On the other hand, once I specify the language available to the players as a κ-algebra, any κ-knowledge space can take into consideration players' interactive knowledge up to the ordinality of κ. The universal κ-knowledge space contains exactly all knowledge hierarchies up to the ordinality of κ. Since κ can be arbitrarily fixed by the outside analysts (for a given set of nature states), choose κ large enough, e.g., $\kappa > |S|$ where nature states S consist of action profiles in a given strategic game (note that such κ would vary with nature states S as the extended game of Table 1 suggests that there is a game in which players may need to engage in an arbitrarily fixed ordinal depth of reasoning). Then, the analysts can obtain exactly all knowledge hierarchies up to κ about the players' actions.

In any belief space augmented with the players' strategies (that map each state of the world to their actions), if the players have (correct) common belief in rationality at a state, then their actions at that state survive any process of IESDA; and for any action profile that survives some process of IESDA, at some state of some belief space, the players have (correct) common belief in rationality and take the given actions.[9] As a game-theoretic application, Section 6.1 shows that any state of a belief space at which the players commonly believe their rationality is uniquely mapped to the corresponding state of the universal space at which the players commonly believe their rationality. Thus, in the universal belief space, if the players commonly believe their rationality at a state, then their actions at the state survive any process of IESDA; and for any action profile that survives some process of IESDA, there exists a state in the universal space at which the players commonly believe their rationality and take the given actions. In the game in Table 1, a universal κ-belief space (where κ is the least uncountable cardinal) can explain any possible ways in which the players interactively reason about their actions up to any countable level of reasoning (which suffices to pin down the prediction).

Going back to the existence of a universal belief (especially, knowledge) space, I turn the previously mentioned negative results into the positive one in two ways (see Section 7 for comparisons). First, I enlarge a class of knowledge spaces by allowing

[9]First, see, for instance, Brandenburger and Dekel (1987), Fukuda (2020), Stalnaker (1994), and Tan and Werlang (1988) for implications of common belief in rationality. The characterization by Fukuda (2020) does not presuppose any property on players' beliefs. On a related point, if players' beliefs may not satisfy certain logical properties, then their common belief may not necessarily be characterized by all the finite iterations of mutual beliefs but by transfinite iterations of mutual beliefs. My construction of a universal space grantees that, by appropriately setting depth of reasoning, the analysts can incorporate all possible ways in which players reason up to the predetermined depth, including common beliefs.

Second, technically, correct common belief in rationality means that whenever the players commonly believe their rationality, they are rational. In order for rationality and common belief in rationality (instead of assuming correct common belief in rationality) to lead to IESDA, the players need to be logical. In contrast, common knowledge of rationality always leads to IESDA, irrespective of underlying properties of beliefs.

the domain of a knowledge space to be a κ-algebra. Since any κ-knowledge space accommodates the knowledge hierarchies up to the ordinal level of κ that the underlying states intend to represent, I show that no κ-knowledge space on an arbitrary κ-algebra misses any interactive reasoning up to the ordinality of κ. Second, I find a universal κ-knowledge space by keeping track of exactly all forms of reasoning up to the depth of κ attained in the given class of κ-knowledge spaces.[10] The set of all knowledge hierarchies up to the depth of κ is well-defined when κ is fixed (in contrast, the set of all knowledge hierarchies of an arbitrary depth is generally too big to be a set), and there exists a natural κ-algebra that captures nature states and players' interactive beliefs.

This paper shows existence hinges on the specification of a domain (i.e., depth of reasoning) rather than on assumptions on beliefs themselves. Thus, the framework applies not only to qualitative belief or knowledge but also to various forms of probabilistic beliefs in a unified manner.[11] Put differently, the facts that a universal probabilistic-belief space has been constructed and that a universal knowledge space has been shown not to exist reduce to whether the framework specifies depth of reasoning. The domain of a (σ-additive) type space is assumed to be a σ-algebra for the rather technical reason that a σ-additive probability measure may not necessarily be defined on the power set.[12] The domain of a type space (σ-algebra) is the language available to the players in reasoning up to countable-level interactive probabilistic-beliefs. The domain of any $\aleph_1$-qualitative-belief space (σ-algebra) is the language available to the players in reasoning up to countable-level interactive qualitative-beliefs. While I construct a universal belief space by keeping track of belief hierarchies up to the ordinality of $\aleph_1$, the continuity (σ-additivity) of beliefs (or the continuity of the operation "Δ") guarantees that the least infinite depth of interactive beliefs can determine any subsequent countable order in a universal type space (e.g., Fagin et al., 1999; Heifetz and Samet, 1998b).[13] Furthermore, Meier (2006)

[10] In a special case in which players possess countably-additive (i.e., "continuous") beliefs, a belief hierarchy that contains any finite level of reasoning uniquely extends to the one that contains countable levels as in the literature on type spaces (Proposition 4). For qualitative beliefs, my universal knowledge (or qualitative-belief) space usually has transfinite (precisely, κ) hierarchies of interactive knowledge (beliefs) incorporating all possible forms of interactive reasoning up to the depth of κ.

[11] Appendix C.3 also discusses how the specification of a domain (i.e., depth of reasoning) sheds light on the constructions of universal unawareness, preference, and expectation spaces.

[12] Suppose that the outside analysts study the strategic game in Table 1 using the framework of (probabilistic) type spaces, as the set of action profiles is countable. The domain of such (probabilistic) type space is a measurable space, and thus coincidentally the analysts can incorporate countable yet transfinite levels of ineractive reasoning, which, in this case, are sufficient to pin down the unique prediction under IESDA.

[13] Moss and Viglizzo (2004, 2006) reformulate σ-additive type spaces as coalgebras for a certain endofunctor F, which is related to the functor Δ. They show that a universal type space (i.e., a terminal coalgebra) is expressed as the set of descriptions of each point (type profile together with a state of nature) in all coalgebras, endowed with measurable and coalgebra structures. Since a terminal coalgebra T is isomorphic to $F(T)$, it is also "(belief-)complete." See Brandenburger

shows, while a universal finitely-additive-belief space does not exist if all subsets are measurable (see also Fagin et al., 1999), it exists once players' beliefs are defined on a κ-algebra.

My existence result on a universal κ-knowledge space is related to the previous two positive results. First, Meier (2008) constructs a universal knowledge-belief space in which players' knowledge operators operate only on a σ-algebra on which players' probabilistic beliefs are defined. The construction in Theorem 1 of Section 3 nests Meier (2008) as a special class of $\aleph_1$-knowledge(-belief) spaces under his assumptions on players' knowledge, which may not necessarily be induced from possibility correspondences. Technically, in addition to nesting his result, this paper shows the existence of a universal κ-belief space for such models as possibility correspondences of fully introspective or non-introspective knowledge and qualitative beliefs. Conceptually, this paper shows the existence of a universal belief space is unrelated to specific nature of beliefs and hinges rather on specifying depth of reasoning. For the game in Table 1, a universal $\aleph_1$-belief space can capture players' countable levels of reasoning. For the game extended from Table 1, the outside analysts would need to choose a large enough κ so that a universal κ-belief space can capture all possible forms of players' interactive reasoning up to the ordinal level of κ.

Second, Aumann (1999) constructs what he calls a canonical knowledge system (of a finitary epistemic $S5$ logic), where each state of the world is a "complete and coherent" set of formulas describing finite levels of players' interactive knowledge.[14] Theorem 2 in Section 4.4 reformulates a universal belief space by generalizing and modifying Aumann (1999)'s canonical knowledge system for any combination of assumptions on players' beliefs and for any domain (i.e., for any κ). In a particular case in which players with fully-introspective knowledge reason about finite levels of interactive knowledge, Theorem 2 formally proves that Aumann (1999)'s canonical space can be taken as a universal $\aleph_0$-(fully-introspective-)knowledge space, contrary to the conjecture of Heifetz and Samet (1998b, Section 6). Generally, Theorem 2 restates that the universal κ-belief space is the largest set (i) consisting of "complete and coherent" sets of formulas describing the players' belief hierarchies; (ii) satisfying the "(collective) coherency" condition on the entire space that induces the players' beliefs in a well-defined manner; and (iii) respecting given assumptions on their beliefs. For example, when players can engage in countable-level interactive reasoning, i.e., $\kappa = \aleph_1$, a universal $\aleph_1$-knowledge space can accommodate countable levels of interactive knowledge including common knowledge, as in the game in Table 1.

(2003) and Brandenburger and Keisler (2006) for (belief-)completeness.

[14] Meier (2012) axiomatizes classes of belief/type spaces and shows that the space of all maximally consistent sets of formulas of his infinitary probability logic (i.e., the canonical space) is a universal space, which is isomorphic to the universal type space constructed by Heifetz and Samet (1998b). Zhou (2010) studies a canonical infinitary finitely-additive probability logic.

2 Belief Spaces

This section starts with providing technical preliminaries. Then, Section 2.1 defines a belief space and properties of beliefs. Section 2.2 defines a universal belief space.

Throughout the paper, denote by I a non-empty set of players. Let S be a non-empty set of *states of nature*, endowed with a sub-collection $\mathcal{S}$ of the power set $\mathcal{P}(S)$. An element of S is regarded as a specification of the exogenous values that are relevant to the strategic interactions among the players. For example, $(S, \mathcal{S})$ is the set of strategies or payoff functions endowed with a topological or measurable structure.

Throughout the paper, I explicitly assume the axiom of choice. Then, associate, with an (infinite) cardinal κ, the least ordinal $\overline{\kappa}$ (called the initial ordinal of κ) with its cardinality $|\overline{\kappa}| = \kappa$. That is, the cardinal κ is also identified with the ordinal $\overline{\kappa}$.

I introduce technical definitions. Let κ be an infinite cardinal. Call a collection $\mathcal{D}$ of subsets of a set Ω (or a pair $(\Omega, \mathcal{D})$ itself) a *κ-complete algebra* (*κ-algebra*, for short) if $\mathcal{D}$ is closed under complementation and is closed under arbitrary union (and consequently intersection) of any sub-collection with cardinality less than κ (i.e., closed under *κ-union* and *κ-intersection*). Note that $\emptyset =: \bigcup \emptyset \in \mathcal{D}$ and $\Omega =: \bigcap \emptyset \in \mathcal{D}$. For example, an $\aleph_0$-algebra is an algebra of sets, because $\aleph_0$ is the least infinite cardinal. An $\aleph_1$-algebra is a σ-algebra, because $\aleph_1$ is the least uncountable cardinal.

Likewise, call the sub-collection $\mathcal{D}$ of $\mathcal{P}(\Omega)$ (or the pair $(\Omega, \mathcal{D})$) a *complete algebra* if $\mathcal{D}$ is closed under complementation and is closed under arbitrary union and intersection. Following Meier (2012), call a complete algebra an *∞-algebra* so that one can conveniently refer to both a κ-algebra (where κ is an infinite cardinal) and a complete algebra. The symbol $\kappa = \infty$ is used only for indicating a complete algebra. However, the symbol ∞ (which is not a cardinal) is informally interpreted as satisfying $\lambda < \infty$ for any cardinal λ, because a complete algebra is closed under λ-union (and λ-intersection) for any cardinal λ.

For an infinite cardinal κ or the symbol $\kappa = \infty$, denote by $\mathcal{A}_\kappa(\cdot)$ the smallest κ-algebra (i.e., the intersection of all κ-algebras) including a given collection. For example, $\mathcal{A}_{\aleph_1}(\cdot) = \sigma(\cdot)$ generates the smallest $\aleph_1$-algebra (i.e., σ-algebra).

In order to specify a language the players are allowed to use in making inferences about nature states S and their interactive beliefs, endow $\mathcal{S}$ with a "logical" (precisely, set-algebraic) structure. Let κ be an infinite cardinal. Call $E \in \mathcal{A}_\kappa(\mathcal{S})$ an *event of nature*. Each $E \in \mathcal{A}_\kappa(\mathcal{S})$ plays a role of a "proposition" regarding nature states S about which players interactively reason. Hence, if E is a nature event, then so is its complement E^c (also denote it by $\neg E$); if each $E \in \mathcal{E}$ with $|\mathcal{E}| < \kappa$ is a nature event, then so are its union $\bigcup \mathcal{E} = \bigcup_{E \in \mathcal{E}} E$ and its intersection $\bigcap \mathcal{E} = \bigcap_{E \in \mathcal{E}} E$. In Section 3, κ (precisely, the ordinal $\overline{\kappa}$) will determine possible depth of players' reasoning.

I remark that, as mentioned in Meier (2006, Remark 1), it is without loss to assume an infinite cardinal κ to be regular. For any infinite cardinal κ which is not regular, $(S, \mathcal{A}_\kappa(\mathcal{S}))$ is indeed a κ^+-algebra, where the successor cardinal κ^+ is known to be regular by the axiom of choice. Hence, if the outside analysts take a non-regular (i.e., singular) infinite cardinal κ, then they are implicitly taking an infinite regular cardinal κ^+. Note that $\aleph_0$ and $\aleph_1$ are regular.

2.1 Belief Spaces

I define a model of players' beliefs in which belief operators on a state space induce players' interactive beliefs regarding nature states $(S, \mathcal{S})$. Call the model a κ-belief space when the underlying state space is a κ-algebra. First, Definition 1 formally defines a κ-belief space. Next, Definition 2 specifies properties of beliefs.

Definition 1 (Belief Space)**.** *A κ-belief space of I on $(S, \mathcal{S})$ (a belief space, for short) is a tuple $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta\rangle$ with the following three properties.*

1. *$(\Omega, \mathcal{D})$ is a κ-algebra. Call Ω the set of states of the world (the state space). Call each $E \in \mathcal{D}$ an event (of the world).*
2. *For each $i \in I$, $B_i : \mathcal{D} \to \mathcal{D}$ is player i's belief operator. For each $E \in \mathcal{D}$, $B_i(E)$ denotes the event that player i believes E. A player $i \in I$ believes an event $E \in \mathcal{D}$ at a state $\omega \in \Omega$ if $\omega \in B_i(E)$.*
3. *$\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_\kappa(\mathcal{S}))$ is a measurable map: $\Theta^{-1}(E) \in \mathcal{D}$ for any $E \in \mathcal{S}$.*

In addition to the underlying state space and the players' belief operators, the mapping Θ associates, with each state of the world, the corresponding state of nature. In Condition (3), since $(\Omega, \mathcal{D})$ is a κ-algebra, Θ is measurable as long as $\Theta^{-1}(\mathcal{S}) \subseteq \mathcal{D}$. By this condition, any set-algebraic ("logical") operations in $\mathcal{A}_\kappa(\mathcal{S})$ are preserved in the domain $\mathcal{D}$. One can regard Θ as a pair of mappings $(\Theta : \Omega \to S, \Theta^{-1} : \mathcal{A}_\kappa(\mathcal{S}) \to \mathcal{D})$ defining also whether a nature event E is true at a state of the world ω: $\omega \in \Theta^{-1}(E)$ in $(\Omega, \mathcal{D})$ if and only if (iff, for short) $\Theta(\omega) \in E$ in $(S, \mathcal{A}_\kappa(\mathcal{S}))$.

While a standard partitional model assumes any subset of underlying states Ω to be an event (i.e., $\mathcal{D} = \mathcal{P}(\Omega)$), specifying the domain as a κ-algebra plays a crucial role in constructing the universal κ-belief space in Section 3 (there, the domain of the universal κ-belief space turns out to be a κ-algebra generated by events corresponding to nature states and players' interactive beliefs up to the ordinal depth $\overline{\kappa}$). A game-theoretic application in Section 6.1 also suggests the importance of specifying possible depth of reasoning. Moreover, the specification of the domain allows for treating both knowledge and beliefs on a κ-algebra (primarily, σ-algebra), for example, without assuming that players' partitions are at most countable (e.g., Aumann, 1976).

Next, I define properties of qualitative beliefs. Theorem 1 (in Section 3) constructs a universal κ-belief space within a given class of κ-belief spaces satisfying an arbitrary combination of properties specified below. As will be seen, the following list of properties covers various classes of possibility correspondence models of (introspective/non-introspective) knowledge and qualitative beliefs. For example, if a belief operator B_i in a κ-belief space is induced by a partition, then B_i satisfies all the properties below (the converse also holds with some redundancies).[15]

Definition 2 (Propeties of Beliefs)**.** *Let $\langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, \Theta\rangle$ be a κ-belief space. Fix $i \in I$.*

1. *Monotonicity: $B_i(E) \subseteq B_i(F)$ for any $E, F \in \mathcal{D}$ with $E \subseteq F$.*
2. *Necessitation: $B_i(\Omega) = \Omega$.*
3. *Non-empty λ-Conjunction ($\lambda \leq \kappa$ is a fixed infinite cardinal or $\lambda = \kappa = \infty$): $\bigcap_{E\in\mathcal{E}} B_i(E) \subseteq B_i(\bigcap \mathcal{E})$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$.*
4. *The Kripke property: for each $(\omega, E) \in \Omega \times \mathcal{D}$, $\omega \in B_i(E)$ if(f) $E \supseteq b_{B_i}(\omega) := \bigcap\{E \in \mathcal{D} \mid \omega \in B_i(E)\}$.*
5. *Consistency: $B_i(E) \subseteq (\neg B_i)(E^c)$ for any $E \in \mathcal{D}$.*
6. *Truth Axiom: $B_i(E) \subseteq E$ for any $E \in \mathcal{D}$.*
7. *Positive Introspection: $B_i(\cdot) \subseteq B_i B_i(\cdot)$.*
8. *Negative Introspection: $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$.*

First, Monotonicity states that if a player believes some event then she believes any of its logical consequences. Second, Necessitation means that a player believes any form of tautology such as $E \cup E^c$ expressed as Ω. Third, Non-empty λ-Conjunction says that a player believes any non-empty conjunction of events (with cardinality less than λ) if she believes each event. Non-empty ∞-Conjunction means that a player believes any non-empty conjunction of events (without any cardinal restriction) whenever she believes each event (it presupposes that the domain is a complete algebra). Necessitation is identified as the empty conjunction since $\Omega = \bigcap \emptyset$. For example, if $B_i(E)$ denotes the event that player i believes E with probability one (assume countable additivity), then B_i satisfies Monotonicity, Necessitation, and Non-empty λ-Conjunction for $\lambda = \aleph_1$ but not necessarily for $\lambda > \aleph_1$.

Fourth, the Kripke property is the condition under which B_i is induced from the possibility correspondence $b_{B_i} : \Omega \to \mathcal{P}(\Omega)$.[16] The information (or possibility) set $b_{B_i}(\omega) = \{\omega' \in \Omega \mid \omega' \in E \text{ for all } E \in \mathcal{D} \text{ with } \omega \in B_i(E)\}$ consists of states i considers possible at ω. The Kripke property implies Monotonicity, Necessitation, and Non-empty κ-Conjunction.[17]

[15] Theorem 1 extends to a class of κ-belief spaces in which belief operators satisfy general set-theoretic properties (given in Lemma A.1 in Appendix A.1) beyond Definition 2. Using this result, Section 5 constructs a universal probabilistic-belief space for various notions of probabilistic beliefs.

[16] If there is $b : \Omega \to \mathcal{P}(\Omega)$ with $B_i(E) = \{\omega \in \Omega \mid b(\omega) \subseteq E\}$ for all $E \in \mathcal{D}$, then B_i satisfies the Kripke property, i.e., $B_i(E) = \{\omega \in \Omega \mid b_{B_i}(\omega) \subseteq E\}$ for all E (the converse trivially holds). See Fukuda (2019, Remark 1).

[17] The converse may not necessarily hold unless $\kappa = \infty$ (e.g., Samet, 2010 when $\kappa = \aleph_0$).

Fifth, Consistency means that, if a player believes an event E then she does not believe its negation E^c. Probability-one belief satisfies Consistency (assuming additivity). Sixth, Truth Axiom says that a player can only "know" what is true. Truth Axiom distinguishes belief and knowledge in that belief can be false while knowledge has to be true. Truth Axiom implies Consistency. Seventh, Positive Introspection states that if a player believes some event then she believes that she believes it. Eighth, Negative Introspection states that if a player does not believe some event then she believes that she does not believe it.

Three remarks are in order. First, one can assume different properties of beliefs for different players. Players may also have multiple kinds of "belief" operators.[18]

Second, my framework nests possibility correspondence models. Assume the Kripke property (and consequently Monotonicity, Necessitation, and Non-empty κ-Conjunction). The other properties can be expressed in terms of the possibility correspondence. First, B_i satisfies Consistency iff b_{B_i} is serial (i.e., $b_{B_i}(\cdot) \neq \emptyset$). Second, B_i satisfies Truth Axiom iff b_{B_i} is reflexive (i.e., $\omega \in b_{B_i}(\omega)$ for all $\omega \in \Omega$). Third, B_i satisfies Positive Introspection iff b_{B_i} is transitive (i.e., $\omega' \in b_{B_i}(\omega)$ implies $b_{B_i}(\omega') \subseteq b_{B_i}(\omega)$). Fourth, B_i satisfies Negative Introspection iff b_{B_i} is Euclidean (i.e., $\omega' \in b_{B_i}(\omega)$ implies $b_{B_i}(\omega) \subseteq b_{B_i}(\omega')$). Thus, b_{B_i} forms a partition iff B_i satisfies Truth Axiom, Positive Introspection, and Negative Introspection (note that Negative Introspection and Truth Axiom imply Positive Introspection). Likewise, one can capture non-partitional models (see footnote 1): b_{B_i} is reflexive and transitive iff B_i satisfies Truth Axiom and Positive Introspection. Also, one can capture qualitative beliefs: b_{B_i} is serial, transitive, and Euclidean iff B_i satisfies Consistency, Positive Introspection, and Negative Introspection. Hence, one can exactly identify various classes of possibility correspondence models on a κ-algebra $(\Omega, \mathcal{D})$. For each of such classes of possibility correspondence models, the main result (Theorem 1 in Section 3) implies the existence of a universal possibility correspondence model in the class.

Third, in order to accommodate Truth Axiom, the state space $(\Omega, \mathcal{D})$ may not necessarily be assumed to be the product κ-algebra of the nature states $(S, \mathcal{A}_\kappa(\mathcal{S}))$ and the players' type spaces $((T_i, \mathcal{T}_i))_{i \in I}$ (all of which form a κ-algebra). If player i's beliefs depend only on her own types, then B_i would violate Truth Axiom.

2.2 A Terminal Belief Space

So far, the previous subsection has defined belief spaces and properties of beliefs. These determine the class of κ-belief spaces that respect certain assumptions on beliefs. The main result (Theorem 1 in Section 3) constructs a universal κ-belief space for any infinite regular cardinal κ and for any combination of assumptions on players' beliefs specified in Definition 2. Section 4 shows that the universal belief space constructed in Theorem 1 (i) encodes players' interactive beliefs about the

[18] Extend the set of players to $\{0,1\} \times I$, where player i's knowledge operator (which satisfies Truth Axiom) is $K_i := B_{(0,i)}$ while her qualitative-belief operator is $B_i := B_{(1,i)}$. See Appendix C.2.

space itself (Proposition 1) and (ii) is the largest "coherent" set describing players' belief hierarchies (Theorem 2).

To that end, Definition 4 defines a universal belief space in a given class of belief spaces as a terminal belief space in the class. It is a belief space to which every belief space in the given class is uniquely mapped in a belief-preserving manner. I start by formalizing the notion of a belief-preserving map, a belief morphism.

Definition 3 (Belief Morphism)**.** *Let* $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta \rangle$ *and* $\overrightarrow{\Omega'} = \langle (\Omega', \mathcal{D}'), (B'_i)_{i \in I}, \Theta' \rangle$ *be belief spaces of a given class. A (belief) morphism* $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ *is a measurable map* $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ *satisfying: (i)* $\Theta' \circ \varphi = \Theta$*; and (ii)* $B_i(\varphi^{-1}(E')) = \varphi^{-1}(B'_i(E'))$ *for each* $(i, E') \in I \times \mathcal{D}'$*.*

Condition (i) requires that the same nature state prevail for two associated belief spaces. By Condition (ii), players' beliefs are preserved from one space to another in that player i believes an event E' at $\varphi(\omega)$ iff she believes $\varphi^{-1}(E')$ at ω.

For any belief space $\overrightarrow{\Omega}$, the identity map id_Ω on Ω is a morphism from $\overrightarrow{\Omega}$ into itself. Denote by $\mathrm{id}_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega}$ the identity (belief) morphism. Next, call a belief morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ a *(belief) isomorphism*, if there is a morphism $\psi : \overrightarrow{\Omega'} \to \overrightarrow{\Omega}$ with $\psi \circ \varphi = \mathrm{id}_{\overrightarrow{\Omega}}$ and $\varphi \circ \psi = \mathrm{id}_{\overrightarrow{\Omega'}}$ (that is, φ is bijective and its inverse φ^{-1} is a morphism). If φ is an isomorphism then its inverse φ^{-1} is unique. Belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ are *isomorphic*, if there is an isomorphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$.

Now, I define a terminal belief space. It "includes" all belief spaces in that any belief space can be mapped to the terminal space by a unique morphism.

Definition 4 (Terminal Belief Space)**.** *Fix a class of* κ*-belief spaces of* I *on* $(S, \mathcal{S})$*. A* κ*-belief space* $\overrightarrow{\Omega^*}$ *in the class is terminal if, for any* κ*-belief space* $\overrightarrow{\Omega}$ *in the class, there is a unique morphism* $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$*.*

Fix a non-empty set of players I, a space of nature states $(S, \mathcal{S})$, an infinite cardinal κ (or the symbol $\kappa = \infty$), and assumptions on players' beliefs. Then, the given class of κ-belief spaces of I on $(S, \mathcal{S})$ forms a *category*, where a belief space $\overrightarrow{\Omega}$ is an *object* and a belief morphism is a *morphism*. In the language of category theory, a terminal κ-belief space in the class is a terminal (final) object in the category of belief spaces. As is well known in category theory, a terminal belief space (in a given class) is unique up to belief isomorphism. Theorem 1 in the next section constructs a terminal κ-belief space of I on $(S, \mathcal{S})$ when κ is an infinite regular cardinal.

3 Construction of a Terminal Belief Space

Throughout this section, fix a category of κ-belief spaces of I on $(S, \mathcal{S})$ that satisfy some given assumptions of beliefs, where κ is an infinite regular cardinal. A belief space refers to a κ-belief space of I on $(S, \mathcal{S})$.

I construct a terminal (κ-)belief space by employing the "expressions-descriptions" approach (Heifetz and Samet, 1998b; Meier, 2006, 2008). The construction in this section demonstrates that the existence of the terminal belief space hinges on the specification of the infinite regular cardinal κ, which determines the depth of players' reasoning, rather than on assumptions on beliefs themselves. Within the category of κ-belief spaces, define logical formulas describing nature states and the players' interactive beliefs about nature states up to "depth" $\overline{\kappa}$. It turns out that, when depth $\overline{\kappa}$ is fixed, the collection of such logical formulas is rich enough to capture any additional interactive reasoning by the players about nature states (Remark 1). Section 4 studies the properties of the terminal space. Since the construction proves the existence of a terminal space for a wide variety of properties of beliefs, Section 5 constructs a terminal belief space for various probabilistic beliefs. Section 6.1 shows that the specification of κ is also important in an epistemic characterization of a game-theoretic solution concept when players have qualitative beliefs. Section 7 studies the role of κ in comparison with the previous negative results on the existence of a terminal knowledge space. Technically, the construction dispenses with any restriction on $(S, \mathcal{S})$.[19]

The construction of a terminal belief space consists of six steps. Figure 1 illustrates the role of the definitions and lemmas in this section in establishing Theorem 1.

The first step is to inductively define *expressions*, syntactic formulas that express events defined solely in terms of nature and interactive beliefs. Since any nature event is an object of beliefs, treat any nature event $E \in \mathcal{A}_\kappa(\mathcal{S})$ as a proposition and call it an expression. Since objects of beliefs are closed under conjunction, disjunction, negation, and the players' beliefs, define the corresponding syntactic (not set-theoretic) operations for expressions. For a set of expressions $\mathcal{E}$, define $(\bigwedge \mathcal{E})$ as a (syntactic) expression denoting the conjunction of expressions $\mathcal{E}$. For an expression e, let $(\neg e)$ be the (syntactic) expression denoting the negation of e, and let $(\beta_i(e))$ be the (syntactic) expression denoting that player i believes e. Formally:

Definition 5 (Expressions)**.** *Let λ be an infinite regular cardinal with $\lambda \leq \kappa$. The set of all λ-expressions $\mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$ is the smallest set satisfying the following.*

1. *Nature: Every nature event $E \in \mathcal{A}_\kappa(\mathcal{S})$ is a λ-expression.*

2. *Conjunction: If $\mathcal{E}$ is a set of λ-expressions with $|\mathcal{E}| < \lambda$, then so is $(\bigwedge \mathcal{E})$, where $S := \bigwedge \emptyset$ and identify $\bigwedge \mathcal{E} := \bigcap \mathcal{E}$ if $\mathcal{E}$ is a subset of $\mathcal{A}_\kappa(\mathcal{S})$ with $|\mathcal{E}| < \lambda$.*

3. *Negation: If e is a λ-expression then so is $(\neg e)$, where identify $(\neg E) := E^c$ for all $E \in \mathcal{A}_\kappa(\mathcal{S})$.*

[19] Meier (2006, 2008) assumes the following "separative" condition on $(S, \mathcal{S})$: for any distinct $s, s' \in S$, there is $E \in \mathcal{S}$ with ($s \in E$ and $s' \notin E$) or ($s' \in E$ and $s \notin E$) (i.e., there is $E \in \mathcal{A}_\kappa(\mathcal{S})$ with $s \in E$ and $s' \notin E$). Then, $\{s\} = \bigcap\{E \in \mathcal{A}_\kappa(\mathcal{S}) \mid s \in E\}$ for each $s \in S$, though it may be the case that $\{s\} \notin \mathcal{A}_\kappa(\mathcal{S})$. Moss and Viglizzo (2004, 2006) also impose the separative condition.

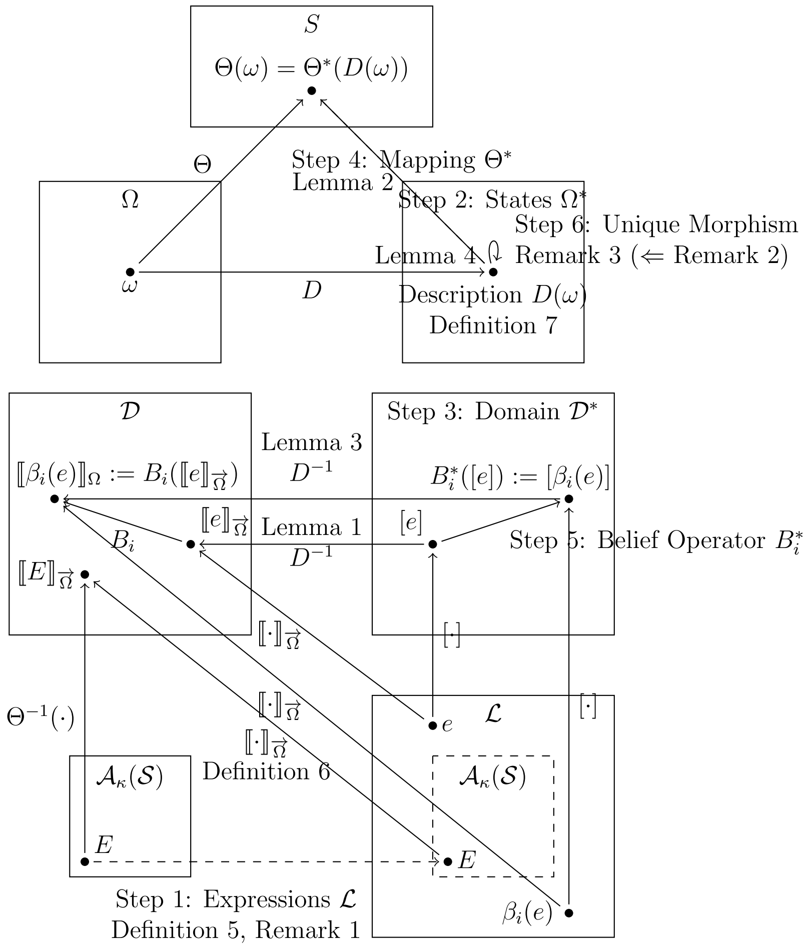

Figure 1: Interrelations among the Definitions and Lemmas for Theorem 1.

4. Beliefs: If e is a λ-expression, then so is $(\beta_i(e))$ for each $i \in I$.

For $\lambda = \kappa$, call each κ-expression an expression, and denote $\mathcal{L} = \mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S}))$.

Four remarks on Definition 5 are in order. First, since κ is fixed, for each λ, the (smallest) set $\mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$ is well-defined by induction. Alternatively, Remark 1 below shows that $\mathcal{L}$ exactly consists of logical formulas regarding nature states and interactive beliefs about nature states of "depth at most $\overline{\kappa}$" (since κ is fixed, $\mathcal{L}$ and consequently $\mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$ are well-defined).

Second, I consider κ-expressions beyond $\lambda = \aleph_0$ because a qualitative-belief hierarchy consisting of all the finite levels of beliefs does not necessarily uniquely extend to the limit.[20]

[20]In contrast, Proposition 4 in Section 5.1 shows that, for countably-additive probabilistic beliefs on $\aleph_1$-algebras, $\aleph_0$-expressions $\mathcal{L}^I_{\aleph_0}(\mathcal{A}_{\aleph_1}(\mathcal{S}))$ suffice to capture countable-level interactive beliefs, consistently with the construction of a terminal type space in the literature.

Third, since Definition 5 treats every $E \in \mathcal{A}_\kappa(\mathcal{S})$ as a logical formula (i.e., a λ-expression) and since $\mathcal{A}_\kappa(\mathcal{S})$ is closed under λ-intersection, identify λ-conjunction (a syntactic operation) with λ-intersection (a semantic operation) in $\mathcal{A}_\kappa(\mathcal{S})$. Similarly, identify negation and complementation in $\mathcal{A}_\kappa(\mathcal{S})$. For example, for $E, F \in \mathcal{A}_\kappa(\mathcal{S})$, the conjunction $\bigwedge\{E, F\}$ is identified with the intersection $E \cap F$.

Fourth, for ease of notation, I often add or omit parentheses in denoting expressions (and in other occurrences). If $\mathcal{E}$ is a set of expressions with $|\mathcal{E}| < \kappa$, then let $(\bigvee \mathcal{E}) := \neg(\bigwedge\{(\neg e) \in \mathcal{L} \mid e \in \mathcal{E}\})$ and identify $\bigvee \emptyset := \emptyset$. Thus, $\bigvee \mathcal{E}$ denotes the disjunction of $\mathcal{E}$. Also, I interchangeably denote, for instance, $e_1 \wedge e_2 = \bigwedge\{e_1, e_2\}$ and $e_1 \vee e_2 = \bigvee\{e_1, e_2\}$.[21] I interchangeably denote $\bigwedge_{j\in J} e_j = \bigwedge\{e_j \mid j \in J\}$ and $\bigvee_{j\in J} e_j = \bigvee\{e_j \mid j \in J\}$ when expressions are indexed by some set J. Denote the implication $(e \to f) := ((\neg e) \vee f)$ and the equivalence $(e \leftrightarrow f) := ((e \to f) \wedge (f \to e))$.

The set $\mathcal{L}$ incorporates all the belief hierarchies regarding nature states (i.e., player i's beliefs about nature $\mathcal{A}_\kappa(\mathcal{S})$, player i's beliefs about player j's beliefs about nature, and so on) up to the ordinal depth $\overline{\kappa}$. For example, if κ is the least infinite cardinal $\aleph_0$, then the set $\mathcal{L} = \mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S}))$ can capture any finite-level belief hierarchies. Likewise, if κ is the least uncountable cardinal $\aleph_1$, then the set $\mathcal{L} = \mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S}))$ can capture any countable-level interactive beliefs.

The following remark shows how the set of expressions $\mathcal{L}$ is inductively generated from the nature states $(S, \mathcal{A}_\kappa(\mathcal{S}))$ in $\overline{\kappa}$ steps.

Remark 1 (Restatement of Expressions $\mathcal{L}$)**.** Let λ be an infinite regular cardinal with $\lambda \leq \kappa$. The following auxiliary ordinal sequence $(\mathcal{L}_\alpha)_{\alpha=0}^{\overline{\lambda}}$ generates the set of expressions $\mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S})) = \mathcal{L}_{\overline{\lambda}}$. In particular, if $\lambda = \kappa$ then $\mathcal{L} = \mathcal{L}_{\overline{\kappa}}$. Namely, let $\mathcal{L}_0 := \mathcal{A}_\kappa(\mathcal{S})$. For any ordinal α with $0 < \alpha \leq \overline{\lambda}$, define

$$\mathcal{L}_\alpha := \mathcal{L}'_\alpha \cup \{(\neg e) \mid e \in \mathcal{L}'_\alpha\} \cup \{\bigwedge \mathcal{F} \mid \mathcal{F} \subseteq \mathcal{L}'_\alpha \text{ and } 0 < |\mathcal{F}| < \lambda\}, \text{ where}$$
$$\mathcal{L}'_\alpha := (\bigcup_{\beta<\alpha} \mathcal{L}_\beta) \cup \bigcup_{i\in I}\{\beta_i(e) \mid e \in \bigcup_{\beta<\alpha} \mathcal{L}_\beta\}.$$

Intuitively, Remark 1 states that $\mathcal{L}$ consists of expressions of "depth at most $\overline{\kappa}$," i.e., logical formulas expressing interactive beliefs regarding $(S, \mathcal{A}_\kappa(\mathcal{S}))$ (indeed, $(S, \mathcal{S})$) up to depth $\overline{\kappa}$. Put differently, the collection $\mathcal{L}_{\overline{\kappa}}$ of logical formulas of depth at most $\overline{\kappa}$ regarding nature and interactive beliefs about nature is rich enough to capture any syntactic logical operations and players' reasoning defined in Definition 5 within itself because $\mathcal{L}_{\overline{\kappa}} = \mathcal{L}$. Still in other words, any additional reasoning of depth up to $\overline{\kappa}$ about nature states and interactive beliefs of depth up to $\overline{\kappa}$ (which can be captured by the expressions $\mathcal{L}_{\overline{\kappa}}$) can be captured within the expressions $\mathcal{L}_{\overline{\kappa}}$. Concretely, if $\kappa = \aleph_0$, then $\mathcal{L}_{\overline{\kappa}} = \mathcal{L}$ can capture any finite-level reasoning about any finite-level reasoning

[21]For example, I simply do not distinguish $e_1 \vee e_2$ and $e_2 \vee e_1$. Similarly, since $\{e, e\} = \{e\}$, I simply identify $(e \wedge e)$ as e. These could be augmented by defining $(\bigwedge \mathcal{F})$ for an ordinal sequence of expressions $\mathcal{F}$ instead of a set of expressions.

about nature states and interactive beliefs, which is still finite-level reasoning about nature states and interactive beliefs. Likewise, if $\kappa = \aleph_1$, then $\mathcal{L}_{\overline{\kappa}} = \mathcal{L}$ can capture any countable-level reasoning about any countable-level reasoning, which is indeed countable-level reasoning.

While expressions themselves are defined independently of any particular belief space, for any belief space $\overrightarrow{\Omega}$, I can recursively identify each expression e with the corresponding event $[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D}$ so that $[\![e]\!]_{\overrightarrow{\Omega}}$ is the set of states of the world in which the expression e holds. Thus, the event $[\![e]\!]_{\overrightarrow{\Omega}}$ gives the semantic meaning of the expression e. Specifically, recalling the discussion on Definition 1 (3), $[\![E]\!]_{\overrightarrow{\Omega}} := \Theta^{-1}(E) \in \mathcal{D}$ is the set of states at which $E \in \mathcal{A}_\kappa(\mathcal{S})$ is true. The set of states $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega}}$ at which an expression $\beta_i(e)$ is true ("i believes e") is defined by $B_i([\![e]\!]_{\overrightarrow{\Omega}})$. Formally:

Definition 6 (Expressions Identified as Events)**.** *Fix a κ-belief space $\overrightarrow{\Omega}$. Inductively define the map $[\![\cdot]\!]_{\overrightarrow{\Omega}} : \mathcal{L} \to \mathcal{D}$, the semantic interpretation function of $\overrightarrow{\Omega}$, as follows.*

1. *Nature: $[\![E]\!]_{\overrightarrow{\Omega}} := \Theta^{-1}(E)$ for every $E \in \mathcal{A}_\kappa(\mathcal{S})$.*
2. *Conjunction: $[\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}} := \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}}$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{L})$ with $|\mathcal{E}| < \kappa$.*
3. *Negation: $[\![\neg e]\!]_{\overrightarrow{\Omega}} := \neg [\![e]\!]_{\overrightarrow{\Omega}} (= ([\![e]\!]_{\overrightarrow{\Omega}})^c)$ for each expression e.*
4. *Beliefs: $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega}} := B_i([\![e]\!]_{\overrightarrow{\Omega}})$ for each $i \in I$ and expression e.*

Call $[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D}$ the denotation of e in $\overrightarrow{\Omega}$.

The semantic interpretation function of a given belief space is, by recursion, uniquely extended from Θ^{-1}.[22] I remark that, by recursion, a morphism preserves semantics. This will be used to see that if two states (say, ω and $\varphi(\omega)$) are linked through a morphism, then they induce the same belief hierarchy because the way that nature states and interactive beliefs are interpreted at these two states is identical.

Remark 2 (Morphism Preserves the Meaning of an Expression)**.** If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism, then $[\![\cdot]\!]_{\overrightarrow{\Omega}} = \varphi^{-1}([\![\cdot]\!]_{\overrightarrow{\Omega'}})$.

The second step is to define *descriptions* by the set of expressions and the nature state that obtain at each state of each belief space. Each description turns out to be a state of the terminal space, i.e., each state in the terminal space describes the nature state and the set of expressions that hold at some state of some belief space. In defining a description, observe that nature states and expressions reside in different spaces. Thus, I define a description to be a subset of the disjoint union $S \sqcup \mathcal{L} := \{(0, s) \in \{0\} \times S \mid s \in S\} \cup \{(1, e) \in \{1\} \times \mathcal{L} \mid e \in \mathcal{L}\}$. While this definition of the description is different from the one in the previous literature, this definition uniquely identifies the corresponding nature state for each description without any condition (i.e., the separative condition in footnote 19) on $(S, \mathcal{S})$. Formally:

[22] While I do not discuss implications of finite-depth reasoning, one could analyze players' finite depth/level-α reasoning in any belief space $\overrightarrow{\Omega}$ by restricting attention to events $[\![e]\!]_{\overrightarrow{\Omega}}$ with $e \in \mathcal{L}_\alpha$.

Definition 7 (Descriptions). *For any belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$, define the description $D(\omega)$ of ω by $D(\omega) := \{\Theta(\omega)\} \sqcup \{e \in \mathcal{L} \mid \omega \in [\![e]\!]_{\overrightarrow{\Omega}}\}$.*

Each description $D(\omega) = \{(0, \Theta(\omega))\} \cup \{(1, e) \in \{1\} \times \mathcal{L} \mid \omega \in [\![e]\!]_{\overrightarrow{\Omega}}\}$ contains the unique nature state $\Theta(\omega) \in S$ associated with ω and the expressions e which are true at ω.[23] Especially, $D(\omega)$ induces the belief hierarchy (player i's beliefs about nature $\mathcal{A}_\kappa(\mathcal{S})$, player i's beliefs about player j's beliefs about nature, and so on) up to depth $\overline{\kappa}$ that holds at ω. For ease of notation, write $s \in_0 D(\omega)$ for $(0, s) \in D(\omega)$. Also, write $e \in_1 D(\omega)$ for $(1, e) \in D(\omega)$. The reader could even read "$s \in D(\omega)$" and "$e \in D(\omega)$" by disregarding the subscripts 0 and 1.

Descriptions have two roles in constructing a terminal belief space. First, I will construct the terminal belief space so that the underlying states Ω^* consist of all descriptions of states of the world ranged over all belief spaces in the given category:

$$\Omega^* := \{\omega^* \in \mathcal{P}(S \sqcup \mathcal{L}) \mid \omega^* = D(\omega) \text{ for some } \overrightarrow{\Omega} \text{ and } \omega \in \Omega\}. \tag{1}$$

Second, regard D as a mapping $D : \Omega \to \Omega^*$ (or $D_{\overrightarrow{\Omega}} : \Omega \to \Omega^*$ to stress its domain) for any belief space $\overrightarrow{\Omega}$, and call D the *description map*. The description map D turns out to be a unique morphism.

Two remarks are in order. First, Ω^* is not empty because there is a belief space $\overrightarrow{\Omega}$ with $\Omega \neq \emptyset$ in the given category. Consider a belief space $\overrightarrow{\{s\}} := \langle(\{s\}, \mathcal{P}(\{s\})), (\mathrm{id}_{\mathcal{P}(\{s\})})_{i \in I}, \Theta\rangle$ where $s \in S$ and $\Theta : \{s\} \ni s \mapsto s \in S$. Each $B_i = \mathrm{id}_{\mathcal{P}(\{s\})}$ satisfies all the properties of beliefs in Definition 2.

Second, Ω^* depends on the choice of a category of belief spaces. The more assumptions on beliefs the outside analysts impose, the smaller Ω^* becomes. Formally, consider two categories of belief spaces where assumptions on players' beliefs in the first are also imposed in the second. Denoting by Ω^{1*} and Ω^{2*} the spaces constructed according to Equation (1), $\Omega^{2*} \subseteq \Omega^{1*}$ holds by construction.

To show that the description map D is a unique morphism (in the sixth step), here I remark that a morphism preserves the descriptions, i.e., nature states and belief hierarchies.

Remark 3 (Morphism Preserves Descriptions). If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism, then $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \varphi$.

To see this, fix belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ and $(\omega, \omega') \in \Omega \times \Omega'$. Then, $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$ iff (i) $\Theta(\omega) = \Theta'(\omega')$ and (ii) $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega' \in [\![e]\!]_{\overrightarrow{\Omega'}}$ for all $e \in \mathcal{L}$. Thus, $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$ means that the outside analysts would consider states ω and ω' to be equivalent in terms of a prevailing nature state and prevailing expressions, abstracting away from physical representations of $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ (see Remark S.1 in

[23] Note that $(0, s) \in D(\omega)$ indicates which nature event belongs to $D(\omega)$ in the following sense: for any $E \in \mathcal{A}_S$, $(1, E) \in D(\omega)$ iff $s \in E$. For ease of exposition, however, I keep track of all expressions including nature events which are true at ω in the description $D(\omega)$.

Appendix B for further discussions). By Remark 2, both conditions are met for any $(\omega, \varphi(\omega)) \in \Omega \times \Omega'$ where $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism. Thus, for any state $\omega \in \Omega$, there is a state $\varphi(\omega) \in \Omega'$ such that ω and $\varphi(\omega)$ induce the same belief hierarchy together with the corresponding nature state.

I discuss two implications of Remark 3. First, call a belief space $\overrightarrow{\Omega}$ *non-redundant* (Mertens and Zamir, 1985, Definition 2.4) (or *non-flabby* (Fagin, 1994)) if its description map D is injective. In other words, for any distinct ω and ω', either $\Theta(\omega) \neq \Theta(\omega')$ or they are separated by (a sub-κ-algebra) $\mathcal{D}_{\overline{\kappa}} := \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$ (it can be expressed solely by the primitives of the belief space by Definition 11 in Section 7).

Second, Remark 3 implies that if $\overrightarrow{\Omega'}$ is non-redundant then there is at most one morphism from a given space $\overrightarrow{\Omega}$ into $\overrightarrow{\Omega'}$.[24] I will show that $D_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is a unique morphism (in the sixth step) by demonstrating that $D_{\overrightarrow{\Omega^*}}$ is the identity.

The third step is to define the domain $\mathcal{D}^*$ of the candidate terminal belief space Ω^*. Since each expression e corresponds to an object of beliefs, define the set $[e]$ of descriptions that make e true (i.e., contain e) to be an event in Ω^*. Formally, for each $e \in \mathcal{L}$, define the set of descriptions $[e] := \{\omega^* \in \Omega^* \mid e \in_1 \omega^*\}$. I show that $\mathcal{D}^* := \{[e] \in \mathcal{P}(\Omega^*) \mid e \in \mathcal{L}\}$ is a legitimate domain.

Lemma 1 (Domain $\mathcal{D}^*$)**.** *$(\Omega^*, \mathcal{D}^*)$ is a κ-algebra. For any belief space $\overrightarrow{\Omega}$, the description map $D : (\Omega, \mathcal{D}) \to (\Omega^*, \mathcal{D}^*)$ is a measurable map with $D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}}$ for any $e \in \mathcal{L}$.*

The property that $D^{-1}_{\overrightarrow{\Omega}}([\cdot]) = [\![\cdot]\!]_{\overrightarrow{\Omega}}$ exhibits duality between the semantic interpretation function $[\![\cdot]\!]_{\overrightarrow{\Omega}} : \mathcal{L} \to \mathcal{D}$ (which, by recursion, is unique) and the description map $D_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ (which turns out to be a unique morphism). Through $[\![\cdot]\!]_{\overrightarrow{\Omega}}$, the meaning of each expression $e \in \mathcal{L}$ regarding nature states and interactive beliefs is interpreted as an event $[\![e]\!]_{\overrightarrow{\Omega}}$ in the given space $\overrightarrow{\Omega}$. In contrast, through $D_{\overrightarrow{\Omega}}$, each state ω in the given space $\overrightarrow{\Omega}$ is re-formulated as the corresponding nature state and expressions (i.e., the description) $D_{\overrightarrow{\Omega}}(\omega)$. Note also that the sub-κ-algebra $\mathcal{D}_{\overline{\kappa}}$ is the one induced by $D_{\overrightarrow{\Omega}}$: $\mathcal{D}_{\overline{\kappa}} = D^{-1}_{\overrightarrow{\Omega}}(\mathcal{D}^*) = \{D^{-1}_{\overrightarrow{\Omega}}([e]) \in \mathcal{D} \mid e \in \mathcal{L}\}$.

Call $\overrightarrow{\Omega}$ *minimal* (Di Tillio, 2008) if $\mathcal{D}_{\overline{\kappa}} = \mathcal{D}$.[25] It will be shown that if a given belief space is non-redundant and minimal, then the belief space is isomorphic to a subspace of the terminal space.

The fourth step is to construct the mapping $\Theta^* : \Omega^* \to S$ that associates with each state $\omega^* \in \Omega^*$ the unique nature state s contained in ω^* (i.e., $s \in_0 \omega^*$).

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[24] *Proof.* If $\varphi, \psi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ are morphisms then $D_{\overrightarrow{\Omega'}} \circ \varphi = D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \psi$. Since $D_{\overrightarrow{\Omega'}}$ is injective, $\varphi = \psi$.

[25] Friedenberg and Meier (2011) call it *strongly measurable* in a related context (recall $\mathcal{D}_{\overline{\kappa}} = D^{-1}_{\overrightarrow{\Omega}}(\mathcal{D}^*)$). See also Appendix E for the characterization of minimality (strong measurability).

Lemma 2 (Mapping Θ^*)**.** *There is a measurable map* $\Theta^* : (\Omega^*, \mathcal{D}^*) \to (S, \mathcal{A}_\kappa(\mathcal{S}))$ *with the following properties: (i)* $\Theta^*(D(\omega)) = \Theta(\omega)$ *for any belief space* $\overrightarrow{\Omega}$ *and* $\omega \in \Omega$*; and (ii)* $(\Theta^*)^{-1}(E) = [E] \in \mathcal{D}^*$ *for all* $E \in \mathcal{A}_\kappa(\mathcal{S})$*.*

The fifth step is to introduce the players' belief operators on $\mathcal{D}^*$ in a way such that player i believes an event $[e]$ at a state ω^* iff ω^* contains $\beta_i(e)$ (i.e., $\beta_i(e) \in_1 \omega^*$). I show that this is well-defined: if expressions e and f are equivalent in the sense that $e \in_1 \omega^*$ iff $f \in_1 \omega^*$, then $\beta_i(e)$ and $\beta_i(f)$ are equivalent in the same sense.[26]

Lemma 3 (Belief Operators B_i^*)**.** *Fix* $i \in I$*. Define* $B_i^* : \mathcal{D}^* \to \mathcal{D}^*$ *by* $B_i^*([e]) := [\beta_i(e)]$ *for each* $e \in \mathcal{L}$*. Then,* B_i^* *is a well-defined belief operator which inherits the properties of beliefs imposed in the given category. Moreover, for any belief space* $\overrightarrow{\Omega}$*,* $D^{-1}(B_i^*([e])) = B_i(D^{-1}([e]))$ *for all* $[e] \in \mathcal{D}^*$*.*

Lemma A.1 in Appendix A.1 shows that each B_i^* inherits various properties satisfied in the given category beyond Definition 2. In other words, this paper shows the existence of a terminal belief space as long as players' beliefs are represented by their belief operators and properties of the belief operators satisfy the set-theoretic conditions of Lemma A.1 in Appendix A.1. In fact, Section 5 extends the construction of a terminal belief space to such cases as probabilistic beliefs.

I remark on two additional results proved in Lemma A.1. First, if there is a belief space $\overrightarrow{\Omega}$ such that B_i fails a given property with respect to $[\![e]\!]_{\overrightarrow{\Omega}}$, then B_i^* fails that property with respect to $[e]$. Thus, B_i^* satisfies the properties of beliefs for player i that are common among all the belief spaces in the given category. That is, B_i^* satisfies the properties that the outside analysts exactly would like to impose.[27]

Second, for belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ which may reside in different categories, if $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ is a surjective measurable map with $B_i \varphi^{-1}(\cdot) = \varphi^{-1} B_i'(\cdot)$, then B_i' inherits the properties of B_i.

So far, $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (B_i^*)_{i \in I}, \Theta^* \rangle$ is a belief space of the given category such that, for any belief space $\overrightarrow{\Omega}$, the description map $D : \Omega \to \Omega^*$ is a morphism.

The sixth step finally establishes that the description map D is a unique morphism. To that end, I show that the description map from $\overrightarrow{\Omega^*}$ into itself is the identity map.

Lemma 4 (Description Map $D_{\overrightarrow{\Omega^*}}$)**.** *The description map* $D_{\overrightarrow{\Omega^*}} : \overrightarrow{\Omega^*} \to \overrightarrow{\Omega^*}$ *is the identity morphism.*

[26]First, this equivalence depends on assumptions on beliefs. For example, if Positive Introspection and Truth Axiom are imposed on player i, then $\beta_i(e)$ and $\beta_i\beta_i(e)$ are equivalent. Section 4.3 examines how assumptions on beliefs are encoded within Ω^* itself. Second, Proposition 2 in Section 4.2 shows if there are a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $[\![e]\!]_{\overrightarrow{\Omega}} \neq [\![f]\!]_{\overrightarrow{\Omega}}$ then e and f are not equivalent (i.e., $[e] \neq [f]$). Thus, such identifications (equivalences) of expressions are minimum in $\overrightarrow{\Omega^*}$.

[27]Roy and Pacuit (2013) define "substantive" and "structural" assumptions in a syntactic interactive epistemic model. In their framework, a terminal structure, if it exists, minimizes substantive assumptions and validates only structural assumptions.

I prove Lemma 4 by showing $[\cdot] = [\![\cdot]\!]_{\overrightarrow{\Omega^*}}$, which means whether an expression e is true at ω^* is determined solely by whether $e \in_1 \omega^*$ within ω^*. The lemma implies the belief space $\overrightarrow{\Omega^*}$ is non-redundant and minimal. Moreover, it implies $D_{\overrightarrow{\Omega}}$ is a unique morphism: if $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is a morphism then $D_{\overrightarrow{\Omega}}(\cdot) = D_{\overrightarrow{\Omega^*}}(\varphi(\cdot)) = \varphi(\cdot)$. Thus:

Theorem 1 ($\overrightarrow{\Omega^*}$ is Terminal)**.** *The space $\overrightarrow{\Omega^*} = \langle(\Omega^*, \mathcal{D}^*), (B_i^*)_{i\in I}, \Theta^*\rangle$ is a terminal κ-belief space of I on $(S, \mathcal{S})$ for a given category of κ-belief spaces.*

As discussed in Section 2.2, a terminal belief space exists uniquely up to isomorphism. Since terminality requires a unique morphism from a given space, a belief space $\overrightarrow{\Omega}$ is terminal iff the description map $D_{\overrightarrow{\Omega}}$ is an isomorphism. Especially, any terminal space is non-redundant and minimal.

I discuss how the belief space Ω^* "includes" all belief spaces. First, for any state ω of any particular space $\overrightarrow{\Omega}$, states $\omega \in \Omega$ and $D(\omega) \in \Omega^*$ are equivalent in that the same state of nature $\Theta(\omega) = \Theta^*(D(\omega))$ prevails and the same set of expressions regarding nature and interactive beliefs obtains. This is because $D(\omega) = D_{\overrightarrow{\Omega^*}}(D(\omega))$ (recall discussions after Remark 3). To restate, for any representation $\overrightarrow{\Omega}$ of interactive beliefs regarding $(S, \mathcal{S})$ and for any state $\omega \in \Omega$, the prevailing nature state and the prevailing set of expressions at ω are encoded in the state $D(\omega)$ in $\overrightarrow{\Omega^*}$. Hence, $\overrightarrow{\Omega^*}$ is also terminal in the sense of Friedenberg (2010): for any state $\omega \in \Omega$ of any belief space $\overrightarrow{\Omega}$, there is a unique state $\omega^* = D(\omega)$ in $\overrightarrow{\Omega^*}$ such that ω and ω^* induce the same belief hierarchies. Especially, the expressions $\{e \in \mathcal{L} \mid \omega \in B_i([\![e]\!]_{\overrightarrow{\Omega}})\}$ that player i believes at ω coincide with those $\{e \in \mathcal{L} \mid \omega^* \in B_i^*([e])\}$ that she believes at ω^*.

Second, a non-redundant belief space $\overrightarrow{\Omega}$ is, by definition, embedded into $\overrightarrow{\Omega^*}$: there is a belief (sub-)space $\overrightarrow{D(\Omega)} := \langle(D(\Omega), \mathcal{D}^* \cap D(\Omega)), (B'_i)_{i\in I}, \Theta^*|_{D(\Omega)}\rangle$ such that $D : \overrightarrow{\Omega} \to \overrightarrow{D(\Omega)}$ is a bijective morphism, where $\mathcal{D}^* \cap D(\Omega) := \{[e] \cap D(\Omega) \mid [e] \in \mathcal{D}^*\}$ and $B'_i([e] \cap D(\Omega)) := B_i^*([e]) \cap D(\Omega)$. If, in addition, $\overrightarrow{\Omega}$ is minimal, then $\overrightarrow{\Omega}$ and $\overrightarrow{D(\Omega)}$ are isomorphic (i.e., the inverse D^{-1} is also a morphism) because any $E \in \mathcal{D}$ is associated with some $[e] \in \mathcal{D}^*$ through $[e] = D^{-1}(E)$. Indeed, $\overrightarrow{\Omega}$ and $\overrightarrow{D(\Omega)}$ are isomorphic iff $\overrightarrow{\Omega}$ is non-redundant and minimal (if this is the case, then observe that a κ-algebra $\mathcal{D}$ is typically not the power set).

4 Characterization of the Terminal Space

The previous section has shown that the terminal space $\overrightarrow{\Omega^*}$ is non-redundant and minimal. Each belief operator B_i^* satisfies the minimum possible assumptions (i.e., the exact assumptions the outside analysts impose) in a given category of belief spaces.

This section studies properties of the terminal belief space $\overrightarrow{\Omega^*}$ constructed in the previous section. The first two subsections examine the sense in which the terminal space captures any reasoning in smaller spaces. The next two subsections study the sense in which the terminal space resolves the issue of self-reference (e.g., Aumann, 1976): each state is supposed to represent the way players interactively reason about each other's beliefs, but players' beliefs are defined on the states.

Section 4.1 shows that the terminal space is (belief-)complete: the space contains, within itself, all possible ways in which the players possess interactive beliefs. Section 4.2 shows: any expression that holds at some state of some belief space holds at the corresponding state of the terminal space; and the terminal space has the minimum possible assumptions on how players' beliefs are modeled in the given category. As a game-theoretic application, Section 6 demonstrates that players' rationality is well-defined in a belief space and that if players are rational at some state of some belief space, then they are rational at the corresponding state of the terminal space.

Section 4.3 demonstrates that each state of the terminal space encodes players' interactive beliefs at that state within itself in a complete and coherent manner. Section 4.4 restates that the terminal belief space $\overrightarrow{\Omega^*}$ is the largest space in which each state encodes players' interactive beliefs at that state within itself in a complete and coherent manner.

4.1 Belief-Completeness

I show that the space Ω^* is (belief-)complete in that, for any profile of sets of expressions that individual players can possibly believe within the framework of belief spaces, there exists a state ω^* in the terminal belief space $\overrightarrow{\Omega^*}$ at which each player believes the corresponding set of expressions. More generally, I show that a non-redundant and minimal belief space is terminal iff it is (belief-)complete.[28]

Let Ω^{**} consist of a nature state that holds and expressions that individual players believe within the framework of belief spaces:

$$\Omega^{**} := \{(s, \Psi) \in S \times \mathcal{P}(\mathcal{L})^I \mid (s, \Psi) = (\Theta(\omega), (\{e \in \mathcal{L} \mid \omega \in B_i [\![e]\!]_{\overrightarrow{\Omega}}\})_{i \in I}) \text{ for some belief space } \overrightarrow{\Omega} \text{ and } \omega \in \Omega\}.$$

The set Ω^{**} consists of a nature state and players' interactive beliefs (players' beliefs about nature states, their beliefs about their beliefs about nature states, and so on) up to the ordinality of $\overline{\kappa}$ that can attain at some state of some belief space. Since assumptions on beliefs are arbitrary, instead of explicitly defining the coherency conditions on sets of expressions Ψ that reflect given assumptions of beliefs, I implicitly provide the coherency conditions by sets of expressions that the players believe at

[28]First, the notion of (belief-)completeness is usually defined in the context of product type spaces. Here, given the different framework, I define (belief-)completeness in terms of a language, as in Brandenburger and Keisler (2006). Second, in the literature on product type spaces, a terminal space is (belief-)complete (Moss and Viglizzo, 2004, 2006), and a (belief-)complete space is terminal given topological conditions (Friedenberg, 2010). Third, see, for instance, Friedenberg (2010) and the references therein for applications of (belief-)completeness to epistemic game theory.

some state of some belief space.[29] Section 4.4 explicitly introduces the coherency conditions on sets of expressions (i.e., descriptions) to restate the terminal space $\overrightarrow{\Omega^*}$.

Call a belief space $\overrightarrow{\Omega}$ (in the given category) *(belief-)complete* if the mapping

$$\chi_{\overrightarrow{\Omega}} : \Omega \ni \omega \mapsto (\Theta(\omega), (\{e \in \mathcal{L} \mid \omega \in B_i([\![e]\!]_{\overrightarrow{\Omega}})\})_{i \in I}) \in \Omega^{**}$$

is surjective. Each mapping $\omega \mapsto \{e \in \mathcal{L} \mid \omega \in B_i([\![e]\!]_{\overrightarrow{\Omega}})\}$ (that constitutes $\chi_{\overrightarrow{\Omega}}$) defines player i's "type mapping" that associates, with each state ω, her beliefs on Ω about the expressions $\mathcal{L}$ (or $\mathcal{D}_{\overline{\kappa}}$).[30]

The following proposition shows that a belief space is terminal iff it is non-redundant, minimal, and (belief-)complete. Especially, the terminal belief space $\overrightarrow{\Omega^*}$ is (belief-)complete so that it exhausts nature and the players' beliefs.

Proposition 1 ((Belief-)Completeness)**.** *1. A belief space $\overrightarrow{\Omega}$ is terminal (in a given category) iff it is minimal, non-redundant, and (belief-)complete.*

2. A belief space $\overrightarrow{\Omega}$ is non-redundant iff $\chi_{\overrightarrow{\Omega}}$ is injective.

3. The mapping $\chi_{\overrightarrow{\Omega^*}} : \Omega^* \to \Omega^{**}$ is bijective. In particular,

$$\Omega^{**} = \{(\Theta^*(\omega^*), (\{e \in \mathcal{L} \mid \omega^* \in B_i^*[e]\})_{i \in I}) \in S \times \mathcal{P}(\mathcal{L})^I \mid \omega^* \in \Omega^*\}. \quad (2)$$

Part (1) implies that a non-redundant and minimal belief space is (belief-)complete iff it is terminal. Since $\overrightarrow{\Omega^*}$ is terminal, it is (belief-)complete: the space Ω^* exhausts all possible forms of interactive beliefs that can realize at some state of some belief space. (Belief-)completeness of $\overrightarrow{\Omega^*}$ also follows directly because

$$(\Theta(\omega), (\{e \in \mathcal{L} \mid \omega \in B_i[\![e]\!]_{\overrightarrow{\Omega}}\})_{i \in I}) = (\Theta^*(D(\omega)), (\{e \in \mathcal{L} \mid D(\omega) \in B_i^*[e]\})_{i \in I}). \quad (3)$$

As Equation (2) shows, Ω^{**} is obtained by restricting attention to the terminal space. For each player i, each state ω^* contains all the relevant information about i's beliefs at ω^* within itself because, for any expression e, whether i believes $[e]$ or not at ω^* is well-defined according to $\{e \in \mathcal{L} \mid \omega^* \in B_i^*[e]\}$.

Moreover, the set Ω^* is in a bijective relation with

$$\{(s, \Psi) \in S \times \mathcal{P}(\mathcal{D}^*)^I \mid (s, \Psi) = (\Theta(\omega^*), (\{[e] \in \mathcal{D}^* \mid \omega^* \in B_i([e])\})_{i \in I}) \text{ for some belief space } \langle (\Omega^*, \mathcal{D}^*), (B_i)_{i \in I}, \Theta \rangle \text{ and } \omega^* \in \Omega^*\}$$

through a bijection $\omega^* \mapsto (\Theta(\omega^*), (\{[e] \in \mathcal{D}^* \mid \omega^* \in B_i^*([e])\})_{i \in I})$. For any profile of nature state that holds and sets of events that individual players believe at some state of some belief structure on $(\Omega^*, \mathcal{D}^*)$, there is a state in $\overrightarrow{\Omega^*}$ that captures the profile.

[29] Also, the space $\Omega^{**} (\subseteq S \times \mathcal{P}(\mathcal{L})^I)$ may not be a product space under Truth Axiom, under which players' truthful beliefs (knowledge) are correlated (that is, it is not the case that one player knows a certain event while another player knows its negation).

[30] Section 5.1 applies the construction in Section 3 to probabilistic beliefs. There, player i's p-belief operators $(B_i^p)_{p \in [0,1]}$ yield her type $t_{B_i}(\omega)$ at a state ω through the maximum probability p with which she believes an event at the state: $t_{B_i}(\omega)(E) := \sup\{p \in [0,1] \mid \omega \in B_i^p(E)\}$ for each $E \in \mathcal{D}$. Hence, the set of expressions that a player believes at a state recovers her beliefs at that state.

4.2 Informational Robustness

Next, I examine a sense in which reasoning in a smaller belief space can be extended to the terminal space. Specifically, I show: for any set of expressions that hold at some state of some belief space, the set of expressions hold at the corresponding state of the terminal space $\overrightarrow{\Omega^*}$; and if some belief space distinguishes expressions e and f, then so does the terminal space $\overrightarrow{\Omega^*}$. To make the claims formal, I introduce:

Definition 8 (Semantic Properties)**.** *1. An expression $e \in \mathcal{L}$ is valid in a belief space $\overrightarrow{\Omega}$ (written $\models_{\overrightarrow{\Omega}} e$) if $[\![e]\!]_{\overrightarrow{\Omega}} = \Omega$. If e is valid in any belief space (of the given category), then e is valid (in the given category) (written $\models e$).*

2. *A set of expressions $\Phi \in \mathcal{P}(\mathcal{L})$ is satisfiable in $\overrightarrow{\Omega}$ if there is $\omega \in \Omega$ with $\omega \in [\![f]\!]_{\overrightarrow{\Omega}}$ for all $f \in \Phi$. Call Φ satisfiable if Φ is satisfiable in some belief space $\overrightarrow{\Omega}$.*

3. *A set of expressions $\Phi \in \mathcal{P}(\mathcal{E})$ is maximally satisfiable if it is satisfiable and if $\Phi = \Psi$ for any satisfiable set Ψ with $\Phi \subseteq \Psi$.*

4. *An expression $e \in \mathcal{L}$ is a semantic consequence of $\Phi \in \mathcal{P}(\mathcal{L})$ in $\overrightarrow{\Omega}$ (written $\Phi \models_{\overrightarrow{\Omega}} e$) if, $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$ holds whenever $\omega \in [\![f]\!]_{\overrightarrow{\Omega}}$ for all $f \in \Phi$. If $\Phi \models_{\overrightarrow{\Omega}} e$ for any belief space $\overrightarrow{\Omega}$, then $e \in \mathcal{L}$ is a semantic consequence of Φ (written $\Phi \models e$).*

Let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ be a morphism. By Remark 2, any valid expression e in $\overrightarrow{\Omega'}$ is also valid in $\overrightarrow{\Omega}$. If Φ is satisfiable in $\overrightarrow{\Omega}$, then so is it in $\overrightarrow{\Omega'}$. Suppose further that φ is surjective. If e is a semantic consequence of Φ in $\overrightarrow{\Omega}$, then so is it in $\overrightarrow{\Omega'}$.

The following proposition shows that such semantic notions as satisfiability, semantic consequence, and validity in $\overrightarrow{\Omega^*}$ are informationally robust in the sense that they do not depend on particular belief spaces.

Proposition 2 (Informational Robustness)**.** *Let $e \in \mathcal{L}$ and $\Phi \in \mathcal{P}(\mathcal{L})$.*

1. *Φ is satisfiable iff Φ is satisfiable in $\overrightarrow{\Omega^*}$.*

2. *e is a semantic consequence of Φ iff e is a semantic consequence of Φ in $\overrightarrow{\Omega^*}$. Also, e is valid iff e is valid in $\overrightarrow{\Omega^*}$.*

3. *$\Omega^* = \{\{s\} \sqcup \Phi \in \mathcal{P}(S \sqcup \mathcal{L}) \mid \Phi$ is maximally satisfiable and, for any $E \in \mathcal{A}_\kappa(\mathcal{S})$, $s \in E$ iff $E \in \Phi\}$.*

Proposition 2 (1) states that the space $\overrightarrow{\Omega^*}$ exhausts all possible sets of satisfiable expressions within $\overrightarrow{\Omega^*}$. Put differently, if expressions Φ hold at some state ω in some belief space $\overrightarrow{\Omega}$, then the expressions Φ hold at $D(\omega)$ in $\overrightarrow{\Omega^*}$. In fact, the set of expressions that hold at a particular state of a particular model is maximally satisfiable, and any maximally satisfiable set can be associated with the set of expressions that hold at a particular state of a particular model. Thus, as Proposition 2 (3) demonstrates, each state of $\overrightarrow{\Omega^*}$ consists of a maximally satisfiable set of expressions and a corresponding nature state.

Proposition 2 (2) implies that if expressions e and f satisfy $[\![e]\!]_{\overrightarrow{\Omega}} \neq [\![f]\!]_{\overrightarrow{\Omega}}$ for some belief space $\overrightarrow{\Omega}$, then $([e] =)[\![e]\!]_{\overrightarrow{\Omega^*}} \neq [\![f]\!]_{\overrightarrow{\Omega^*}}(= [f])$. Suppose for instance that expressions e and $\beta_i(f)$ happen to satisfy $[\![e]\!]_{\overrightarrow{\Omega'}} = B'_i[\![f]\!]_{\overrightarrow{\Omega'}}$ in a particular representation $\overrightarrow{\Omega'}$. If another belief space $\overrightarrow{\Omega}$ distinguishes these expressions (i.e., $[\![e]\!]_{\overrightarrow{\Omega}} \neq B_i[\![f]\!]_{\overrightarrow{\Omega}}$), it follows in the belief space $\overrightarrow{\Omega^*}$ that $[\![e]\!]_{\overrightarrow{\Omega^*}} \neq B^*_i[\![f]\!]_{\overrightarrow{\Omega^*}}$.

Put differently, if $[\![e]\!]_{\overrightarrow{\Omega^*}} = B^*_i[\![f]\!]_{\overrightarrow{\Omega^*}}$, then it is always the case in any belief space $\overrightarrow{\Omega}$ that $[\![e]\!]_{\overrightarrow{\Omega}} = B_i[\![f]\!]_{\overrightarrow{\Omega}}$. Thus, the space $\overrightarrow{\Omega^*}$ makes the minimum possible assumptions on how the players' interactive beliefs about nature states are modeled, or how expressions $\mathcal{L}$ (i.e., nature states and interactive beliefs) are interpreted and identified (recall the discussions in the fifth step of the construction of $\overrightarrow{\Omega^*}$ in Section 3).

Proposition 2 provides a formal sense in which reasoning in a smaller belief space can be extended to the terminal space. Section 6.1 shows that players' rationality (and common belief in rationality) are well-defined in every qualitative-belief space. If players are rational at some state of some belief space, then they are rational at the corresponding state of the terminal space. Moving on to probabilistic beliefs, Appendix D.1 studies correlated equilibria. One epistemic characterization is Bayesian rationality: a probabilistic-belief space in which each player is Bayesian rational at every state (and thus players commonly believe their Bayesian rationality) is a correlated equilibrium (Aumann, 1987). Within such category of probabilistic-belief spaces, Bayesian rationality is valid. A terminal space exists, and at every state of the terminal space, players are Bayesian rational (and hence they commonly believe their Bayesian rationality).

In the context of Bayesian games, Friedenberg and Meier (2017) nevertheless show that a (smaller) type space may have a Bayesian equilibrium that cannot be extended to a canonical (larger) type space. In the above applications, players interactively reason only about their actions of an underlying strategic game (i.e., no interactive reasoning about payoffs), that is, a belief space $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta\rangle$ encodes the "strategy" choices Θ as part of the primitives. Appendix D.1 shows that if a correlated equilibrium as a probabilistic-belief space is non-redundant and minimal then the underlying state space Ω of the correlated equilibrium can be replaced by a subspace $D(\Omega)$ of the terminal space which consists of players' belief hierarchies. Also, the "strategies" Θ are always assumed to be measurable.

To sum up this subsection, Bjorndahl, Halpern, and Pass (2013) study psychological games in which players' interactive qualitative beliefs themselves enter into their preferences. Bjorndahl, Halpern, and Pass (2013) introduce the language which describes players' strategy choices and their interactive beliefs (in the framework of this paper, $\mathcal{L}_{\aleph_0}(\mathcal{A}_{\aleph_0}(\mathcal{S}))$), and a player's utility function is defined on the set of situations, where a situation is a maximally satisfiable set of formulas from the language. By Proposition 2 (3), a maximally satisfiable set of expressions, together with a corresponding nature state (i.e., the players' strategy choices), constitute a state of the terminal space $\overrightarrow{\Omega^*}$. Hence, each player i's payoff function is defined over the terminal space Ω^* (as standard psychological games consider probabilistic beliefs or probabilistic conditional beliefs, Section 5 establishes a terminal space for probabilistic beliefs, and Appendix C.1 establishes a terminal space for probabilistic conditional beliefs). For solution concepts, Bjorndahl, Halpern, and Pass (2013) define, for example, a rationalizable solution concept by its epistemic characterization. As discussed above, Section 6.1 studies an implication of common belief in rationality.

4.3 Self-Reference

One of the conceptual issues in modeling players' beliefs on a state space since Aumann (1976) is the interpretation of each state as a "full" description of players' beliefs held at the state. While each state is supposed to be a full description of players' beliefs, their beliefs are defined on the states.

Each state ω^* of the terminal space $\overrightarrow{\Omega^*}$ contains expressions that hold at ω^* (i.e., $e \in_1 \omega^*$ iff $\omega^* \in [e]$) as well as the corresponding nature state $s = \Theta^*(\omega^*)$. I examine how each ω^* describes the nature state and players' beliefs in two formal senses.

The first is its logical structure, following Aumann (1999). Each ω^* is *coherent*: if ω^* contains an expression e then it does not contain $(\neg e)$. Each ω^* is *complete*: if ω^* does not contain e then it contains $(\neg e)$.[31] Every ω^* is logically closed. Especially, it contains such expressions as S that hold in any belief space of the given class.

Proposition 3 (Logical Properties of Each State)**.** *Fix $\omega^* \in \Omega^*$, $e \in \mathcal{L}$, $f \in \mathcal{L}$, and $\mathcal{E} \in \mathcal{P}(\mathcal{L})$ with $|\mathcal{E}| < \kappa$.*

1. *Coherence and Completeness: $e \notin_1 \omega^*$ iff $(\neg e) \in_1 \omega^*$.*
2. *Closure under Implication: If $e \in_1 \omega$ and $(e \to f) \in_1 \omega$, then $f \in_1 \omega$.*
3. *Closure under Conjunction: $\bigwedge \mathcal{E} \in_1 \omega^*$ iff $e \in_1 \omega^*$ for all $e \in \mathcal{E}$.*

Second, the terminal space $\overrightarrow{\Omega^*}$ resolves self-reference: players' beliefs are defined on states while states are supposed to describe the world. Recall the players' beliefs at each state ω^* are built within ω^* itself: $\omega^* \in B_i^*([e])$ iff $\beta_i(e) \in_1 \omega^*$. Since each ω^* satisfies the logical requirement postulated in Proposition 3 and since B_i^* inherits the properties of beliefs assumed in a given category, the following corollary shows that the players' beliefs at each state are encoded within the state itself.

Corollary 1 (Beliefs within Each State)**.** *Fix $\omega^* \in \Omega^*$, $i \in I$, and $e \in \mathcal{L}$.*

[31] In relation to the semantic notions studied in Section 4.2, a satisfiable set is maximally satisfiable iff it is coherent and complete.

1. *Exactly one of* $\beta_i(e) \in_1 \omega^*$ *or* $(\neg\beta_i)(e) \in_1 \omega^*$ *holds.*

2. *At least one of* $\beta_i(e) \in_1 \omega^*$, $\beta_i(\neg e) \in_1 \omega^*$, *or* $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e) \in_1 \omega^*$ *holds. Exactly one of them always holds (for any* (e, ω^*)*) iff Consistency holds for* i.

3. *Exactly one of* $\beta_i(e) \in_1 \omega^*$, $(\neg\beta_i)(e) \wedge \beta_i(\neg\beta_i)(e) \in_1 \omega^*$, *or* $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg\beta_i)(e) \in_1 \omega^*$ *holds. The third condition never occurs (for any* (e, ω^*)*) iff Negative Introspection holds for* i.

In the first part of Corollary 1, each ω^* fully describes i's beliefs: for any $e \in \mathcal{L}$, the state ω^* contains exactly one of the two expressions denoting "i believes e" or "i does not believe e." The second and third parts characterize how Ω^* encodes such properties as Consistency and Negative Introspection. Put differently, the space Ω^* itself encodes whether such assumptions on beliefs are made. If player i's beliefs violate Consistency, then some state ω^* contains $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e)$ for some e, i.e., it is written in the state ω^* that player i is "ignorant" of e. In fact, $\omega^* \in (\neg B_i^*)([e]) \cap (\neg B_i^*)(\neg[e])$. Likewise, if player i's beliefs violate Negative Introspection, then some state ω^* contains $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg\beta_i)(e)$ for some e, i.e., it is written in the state ω^* that player i is "unaware" of e. In fact, $\omega^* \in (\neg B_i^*)([e]) \cap (\neg B_i^*)(\neg B_i^*)([e])$. Corollary 1 is related to some consistency conditions of Gilboa (1988) for a state to fully describe the world. The next subsection characterizes how states in Ω^* encode all the properties of beliefs specified in Definition 2 by demonstrating that properties imposed on player i's beliefs by the outside analysts are expressed within Ω^*.

4.4 Largest Coherent Set of Descriptions Ω^*

Each state of the terminal belief space $\overrightarrow{\Omega^*}$ is a maximally satisfiable set of expressions (together with a state of nature).[32] This subsection instead explicitly characterizes the space $\overrightarrow{\Omega^*}$ as the largest *coherent set of descriptions.*

Call a subset Ω of $\mathcal{P}(S \sqcup \mathcal{L})$ a coherent set of descriptions if: (i) each $\omega \in \Omega$ is a complete and coherent set of expressions together with a unique nature state; (ii) the set Ω as a whole induces the players' beliefs in a well-defined manner; and (iii) each ω reflects assumptions on players' beliefs. This characterization holds for any infinite regular cardinal κ and assumptions on beliefs. Formally:

Definition 9 (Coherent Set of Descriptions)**.** *Call a subset* Ω *of* $\mathcal{P}(S \sqcup \mathcal{L})$ *a coherent set of descriptions if it satisfies the following conditions.*

[32]When infinitary operations are allowed (i.e., $\kappa > \aleph_0$), states in the terminal κ-belief space would generally be different from the collection of "maximally consistent" sets of expressions (together with a state of nature) in some syntax system, which are often used to prove the "completeness" theorem of the syntax system. See Karp (1964) for the discrepancy between a semantic notion of satisfiability and a syntactic notion of maximal consistency when infinitary operations are allowed. See also Heifetz (1997), Meier (2012), Moss and Viglizzo (2004, 2006), and Zhou (2010).

1. *Each $\omega \in \Omega$ satisfies the following.*

   (a) *Nature State: There is a unique $s \in S$ with $s \in_0 \omega$. Moreover, $E \in_1 \omega$ for all $E \in \mathcal{A}_\kappa(\mathcal{S})$ with $s \in E$.*

   (b) *Closure under Implication: If $e \in_1 \omega$ and $(e \to f) \in_1 \omega$ then $f \in_1 \omega$.*

   (c) *Closure under Conjunction: For any $\mathcal{E} \in \mathcal{P}(\mathcal{L})$ with $|\mathcal{E}| < \kappa$, $\bigwedge \mathcal{E} \in_1 \omega$ iff $e \in_1 \omega$ for all $e \in \mathcal{E}$.*

   (d) *Coherency: For each $e \in \mathcal{L}$, if $(\neg e) \in_1 \omega$ then $e \notin_1 \omega$.*

   (e) *Completeness: For each $e \in \mathcal{L}$, if $e \notin_1 \omega$ then $(\neg e) \in_1 \omega$.*

2. *The set Ω satisfies the following ((2b) and (2c) depend on assumptions on beliefs).*

   (a) *Equivalence: If $e, f \in \mathcal{L}$ satisfy $(e \leftrightarrow f) \in_1 \omega$ for all $\omega \in \Omega$, then $(\beta_i(e) \leftrightarrow \beta_i(f)) \in_1 \omega$ for all $\omega \in \Omega$.*

   (b) *Let Monotonicity be assumed for player i. If $e, f \in \mathcal{L}$ satisfy $(e \to f) \in_1 \omega$ for all $\omega \in \Omega$, then $(\beta_i(e) \to \beta_i(f)) \in_1 \omega$ for all $\omega \in \Omega$.*

   (c) *Let the Kripke property be assumed for player i. Then, $\beta_i(e) \in_1 \omega$ for any $(e, \omega) \in \mathcal{L} \times \Omega$ with the following condition: if $\omega' \in \Omega$ satisfies $f \in_1 \omega'$ for all $f \in \mathcal{L}$ with $\beta_i(f) \in_1 \omega$, then $e \in_1 \omega'$.*

3. *Depending on assumptions on beliefs, each $\omega \in \Omega$ contains any instance of the following expressions.*

   (a) *Necessitation: $\beta_i(S)$.*

   (b) *Non-empty λ-Conjunction: $((\bigwedge_{e \in \mathcal{E}} \beta_i(e)) \to \beta_i(\bigwedge \mathcal{E}))$ with $0 < |\mathcal{E}| < \lambda$.*

   (c) *Consistency: $(\beta_i(e) \to (\neg \beta_i)(\neg e))$.*

   (d) *Truth Axiom: $(\beta_i(e) \to e)$.*

   (e) *Positive Introspection: $(\beta_i(e) \to \beta_i \beta_i(e))$.*

   (f) *Negative Introspection: $((\neg \beta_i)(e) \to \beta_i(\neg \beta_i)(e))$.*

First, Condition (1a) states that each state of the world ω describes a corresponding nature state s in a well-defined manner. Also, the state ω contains those nature events $E \in \mathcal{A}_\kappa(\mathcal{S})$ that are true at s (i.e., $s \in E$).[33] Conditions (1b) through (1e) are logical requirements (including coherency and completeness) on each state of the world (recall Proposition 3 for each $\omega^* \in \Omega^*$).

Second, Condition (2a) requires that if two expressions e and f are equivalent in the sense that every ω contains $(e \leftrightarrow f)$, then expressions $\beta_i(e)$ and $\beta_i(f)$ are

[33]By Condition (1d), Condition (1a) implies that, for a unique $s \in S$ with $s \in_0 \omega$ and for any $E \in \mathcal{A}_\kappa(\mathcal{S})$, $E \in_1 \omega$ iff $s \in E$. Especially, $S \in_1 \omega$ and $\emptyset \notin_1 \omega$.

equivalent in the same sense. This condition allows one to define the players' belief operators in a way such that if two expressions e and f correspond to the same event then the events associated with the beliefs in e and f are the same.

Third, Conditions (2b), (2c), and (3) describe how states Ω describe properties of beliefs. Corollary 1 has provided related characterizations for Consistency and Negative Introspection for Ω^*. Assumptions on beliefs are encoded within Ω itself in that the resulting belief operators satisfy given assumptions.

Now, I restate the terminal space Ω^* as the largest set of coherent descriptions.

Theorem 2 (Ω^* as Largest Coherent Set of Descriptions)**.** *The set Ω^* constructed in Section 3 is the largest coherent set of descriptions: for any set Ω of coherent descriptions, there is a (non-redundant and minimal) belief space $\overrightarrow{\Omega}$ such that its description map $D_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map and thus $\Omega \subseteq \Omega^*$.*

Two remarks are in order. First, Theorem 2 restates the terminal space $\overrightarrow{\Omega^*}$ as the largest set of coherent descriptions irrespective of properties of beliefs. In a particular case in which the players with fully-introspective knowledge reason about their finite-level interactive knowledge, Theorem 2 formally proves that Aumann (1999)'s canonical space can be taken as a universal space within the particular class, contrary to the conjecture of Heifetz and Samet (1998b, Section 6).

Second, the two constructions of the terminal space $\overrightarrow{\Omega^*}$ (in Sections 3 and 4.4) are analogous to two constructions of a terminal type space: one (Heifetz and Samet, 1998b) in which a terminal type space consists of belief hierarchies induced by some type of some type space, and the other (Brandenburger and Dekel, 1993) in which a terminal type space is the largest set consisting of coherent finite-level belief hierarchies and satisfying a coherency condition on the set that induces players' type mappings (which leads to "common certainty of coherency").[34] In the construction in Section 3, each state in $\overrightarrow{\Omega^*}$ is a belief hierarchy (precisely, a description) induced by some state of some belief space. In the construction in this subsection, each state in $\overrightarrow{\Omega^*}$ is a coherent belief hierarchy (precisely, a coherent description that satisfies Definition 9 (1) and (3)), and Ω^* is the largest set of coherent descriptions that induces (coherent) beliefs on itself (Definition 9 (2), i.e., Ω^* is "closed under common certainty of coherency").

[34]Technically, the definition of coherency here is different from the one in the type space literature. In Brandenburger and Dekel (1993), a belief hierarchy (consisting of all finite-level beliefs) is coherent if no finite levels of beliefs contradict with each other (by their topological assumption, such coherent belief hierarchies extend to countable levels). Here, each belief hierarchy is of depth $\overline{\kappa}$, and the coherency condition requires no different levels of a belief hierarchy to contradict with each other.

5 Applications to Other Forms of Beliefs

The construction of a terminal belief space in Section 3 has demonstrated that a terminal belief space exists when the players' beliefs are represented by their belief operators and when an infinite regular cardinal κ determines the possible depth of players' reasoning through the restriction on the domain. Hence, the framework of this paper (especially, the existence and characterizations of a terminal belief space) applies to richer forms of beliefs such as probabilistic beliefs because probabilistic beliefs are represented by p-belief operators on a κ-algebra.

Section 5.1 constructs a terminal space for countably-additive, finitely-additive, and non-additive (not-necessarily-additive) beliefs. Conceptually, I establish the existence of a terminal belief space under the same condition (i.e., the domain specification) irrespective of whether beliefs are probabilistic or qualitative (or knowledge). Technically, the framework nests, for example, Heifetz and Samet (1998b), Meier (2006), and Pintér (2012), and establishes the existence of a terminal non-additive belief space irrespective of any continuity property on beliefs.[35] Moreover, in the terminal countably-additive belief space, a belief hierarchy consisting of all the finite levels of beliefs uniquely extends to the limit belief hierarchy.

While the main text focuses on a terminal probabilistic-belief space, the Supplementary Appendix discusses further applications. Appendix C.1 discusses a terminal space for conditional probability systems (CPSs) (Battigalli and Siniscalchi, 1999; Guarino, 2017) using conditional p-belief operators (Di Tillio, Halpern, and Samet, 2014). In Appendix C.2, players' knowledge and qualitative beliefs are indexed by time as in Battigalli and Bonanno (1997). Note that one can also combine knowledge and probabilistic beliefs as in Meier (2008). Appendix C.3 briefly discusses further possible applications, namely, terminal knowledge-unawareness, preference, and expectation spaces.

5.1 Terminal Probabilistic-Belief Space

I formulate a probabilistic-belief space in terms of p-belief operators using the equivalence between a type mapping and p-belief operators established by Samet (2000). To accommodate countable-level interactive probabilistic-beliefs, throughout this subsection, let $\kappa = \aleph_1$. Denote by $\Delta(\Omega)$ the set of countably-additive probability measures on an $\aleph_1$-algebra $(\Omega, \mathcal{D})$. Let $\mathcal{D}_\Delta$ be the $\aleph_1$-algebra on $\Delta(\Omega)$ generated by $\{\{\mu \in \Delta(\Omega) \mid \mu(E) \geq p\} \in \mathcal{P}(\Delta(\Omega)) \mid (E, p) \in \mathcal{D} \times [0, 1]\}$ as in Heifetz and Samet (1998b). Then:

[35] Heifetz and Samet (1998b) employ a product of players' type spaces while this paper does a single non-product state space Ω (if each player is always "certain" of her beliefs, then the non-product terminal belief space is isomorphic to the product of nature states and players' type spaces; see, for example, Mertens and Zamir, 1985). In the literature, the latter non-product structure is referred to as a "belief space" (Mertens and Zamir, 1985). These remarks also apply to a terminal space for conditional probability systems (CPSs) in Appendix C.1.

Definition 10 (Probabilistic-Belief Space)**.** *A probabilistic-belief space of I on $(S, \mathcal{S})$ is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (B_i^p)_{(i,p) \in I \times [0,1]}, \Theta \rangle$ with the following properties.*

1. *$(\Omega, \mathcal{D})$ is an $\aleph_1$-algebra and the map $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_{\aleph_1}(\mathcal{S}))$ is measurable.*

2. *Player i's p-belief operators $B_i^p : \mathcal{D} \to \mathcal{D}$ satisfy the properties below. For each $E \in \mathcal{D}$, $B_i^p(E)$ is the event that player i believes E with probability at least p (i.e., she p-believes E).*

   (a) *Non-Negativity: $B_i^0(\cdot) = \Omega$.*

   (b) *p-Regularity: If $p_n \uparrow p$ then $B_i^{p_n}(\cdot) \downarrow B_i^p(\cdot)$.*

   (c) *Monotonicity: If $E \subseteq F$ then $B_i^p(E) \subseteq B_i^p(F)$.*

   (d) *Normalization: $B_i^1(\Omega) = \Omega$.*

   (e) *Super-additivity: $B_i^p(E \cap F) \cap B_i^q(E \cap (\neg F)) \subseteq B_i^{p+q}(E)$ for $p + q \leq 1$.*

   (f) *Sub-additivity: $(\neg B_i^p)(E) \cap (\neg B_i^q)(F) \subseteq (\neg B_i^{p+q})(E \cup F)$ for $p + q \leq 1$.*

   (g) *Continuity-from-above: If $E_n \downarrow E \in \mathcal{D}$ then $B_i^p(E_n) \downarrow B_i^p(E)$.*

   (h) *Continuity-from-below: If $E_n \uparrow E \in \mathcal{D}$ then $B_i^p(E) = \bigcap_{r \in \mathbb{N}: p - \frac{1}{r} \geq 0} \bigcup_{n \in \mathbb{N}} B_i^{p - \frac{1}{r}}(E_n)$.*

   (i) *Certainty-of-Beliefs: If $[t_{B_i}(\omega)] \subseteq E$, then $\omega \in B_i^1(E)$, where*

$$[t_{B_i}(\omega)] := \Big(\bigcap_{\substack{(p,E) \in [0,1] \times \mathcal{D} \\ \omega \in B_i^p(E)}} B_i^p(E) \Big) \cap \Big(\bigcap_{\substack{(p,E) \in [0,1] \times \mathcal{D} \\ \omega \in (\neg B_i^p)(E)}} (\neg B_i^p)(E) \Big).$$

In a probabilistic-belief space, player i's p-belief operators induce her *type mapping* $t_{B_i} : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, a measurable map defined by $t_{B_i}(\omega)(E) := \sup\{p \in [0,1] \mid \omega \in B_i^p(E)\}$ for each $(\omega, E) \in \Omega \times \mathcal{D}$. At each state ω, a countably-additive probability measure $t_{B_i}(\omega)$ dictates i's beliefs at that state. Since t_{B_i} is measurable, $B_i^p(E) = \{\omega \in \Omega \mid t_{B_i}(\omega)(E) \geq p\} \in \mathcal{D}$ for each $E \in \mathcal{D}$.

Conditions (2), slightly different from Samet (2000), axiomatize type mappings. Conditions (2a) and (2b) guarantee that the map t_{B_i} is well-defined, irrespective of properties of probabilistic beliefs. By (2c), each $t_{B_i}(\omega)$ is monotonic (i.e., $E \subseteq F$ implies $t_{B_i}(\omega)(E) \leq t_{B_i}(\omega)(F)$). Condition (2d) is a normalization: $t_{B_i}(\cdot)(\Omega) = 1$. Thus, (2a)-(2d) yield non-additive beliefs (or capacities).

By (2e), each $t_{B_i}(\omega)$ is super-additive: $t_{B_i}(\omega)(E \cap F) + t_{B_i}(\omega)(E \cap (\neg F)) \leq t_{B_i}(\omega)(E)$. Note that (2a) and (2e) imply (2c). By (2f), each $t_{B_i}(\omega)$ is sub-additive: $t_{B_i}(\omega)(E \cup F) \leq t_{B_i}(\omega)(E) + t_{B_i}(\omega)(F)$. Thus, (2a)-(2f) yield finitely-additive beliefs. By (2g) or (2h), a finitely-additive probability measure $t_{B_i}(\omega)$ becomes countably additive. I have presented both (2g) and (2h) to accommodate non-additive beliefs.

Condition (2i) is the introspective property that requires player i to be certain of her own beliefs. The set $[t_{B_i}(\omega)]$ consists of states ω' that player i cannot distinguish from ω based on her probabilistic beliefs, as $[t_{B_i}(\omega)] = \{\omega' \in \Omega \mid t_{B_i}(\omega') = t_{B_i}(\omega)\}$.

Thus, player i is certain of her beliefs in that she believes E with probability one if E is implied by $[t_{B_i}(\omega)]$. Especially, $B_i^p(E) \subseteq B_i^1 B_i^p(E)$ and $(\neg B_i^p)(E) \subseteq B_i^1(\neg B_i^p)(E)$ hold: if player i p-believes (does not p-believe) an event E, then she believes with probability one that she p-believes (does not p-believe) the event E.

A *(probabilistic belief) morphism* from $\overrightarrow{\Omega}$ to $\overrightarrow{\Omega'}$ is a measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ satisfying: (i) $\Theta' \circ \varphi = \Theta$; and (ii) $B_i^p(\varphi^{-1}(\cdot)) = \varphi^{-1}(B_i'^p(\cdot))$ for all $(i,p) \in I \times [0,1]$. A probabilistic-belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S})$ is *terminal* if, for any probabilistic-belief space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{S})$, there is a unique morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. By Theorem 1, a terminal probabilistic-belief space exists. One can also extend it to various notions of beliefs by dropping corresponding conditions in Definition 10 (2).

Corollary 2 (Terminal Probabilistic-Belief Space)**.** *There exists a terminal probabilistic-belief space* $\overrightarrow{\Omega^*}$ *of* I *on* $(S, \mathcal{S})$.

In the construction of the terminal belief space for a generic class of belief spaces in Section 3, recall that the set $\mathcal{L} = \mathcal{L}_\kappa^I(\mathcal{A}_\kappa(\mathcal{S}))$ of syntactic formulas describing nature and interactive beliefs is infinitary when an infinite regular cardinal κ satisfies $\kappa > \aleph_0$. Hence, each belief hierarchy in the terminal space $\overrightarrow{\Omega^*}$ in Corollary 2 consists of all countable-level interactive beliefs.

Here, I show that, as in the literature on type spaces, the terminal probabilistic-belief space is also constructed by taking care only of finite-level interactive beliefs. By the continuity (countable-additivity) of beliefs, finite-level interactive beliefs uniquely extend to countable levels. For the rest of this subsection, let $(\kappa, \lambda) = (\aleph_1, \aleph_0)$. Recalling Definition 5 and Remark 1, consider λ-expressions $\mathcal{L}_\lambda^I(\mathcal{A}_\kappa(\mathcal{S}))$: syntactic formulas that express nature and finite-level interactive beliefs. While the players reason about countable-level interactive beliefs, the language available to the players is finite. I show:

Proposition 4 (Extesion of Finitary Language)**.** $\mathcal{D}^* = \sigma(\{[e] \in \mathcal{D}^* \mid e \in \mathcal{L}_\lambda^I(\mathcal{A}_\kappa(\mathcal{S}))\})$.

For any probabilistic-belief space $\overrightarrow{\Omega}$, there is a unique morphism $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. In contrast, one could also construct a terminal probabilistic-belief space $\overrightarrow{\Omega^{**}}$ in a way such that a unique morphism $h : \overrightarrow{\Omega} \to \overrightarrow{\Omega^{**}}$ associates, with each state $\omega \in \Omega$, the profile of the corresponding nature state and the players' belief hierarchies (each player i's first-order beliefs $t_{B_i}(\omega) \circ \Theta^{-1}$ over S, each player's second-order beliefs, and so on). Since both spaces are terminal, there exists a unique isomorphism $\varphi : \overrightarrow{\Omega^*} \to \overrightarrow{\Omega^{**}}$ such that $h = \varphi \circ D$. Hence, D indeed yields the players' belief hierarchies.

The proposition implies that, once the outside analysts specify finite-level interactive beliefs by the language $\mathcal{L}_\lambda^I(\mathcal{A}_\kappa(\mathcal{S}))$, finite-level interactive beliefs uniquely extend to countable levels. Applying the construction of a generic terminal space in Section 3 to the current context, Lemma 1 implies that $(\Omega^*, \{[e] \in \mathcal{D}^* \mid e \in \mathcal{L}_\lambda^I(\mathcal{A}_\kappa(\mathcal{S}))\})$ is a λ-algebra. One can define each player's p-belief operator B_i^{*p} from $\{[e] \in \mathcal{D}^* \mid e \in \mathcal{L}_\lambda^I(\mathcal{A}_\kappa(\mathcal{S}))\}$ into itself. By countable additivity, each B_i^{*p} admits a unique extension to $\mathcal{D}^* = \sigma(\{[e] \in \mathcal{D}^* \mid e \in \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))\})$. Corollary 2 and Proposition 4 imply that the question of whether there exists a terminal space (Corollary 2) and that of whether the terminal space can be constructed by keeping track (only) of finite-level reasoning (Proposition 4) can be separately asked.

6 Game-Theoretic Applications

The key insight behind the construction of a terminal belief space is to specify possible depth of reasoning. Once the terminal space is constructed, any logical statement captured in a particular space can be extended to the terminal space. This section shows that the specification of possible depth of reasoning is also important in game-theoretic applications. This section also demonstrates that players' rationality and common belief in rationality captured in a smaller space can be extended to a terminal space in both probabilistic and non-probabilistic contexts.

Section 6.1 studies qualitative beliefs, and characterizes iterated elimination of strictly dominated actions (IESDA) as the implication of common belief in rationality in a terminal qualitative-belief space. The subsection also shows that, for any given ordinal, there exists a game in which the players need to reason beyond the given ordinal level to reach the unique prediction under IESDA. Hence, in a situation in which the players reason about their actions in a given game, the outside analysts would need to fix an appropriate level $\overline{\kappa}$ with which to accommodate all possible reasoning about the given game.

While the main text focuses on qualitative belief in game-theoretic applications, Appendix D studies correlated equilibria. Since a correlated equilibrium itself is a belief space, if it is non-redundant and minimal then it is embedded into a subspace of the terminal space. Put differently, the underlying state space of the correlated equilibrium (i.e., the correlating device) can be replaced with players' belief hierarchies about their play.

6.1 Implication of Common Belief in Rationality

This subsection incorporates common belief, and studies the solution concept of iterated elimination of strictly dominated actions (IESDA) in a terminal belief space as an implication of common belief in rationality for a game with ordinal payoffs and qualitative beliefs. Section 6.1.1 demonstrates that, in a category of belief spaces in which common belief is well-defined, a terminal space exists (i.e., common belief is well-defined in the terminal space). Section 6.1.2 shows: within a class of (qualitative-)belief spaces with common belief, (i) players' rationality is well-defined in the terminal space, and (ii) common belief in rationality in any given belief space is preserved in the terminal space. Especially, the terminal space itself characterizes the solution concept of IESDA as an implication of common belief in rationality. Finally, Section 6.1.3 provides an example of a strategic game and a belief space in which the players need to reason beyond a given ordinal level $\overline{\kappa}$ to pin down the unique prediction under IESDA. Hence, the outside analysts would need to specify an infinite regular cardinal κ, which determines depth of reasoning, when players possess qualitative belief about a given space of nature states, depending on nature states.

6.1.1 Common Belief

I show that a terminal space exists in a class of belief spaces with common belief. To that end, I incorporate the notion of common belief, irrespective of a choice of κ and assumptions on beliefs, following Fukuda (2020). The definition of common belief does not resort to the chain of mutual beliefs. Thus, one can analyze players who fail logical reasoning (e.g., Monotonicity or Non-empty λ-Conjunction) or players who reason only about finite levels of interactive beliefs (i.e., $\kappa = \aleph_0$).

Fix a non-empty set I of players, and let κ be an infinite regular cardinal with $\kappa > |I|$. By this assumption, in any κ-belief space $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta \rangle$, define the *mutual belief operator* $B_I : \mathcal{D} \to \mathcal{D}$ by $B_I(\cdot) = \bigcap_{i \in I} B_i(\cdot)$.

Call an event E a *common basis* if everybody believes any logical implication of E whenever E is true: $E \subseteq F$ implies $E \subseteq B_I(F)$ (Fukuda, 2020). If the mutual belief operator B_I satisfies Monotonicity, then E is a common basis if(f) it is publicly-evident: $E \subseteq B_I(E)$. Denote by $\mathcal{J}_I$ the collection of common bases. Now, an event E is *common belief* at a state ω if there is a common basis $F \in \mathcal{J}_I$ which is true at ω and which implies the mutual belief in E: $\omega \in F \subseteq B_I(E)$. If B_I satisfies Monotonicity, then this definition of common belief reduces to Monderer and Samet (1989).

The common belief in E at a state ω implies the chain of mutual beliefs in E at that state: at ω, everybody believes E (i.e., $\omega \in B_I(E)$), everybody believes that everybody believes E (i.e., $\omega \in B_I B_I(E)$), and so on *ad infinitum*. The converse holds (i.e., common belief reduces to the chain of mutual beliefs) when, for example, B_I satisfies Monotonicity and Non-empty $\aleph_1$-Conjunction as in a possibility correspondence model (Fukuda, 2020).[36]

A *κ-belief space with a common belief operator* is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, C, \Theta \rangle$ such (i) that $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta \rangle$ is a κ-belief space (satisfying given assumptions on beliefs) and (ii) that $C : \mathcal{D} \to \mathcal{D}$ satisfies $C(E) = \max\{F \in \mathcal{J}_I \mid F \subseteq B_I(E)\}$ for each $E \in \mathcal{D}$, where "max" is taken with respect to the set inclusion. I show a terminal space exists within the class of κ-belief spaces with a common belief operator irrespective of κ and assumptions on beliefs.

Corollary 3 (Terminal Belief Space with a Common Belief Operator)**.** *There exists a terminal κ-belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S})$ with a common belief operator.*

[36]Also, in a probabilistic-belief space (recall Definition 10), let $B_I^p(\cdot) := \bigcap_{i \in I} B_i^p(\cdot)$ be the mutual p-belief operator. The common p-belief operator C^p is well-defined and coincides with the iteration of mutual p-beliefs: $C^p(\cdot) = \bigcap_{n \in \mathbb{N}} (B_I^p)^n(\cdot)$, as $(B_I^p)_{n \in \mathbb{N}}$ is decreasing.

Any property of beliefs satisfied in a belief space $\overrightarrow{\Omega}$ holds in $\overrightarrow{D(\Omega)} = \langle (D(\Omega), \mathcal{D}^* \cap D(\Omega)), (B'_i)_{i \in I}, \Theta^*|_{D(\Omega)} \rangle$ (defined in Section 3). If $\overrightarrow{\Omega}$ admits a common belief operator, then so does $\overrightarrow{D(\Omega)}$. If Necessitation holds for every player, $\overrightarrow{D(\Omega)}$ is *belief-closed* (Mertens and Zamir, 1985): $B'_i(D(\Omega)) = D(\Omega)$ for all $i \in I$. Consequently, $D(\Omega)$ is commonly believed in $\overrightarrow{D(\Omega)}$.

6.1.2 Common Belief in Rationality

To study an implication of common belief in rationality, I move on to formalizing rationality. I show: within a given class of belief spaces with a common belief operator, (i) players' rationality is well-defined in the terminal space; and (ii) common belief in rationality in any given belief space is preserved in the terminal space.

Fix a strategic game $\Gamma := \langle (A_i)_{i \in I}, (\succcurlyeq_i)_{i \in I} \rangle$ with ordinal payoffs: each A_i is a player i's (non-empty) action set, and each $\succcurlyeq_i$ is player i's (complete and transitive) preference relation on $A := \prod_{i \in I} A_i$. Denote by $\sim_i$ and $\succ_i$ the indifference and strict preference relations, respectively. Since the players reason about their actions, let $S = A$. Fix an infinite regular cardinal κ with $\max(|I|, |A|) < \kappa$.[37] In order for each player to be able to reason about her own actions, assume $\{a_i\} \times A_{-i} \in \mathcal{S}$ for each $i \in I$ and $a_i \in A_i$. Since $\kappa > |A|$, the assumption amounts to $\mathcal{A}_\kappa(\mathcal{S}) = \mathcal{P}(S)$.

In a κ-belief space (with a common belief operator) $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, C, \Theta \rangle$, the measurable map Θ is decomposed into $\Theta = (\Theta_i)_{i \in I}$, where $\Theta_i = \pi_i \circ \Theta$ and $\pi_i : A \to A_i$ is the projection. The mapping Θ_i is interpreted as player i's (behavioral) strategy: it associates, with each state ω, the corresponding action $\Theta_i(\omega) \in A_i$. For each $a_i \in A_i$, $\Theta_i^{-1}(\{a_i\}) = [\![\{a_i\} \times A_{-i}]\!]_{\overrightarrow{\Omega}} \in \mathcal{D}$ is the event that player i plays a_i. For each $(a'_i, a_i) \in A_i^2$, define $[a'_i \succ_i a_i] := \{a_{-i} \in A_{-i} \mid (a'_i, a_{-i}) \succ_i (a_i, a_{-i})\}$ and $[\![a'_i \succ_i a_i]\!]_{\overrightarrow{\Omega}} := (\Theta_{-i})^{-1}([a'_i \succ_i a_i]) \in \mathcal{D}$. The set $[\![a'_i \succ_i a_i]\!]_{\overrightarrow{\Omega}}$ is the event that player i strictly prefers a'_i to a_i given the other players' strategies.

Define the set RAT_i (or $\mathrm{RAT}_i^{\overrightarrow{\Omega}}$) of states at which player i is rational: $\mathrm{RAT}_i := \{\omega \in \Omega \mid \omega \in B_i [\![a'_i \succ_i \Theta_i(\omega)]\!]_{\overrightarrow{\Omega}}$ for no $a'_i \in A_i\}$ (e.g., Bonanno, 2008; Chen, Long, and Luo, 2007). Player i is rational at ω if, for no action a'_i, she believes playing a'_i is strictly better than $\Theta_i(\omega)$ given the opponents' strategies. It can be seen:

$$\mathrm{RAT}_i = \bigcap_{a_i \in A_i} ((\Theta_i^{-1}(\{a_i\}))^c \cup \bigcap_{a'_i \in A_i} (\neg B_i)([\![a'_i \succ_i a_i]\!]_{\overrightarrow{\Omega}})) \in \mathcal{D}.$$

Let $\mathrm{RAT}_I := \bigcap_{i \in I} \mathrm{RAT}_i \in \mathcal{D}$. Hence, in the terminal space $\overrightarrow{\Omega^*}$ (indeed, in any belief space), the players' rationality is well-defined.

If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism, then $\varphi^{-1}([\![a'_i \succ_i a_i]\!]_{\overrightarrow{\Omega'}}) = [\![a'_i \succ_i a_i]\!]_{\overrightarrow{\Omega}}$ and $\varphi^{-1}(\mathrm{RAT}_i^{\overrightarrow{\Omega'}}) = \mathrm{RAT}_i^{\overrightarrow{\Omega}}$. Especially, $D^{-1}(C^*(\mathrm{RAT}_I^{\overrightarrow{\Omega^*}})) = C(\mathrm{RAT}_I^{\overrightarrow{\Omega}})$: if the players

[37] Recalling the technical preliminaries in Section 2, one can always take the successor cardinal $(\max(|I|, |A|))^+$.

commonly believe their rationality at ω in a given belief space $\overrightarrow{\Omega}$, then so do they at $\omega^* = D(\omega)$ in the terminal space.

This implies that the terminal space $\overrightarrow{\Omega^*}$ itself can characterize the solution concept of IESDA as an implication of common belief in rationality. As discussed in the Introduction, common belief in rationality characterizes IESDA. To define a process of IESDA, for the game Γ, identify any subset $A' = \prod_{i \in I} A'_i$ of A with a (sub-)game. A process of *iterated elimination of strictly dominated actions (IESDA)* is an ordinal sequence of $A^\alpha = \prod_{i \in I} A_i^\alpha$ (with $|\alpha| \leq |A|$) defined as follows: (i) $A^0 = A$; (ii) for a non-zero ordinal α, if the game $\bigcap_{\beta < \alpha} A^\beta$ has strictly dominated actions, then A^α is obtained by eliminating *at least one* such action from $\bigcap_{\beta < \alpha} A^\beta$; and (iii) if the game $\bigcap_{\beta < \alpha} A^\beta$ does not have any strictly dominated action, then let $A^\alpha = \bigcap_{\beta < \alpha} A^\beta$. Since $(A^\alpha)_\alpha$ is a decreasing sequence, take the smallest ordinal α (with $|\alpha| \leq |A|$) with $A^\alpha = \bigcap_{\beta < \alpha} A^\beta$. An action profile $a \in A$ *survives* the process of IESDA if $a \in A^{\text{IESDA}} := A^\alpha$. Note that a process of IESDA may depend on the order of elimination (e.g., Chen, Long, and Luo, 2007; Dufwenberg and Stegeman, 2002).

Consider a category of κ-belief spaces (with $\kappa > \max(|I|, |A|)$) such that the players have correct common belief in rationality: $C(\text{RAT}_I) \subseteq \text{RAT}_I$. Take any process of IESDA. One characterization result states:[38]

1. For any κ-belief space $\overrightarrow{\Omega}$, if $\omega \in C(\text{RAT}_I)$ then $\Theta(\omega) \in A^{\text{IESDA}}$.

2. Conversely, for any $a \in A^{\text{IESDA}}$, there exist a κ-belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $a = \Theta(\omega)$ and $\omega \in C(\text{RAT}_I)$.

In the terminal κ-belief space $\overrightarrow{\Omega^*}$ (of the category in question):

3. For a κ-belief space $\overrightarrow{\Omega}$ with $\omega \in C(\text{RAT}_I^{\overrightarrow{\Omega}})$, $D_{\overrightarrow{\Omega}}(\omega) \in C^*(\text{RAT}_I^{\overrightarrow{\Omega^*}})$ and $\Theta^*(D_{\overrightarrow{\Omega}}(\omega)) = \Theta(\omega) \in A^{\text{IESDA}}$.

4. Conversely, for any $a \in A^{\text{IESDA}}$, there exists a state $\omega \in \Omega^*$ with $a = \Theta^*(\omega^*)$ and $\omega^* \in C^*(\text{RAT}_I^{\overrightarrow{\Omega^*}})$.

Part (3) states that for any state of a belief space in which the players have (correct) common belief in rationality, they have (correct) common belief in rationality and their actions survive any process of IESDA in the corresponding state of the terminal space. Part (4) says that, for any action profile that survives some process of IESDA, there exists a state in the terminal space at which the players, who commonly believe

[38] First, if some player's belief operator B_i satisfies Truth Axiom, then common belief satisfies Truth Axiom. Second, if each B_i satisfies the Kripke property and Consistency and if each player is certain of her own strategy (i.e., $\Theta_i^{-1}(\{a_i\}) \subseteq B_i(\Theta_i^{-1}(\{a_i\}))$), then (each player correctly believes her own rationality and consequently) the players have correct common belief in rationality. Third, an alternative characterization result in terms of rationality and common belief in rationality ($\text{RAT}_I \cap C(\text{RAT}_I)$) also holds when each B_i satisfies Monotonicity and Non-empty $\aleph_0$-Conjunction. See Fukuda (2020).

their rationality, take the given actions. Unlike Part (2), it is always sufficient to consider the terminal space. Hence, the terminal space can capture any prediction under the solution concept of IESDA as an implication of common belief in rationality.

6.1.3 Example of a Game with a Transfinite Process of IESDA

It has been shown that the terminal κ-belief space (with $\kappa > \max(|I|, |A|)$) can capture any prediction under IESDA as an implication of common belief in rationality. To conclude, I discuss an example of a two-player strategic game in which the players would need to engage in an arbitrary ordinal level of reasoning to reach a unique prediction under IESDA. Hence, for a given strategic game, the outside analysts need to choose an infinite regular cardinal κ (with $\kappa > \max(|I|, |A|)$) to exhaust predictions under IESDA about the given game in the terminal κ-belief space.

Define a strategic game $\langle (A_i)_{i \in I}, (u_i)_{i \in I} \rangle$ (for ease of exposition, in terms of payoff functions) as follows. Let α be a non-zero limit ordinal. Let $A_i := \alpha + 2$ (i.e., $A_i = \{0, 1, \ldots, \alpha, \alpha + 1\}$) be the set of actions available to $i \in I := \{1, 2\}$. Define i's payoff function $u_i : A_i \times A_{-i} \to \mathbb{R}$ as

$$u_i(a_i, a_{-i}) := \begin{cases} 0 & \text{if } a_i < a_{-i} \text{ or } a_i = a_{-i} \neq \alpha + 1 \\ 2 & \text{if } a_{-i} < a_i \neq \alpha + 1 \\ 1 & \text{if } a_i = \alpha + 1 \end{cases}.$$

Table 1 in Section 1.1 depicts $u_i(a_i, a_{-i})$ when α is the least infinite ordinal (i.e., the set of non-negative integers). Action $\alpha + 1$ always yields a payoff of 1 irrespective of the opponent's action. For any other action, player i obtains a payoff of 2 if her action a_i is (strictly) higher than the opponent's, and she obtains a payoff of 0 otherwise. Any process of IESDA yields a unique action profile $(a_1, a_2) = (\alpha + 1, \alpha + 1)$ (e.g., the process of eliminating every strictly dominated action at each step leads to the prediction after the $\alpha + 1$ round of elimination). Hence, the process requires the players to engage in a transfinite process of reasoning.[39]

To capture the reasoning processes of the players facing a strategic game such as the one given here, the outside analysts would need a model which allows for the players' transfinite reasoning processes. The analysts would need to consider a class of κ-belief spaces with $\kappa > |A|$ in which (correct) common belief in rationality always leads to a prediction under IESDA (or in which (correct) common belief in rationality in the terminal space exhausts the predictions under IESDA).

[39]When α is countable, one can consider mixed strategies over each player's action space. In such probabilistic-belief setting, even when the definition of strictly dominated actions allows for mixed strategies, the strategic game defined here calls for a transfinite process of reasoning (Lipman (1994) is the first paper that points out the need for transfinite levels of reasoning in a standard probabilistic-belief framework). In such probabilistic-belief setting, however, the outside analysts implicitly assume that the belief (type) space has a measurable structure, i.e., is an $\aleph_1$-belief space, and thus the belief space can accommodate any countable yet transfinite level of interactive reasoning.

For the rest of this subsection, to complement the argument that the outside analysts would need to consider a class of κ-belief spaces, I construct a belief space $\langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, C, \Theta\rangle$ in which, for the given limit ordinal α, common belief in rationality is attained exactly as the $\alpha + 1$ iterations of mutual beliefs in rationality: $C(\mathrm{RAT}_I) = \bigcap_{\beta:1\leq\beta\leq\alpha+1} B^{\beta}(\mathrm{RAT}_I)$.[40]

First, let $(\Omega, \mathcal{D}) := (\alpha+3, \mathcal{P}(\Omega))$. For any ordinal $\beta \leq \alpha+2$, denote by $[\beta, \alpha+2]$ the set of ordinals γ with $\beta \leq \gamma \leq \alpha + 2$. Second, define B_i as (i) $B_i(E) := E$ if $E \in \{\emptyset, \{\alpha + 2\}, \Omega\}$; and (ii) $B_i(E) := E \setminus \{\min(E)\}$ otherwise. Note that, since Ω is well-ordered, $\min(E)$ is well-defined for any non-empty subset E of Ω. Each player does not believe a contradiction of the form $\emptyset$, always believes a tautology of the form Ω, and believes $\{\alpha + 2\}$ at $\alpha + 2$. Each player believes any other event E at any state in E except at $\min E$. Note that each B_i satisfies Truth Axiom and Monotonicity but fails Non-empty $\aleph_0$-Conjunction (and consequently fails the Kripke property). Third, the common belief (or, common knowledge) operator $C : \mathcal{D} \to \mathcal{D}$, by definition, satisfies: (i) $C(\Omega) = \Omega$; (ii) $C(E) = \{\alpha + 2\}$ if $\alpha + 2 \in E$ and $E \neq \Omega$; and (iii) $C(E) = \emptyset$ if $\alpha + 2 \notin E$. Fourth, define $\Theta = (\Theta_i)_{i\in I}$ as: (i) $\Theta_i(\omega) := \alpha + 1$ if $\omega \neq 0$; and (ii) $\Theta_i(0) := 0$. Note that while action $\alpha + 1$ is the action that survives any process of IESDA, action 0 is the unique strictly dominated action in the original game. Each player is rational at each state except at 0: $\mathrm{RAT}_i = \Omega \setminus \{0\}$. For any ordinal β with $1 \leq |\beta| < \kappa$, $B_I^{\beta}(\mathrm{RAT}_I) = [\beta, \alpha + 2]$ if $\beta < \alpha + 1$, and $B_I^{\beta}(\mathrm{RAT}_I) = \{\alpha + 2\}$ if $\beta \geq \alpha + 1$. Hence, $C(\mathrm{RAT}_I) = \bigcap_{\beta:1\leq\beta\leq\alpha+1} B_I^{\beta}(\mathrm{RAT}_I)$. That is, the chain of mutual beliefs in rationality converges to common belief in rationality at the $\alpha+1$ round. Then, common belief in rationality $C(\mathrm{RAT}_I)$ captures IESDA: $\Theta(\alpha + 2) = (\alpha + 1, \alpha + 1) \in A^{\mathrm{IESDA}}$. Also, in a category of κ-belief spaces (where $\kappa > |A|$ and to which the example model belongs), the terminal κ-belief space "include" the example model.

[40] In any possibility correspondence model on a κ-algebra, each player's belief operator satisfies Monotonicity and Non-empty $\aleph_1$-Conjunction. Consequently, while $C(\mathrm{RAT}_I) = \bigcap_{\beta:1\leq\beta\leq\alpha+1} B_I^{\beta}(\mathrm{RAT}_I)$ always holds, the least infinite chain of mutual beliefs $\bigcap_{n\in\mathbb{N}} B_I^n(\mathrm{RAT}_I)$ already converges to common belief in rationality $C(\mathrm{RAT}_I)$. However, even in a possibility correspondence model in which the common belief operator always coincides with all the finite iterations of mutual beliefs, the model may need to be defined on a κ-algebra (with $\kappa > |A|$) to accommodate the $\alpha+1$ iterations of mutual beliefs to reflect processes of IESDA. In other words, common belief in rationality always leads to a prediction under IESDA irrespective of properties of beliefs in any κ-belief space (with $\kappa > |A|$). Here, I provide a belief space in which $C(\mathrm{RAT}_I) = \bigcap_{\beta:1\leq\beta\leq\alpha+1} B_I^{\beta}(\mathrm{RAT}_I)$ but $C(\mathrm{RAT}_I) \subsetneq \bigcap_{\beta:1\leq\beta\leq\alpha} B_I^{\beta}(\mathrm{RAT}_I)$ by dropping Non-empty $\aleph_1$-Conjunction. That is, once the players fail to satisfy certain logical properties such as Non-empty $\aleph_1$-Conjunction, then even the common belief operator may not be characterized by all the finite iterations of the mutual belief operator. The framework of this paper makes it possible to provide the epistemic characterization of IESDA when players' beliefs may fail to satisfy certain logical (or introspective) properties or when they face a general infinite game.

7 Comparison with the Previous Negative Results

Throughout this section, unless otherwise stated, fix an infinite regular cardinal κ. Recall that the framework of this paper admits the category of κ-knowledge spaces $\overrightarrow{\Omega}$ (in which $(\Omega, \mathcal{D})$ is a κ-algebra and) in which each B_i is induced by a partitional possibility correspondence (recall Definition 2 and its discussions). Theorem 1 constructs a terminal κ-knowledge space $\overrightarrow{\Omega^*}$ of this category (i.e., $(\Omega^*, \mathcal{D}^*)$ is a κ-algebra and each B_i^* is induced by a partitional possibility correspondence): for any κ-knowledge space $\overrightarrow{\Omega}$ in this category, there is a unique morphism $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$.[41]

To conclude the discussions on the existence of a terminal κ-belief (especially, knowledge) space, this section compares the existence of a terminal knowledge space with the previous non-existence results (e.g., Fagin et al., 1999; Heifetz and Samet, 1998a; Meier, 2005). This section asserts and examines the claim that, for the fixed infinite regular cardinal κ, the terminal space $\overrightarrow{\Omega^*}$ (of a given category of κ-belief spaces) describes all possible belief hierarchies of depth up to $\overline{\kappa}$ but only up to $\overline{\kappa}$.

Specifically, Section 7.1 examines whether the domain specification of a κ-belief space $\overrightarrow{\Omega}$ not as the power set but as a κ-algebra may neglect any reasoning regarding the underlying states Ω. It shows that, as long as nature states and the players' belief hierarchies up to $\overline{\kappa}$ are concerned, the κ-algebra $\mathcal{D}$ can always accommodate the belief hierarchies up to $\overline{\kappa}$ that the underlying states Ω intend to represent (however, the particular set Ω may not represent all belief hierarchies up to $\overline{\kappa}$).

Section 7.2 shows that the terminal space $\overrightarrow{\Omega^*}$ contains all belief hierarchies up to $\overline{\kappa}$. By defining the notion of a κ-rank, it shows that the terminal space attains the highest κ-rank (which is indeed $\overline{\kappa}$ in a category of κ-qualitative-belief spaces).

Section 7.3 shows that, while the cardinality of a terminal λ-belief space (where $\lambda \geq \kappa$) has at least as high as that of the terminal κ-belief space, the terminal λ-belief space may be redundant as a κ-belief space. Hence, the terminal κ-belief space has the set of exactly all possible belief hierarchies up to $\overline{\kappa}$, and nothing more.

7.1 Informational Content of the Domain

For the terminal κ-belief space constructed in Section 3, the domain $\mathcal{D}^*$ is generically not the power set of the underlying states Ω^*.[42] Does the domain $\mathcal{D}^*$ have some limitation on the representation of players' interactive beliefs? This subsection shows that any κ-belief space $\overrightarrow{\Omega}$ can accommodate belief hierarchies of depth up to $\overline{\kappa}$ that

[41] Also, in the category of κ-belief spaces $\overrightarrow{\Omega}$ in which each B_i is induced by a serial, transitive, and Euclidean possibility correspondence (i.e., B_i satisfies Consistency, Positive Introspection, Negative Introspection, and the Kripke property), a terminal κ-belief space exists.

[42] While the domain of a standard possibility correspondence model in the previous literature is often assumed to be the power set, the domain of an arbitrary (probabilistic) type space is implicitly assumed to be a σ-algebra. As argued in the Introduction, if one accommodates both probabilistic beliefs and knowledge, then one needs to define them on a σ-algebra.

the underlying states Ω intend to represent as long as the domain $\mathcal{D}$ is a κ-algebra because $\mathcal{D}$ contains all the events generated by the expressions $\mathcal{L} = \mathcal{L}_{\overline{\kappa}}$.

On the one hand, the syntactic formulas $\mathcal{L} = \mathcal{L}_{\overline{\kappa}}$ express nature $(S, \mathcal{S})$ and interactive beliefs up to $\overline{\kappa}$. On the other hand, an arbitrary κ-belief space $\overrightarrow{\Omega}$ represents how each nature state s, each nature event E, and players' interactive beliefs about nature events are described. Formally, the semantic interpretation function $[\![\cdot]\!]_{\overrightarrow{\Omega}}$ associates, with each expression $e \in \mathcal{L}$ that describes nature states and interactive beliefs, the corresponding event $[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D}$ in the space $\overrightarrow{\Omega}$. Hence, as long as exogenously given nature states $(S, \mathcal{S})$ and interactive beliefs of depth $\overline{\kappa}$ are concerned, $\mathcal{D}_{\overline{\kappa}} = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$ contains sufficient information about how much $\overrightarrow{\Omega}$ can describe nature states and interactive beliefs determined by $\mathcal{L}$; and any event $E \in \mathcal{D} \setminus \mathcal{D}_{\overline{\kappa}}$, if there is any, cannot be captured by the given language $\mathcal{L}$.[43]

Since $\mathcal{D}_{\overline{\kappa}} = D_{\overrightarrow{\Omega}}^{-1}(\mathcal{D}^*)$ (recall the discussion in Section 3), the sufficient information that the given belief space $\overrightarrow{\Omega}$ can capture takes a form of a κ-algebra. Formally:

Remark 4. If $\overrightarrow{\Omega}$ is a κ-belief space in a given category, then $\overrightarrow{\Omega_{\overline{\kappa}}} := \langle(\Omega, \mathcal{D}_{\overline{\kappa}}), (B_i|_{\mathcal{D}_{\overline{\kappa}}})_{i\in I}, \Theta\rangle$ is a κ-belief space in the same category with the following properties: (i) the identity map $\mathrm{id}_\Omega : \overrightarrow{\Omega} \to \overrightarrow{\Omega_{\overline{\kappa}}}$ is a morphism (an isomorphism iff $\overrightarrow{\Omega}$ is minimal); and (ii) $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega_{\overline{\kappa}}}} \circ \mathrm{id}_\Omega$. Moreover, by construction, $\overrightarrow{\Omega_{\overline{\kappa}}}$ is minimal: $\mathcal{D}_{\overline{\kappa}} = D_{\overrightarrow{\Omega_{\overline{\kappa}}}}^{-1}(\mathcal{D}^*)$.

Remark 4 states that any belief hierarchy up to $\overline{\kappa}$ generated by a κ-belief space $\overrightarrow{\Omega}$ is also generated by the minimal κ-belief space $\overrightarrow{\Omega_{\overline{\kappa}}}$. For any $\omega^* \in \Omega^*$, there are a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $\omega^* = D_{\overrightarrow{\Omega}}(\omega)$. Then, one can always take the space $\overrightarrow{\Omega_{\overline{\kappa}}}$ and $\omega \in \Omega$ with $\omega^* = D_{\overrightarrow{\Omega_{\overline{\kappa}}}}(\omega)$.

Remark 4 can also be used to characterize minimality: Proposition S.1 in Appendix E characterizes minimality as in Friedenberg and Meier (2011, Theorem 5.1).

7.2 Belief Hierarchies of Depth $\overline{\kappa}$

The previous subsection has shown that any κ-belief space can accommodate belief hierarchies up to $\overline{\kappa}$ that the underlying states Ω intend to represent. For any expression $e \in \mathcal{L}$ concerning nature states and the players' interactive beliefs up to $\overline{\kappa}$, the domain represents the expression as an event $[\![e]\!]_{\overrightarrow{\Omega}}$. Thus, in the particular belief space $\overrightarrow{\Omega}$, two expressions e and f may be identified as the same event $[\![e]\!]_{\overrightarrow{\Omega}} = [\![f]\!]_{\overrightarrow{\Omega}}$ even though their denotations are different. In contrast, the terminal κ-belief space $\overrightarrow{\Omega^*}$ has the minimum possible assumptions on how nature states and interactive beliefs up to $\overline{\kappa}$ are interpreted as $[\![\cdot]\!]_{\overrightarrow{\Omega^*}} = [\cdot]$. Since the space $\overrightarrow{\Omega^*}$ consists of belief hierarchies up to $\overline{\kappa}$ induced by some state of some κ-belief space, it contains *all* possible

[43] Such $E \in \mathcal{D} \setminus \mathcal{D}_{\overline{\kappa}}$ might be captured by a richer language, i.e., $E \in \mathcal{D}_{\overline{\lambda}}$ with $\overline{\lambda} > \overline{\kappa}$.

belief hierarchies of depth $\overline{\kappa}$ within the category of κ-belief spaces, as long as belief hierarchies up to $\overline{\kappa}$ are concerned.

This subsection defines the *κ-rank* of a κ-belief space, which represents the maximal ordinality of the non-trivial belief hierarchies up to $\overline{\kappa}$ that a given κ-belief space can generate. The κ-rank is extended from the rank of a standard partitional knowledge space (where the domain is the power set of underlying states) studied by Heifetz and Samet (1998a). This subsection shows that the terminal κ-belief space attains the highest κ-rank, which is $\overline{\kappa}$ in the category of κ-qualitative-belief spaces.

More generally, I define the κ-rank of a λ-belief space, where λ is an infinite regular cardinal with $\lambda \geq \kappa$ or the symbol $\lambda = \infty$. I allow for $\lambda \geq \kappa$ because any λ-belief space can accommodate the belief hierarchies of depth up to $\overline{\kappa}$ and because a given κ-belief space $\overrightarrow{\Omega}$ may be a λ-belief space. In the latter case, the κ-rank of $\overrightarrow{\Omega}$ as a κ-belief space is equal to the κ-rank of $\overrightarrow{\Omega}$ as a λ-belief space.

Definition 11 (κ-Rank)**.** *The κ-rank of a λ-belief space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{S})$ is the least ordinal α with $\mathcal{C}_\alpha = \mathcal{C}_{\alpha+1}$, where the sequence $(\mathcal{C}_\alpha)_\alpha$ is defined as follows:*

$$\mathcal{C}_\alpha := \begin{cases} \mathcal{A}_\kappa(\{\Theta^{-1}(E) \in \mathcal{D} \mid E \in \mathcal{S}\}) = \Theta^{-1}(\mathcal{A}_\kappa(\mathcal{S})) & \text{if } \alpha = 0 \\ \mathcal{A}_\kappa((\bigcup_{\beta<\alpha} \mathcal{C}_\beta) \cup \bigcup_{i \in I} \{B_i(E) \in \mathcal{D} \mid E \in \bigcup_{\beta<\alpha} \mathcal{C}_\beta\}) & \text{if } \alpha > 0 \end{cases}.$$

The notion of rank is extended from Heifetz and Samet (1998a), which corresponds to the ∞-rank of an ∞-knowledge space of I on $(S, \mathcal{P}(S))$ with $|S| \geq 2$ and $|I| \geq 2$.[44] Heifetz and Samet (1998a) demonstrate that there is no terminal standard partitional knowledge space on the following two grounds using the notion of rank. First, a morphism preserves the (∞-)ranks. Second, there is a standard partitional knowledge space with arbitrarily high (∞-)rank. Hence, for any candidate terminal standard partitional knowledge space, there exists a standard partitional knowledge space with a higher (∞-)rank, and thus the candidate space must not be terminal.

The next proposition first shows that a morphism preserves the κ-ranks (as long as nature states and the players' belief hierarchies up to $\overline{\kappa}$ are concerned). Second, the proposition shows that the κ-rank of any λ-belief space (with $\lambda \geq \kappa$) is at most $\overline{\kappa}$. While one can construct a non-trivial belief hierarchy of an arbitrary length in some belief space $\overrightarrow{\Omega}$, such belief space may not contain all possible ways in which the players reason about their interactive beliefs up to $\overline{\kappa}$. In contrast, for a fixed infinite regular cardinal κ, the terminal κ-belief space Ω^* contains all possible belief hierarchies of depth $\overline{\kappa}$ attained in some state of some κ-belief space. Especially, the terminal κ-belief space Ω^* contains all belief hierarchies of depth $\overline{\kappa}$ attained in $\overrightarrow{\Omega}$.

Proposition 5 (κ-Rank)**.** *1. If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism between λ-belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$, then the κ-rank of $\overrightarrow{\Omega'}$ is at least as high as that of $\overrightarrow{\Omega}$.*

[44] Definition 11 is also well-defined for $\kappa = \lambda = \infty$. Also, if $\mathcal{S}$ satisfies the separative property (see footnote 19) then $\mathcal{A}_\infty(\mathcal{S}) = \mathcal{P}(S)$.

2. The κ-rank of any λ-belief space $\overrightarrow{\Omega}$ is at most $\overline{\kappa}$.

By Proposition 5 (1), any two isomorphic κ-belief spaces have the same κ-rank. Especially, the κ-rank of a terminal κ-belief space is unique. Part (2) hinges on the fact that the expressions $\mathcal{L} = \mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S}))$ involve the player's interactive beliefs up to $\overline{\kappa}$ (Remark 1). Specifically, define $\mathcal{D}_\alpha := \mathcal{A}_\kappa(\{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}_\alpha\})$ for each ordinal $\alpha \leq \overline{\kappa}$, where $(\mathcal{L}_\alpha)_{\alpha=0}^{\overline{\kappa}}$ is defined as in Remark 1 so that $\mathcal{L} = \mathcal{L}_{\overline{\kappa}}$. Note that $\mathcal{D}_{\overline{\kappa}} = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$. I show in the proof that $\mathcal{D}_\alpha = \mathcal{C}_\alpha$ for each ordinal $\alpha \leq \overline{\kappa}$. Then, $\mathcal{D}_{\overline{\kappa}} = \mathcal{C}_{\overline{\kappa}} = \mathcal{C}_{\overline{\kappa}+1}$, i.e., the κ-rank of $\overrightarrow{\Omega}$ is at most $\overline{\kappa}$. Also, since $\{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\} = \mathcal{D}_{\overline{\kappa}} = \mathcal{C}_{\overline{\kappa}}$, one can check whether a given κ-belief space $\overrightarrow{\Omega}$ is non-redundant through its primitives alone (i.e., $\mathcal{C}_{\overline{\kappa}}$).

Two remarks on Proposition 5 (2) are in order. By Heifetz and Samet (1998a), the κ-rank of the terminal partitional κ-knowledge space $\overrightarrow{\Omega^*}$ is generically $\overline{\kappa}$. By the construction of Heifetz and Samet (1998a), there exists a standard partitional (κ-)knowledge space with κ-rank $\overline{\kappa}$, but the κ-rank of such space never exceeds $\overline{\kappa}$ (for the fixed infinite regular cardinal κ). Also, such a particular space does not contain all possible belief hierarchies of depth $\overline{\kappa}$. Thus, for the infinite regular cardinal κ, Heifetz and Samet (1998a)'s non-existence argument does not apply to a given class of κ-belief spaces.[45] Note that the same argument works for a category of ∞-belief spaces which includes, as a subclass, the category of partitional ∞-knowledge spaces.

Second, one needs to take care of all the $\overline{\kappa}$ levels of interactive beliefs in order to incorporate possible "discontinuity" of qualitative beliefs (knowledge). However, contrary to the claims of the previous literature (e.g., Fagin, 1994; Fagin, Halpern, and Vardi, 1991; Fagin et al., 1999; Heifetz and Samet, 1998a,b, 1999), the existence of a terminal qualitative belief (knowledge) space does not directly hinge on the continuity of beliefs (knowledge) itself by keeping track of all possible transfinite belief hierarchies (see also Zhou (2010) in the context of finitely-additive beliefs). While the lack of continuity makes it impossible to construct an $\aleph_1$-belief space with the finitary language, this paper rather shows that the non-existence hinges on the fact that depth of reasoning κ is not specified.

In contrast, one only needs to take care of all possible finite-level interactive beliefs for countably-additive probabilistic beliefs. Section 5.1 has shown that probabilistic beliefs are represented by p-belief operators $(B_i^p)_{(i,p)\in I\times[0,1]}$. Countably-additive beliefs are continuous with respect to a monotone sequence of events (see Definition 10 (2g) and (2h)).[46] Proposition 4 shows that, within the class of probabilistic-belief spaces, the terminal probabilistic-belief space has the $\aleph_1$-rank $\aleph_0$.[47]

[45] On the contrary, if a terminal partitional ∞-knowledge space existed, then, for any (infinite regular) cardinal κ, its ∞-rank is at least $\overline{\kappa}$, which is a contradiction because the ∞-rank of the terminal space is a fixed ordinal.

[46] Unlike qualitative or probability-one belief alone, the continuity of beliefs with respect to an increasing sequence of events (Definition 10 (2h)) requires degrees of p-beliefs.

[47] In the terminal probabilistic-belief space, a belief hierarchy consisting of all the finite-level beliefs

By fixing the language that the players are allowed to use in reasoning about their interactive beliefs (within the ordinality of κ), or by making the domain of a belief space explicit, a morphism (a description map) preserves interactive beliefs in a given κ-belief space to the terminal κ-belief space. At the same time, such preservation concerns only to the extent that belief hierarchies of depth $\overline{\kappa}$ are preserved.

7.3 Comparison of Terminal κ-Belief and λ-Belief Spaces

The previous subsection has examined the sense in which the terminal κ-belief space contains all possible belief hierarchies up to $\overline{\kappa}$. This subsection compares the terminal κ-belief space and the terminal λ-belief space with $\lambda > \kappa$.

Throughout the subsection, let κ and λ be infinite regular cardinals with $\kappa < \lambda$. Fix I, $(S, \mathcal{S})$, and some properties in Definition 2 for the players' beliefs. Denote by $\overrightarrow{\Omega^*_\kappa}$ and $\overrightarrow{\Omega^*_\lambda}$ the terminal κ-space and the terminal λ-space, respectively.

Then, the cardinality of the terminal κ-space $\overrightarrow{\Omega^*_\kappa}$ is at least as small as that of $\overrightarrow{\Omega^*_\lambda}$, because the description map $D_{\overrightarrow{\Omega^*_\lambda}}$ from $\overrightarrow{\Omega^*_\lambda}$ into the terminal κ-space $\overrightarrow{\Omega^*_\kappa}$ is surjective (note that both spaces reside in the category of κ-belief spaces at hand). However, as a κ-belief space, $\overrightarrow{\Omega^*_\lambda}$ is redundant. Formally:

Proposition 6 (Cardinality of a Terminal Space)**.** *1. $D_{\overrightarrow{\Omega^*_\lambda}} : \overrightarrow{\Omega^*_\lambda} \to \overrightarrow{\Omega^*_\kappa}$ is a surjective morphism so that $|\Omega^*_\kappa| \leq |\Omega^*_\lambda|$.*

2. Let $|I| \geq 2$ and $|S| \geq 2$, and suppose $\mathcal{S}$ contains E with $\emptyset \subsetneq E \subsetneq S$. Then, $\max(2^{\aleph_0}, \kappa) \leq |\Omega^*_\kappa|$.

Proposition 6 (1) shows that the terminal λ-belief space $\overrightarrow{\Omega^*_\lambda}$ is at least as rich as $\overrightarrow{\Omega^*_\kappa}$ in cardinality: $|\Omega^*_\kappa| \leq |\Omega^*_\lambda|$. When it comes to belief hierarchies up to $\overline{\kappa}$, however, the space $\overrightarrow{\Omega^*_\lambda}$ is redundant (when $\lambda > \kappa$) in that there would be two states which induce the same belief hierarchy up to $\overline{\kappa}$. Thus, when it suffices for the outside analysts to analyze players' belief hierarchies up to $\overline{\kappa}$ (e.g., recall the epistemic characterization of IESDA in Section 6.1), the terminal space $\overrightarrow{\Omega^*_\kappa}$ accommodates exactly the set of all possible belief hierarchies up to $\overline{\kappa}$.

Proposition 6 sheds light on the difference between the existence and non-existence of a terminal ∞-belief space: while a terminal κ-belief space exists for the fixed infinite regular cardinal κ (however, κ can be fixed arbitrarily), the cardinality of the ∞-belief space blows up as κ arbitrarily increases.

Somewhat informally, define the class $\mathcal{L}_\infty$ of expressions as in Definition 5.[48] Define the class Ω^*_∞ as in Equation (1), and let $\overrightarrow{\Omega^*_\infty}$ be the terminal space. Since $\overrightarrow{\Omega^*_\infty}$

uniquely extends to the limit belief hierarchy (the $\aleph_1$-rank is $\aleph_0 < \aleph_1$), but does not uniquely extend to an uncountable belief hierarchy ($\kappa = \aleph_1$).

[48]The class $\mathcal{L}_\infty$ is too big to be a set in the standard set theory (e.g., for any ordinal α, there is $e_\alpha \in \mathcal{L}_\infty$ such that if $\alpha > 0$ then $e_\alpha \neq e_\beta$ for all $\beta < \alpha$). In contrast, for an infinite regular cardinal κ, recall that $\mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S}))$ (Definition 5) is a well-defined set.

would be non-redundant, $\mathcal{D}^*_\infty$ is a complete algebra that separates any two states, i.e., the power class of Ω^*_∞. Proposition 6 suggests that $\kappa \leq |\Omega^*_\infty|$ for any (infinite regular) cardinal, meaning that Ω^*_∞ is too big to be a set.

Or, consider the terminal κ-belief space $\overrightarrow{\Omega^*_\kappa}$ of the category of κ-belief spaces that satisfy at least the Kripke property (recall Definition 2). One can introduce the players' beliefs about any subset of $\overrightarrow{\Omega^*_\kappa}$ (formally, see Remark A.1 in Appendix A.5). In the extended space $\langle (\Omega^*_\kappa, \mathcal{P}(\Omega^*_\kappa)), (\overline{B}^*_i)_{i \in I}, \Theta^* \rangle$, every state ω^*_κ induces a belief hierarchy of an arbitrary length uniquely extended from the original hierarchy (of depth $\overline{\kappa}$). If the terminal ∞-belief space $\overrightarrow{\Omega^*_\infty}$ existed (as a set) and contained any such belief hierarchy, then there would be an injection from Ω^*_κ into Ω^*_∞, asserting again that $\kappa \leq |\Omega^*_\infty|$ for any κ.

8 Concluding Remarks

The main result of this paper (Theorem 1) is the construction of the terminal belief space $\overrightarrow{\Omega^*}$ for varieties of assumptions on beliefs. In a κ-belief space where κ is a fixed infinite regular cardinal, the players can reason about their interactive beliefs up to depth $\overline{\kappa}$. The space Ω^* exhausts all possible interactive beliefs up to $\overline{\kappa}$ that can realize at some state of some belief space. The space Ω^* encodes interactive beliefs within itself (Proposition 1) and exhausts any statement regarding interactive beliefs that holds at some state of some belief space (Proposition 2). In Ω^*, only the explicit assumptions on beliefs made by the outside analysts are imposed. That is, Ω^* is free from implicit assumptions imposed by how the model is represented. Each state in Ω^* coherently and completely describes the corresponding nature state and interactive beliefs (Proposition 3 and Corollary 1). Explicitly, Theorem 2 shows that $\overrightarrow{\Omega^*}$ is the largest set of coherent descriptions that reflects assumptions on beliefs.

This paper shows that a terminal belief space exists irrespective of nature of beliefs, i.e., under the common framework regardless of whether beliefs are qualitative or probabilistic (Corollary 2). Proposition 4 shows, under the framework of this paper, finite-level belief hierarchies uniquely extend to countable levels for countably-additive probabilistic beliefs, as in the literature on universal type spaces. Appendix C shows that the framework of this paper also applies to richer forms of beliefs such as conditional beliefs. Appendix C also discusses the implication of this paper (i.e., the specification of depth reasoning) to terminal knowledge-unawareness, preference, and expectation spaces.

This paper circumvents the previous non-existence of a terminal knowledge space by restricting attention to all possible knowledge (belief) hierarchies of depth $\overline{\kappa}$. The κ-algebra $\mathcal{D}^*$ accommodates all possible interactive beliefs of depth $\overline{\kappa}$ (Proposition 5 and Remark 4). While an infinite regular cardinal κ can be taken arbitrarily for given nature states $(S, \mathcal{S})$, the terminal κ-belief space $\overrightarrow{\Omega^*}$ contains exactly all belief hierarchies up to depth $\overline{\kappa}$ (Proposition 6).

The paper demonstrates that the existence of a terminal belief space hinges on the specification of an infinite regular cardinal κ, which determines the possible depth of reasoning, rather than on properties of beliefs themselves (Remark 1). Section 6.1 shows that the specification of $\overline{\kappa}$ is crucial also for an epistemic characterization of a game-theoretic solution concept, i.e., iterated elimination of strictly dominated actions. Thus, the specification of depth of reasoning also plays an important role in such game-theoretic application especially when players face a general infinite game or when their beliefs may not satisfy logical properties. Appendix D.1 shows that if a correlated equilibrium is non-redundant and minimal then the correlating device (the underlying state space) can be replaced with players' belief hierarchies about their play.

Hence, given a strategic game, this paper shows that there exists a terminal belief space with the following properties: (i) the players can engage in interactive reasoning up to some predetermined depth (i.e., the ordinality of κ) which is sufficient to reason about their interactive beliefs about their play; (ii) rationality, beliefs in rationality, and common belief in rationality are expressible within the terminal space; (iii) common belief in rationality characterizes iterated elimination of strictly dominated actions within the terminal space; and (iv) the above (i)-(iii) hold irrespective of assumptions on the players' beliefs.

A Appendix

A.1 Section 3

Proof of Remark 1. First, I show, by induction on $\mathcal{L}_\alpha$, that $\mathcal{L}_{\overline{\lambda}} \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$; (i) $\mathcal{L}_0 \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$; and (ii) if $\mathcal{L}_\beta \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$ for all $\beta < \alpha(\leq \overline{\lambda})$ then $\mathcal{L}_\alpha \subseteqq \mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S}))$. Conversely, it can be seen that if $e \in \mathcal{L}_{\overline{\lambda}}$ then $e \in \mathcal{L}_\alpha$ for some $\alpha < \overline{\lambda}$. I show, by induction on λ-expressions, that $\mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S})) \subseteq \mathcal{L}_{\overline{\lambda}}$. First, $\mathcal{A}_\kappa(\mathcal{S}) \subseteq \mathcal{L}_{\overline{\lambda}}$. Second, if $e \in \mathcal{L}_{\overline{\lambda}}$ then $e \in \mathcal{L}_\alpha$ for some $\alpha < \overline{\lambda}$ and thus $(\neg e), (\beta_i(e)) \in \mathcal{L}_{\alpha+1} \subseteq \mathcal{L}_{\overline{\lambda}}$. Third, take $\mathcal{F} \subseteq \mathcal{L}_{\overline{\lambda}}$ with $0 < |\mathcal{F}| < \lambda$. For each $f \in \mathcal{F}$, there is $\alpha_f < \overline{\lambda}$ with $f \in \mathcal{L}_{\alpha_f}$. Since $|\mathcal{F}| < \overline{\lambda}$, the definition of an infinite regular cardinal yields $\gamma := \sup_{f \in \mathcal{F}} \alpha_f < \overline{\lambda}$. Since $\mathcal{F} \subseteq \mathcal{L}_\gamma \subseteq \mathcal{L}_{\overline{\lambda}}$, it follows $\bigwedge \mathcal{F} \in \mathcal{L}_{\overline{\lambda}}$. Hence, $\mathcal{L}^I_\lambda(\mathcal{A}_\kappa(\mathcal{S})) \subseteq \mathcal{L}_{\overline{\lambda}}$. □

Proof Sketch of Remark 2. The proof is similar to Heifetz and Samet (1998b, Proposition 4.1) and Meier (2006, Proposition 2). For nature events, since φ is a morphism, $[\![\cdot]\!]_{\overrightarrow{\Omega}} = \Theta^{-1}(\cdot) = \varphi^{-1}(\Theta')^{-1}(\cdot) = \varphi^{-1}([\![\cdot]\!]_{\overrightarrow{\Omega'}})$. Then, use the property that φ^{-1} commutes with set-algebraic operations and belief operators. □

Proof of Lemma 1. The proof consists of three steps. The first step establishes the following correspondence between syntactic and semantic operations.

1. $[(\neg e)] = \neg[e](= [e]^c)$ for any $e \in \mathcal{L}$.

2. $[S] = \Omega^*$ (and $[\emptyset] = \emptyset$). In other words, $[\bigwedge \emptyset] = \Omega^*$ (and $[\bigvee \emptyset] = \emptyset$).

3. $[\bigwedge \mathcal{E}] = \bigcap_{e \in \mathcal{E}} [e]$ (and $[\bigvee \mathcal{E}] = \bigcup_{e \in \mathcal{E}} [e]$) for any $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$.

To prove (1), fix $e \in \mathcal{L}$. Then, $\omega^* \in [(\neg e)]$ iff $(\neg e) \in_1 \omega^* = D(\omega)$ iff $\omega \in [\![\neg e]\!]_{\overrightarrow{\Omega}} = \neg [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega \notin [\![e]\!]_{\overrightarrow{\Omega}}$ iff $e \notin_1 D(\omega) = \omega^*$ iff $\omega^* \notin [e]$ iff $\omega^* \in (\neg [e])$, where a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. Thus, $[(\neg e)] = \neg [e]$.

To prove (2), if $\omega^* \in \Omega^*$ then $\omega^* = D(\omega)$ for some belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$. Since $\omega \in \Omega = \Theta^{-1}(S) = [\![S]\!]_{\overrightarrow{\Omega}}$, I get $S \in_1 D(\omega) = \omega^*$ and thus $\omega^* \in [S]$.

For (3), $\omega^* \in [\bigwedge \mathcal{E}]$ iff $\bigwedge \mathcal{E} \in_1 \omega^* = D(\omega)$ iff $\omega \in [\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}} = \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega^* \in \bigcap_{e \in \mathcal{E}} [e]$, where a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$.

The second step establishes that $\mathcal{D}^*$ is a κ-algebra on Ω^*. If $[e] \in \mathcal{D}^*$, then it follows from the first step and $(\neg e) \in \mathcal{L}$ that $\neg [e] = [(\neg e)] \in \mathcal{D}^*$. Next, take $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. It follows from the first step and $\bigwedge \mathcal{E} \in \mathcal{L}$ that $\bigcap_{e \in \mathcal{E}} [e] = [\bigwedge \mathcal{E}] \in \mathcal{D}^*$. For the case that $\mathcal{E} = \emptyset$, observe $\Omega^* = [S] \in \mathcal{D}^*$.

The third step establishes $D^{-1}([\cdot]) = [\![\cdot]\!]_{\overrightarrow{\Omega}}$ for any belief space $\overrightarrow{\Omega}$. For any $e \in \mathcal{L}$, $\omega \in D^{-1}([e])$ iff $D(\omega) \in [e]$ iff $e \in_1 D(\omega)$ iff $\omega \in [\![e]\!]_{\overrightarrow{\Omega}}$. □

Proof of Lemma 2. For any $\omega^* \in \Omega^*$, choose a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = D(\omega)$, and define $\Theta^*(\omega^*) := \Theta(\omega)$, where $\Theta(\omega) \in_0 D(\omega)$. I show $\Theta^*(\omega^*)$ does not depend on a particular choice of $\overrightarrow{\Omega}$ and ω (i.e., $\Theta^* : \Omega^* \to S$ is well-defined). If $\omega^* = D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$ for some $\omega \in \Omega$ and $\omega' \in \Omega'$, then $(0, \Theta(\omega)) = (0, \Theta'(\omega'))$.

Next, for each $E \in \mathcal{A}_\kappa(\mathcal{S})$, $\omega^* \in (\Theta^*)^{-1}(E)$ iff $\Theta^*(\omega^*) = \Theta(\omega) \in E$ iff $\omega \in \Theta^{-1}(E) = [\![E]\!]_{\overrightarrow{\Omega}}$ iff $E \in_1 D(\omega) = \omega^*$ iff $\omega^* \in [E]$, where $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. □

To establish Lemma 3, I provide Lemma A.1 below. Suppose that a certain property of beliefs is represented by operators $f_{\overrightarrow{\Omega}} : \mathcal{D} \to \mathcal{D}$ in each belief space $\overrightarrow{\Omega}$. Operators would be generated by composing belief operators $(B_i)_{i \in I}$ and set-algebraic as well as constant and identity operations. For example, let $f_{\overrightarrow{\Omega}}(\cdot) = B_i(\cdot)$ and $g_{\overrightarrow{\Omega}}(\cdot) = B_i B_i(\cdot)$. Positive Introspection is characterized by $f_{\overrightarrow{\Omega}}(\cdot) \subseteq g_{\overrightarrow{\Omega}}(\cdot)$. Truth Axiom is characterized by $f_{\overrightarrow{\Omega}}(\cdot) \subseteq \mathrm{id}_{\mathcal{D}}(\cdot)$. Monotonicity is expressed as $f_{\overrightarrow{\Omega}}$ being *monotone*: $f_{\overrightarrow{\Omega}}(E) \subseteq f_{\overrightarrow{\Omega}}(F)$ for all $E, F \in \mathcal{D}$ with $E \subseteq F$. Likewise, Non-empty λ-Conjunction is expressed as $f_{\overrightarrow{\Omega}}$ satisfying *non-empty λ-conjunction*: $\bigcap_{E \in \mathcal{E}} f_{\overrightarrow{\Omega}}(E) \subseteq f_{\overrightarrow{\Omega}}(\bigcap \mathcal{E})$ for all $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$. Abusing the notation, denote by $f_{\overrightarrow{\Omega^*}}$ the corresponding operation in $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (B_i^*)_{i \in I}, \Theta^* \rangle$ (note that B_i^* is shown to be well-defined on $(\Omega^*, \mathcal{D}^*)$ irrespective of Lemma A.1 below).

Lemma A.1 (Preservation of Properties of Beliefs)**.** *Suppose that $f_{\overrightarrow{\Omega}} : \mathcal{D} \to \mathcal{D}$ and $g_{\overrightarrow{\Omega}} : \mathcal{D} \to \mathcal{D}$ are defined in each κ-belief space $\overrightarrow{\Omega}$. Suppose further that if $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ is measurable then $\varphi^{-1} f_{\overrightarrow{\Omega'}}(\cdot) = f_{\overrightarrow{\Omega}} \varphi^{-1}(\cdot)$ and $\varphi^{-1} g_{\overrightarrow{\Omega'}}(\cdot) = g_{\overrightarrow{\Omega}} \varphi^{-1}(\cdot)$. Then:*

1. (a) *If* $f_{\overrightarrow{\Omega}}(\cdot) \subseteq g_{\overrightarrow{\Omega}}(\cdot)$ *holds for every belief space* $\overrightarrow{\Omega}$*, then* $f_{\overrightarrow{\Omega^*}}(\cdot) \subseteq g_{\overrightarrow{\Omega^*}}(\cdot)$.

   (b) *If* $f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq g_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}})$ *for some belief space* $\overrightarrow{\Omega}$ *and some* $e \in \mathcal{L}$*, then* $f_{\overrightarrow{\Omega^*}}([e]) \not\subseteq g_{\overrightarrow{\Omega^*}}([e])$.

   (c) *If there exits a surjective measurable map* $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$*, then* $f_{\overrightarrow{\Omega}}(\cdot) \subseteq g_{\overrightarrow{\Omega}}(\cdot)$ *implies* $f_{\overrightarrow{\Omega'}}(\cdot) \subseteq g_{\overrightarrow{\Omega'}}(\cdot)$.

2. (a) *If* $f_{\overrightarrow{\Omega}}$ *is monotone for every belief space* $\overrightarrow{\Omega}$*, then so is* $f_{\overrightarrow{\Omega^*}}$.

   (b) *Suppose* $f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq f_{\overrightarrow{\Omega}}([\![\hat{e}]\!]_{\overrightarrow{\Omega}})$ *for some belief space* $\overrightarrow{\Omega}$ *and some* $e, \hat{e} \in \mathcal{L}$ *with* $[\![e]\!]_{\overrightarrow{\Omega}} \subseteq [\![\hat{e}]\!]_{\overrightarrow{\Omega}}$*. Then,* $f_{\overrightarrow{\Omega^*}}([e]) \not\subseteq f_{\overrightarrow{\Omega^*}}([\hat{e}])$.

   (c) *Suppose there exits a surjective measurable map* $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$*. If* $f_{\overrightarrow{\Omega}}$ *is monotone, then so is* $f_{\overrightarrow{\Omega'}}$.

3. (a) *If* $f_{\overrightarrow{\Omega}}$ *satisfies non-empty* λ*-conjunction for every belief space* $\overrightarrow{\Omega}$*, then so does* $f_{\overrightarrow{\Omega^*}}$.

   (b) *Suppose* $\bigcap_{e\in\mathcal{E}} f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) \not\subseteq f_{\overrightarrow{\Omega}}(\bigcap_{e\in\mathcal{E}}[\![e]\!]_{\overrightarrow{\Omega}})$ *for some belief space* $\overrightarrow{\Omega}$ *and some* $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ *with* $|\mathcal{E}| < \lambda$*. Then,* $\bigcap_{e\in\mathcal{E}} f_{\overrightarrow{\Omega^*}}([e]) \not\subseteq f_{\overrightarrow{\Omega^*}}(\bigcap_{e\in\mathcal{E}}[e])$.

   (c) *Suppose there exits a surjective measurable map* $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$*. If* $f_{\overrightarrow{\Omega}}$ *satisfies non-empty* λ*-conjunction, then so does* $f_{\overrightarrow{\Omega'}}$.

Proof of Lemma A.1. 1. I start with an intermediate result. Let $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ be measurable. Suppose that, for all $E \in \mathcal{D}$, if $\omega \in f_{\overrightarrow{\Omega}}(E)$ then $\omega \in g_{\overrightarrow{\Omega}}(E)$. Then, for any $E' \in \mathcal{D}'$, $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(E')$ (i.e., $\omega \in f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'))$) implies $\varphi(\omega) \in g_{\overrightarrow{\Omega'}}(E')$. With this in mind:

   (a) If $\omega^* \in f_{\overrightarrow{\Omega^*}}([e])$, then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = D(\omega)$. Now, $\omega^* = D(\omega) \in g_{\overrightarrow{\Omega^*}}([e])$.

   (b) By hypothesis, there is $\omega \in \Omega$ such that $\omega \in f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1}f_{\overrightarrow{\Omega^*}}([e])$ and $\omega \notin g_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1}g_{\overrightarrow{\Omega^*}}([e])$.

   (c) If $\omega' \in f_{\overrightarrow{\Omega'}}(E')$, then $\omega' = \varphi(\omega)$ for some $\omega \in \Omega$. Now, $\omega' = \varphi(\omega) \in g_{\overrightarrow{\Omega'}}(E')$.

2. I start with an intermediate result. Let $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ be measurable, and let $f_{\overrightarrow{\Omega}}$ be monotone. For any $E', F' \in \mathcal{D}'$ with $E' \subseteq F'$, if $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(E')$ (i.e., $\omega \in f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'))$) then $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(F')$. With this in mind:

   (a) Let $[e] \subseteq [\hat{e}]$. If $\omega^* \in f_{\overrightarrow{\Omega^*}}([e])$, then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = D(\omega)$. Now, $\omega^* = D(\omega) \in f_{\overrightarrow{\Omega^*}}([\hat{e}])$.

   (b) By hypothesis, there is $\omega \in \Omega$ such that $\omega \in f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1}f_{\overrightarrow{\Omega^*}}([e])$ and $\omega \notin f_{\overrightarrow{\Omega}}([\![\hat{e}]\!]_{\overrightarrow{\Omega}}) = D^{-1}f_{\overrightarrow{\Omega^*}}([\hat{e}])$.

(c) Let $E' \subseteq F'$. If $\omega' \in f_{\overrightarrow{\Omega'}}(E')$, then $\omega' = \varphi(\omega)$ for some $\omega \in \Omega$. Now, $\omega' = \varphi(\omega) \in f_{\overrightarrow{\Omega'}}(F')$.

3. I start with an intermediate result. Let $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ be measurable. If $f_{\overrightarrow{\Omega}}$ satisfies non-empty λ-conjunction, then $\varphi(\omega) \in \bigcap_{E' \in \mathcal{E}'} f_{\overrightarrow{\Omega'}}(E')$ implies $\varphi(\omega) \in f_{\overrightarrow{\Omega'}}(\bigcap \mathcal{E}')$ for all $\mathcal{E}' \in \mathcal{P}(\mathcal{D}') \setminus \{\emptyset\}$ with $|\mathcal{E}'| < \lambda$. With this in mind:

   (a) Fix $\mathcal{E}^* \in \mathcal{P}(\mathcal{D}^*) \setminus \{\emptyset\}$ with $|\mathcal{E}^*| < \lambda$. If $\omega^* \in \bigcap_{[e] \in \mathcal{E}^*} f_{\overrightarrow{\Omega^*}}([e])$, then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = D(\omega)$. Now, $\omega^* = D(\omega) \in f_{\overrightarrow{\Omega^*}}(\bigcap \mathcal{E}^*)$.

   (b) By hypothesis, there is $\omega \in \Omega$ with $\omega \in \bigcap_{e \in \mathcal{E}} f_{\overrightarrow{\Omega}}([\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1}(\bigcap_{e \in \mathcal{E}} f_{\overrightarrow{\Omega^*}}([e]))$ and $\omega \notin f_{\overrightarrow{\Omega}}(\bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}}) = D^{-1}(f_{\overrightarrow{\Omega^*}}(\bigcap_{e \in \mathcal{E}} [e]))$.

   (c) Fix $\mathcal{E}' \in \mathcal{P}(\mathcal{D}') \setminus \{\emptyset\}$ with $|\mathcal{E}'| < \lambda$. If $\omega' \in \bigcap_{E' \in \mathcal{E}'} f_{\overrightarrow{\Omega'}}(E')$, then there is $\omega \in \Omega$ with $\omega' = \varphi(\omega)$. Now, $\omega' = \varphi(\omega) \in f_{\overrightarrow{\Omega'}}(\bigcap \mathcal{E}')$.

□

Two remarks on Lemma A.1 are in order. First, B_i^* violates some property of beliefs if there exists a belief space $\overrightarrow{\Omega}$ such that B_i violates the corresponding property with respect to $\mathcal{D}_{\overline{\kappa}} = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$. Second, for belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$, if there is a surjective measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ with $B_i \varphi^{-1}(\cdot) = \varphi^{-1} B_i'(\cdot)$, then B_i' inherits the properties of B_i. Now, I prove Lemma 3.

Proof of Lemma 3. Fix $i \in I$. I show that B_i^* is well-defined and inherits all the properties imposed in the given category. Then, for any $e \in \mathcal{L}$, $B_i(D^{-1}([e])) = B_i([\![e]\!]_{\overrightarrow{\Omega}}) = [\![\beta_i(e)]\!]_{\overrightarrow{\Omega}} = D^{-1}([\beta_i(e)]) = D^{-1}(B_i^*([e]))$.

To show B_i^* is well-defined, take $e, f \in \mathcal{D}$ with $[e] = [f]$. If $\omega^* \in B_i^*([e]) = [\beta_i(e)]$, then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $D(\omega) \in [\beta_i(e)]$, i.e., $\beta_i(e) \in_1 D(\omega)$. Thus, $\omega \in [\![\beta_i(e)]\!]_{\overrightarrow{\Omega}} = B_i([\![e]\!]_{\overrightarrow{\Omega}}) = B_i(D^{-1}([e]))$. Since $[e] = [f]$, $\omega \in [\![\beta_i(f)]\!]_{\overrightarrow{\Omega}}$, i.e., $\omega^* = D(\omega) \in [\beta_i(f)] = B_i^*([f])$. By changing the role of e and f, $B_i^*([e]) = B_i^*([f])$.

Next, I show B_i^* inherits properties specified in Definition 2. For Monotonicity, apply Lemma A.1 (2a) by taking $f_{\overrightarrow{\Omega}} = B_i$. For Non-empty λ-Conjunction, apply Lemma A.1 (3a) by taking $f_{\overrightarrow{\Omega}} = B_i$. Next, apply Lemma A.1 (1a) to the following. For Necessitation, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (\Omega, B_i(\Omega))$. For Consistency, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i(\cdot) \cap (\neg B_i)(\cdot), \emptyset)$. For Truth Axiom, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i, \mathrm{id}_{\mathcal{D}})$. For Positive Introspection, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i, B_i B_i)$. For Negative Introspection, take $(f_{\overrightarrow{\Omega}}, g_{\overrightarrow{\Omega}}) = (B_i, B_i(\neg B_i))$.

Finally, consider the Kripke property. If $b^*_{B_i^*}(\omega^*) \subseteq [e]$ then $b_{B_i}(\omega) = \bigcap_{F \in \mathcal{D} : \omega \in B_i(F)} F \subseteq D^{-1} \bigcap_{[f] \in \mathcal{D}^* : \omega^* \in B_i^*[f]} [f] \subseteq D^{-1}[e]$, where a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. Since B_i satisfies the Kripke property, $\omega \in B_i D^{-1}[e] = D^{-1} B_i^*[e]$, as desired. □

Proof of Lemma 4. First, if $s \in_0 \omega^*$ and $s' \in_0 D(\omega^*)$, then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $s = \Theta(\omega) = \Theta^*(D(\omega)) = \Theta^*(\omega^*) = s'$. Note that the argument does not depend on a particular choice of belief spaces. Second, similarly to Heifetz and Samet (1998b, Lemma 4.6) and Meier (2006, Lemma 6), below I show by induction that $[\![\cdot]\!]_{\overrightarrow{\Omega^*}} = [\cdot]$. Then, $\omega^* = \{s\} \sqcup \{e \in \mathcal{L} \mid e \in_1 \omega^*\} = \{s\} \sqcup \{e \in \mathcal{L} \mid \omega^* \in [e]\} = \{s\} \sqcup \{e \in \mathcal{L} \mid \omega^* \in [\![e]\!]_{\overrightarrow{\Omega^*}}\} = D(\omega^*)$ for any $\omega^* \in \Omega^*$.

To establish $[\![\cdot]\!]_{\overrightarrow{\Omega^*}} = [\cdot]$, start from $E \in \mathcal{A}_\kappa(\mathcal{S})$. In fact, $\omega^* \in [\![E]\!]_{\overrightarrow{\Omega^*}} = (\Theta^*)^{-1}(E)$ iff $\Theta^*(\omega^*) = \Theta^*(D(\omega)) = \Theta(\omega) \in E$ iff $\omega \in \Theta^{-1}(E) = [\![E]\!]_{\overrightarrow{\Omega}}$ iff $E \in_1 D(\omega)$ iff $\omega^* = D(\omega) \in [E]$, where $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$.

Next, let $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. Assume the induction hypothesis that $[\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$ for all $e \in \mathcal{E}$. Then, $[\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega^*}} = \bigcap_{e\in\mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega^*}} = \bigcap_{e\in\mathcal{E}} [e] = [\bigwedge \mathcal{E}]$.

Next, assume the induction hypothesis that $[\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$. By definition, $[\beta_i(e)] = B_i^*([e]) = B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = [\![\beta_i(e)]\!]_{\overrightarrow{\Omega^*}}$. Also, $[(\neg e)] = \neg[e] = \neg[\![e]\!]_{\overrightarrow{\Omega^*}} = [\![\neg e]\!]_{\overrightarrow{\Omega^*}}$. □

Proof of Theorem 1. I have already shown that $\overrightarrow{\Omega^*} = \langle(\Omega^*, \mathcal{D}^*), (B_i^*)_{i\in I}, \Theta^*\rangle$ is a belief space of I on $(S, \mathcal{S})$ of the given category such that, for any belief space $\overrightarrow{\Omega}$, the description map $D_{\overrightarrow{\Omega}}$ is a morphism. If $\varphi : \Omega \to \Omega^*$ is a morphism, then Remark 3 and Lemma 4 imply $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega^*}} \circ \varphi = \varphi$. Thus, $D_{\overrightarrow{\Omega}}$ is a unique morphism. □

A.2 Section 4

Proof of Proposition 1. I show Part (2) first and then Part (1). Part (3) immediately follows from the first two parts.

Part (2). For the "only if" part, let $\overrightarrow{\Omega}$ be non-redundant. Assume $\chi_{\overrightarrow{\Omega}}(\omega) = \chi_{\overrightarrow{\Omega}}(\omega')$. Since $D_{\overrightarrow{\Omega}}$ is injective, it suffices to show, by induction, that $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega}}(\omega')$. First, $\Theta(\omega) = \Theta(\omega')$. Second, for any $E \in \mathcal{A}_\kappa(\mathcal{S})$, $E \in D_{\overrightarrow{\Omega}}(\omega)$ iff $\Theta(\omega) = \Theta(\omega') \in E$ iff $E \in D_{\overrightarrow{\Omega}}(\omega')$. Third, assume $e \in D_{\overrightarrow{\Omega}}(\omega)$ iff $e \in D_{\overrightarrow{\Omega}}(\omega')$. Then, $(\neg e) \in D_{\overrightarrow{\Omega}}(\omega)$ iff $e \notin D_{\overrightarrow{\Omega}}(\omega)$ iff $e \notin D_{\overrightarrow{\Omega}}(\omega')$ iff $(\neg e) \in D_{\overrightarrow{\Omega}}(\omega')$. Fourth, assume $e \in D_{\overrightarrow{\Omega}}(\omega)$ iff $e \in D_{\overrightarrow{\Omega}}(\omega')$ for all $e \in \mathcal{E}$, where $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ and $|\mathcal{E}| < \kappa$. Then, $\bigwedge \mathcal{E} \in D_{\overrightarrow{\Omega}}(\omega)$ iff $e \in D_{\overrightarrow{\Omega}}(\omega)$ for all $e \in \mathcal{E}$ iff $e \in D_{\overrightarrow{\Omega}}(\omega')$ for all $e \in \mathcal{E}$ iff $\bigwedge \mathcal{E} \in D_{\overrightarrow{\Omega}}(\omega')$. Fifth, $\beta_i(e) \in D_{\overrightarrow{\Omega}}(\omega)$ iff $\omega \in B_i([\![e]\!]_{\overrightarrow{\Omega}})$ iff $\omega' \in B_i([\![e]\!]_{\overrightarrow{\Omega}})$ iff $\beta_i(e) \in D_{\overrightarrow{\Omega}}(\omega')$. Hence, $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega}}(\omega')$.

For the "if" part, let $\chi_{\overrightarrow{\Omega}}$ be injective. Assume $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega}}(\omega')$. It suffices to show $\chi_{\overrightarrow{\Omega}}(\omega) = \chi_{\overrightarrow{\Omega}}(\omega')$. First, $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega}}(\omega')$ yields $\Theta(\omega) = \Theta(\omega')$. Second, if $\omega \in B_i([\![e]\!]_{\overrightarrow{\Omega}}) = D_{\overrightarrow{\Omega}}^{-1}([\beta_i(e)])$, then $D(\omega') = D(\omega) \in [\beta_i(e)]$ and thus $\omega' \in B_i([\![e]\!]_{\overrightarrow{\Omega}})$. Likewise, if $\omega' \in B_i([\![e]\!]_{\overrightarrow{\Omega}})$ then $\omega \in B_i([\![e]\!]_{\overrightarrow{\Omega}})$. Hence, $\chi_{\overrightarrow{\Omega}}(\omega) = \chi_{\overrightarrow{\Omega}}(\omega')$.

Part (1). For the "only if" part, a terminal space $\overrightarrow{\Omega}$ is non-redundant and minimal. To show that $\chi_{\overrightarrow{\Omega}}$ is surjective, observe that

$$\begin{aligned}\chi_{\overrightarrow{\Omega}}(\omega) &= (\Theta(\omega), (\{e \in \mathcal{L} \mid \omega \in B_i([\![e]\!]_{\overrightarrow{\Omega}})\})_{i\in I}) \\ &= (\Theta(D(\omega)), (\{e \in \mathcal{L} \mid D(\omega) \in B_i([e])\})_{i\in I}) = \chi_{\overrightarrow{\Omega^*}}(D_{\overrightarrow{\Omega}}(\omega)).\end{aligned}$$

Since $\overrightarrow{\Omega}$ is terminal, $D_{\overrightarrow{\Omega}}$ is bijective. As discussed in the main text (recall Expression (3)), $\chi_{\overrightarrow{\Omega^*}}$ is surjective (also injective because $\overrightarrow{\Omega^*}$ is non-redundant by Part (2)).

For the "if" part, I show that $D_{\overrightarrow{\Omega}}$ is an isomorphism. Since $\overrightarrow{\Omega}$ is minimal (i.e., $\mathcal{D} = D_{\overrightarrow{\Omega}}^{-1}(\mathcal{D}^*)$), if $D_{\overrightarrow{\Omega}}$ is bijective then $D_{\overrightarrow{\Omega}}^{-1} : \Omega^* \to \Omega$ is measurable. By Part (2), $D_{\overrightarrow{\Omega}}$ is injective. Thus, I show that $D_{\overrightarrow{\Omega}}$ is surjective. For any $\omega^* \in \Omega^*$, consider $\chi_{\overrightarrow{\Omega^*}}(\omega^*) \in \Omega^{**}$. Since $\chi_{\overrightarrow{\Omega}}$ is surjective, there is $\omega \in \Omega$ such that $\Theta(\omega) = \Theta^*(\omega^*)$ and $\{e \in \mathcal{L} \mid \omega^* \in B_i^*([e])\} = \{e \in \mathcal{L} \mid \omega \in B_i([\![e]\!]_{\overrightarrow{\Omega}})\}$ for each $i \in I$. Hence, to show that $\omega^* = D_{\overrightarrow{\Omega}}(\omega)$, I show $e \in_1 \omega^*$ iff $e \in_1 D_{\overrightarrow{\Omega}}(\omega)$. For any $E \in \mathcal{A}_\kappa(\mathcal{S})$, $E \in_1 \omega^*$ iff $\Theta^*(\omega^*) \in E$ iff $\Theta(\omega) \in E$ iff $E \in_1 D_{\overrightarrow{\Omega}}(\omega)$. Next, assume $e \in_1 \omega^*$ iff $e \in_1 D_{\overrightarrow{\Omega}}(\omega)$. Then, $(\neg e) \in_1 \omega^*$ iff $e \notin_1 \omega^*$ iff $e \notin_1 D_{\overrightarrow{\Omega}}(\omega)$ iff $(\neg e) \in_1 D_{\overrightarrow{\Omega}}(\omega)$. Next, assume $e \in_1 \omega^*$ iff $e \in_1 D_{\overrightarrow{\Omega}}(\omega)$ for all $e \in \mathcal{E}$, where $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ and $|\mathcal{E}| < \kappa$. Then, $\bigwedge \mathcal{E} \in_1 \omega^*$ iff $e \in_1 \omega^*$ for all $e \in \mathcal{E}$ iff $e \in_1 D_{\overrightarrow{\Omega}}(\omega)$ for all $e \in \mathcal{E}$ iff $\bigwedge \mathcal{E} \in_1 D_{\overrightarrow{\Omega}}(\omega)$. Next, $\beta_i(e) \in \omega^*$ iff $\omega^* \in B_i^*([e])$ iff $\omega \in B_i([\![e]\!]_{\overrightarrow{\Omega}})$ iff $\beta_i(e) \in_1 D_{\overrightarrow{\Omega}}(\omega)$. □

Proof of Proposition 2. For Part (1), it suffices to show that if Φ is satisfiable then it is satisfiable in $\overrightarrow{\Omega^*}$. If there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega \in [\![f]\!]_{\overrightarrow{\Omega}} = D^{-1}([f])$ for all $f \in \Phi$, then $D(\omega) \in [f] = [\![f]\!]_{\overrightarrow{\Omega^*}}$ for all $f \in \Phi$.

For the first assertion of Part (2), it is enough to show that $\Phi \models_{\overrightarrow{\Omega^*}} e$ implies $\Phi \models e$. Let $\overrightarrow{\Omega}$ be a belief space. If $\omega \in [\![f]\!]_{\overrightarrow{\Omega}} = D^{-1}([f])$ for all $f \in \Phi$, then $D(\omega) \in [f] = [\![f]\!]_{\overrightarrow{\Omega^*}}$ for all $f \in \Phi$. By assumption, $D(\omega) \in [\![e]\!]_{\overrightarrow{\Omega^*}} = [e]$, i.e., $\omega \in D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}}$. Thus, $\Phi \models e$. The second assertion can be seen as a special case of the first. Or, for any belief space $\overrightarrow{\Omega}$, $[\![e]\!]_{\overrightarrow{\Omega}} = D^{-1}([e]) = D^{-1}([\![e]\!]_{\overrightarrow{\Omega^*}}) = D^{-1}(\Omega^*) = \Omega$.

For Part (3), let Ω consist of $\{s\} \sqcup \Phi \in \mathcal{P}(S \sqcup \mathcal{L})$ such that Φ is maximally satisfiable and that, for any $E \in \mathcal{A}_\kappa(\mathcal{S})$, $s \in E$ iff $E \in \Phi$. I show $\Omega^* = \Omega$ in two steps. The first step establishes $\Omega^* \subseteq \Omega$. The second step proves $\Omega \subseteq \Omega^*$ by showing that there exists a belief space on Ω such that its description map is an inclusion map.

Step 1. Take $\omega^* \in \Omega^*$. Denote $\omega^* = \{s\} \cup \Phi$, i.e., $s \in_0 \omega^*$ and $\Phi = \{e \in \mathcal{L} \mid e \in_1 \omega^*\}$. Since $\omega^* \in [e] = [\![e]\!]_{\overrightarrow{\Omega^*}}$ for all $e \in \Phi$, the set Φ is satisfiable. To show it is maximally satisfiable, take a satisfiable set of expressions Ψ with $\Phi \subseteq \Psi$. If there is $e \in \Psi \setminus \Phi$, then $(\neg e) \in \Phi \subseteq \Psi$. Then, Ψ is not satisfiable, a contradiction. Thus, $\Phi = \Psi$, i.e., Φ is maximally satisfiable. Next, $\Theta^*(\omega^*) = s$. For any $E \in \mathcal{A}_\kappa(\mathcal{S})$, $s \in E$ iff $E \in_1 \omega^*$, i.e., $E \in \Phi$.

Step 2. I construct, in four substeps, a belief space $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta\rangle$ on Ω such that the description map $D_{\overrightarrow{\Omega}}$ is an inclusion map. To that end, observe that, for any maximally satisfiable set Φ, there are a belief space $\overrightarrow{\Omega'}$ and $\omega' \in \Omega'$ with $\Phi = \{e \in \mathcal{L} \mid \omega' \in [\![e]\!]_{\overrightarrow{\Omega'}}\}$ (i.e., $e \in \Phi$ iff $\omega' \in [\![e]\!]_{\overrightarrow{\Omega'}}$).

Step 2.1. Slightly abusing the notation, define $[e]_{\overrightarrow{\Omega}} := \{\omega \in \Omega \mid e \in_1 \omega\}$ for each $e \in \mathcal{L}$ (Step 2.4 establishes $[\cdot]_{\overrightarrow{\Omega}} = [\cdot]$). I show that $\mathcal{D} := \{[e]_{\overrightarrow{\Omega}} \in \mathcal{P}(\Omega) \mid e \in \mathcal{L}\}$ is a κ-algebra. First, $\Omega = [S]_{\overrightarrow{\Omega}} \in \mathcal{D}$. Second, I show $[\neg e]_{\overrightarrow{\Omega}} = \neg[e]_{\overrightarrow{\Omega}}$. In fact, $\omega \in [\neg e]_{\overrightarrow{\Omega}}$ iff $(\neg e) \in_1 \omega$ iff there are a belief space $\overrightarrow{\Omega'}$ and $\omega' \in \Omega'$ with $\omega' \in [\![\neg e]\!]_{\overrightarrow{\Omega'}} = \neg[\![e]\!]_{\overrightarrow{\Omega'}}$

iff $e \notin_1 \omega$ iff $\omega \notin [e]_{\overrightarrow{\Omega}}$ iff $\omega \in (\neg[e]_{\overrightarrow{\Omega}})$. Now, $\neg[e]_{\overrightarrow{\Omega}} = [\neg e]_{\overrightarrow{\Omega}} \in \mathcal{D}$. Third, I show $\bigcap_{e\in\mathcal{E}}[e]_{\overrightarrow{\Omega}} = [\bigwedge\mathcal{E}]_{\overrightarrow{\Omega}}$, where $\mathcal{E}\in\mathcal{P}(\mathcal{L})\setminus\{\emptyset\}$ and $|\mathcal{E}|<\kappa$. By definition, $\omega\in\bigcap_{e\in\mathcal{E}}[e]_{\overrightarrow{\Omega}}$ iff $\omega\in[e]_{\overrightarrow{\Omega}}$, i.e., $e\in_1\omega$, for all $e\in\mathcal{E}$ iff there are a belief space $\overrightarrow{\Omega'}$ and $\omega'\in\Omega'$ with $\omega'\in\bigcap_{e\in\mathcal{E}}[\![e]\!]_{\overrightarrow{\Omega'}} = [\![\bigwedge\mathcal{E}]\!]_{\overrightarrow{\Omega'}}$ iff $\bigwedge\mathcal{E}\in_1\omega$ iff $\omega\in[\bigwedge\mathcal{E}]_{\overrightarrow{\Omega}}$. Now, $\bigcap_{e\in\mathcal{E}}[e]_{\overrightarrow{\Omega}} = [\bigwedge\mathcal{E}]_{\overrightarrow{\Omega}}\in\mathcal{D}$.

Step 2.2. For each $[e]_{\overrightarrow{\Omega}}\in\mathcal{D}$, define $B_i([e]_{\overrightarrow{\Omega}}) := [\beta_i(e)]_{\overrightarrow{\Omega}}$. First, I show that B_i is well-defined. Assume $[e]_{\overrightarrow{\Omega}} = [f]_{\overrightarrow{\Omega}}$: for any $\omega\in\Omega$, $e\in_1\omega$ iff $f\in_1\omega$. I show $[\![e]\!]_{\overrightarrow{\Omega'}} = [\![f]\!]_{\overrightarrow{\Omega'}}$. If $\tilde\omega\in[\![e]\!]_{\overrightarrow{\Omega'}}$, then $e\in\Phi := \{\hat e\in\mathcal{L}\mid\tilde\omega\in[\![\hat e]\!]_{\overrightarrow{\Omega'}}\}$. Since Φ is maximally satisfiable, $f\in\Phi$, i.e., $\tilde\omega\in[\![f]\!]_{\overrightarrow{\Omega'}}$. Similarly, $\tilde\omega\in[\![f]\!]_{\overrightarrow{\Omega'}}$ implies $\tilde\omega\in[\![e]\!]_{\overrightarrow{\Omega'}}$. Now, if $\omega\in B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$, then there are a belief space $\overrightarrow{\Omega'}$ and $\omega'\in\Omega'$ with $\omega'\in[\![\beta_i(e)]\!]_{\overrightarrow{\Omega'}} = B_i'([\![e]\!]_{\overrightarrow{\Omega'}}) = B_i'([\![f]\!]_{\overrightarrow{\Omega'}}) = [\![\beta_i(f)]\!]_{\overrightarrow{\Omega'}}$. Hence, $\omega\in[\beta_i(f)]_{\overrightarrow{\Omega}} = B_i([f]_{\overrightarrow{\Omega}})$, establishing $B_i([e]_{\overrightarrow{\Omega}})\subseteq B_i([f]_{\overrightarrow{\Omega}})$. Similarly, $B_i([f]_{\overrightarrow{\Omega}})\subseteq B_i([e]_{\overrightarrow{\Omega}})$.

Second, I show that each B_i inherits the properties of beliefs in Definition 2 imposed in a given category of belief spaces. For Monotonicity, let $\omega\in B_i([e]_{\overrightarrow{\Omega}})$ and $[e]_{\overrightarrow{\Omega}}\subseteq[f]_{\overrightarrow{\Omega}}$. There are a belief space $\overrightarrow{\Omega'}$ and $\omega'\in\Omega'$ with $\omega'\in[\![\beta_i(e)]\!]_{\overrightarrow{\Omega'}} = B_i'([\![e]\!]_{\overrightarrow{\Omega'}})$. Since $[e]_{\overrightarrow{\Omega'}}\subseteq[f]_{\overrightarrow{\Omega'}}$ implies $[\![e]\!]_{\overrightarrow{\Omega'}}\subseteq[\![f]\!]_{\overrightarrow{\Omega'}}$, it follows from Monotonicity of B_i' that $\omega'\in B_i'([\![f]\!]_{\overrightarrow{\Omega'}}) = [\![\beta_i(f)]\!]_{\overrightarrow{\Omega'}}$. Then, $\omega\in[\beta_i(f)]_{\overrightarrow{\Omega}} = B_i([f]_{\overrightarrow{\Omega}})$.

For Necessitation, if $\omega\in\Omega$ then there are a belief space $\overrightarrow{\Omega'}$ and $\omega'\in\Omega'$ such that $\omega'\in[\![e]\!]_{\overrightarrow{\Omega'}}$ for all $e\in_1\omega$. Then, $\omega'\in[\![S]\!]_{\overrightarrow{\Omega'}} = B_i'([\![S]\!]_{\overrightarrow{\Omega'}}) = [\![\beta_i(S)]\!]_{\overrightarrow{\Omega'}}$, where the first equality follows from Necessitation of B_i'. Hence, $(\beta_i(S))\in_1\omega$, i.e., $\omega\in[\beta_i(S)]_{\overrightarrow{\Omega'}} = B_i([S]_{\overrightarrow{\Omega'}})$. Thus, $\Omega = B_i([S]_{\overrightarrow{\Omega}}) = B_i(\Omega)$.

For Non-empty λ-Conjunction, if $\omega\in\bigcap_{e\in\mathcal{E}}B_i([e]_{\overrightarrow{\Omega}})$ (where $\mathcal{E}\in\mathcal{P}(\mathcal{L})\setminus\{\emptyset\}$ and $|\mathcal{E}|<\lambda$), then there are a belief space $\overrightarrow{\Omega'}$ and $\omega'\in\Omega'$ such that $\omega'\in B_i'([\![e]\!]_{\overrightarrow{\Omega'}})$ for all $e\in\mathcal{E}$. Since B_i' satisfies Non-empty λ-Conjunction, $\omega'\in\bigcap_{e\in\mathcal{E}}B_i'([\![e]\!]_{\overrightarrow{\Omega'}})\subseteq B_i'(\bigcap_{e\in\mathcal{E}}[\![e]\!]_{\overrightarrow{\Omega'}}) = B_i'([\![\bigwedge\mathcal{E}]\!]_{\overrightarrow{\Omega'}})$. Then, $\omega\in B_i([\bigwedge\mathcal{E}]_{\overrightarrow{\Omega'}})$.

For the Kripke property, let $\bigcap_{[e]_{\overrightarrow{\Omega}}\in\mathcal{D}:\omega\in B_i([e]_{\overrightarrow{\Omega}})}[e]_{\overrightarrow{\Omega}}\subseteq[f]_{\overrightarrow{\Omega}}$. There are a belief space $\overrightarrow{\Omega'}$ and $\omega'\in\Omega'$ with $\{e\in\mathcal{L}\mid\omega\in B_i([e]_{\overrightarrow{\Omega}})\} = \{e\in\mathcal{L}\mid\omega'\in B_i'([\![e]\!]_{\overrightarrow{\Omega'}})\}$. Then, since $\bigcap_{[\![e]\!]_{\overrightarrow{\Omega'}}\in\mathcal{D}':\omega'\in B_i'([\![e]\!]_{\overrightarrow{\Omega'}})}[\![e]\!]_{\overrightarrow{\Omega'}}\subseteq[\![f]\!]_{\overrightarrow{\Omega'}}$ and since B_i' satisfies the Kripke property, $\omega'\in B_i'([\![f]\!]_{\overrightarrow{\Omega'}})$. Then, $\omega\in B_i([f]_{\overrightarrow{\Omega}})$.

For Consistency, if $\omega\in B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$, then there are a belief space $\overrightarrow{\Omega'}$ and $\omega'\in\Omega'$ with $\omega'\in B_i'([\![e]\!]_{\overrightarrow{\Omega'}})\subseteq(\neg B_i')([\![\neg e]\!]_{\overrightarrow{\Omega'}})$, where the set inclusion follows from Consistency of B_i'. Then, $\omega'\in[\![(\neg\beta_i)(\neg e)]\!]_{\overrightarrow{\Omega'}}$, and thus $\omega\in[(\neg\beta_i)(\neg e)]_{\overrightarrow{\Omega}} = (\neg B_i)(\neg[e]_{\overrightarrow{\Omega}})$. For Truth Axiom, if $\omega\in B_i([e]_{\overrightarrow{\Omega}})$ then there are a belief space $\overrightarrow{\Omega'}$ and $\omega'\in\Omega'$ such that $\omega'\in[\![\beta_i(e)]\!]_{\overrightarrow{\Omega'}} = B_i'([\![e]\!]_{\overrightarrow{\Omega'}})\subseteq[\![e]\!]_{\overrightarrow{\Omega'}}$. Thus, $\omega\in[e]_{\overrightarrow{\Omega}}$.

For Positive Introspection, if $\omega\in B_i([e]_{\overrightarrow{\Omega}})$, then there are a belief space $\overrightarrow{\Omega'}$ and $\omega'\in\Omega'$ with $\omega'\in[\![\beta_i(e)]\!]_{\overrightarrow{\Omega'}} = B_i'([\![e]\!]_{\overrightarrow{\Omega'}})\subseteq B_i'B_i'([\![e]\!]_{\overrightarrow{\Omega'}}) = [\![\beta_i\beta_i(e)]\!]_{\overrightarrow{\Omega'}}$, where the set inclusion follows from Positive Introspection of B_i'. Then, $\omega\in[\beta_i\beta_i(e)]_{\overrightarrow{\Omega}} = B_iB_i([e]_{\overrightarrow{\Omega}})$. The proof for Negative Introspection is similar.

Step 2.3. For each $\omega \in \Omega$, let $\Theta(\omega)$ be the unique $s \in S$ with $s \in_0 \omega$. Since $\Theta^{-1}(E) = [E]_{\overrightarrow{\Omega}} \in \mathcal{D}$ for all $E \in \mathcal{A}_\kappa(\mathcal{S})$, $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_\kappa(\mathcal{S}))$ is measurable.

Step 2.4. So far, I have constructed a belief space $\overrightarrow{\Omega}$. To show that the description map $D_{\overrightarrow{\Omega}}$ is an inclusion map, I start with showing that $[\cdot]_{\overrightarrow{\Omega}} = [\![\cdot]\!]_{\overrightarrow{\Omega}}$ (once $\overrightarrow{\Omega} = \overrightarrow{\Omega^*}$ is established, $[\cdot]_{\overrightarrow{\Omega}} = [\![\cdot]\!]_{\overrightarrow{\Omega^*}} = [\cdot]$). For each $E \in \mathcal{A}_\kappa(\mathcal{S})$, $[E]_{\overrightarrow{\Omega}} = \Theta^{-1}(E) = [\![E]\!]_{\overrightarrow{\Omega}}$. If $[e]_{\overrightarrow{\Omega}} = [\![e]\!]_{\overrightarrow{\Omega}}$, then $[\neg e]_{\overrightarrow{\Omega}} = \neg[e]_{\overrightarrow{\Omega}} = \neg[\![e]\!]_{\overrightarrow{\Omega}} = [\![\neg e]\!]_{\overrightarrow{\Omega}}$ and $[\beta_i(e)]_{\overrightarrow{\Omega}} = B_i([e]_{\overrightarrow{\Omega}}) = B_i([\![e]\!]_{\overrightarrow{\Omega}}) = [\![\beta_i(e)]\!]_{\overrightarrow{\Omega}}$. Assume $[e]_{\overrightarrow{\Omega}} = [\![e]\!]_{\overrightarrow{\Omega}}$ for each $e \in \mathcal{E}$, where $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ and $|\mathcal{E}| < \kappa$. Then, $[\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}} = \bigcap_{e \in \mathcal{E}} [e]_{\overrightarrow{\Omega}} = \bigcap_{e \in \mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega}} = [\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}}$.

Finally, I establish $D_{\overrightarrow{\Omega}}(\omega) = \omega$ for all $\omega \in \Omega$. First, for any ω, $(\Theta(\omega), 0) \in D_{\overrightarrow{\Omega}}(\omega)$ and $(s, 0) \in \omega$ satisfy $s = \Theta(\omega)$. Second, $e \in_1 D_{\overrightarrow{\Omega}}(\omega)$ iff $D_{\overrightarrow{\Omega}}(\omega) \in [e]$ iff $\omega \in D^{-1}_{\overrightarrow{\Omega}}([e]) = [\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$ iff $e \in_1 \omega$. □

Proof of Proposition 3. For Part (1), $e \notin_1 \omega^*$ iff $\omega^* \notin [e]$ iff $\omega^* \in \neg[e] = [\neg e]$ iff $(\neg e) \in_1 \omega^*$. For Part (2), let $e \in_1 \omega^*$ and $(e \to f) \in_1 \omega^*$. Then, $\omega^* \in [e]$ and $\omega^* \in [e \to f] = [(\neg e) \vee f] = \neg[e] \cup [f]$. Thus, $\omega^* \in [f]$, i.e., $f \in_1 \omega^*$. For Part (3), $\bigwedge \mathcal{E} \in_1 \omega^*$ iff $\omega^* \in [\bigwedge \mathcal{E}] = \bigcap_{e \in \mathcal{E}} [e]$ iff, for all $e \in \mathcal{E}$, $\omega^* \in [e]$, i.e., $e \in_1 \omega^*$. □

Proof of Corollary 1. Part (1) follows from Proposition 3. Part (3) follows from Proposition 3 and the fact that $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg\beta_i)(e) \in_1 \omega^*$ iff $\omega^* \in (\neg B_i^*)([e]) \cap (\neg B_i^*)(\neg B_i^*)([e])$. Thus, I prove Part (2).

First, if $\beta_i(e) \notin_1 \omega^*$ and $\beta_i(\neg e) \notin_1 \omega^*$, then by Proposition 3, $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e) \in_1 \omega^*$. Second, assume Consistency on i's beliefs. Since $B_i^*([e]) \cap B_i^*(\neg[e]) = \emptyset$, $\beta_i(e) \in_1 \omega^*$ and $\beta_i(\neg e) \in_1 \omega^*$ do not hold simultaneously. If $\beta_i(e) \in_1 \omega^*$ then $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e) \notin_1 \omega^*$. If $\beta_i(\neg e) \in_1 \omega^*$ then $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e) \notin_1 \omega^*$. Also, $(\neg\beta_i)(e) \wedge (\neg\beta_i)(\neg e) \in_1 \omega^*$ implies $\beta_i(e) \notin_1 \omega^*$ and $\beta_i(\neg e) \notin_1 \omega^*$. Conversely, suppose exactly one of the three conditions holds. If $\omega^* \in B_i^*([e])$ then $\beta_i(e) \in_1 \omega^*$. Then, $\beta_i(\neg e) \notin_1 \omega^*$, i.e., $\omega^* \in (\neg B_i^*)([\neg e]) = (\neg B_i^*)([e]^c)$, establishing Consistency. □

Proof of Theorem 2. Step 1. The proof consists of two steps. The first step shows that Ω^* is a coherent set of descriptions. For (1a), take $\omega^* \in \Omega^*$. There is a unique nature state $s = \Theta^*(\omega^*)$ with $s \in_0 \omega^*$. For any $E \in \mathcal{A}_\kappa(\mathcal{S})$ with $s \in E$, $\omega^* \in (\Theta^*)^{-1}(E) = [E]$, i.e., $E \in_1 \omega^*$. Conditions (1b) to (1e) follow from Proposition 3.

Next, consider (2a). If $(e \leftrightarrow f)$ is valid in $\overrightarrow{\Omega^*}$, then $[\![e]\!]_{\overrightarrow{\Omega^*}} = [\![f]\!]_{\overrightarrow{\Omega^*}}$ and $B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = B_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})$, i.e., $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega^*}} = [\![\beta_i(f)]\!]_{\overrightarrow{\Omega^*}}$. It follows that $[\![\beta_i(e) \leftrightarrow \beta_i(f)]\!]_{\overrightarrow{\Omega}} = \Omega^*$, i.e., $(\beta_i(e) \leftrightarrow \beta_i(f))$ is valid in $\overrightarrow{\Omega^*}$.

For (2b), similarly to the above argument, if $(e \to f)$ is valid in $\overrightarrow{\Omega^*}$, then $[\![e]\!]_{\overrightarrow{\Omega^*}} \subseteq [\![f]\!]_{\overrightarrow{\Omega^*}}$ and thus $B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) \subseteq B_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})$, i.e., $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega^*}} \subseteq [\![\beta_i(f)]\!]_{\overrightarrow{\Omega^*}}$. Then, $[\![\beta_i(e) \to \beta_i(f)]\!]_{\overrightarrow{\Omega^*}} = \Omega^*$, i.e., $(\beta_i(e) \to \beta_i(f))$ is valid in $\overrightarrow{\Omega^*}$.

For (2c), by supposition, $\bigcap\{[\![f]\!]_{\overrightarrow{\Omega^*}} \in \mathcal{D}^* \mid \omega^* \in B_i^*([\![f]\!]_{\overrightarrow{\Omega^*}})\} \subseteq [\![e]\!]_{\overrightarrow{\Omega^*}}$. By the Kripke property of $\overrightarrow{\Omega^*}$, $\omega^* \in B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = [\beta_i(e)]$, i.e., $\beta_i(e) \in_1 \omega^*$.

Next, I show (3). Fix $\omega^* \in \Omega^*$. It is enough to show that each of the following expressions is valid in $\overrightarrow{\Omega^*}$. For Necessitation, consider $[\![\beta_i(S)]\!]_{\overrightarrow{\Omega^*}} = B_i^*([\![S]\!]_{\overrightarrow{\Omega^*}}) = B_i^*(\Omega^*) = \Omega^*$. For Non-empty λ-Conjunction, consider: $[\![(\bigwedge_{e\in\mathcal{E}} \beta_i(e)) \to \beta_i(\bigwedge \mathcal{E})]\!]_{\overrightarrow{\Omega^*}} = (\neg \bigcap_{e\in\mathcal{E}} B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}})) \cup B_i^*(\bigcap_{e\in\mathcal{E}} [\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*$. For Consistency, consider $[\![\beta_i(e) \to (\neg\beta_i)(\neg e)]\!]_{\overrightarrow{\Omega^*}} = (\neg B_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup (\neg B_i^*)(\neg [\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*$. For Truth Axiom, consider $[\![\beta_i(e) \to e]\!]_{\overrightarrow{\Omega^*}} = (\neg B_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup [\![e]\!]_{\overrightarrow{\Omega^*}} = \Omega^*$. For Positive Introspection, consider $[\![\beta_i(e) \to \beta_i\beta_i(e)]\!]_{\overrightarrow{\Omega^*}} = (\neg B_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup B_i^* B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*$. For Negative Introspection, consider $[\![(\neg\beta_i)(e) \to \beta_i(\neg\beta_i)(e)]\!]_{\overrightarrow{\Omega^*}} = B_i^*([\![e]\!]_{\overrightarrow{\Omega^*}}) \cup B_i^*(\neg B_i^*)([\![e]\!]_{\overrightarrow{\Omega^*}}) = \Omega^*$.

Step 2. The second step shows that $\Omega \subseteq \Omega^*$ for any set Ω of coherent descriptions. To that end, I introduce a belief space $\overrightarrow{\Omega}$ on Ω, and show that the description map $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map.

Step 2.1. By slightly abusing the notation, let $[e]_{\overrightarrow{\Omega}} := \{\omega \in \Omega \mid e \in_1 \omega\}$ for each $e \in \mathcal{L}$ (it turns out that $[\cdot]_{\overrightarrow{\Omega}} = [\cdot] \cap \Omega$). Let $\mathcal{D} := \{[e]_{\overrightarrow{\Omega}} \in \mathcal{P}(\Omega) \mid e \in \mathcal{L}\}$. I show $(\Omega, \mathcal{D})$ is a κ-algebra. First, $\Omega = [S]_{\overrightarrow{\Omega}} \in \mathcal{D}$. Second, $[\neg e]_{\overrightarrow{\Omega}} = \neg [e]_{\overrightarrow{\Omega}}$ follows because $\omega \in [\neg e]_{\overrightarrow{\Omega}}$ iff $(\neg e) \in_1 \omega$ iff $e \notin_1 \omega$ iff $\omega \in \neg[e]_{\overrightarrow{\Omega}}$. Third, I show $[\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}} = \bigcap_{e\in\mathcal{E}} [e]_{\overrightarrow{\Omega}}$ for any $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \kappa$. Indeed, $\omega \in \bigcap_{e\in\mathcal{E}} [e]_{\overrightarrow{\Omega}}$ iff $e \in_1 \omega$ for all $e \in \mathcal{E}$ iff $\bigwedge \mathcal{E} \in_1 \omega$ iff $\omega \in [\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}}$.

Step 2.2. Define $\Theta : \Omega \to S$ as follows: for each $\omega \in \Omega$, let $\Theta(\omega)$ be the unique $s \in S$ with $s \in_0 \omega$. The map Θ is a well-defined measurable map such that $(\Theta)^{-1}(E) = [E]_{\overrightarrow{\Omega}}$ for each $E \in \mathcal{A}_\kappa(\mathcal{S})$. If $\omega \in [E]_{\overrightarrow{\Omega}}$, then $E \in_1 \omega$. Hence, $\Theta(\omega) \in E$, i.e., $\omega \in \Theta^{-1}(E)$. Conversely, if $\omega \in \Theta^{-1}(E)$ then $\Theta(\omega) \in E$, and thus $E \in_1 \omega$. Hence, $\omega \in [E]_{\overrightarrow{\Omega}}$.

Step 2.3. Fix $i \in I$, and define i's belief operator $B_i : \mathcal{D} \to \mathcal{D}$ by: $B_i([e]_{\overrightarrow{\Omega}}) := [\beta_i(e)]_{\overrightarrow{\Omega}}$ for each $[e] \in \mathcal{D}$. I show B_i is well-defined. If $[e]_{\overrightarrow{\Omega}} = [f]_{\overrightarrow{\Omega}}$, then $[(e \leftrightarrow f)]_{\overrightarrow{\Omega}} = \Omega$. This implies $[(\beta_i(e) \leftrightarrow \beta_i(f))]_{\overrightarrow{\Omega}} = \Omega$. Thus, $[\beta_i(e)] = [\beta_i(f)]$.

Next, I show that B_i reflects assumptions on beliefs. For Necessitation, $\Omega = [\beta_i(S)]_{\overrightarrow{\Omega}} = B_i([S]_{\overrightarrow{\Omega}}) = B_i(\Omega)$. For Monotonicity, take $[e]_{\overrightarrow{\Omega}}, [f]_{\overrightarrow{\Omega}} \in \mathcal{D}$ with $[e]_{\overrightarrow{\Omega}} \subseteq [f]_{\overrightarrow{\Omega}}$. Then, $[e \to f]_{\overrightarrow{\Omega}} = \Omega$. It follows $[\beta_i(e) \to \beta_i(f)]_{\overrightarrow{\Omega}} = \Omega$, i.e., $[\beta_i(e)]_{\overrightarrow{\Omega}} \subseteq [\beta_i(f)]_{\overrightarrow{\Omega}}$. Thus, $B_i([e]_{\overrightarrow{\Omega}}) \subseteq B_i([f]_{\overrightarrow{\Omega}})$.

For Non-empty λ-Conjunction, take $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ with $|\mathcal{E}| < \lambda$. If $\omega \in \bigcap_{e\in\mathcal{E}} B_i([e]_{\overrightarrow{\Omega}}) = [\bigwedge_{e\in\mathcal{E}} \beta_i(e)]_{\overrightarrow{\Omega}}$ then $\bigwedge_{e\in\mathcal{E}} \beta_i(e) \in_1 \omega$. Since $(\bigwedge_{e\in\mathcal{E}} \beta_i(e) \to \beta_i(\bigwedge \mathcal{E})) \in_1 \omega$, it follows $\beta_i(\bigwedge \mathcal{E}) \in_1 \omega$, i.e., $\omega \in [\beta_i(\bigwedge \mathcal{E})]_{\overrightarrow{\Omega}} = B_i(\bigcap_{e\in\mathcal{E}} [e]_{\overrightarrow{\Omega}})$. For the Kripke property, $\omega \in B_i([e]_{\overrightarrow{\Omega}})$ for any $(\omega, [e]_{\overrightarrow{\Omega}}) \in \Omega \times \mathcal{D}$ with $\bigcap \{[f]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid \omega \in B_i([f]_{\overrightarrow{\Omega}})\} \subseteq [e]_{\overrightarrow{\Omega}}$.

For Consistency, if $\omega \in B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$ then $\beta_i(e) \in_1 \omega$. Since $(\beta_i(e) \to (\neg\beta_i)(\neg e)) \in_1 \omega$, it follows $(\neg\beta_i)(\neg e) \in_1 \omega$, i.e., $\omega \in [(\neg\beta_i)(\neg e)]_{\overrightarrow{\Omega}} = (\neg B_i)(\neg [e]_{\overrightarrow{\Omega}})$. For Truth Axiom, if $\omega \in B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$ then $\beta_i(e) \in_1 \omega$. Since $(\beta_i(e) \to e) \in_1 \omega$, it follows $e \in_1 \omega$, i.e., $\omega \in [e]_{\overrightarrow{\Omega}}$. For Positive Introspection, if $\omega \in B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$ then $\beta_i(e) \in_1 \omega$. Since $(\beta_i(e) \to \beta_i\beta_i(e)) \in_1 \omega$, it follows $\beta_i\beta_i(e) \in_1 \omega$, i.e., $\omega \in [\beta_i\beta_i(e)]_{\overrightarrow{\Omega}} = B_i B_i([e]_{\overrightarrow{\Omega}})$. The proof for Negative Introspection is similar.

Step 2.4. So far, $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, \Theta\rangle$ is shown to be a belief space (of the given category). Finally, I demonstrate that the description map $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map (consequently, $\overrightarrow{\Omega}$ is non-redundant and $\Omega \subseteq \Omega^*$) by showing that $[\cdot]_{\overrightarrow{\Omega}} : \mathcal{L} \to \mathcal{D}$, viewed as a mapping, coincides with the semantic interpretation function $[\![\cdot]\!]_{\overrightarrow{\Omega}}$ (consequently, $\overrightarrow{\Omega}$ is minimal).

I show by induction that $[\cdot]_{\overrightarrow{\Omega}} = [\![\cdot]\!]_{\overrightarrow{\Omega}}$. First, fix $E \in \mathcal{A}_\kappa(\mathcal{S})$. Then, $\omega \in [\![E]\!]_{\overrightarrow{\Omega}} = \Theta^{-1}(E)$ iff $\Theta(\omega) \in E$ iff $\omega \in [E]_{\overrightarrow{\Omega}}$. Second, suppose $[\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$. Then, $\omega \in [\![\neg e]\!]_{\overrightarrow{\Omega}} = \neg[\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega \not\in [\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega \not\in [e]_{\overrightarrow{\Omega}}$ iff $\omega \in [\neg e]_{\overrightarrow{\Omega}}$. Also, $[\![\beta_i(e)]\!]_{\overrightarrow{\Omega}} = B_i([\![e]\!]_{\overrightarrow{\Omega}}) = B_i([e]_{\overrightarrow{\Omega}}) = [\beta_i(e)]_{\overrightarrow{\Omega}}$. Third, suppose $[\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$ for all $e \in \mathcal{E}$ with $\mathcal{E} \in \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ and $|\mathcal{E}| < \kappa$. Then, $\omega \in [\![\bigwedge \mathcal{E}]\!]_{\overrightarrow{\Omega}} = \bigcap_{e\in\mathcal{E}}[\![e]\!]_{\overrightarrow{\Omega}}$ iff $\omega \in \bigcap_{e\in\mathcal{E}}[e]_{\overrightarrow{\Omega}} = [\bigwedge \mathcal{E}]_{\overrightarrow{\Omega}}$.

I show that $D(\omega) = \omega$ for all $\omega \in \Omega$. First, $e \in_1 D(\omega)$ iff $D(\omega) \in [e]$ iff $\omega \in D^{-1}([e]) = [\![e]\!]_{\overrightarrow{\Omega}} = [e]_{\overrightarrow{\Omega}}$ iff $e \in_1 \omega$. Second, $s \in_0 \omega$ iff $s = \Theta(\omega) = \Theta^*(D(\omega))$ iff $s \in_0 D(\omega)$. Hence, D is an inclusion map. □

A.3 Section 5

Proof of Corollary 2. Construct $\overrightarrow{\Omega^*} := \langle(\Omega^*, \mathcal{D}^*), (B_i^{*p})_{(i,p)\in I\times[0,1]}, \Theta^*\rangle$ as in the proof of Theorem 1. The set Ω^* is not empty (consider $\overrightarrow{\{s\}}$). To see that $\overrightarrow{\Omega^*}$ is terminal, it suffices to show that the p-belief operators B_i^{*p} satisfy the properties specified in Definition 10 (2).

First, (2a), (2b), (2d), and (2h) follow from Lemma A.1 (1a). Next, (2c) follows from Lemma A.1 (2a). Next, (2g) follows from Lemma A.1 (2a) and (3a).

Next, (2e) and (2f) follow from the following variant of Lemma A.1 (1a). Let $f_{\overrightarrow{\Omega}} : \mathcal{D}^2 \to \mathcal{D}$ and $g_{\overrightarrow{\Omega}} : \mathcal{D}^2 \to \mathcal{D}$ be defined in each probabilistic-belief space $\overrightarrow{\Omega}$ and satisfy, for any measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$, $\varphi^{-1} f_{\overrightarrow{\Omega'}}(E', F') = f_{\overrightarrow{\Omega}}(\varphi^{-1}(E'), \varphi^{-1}(F'))$ and $\varphi^{-1} g_{\overrightarrow{\Omega'}}(E', F') = g_{\overrightarrow{\Omega}}(\varphi^{-1}(E'), \varphi^{-1}(F'))$ for all $E', F' \in \mathcal{D}'$. If $f_{\overrightarrow{\Omega}}(E, F) \subseteq g_{\overrightarrow{\Omega}}(E, F)$ (for all $E, F \in \mathcal{D}$) for every probabilistic-belief space $\overrightarrow{\Omega}$, then $f_{\overrightarrow{\Omega^*}}([e], [f]) \subseteq g_{\overrightarrow{\Omega^*}}([e], [f])$ for all $[e], [f] \in \mathcal{D}^*$.

For (2i), if $[t^*_{B_i^*}(\omega^*)] \subseteq [e]$ then $[t_{B_i}(\omega)] \subseteq D^{-1}[e]$ and thus $\omega \in B_i^1(D^{-1}[e]) = D^{-1}B_i^{*1}([e])$, where a probabilistic-belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ satisfy $\omega^* = D(\omega)$. □

To prove Proposition 4, I show the following preliminary result.

Lemma A.2 (Extension of p-Belief Operators)**.** *Let $(\Omega, \mathcal{D})$ be an $\aleph_0$-algebra, and let $(B_i^p)_{p\in[0,1]}$ be a collection of player i's p-belief operators $B_i^p : \mathcal{D} \to \mathcal{D}$ satisfying Definition 10 (2). Then, there is a unique collection $(\overline{B}_i^p)_{p\in[0,1]}$ of p-belief operators $\overline{B}_i^p : \sigma(\mathcal{D}) \to \sigma(\mathcal{D})$ satisfying Definition 10 (2) and $\overline{B}_i^p|_{\mathcal{D}} = B_i^p$.*

Proof of Lemma A.2. In this proof, denote by $\Delta(\Omega, \mathcal{D})$ the set of countably-additive probability measures on $(\Omega, \mathcal{D})$. Denote by $t_{B_i} : (\Omega, \mathcal{D}) \to (\Delta(\Omega, \mathcal{D}), \mathcal{A}_{\aleph_0}\{\{\mu \in \Delta(\Omega, \mathcal{D}) \mid \mu(E) \geq p\} \mid (E, p) \in \mathcal{D} \times [0, 1]\})$ the measurable type mapping associated with $(B_i^p)_{p\in[0,1]}$. To construct $\overline{B}_i^p$, let $\bar{t}_{B_i}(\omega)$ be the unique Carathéodory extension on $\Delta(\Omega, \sigma(\mathcal{D}))$ from $t_{B_i}(\omega)$. Denote by $\Sigma := \sigma(\{\{\mu \in \Delta(\Omega, \sigma(\mathcal{D})) \mid \mu(E) \geq p\} \mid (E, p) \in \sigma(\mathcal{D}) \times [0, 1]\})$ the σ-algebra on $\Delta(\Omega, \sigma(\mathcal{D}))$. By Heifetz and Samet (1998b, Lemma 4.5), $\Sigma = \sigma(\{\{\mu \in \Delta(\Omega, \sigma(\mathcal{D})) \mid \mu(E) \geq p\} \mid (E, p) \in \mathcal{D} \times [0, 1]\})$.

I show $\bar{t}_{B_i} : (\Omega, \sigma(\mathcal{D})) \to (\Delta(\Omega, \sigma(\mathcal{D})), \Sigma)$ is measurable. For each $(E, p) \in \mathcal{D} \times [0, 1]$, it follows from $\{\omega \in \Omega \mid \bar{t}_{B_i}(\omega)(E) \geq p\} = \{\omega \in \Omega \mid t_{B_i}(\omega)(E) \geq p\}$ that $\bar{t}_{B_i}^{-1}(\{\mu \in \Delta(\Omega, \sigma(\mathcal{D})) \mid \mu(E) \geq p\}) \in \mathcal{D}$. Hence, denote $\overline{B}_i^p(E) := \{\omega \in \Omega \mid \bar{t}_{B_i}(\omega)(E) \geq p\}$ for each $(E, p) \in \sigma(\mathcal{D}) \times [0, 1]$.

To show uniqueness, let $\tilde{B}_i^p$ be an extension. If $\omega \in \tilde{B}_i^p(E)$, then $\bar{t}_{B_i}(\omega)(E) = t_{\tilde{B}_i}(\omega)(E) \geq p$. Thus, $\omega \in \overline{B}_i^p(E)$. The converse holds similarly. □

Proof of Proposition 4. For ease of notation, denote $\mathcal{L}_\lambda^I = \mathcal{L}_\lambda^I(\mathcal{A}_\kappa(\mathcal{S}))$. Also, denote $\mathcal{L} = \mathcal{L}_\kappa^I$. Let $(\mathcal{L}_\alpha)_{\alpha=0}^{\overline{\kappa}}$ be the auxiliary sequence that generates $\mathcal{L}$ as in Remark 1.

Define $[\mathcal{L}_\lambda^I] := \{[e] \in \mathcal{D}^* \mid e \in \mathcal{L}_\lambda^I\}$. As in Lemma 1, $[\mathcal{L}_\lambda^I]$ is an algebra on Ω^*. Since $\mathcal{L}_\lambda^I \subseteq \mathcal{L}$ and since $\mathcal{D}^*$ is a σ-algebra, $\sigma([\mathcal{L}_\lambda^I]) \subseteq \mathcal{D}^*$. To prove the converse set inclusion, I show $[e] \in \sigma([\mathcal{L}_\lambda^I])$ for all $e \in \mathcal{L}$. To that end, I first establish $B_i^{*p}([e]) \in \sigma([\mathcal{L}_\lambda^I])$ for any $[e] \in \sigma([\mathcal{L}_\lambda^I])$, i.e., $B_i^{*p}|_{\sigma([\mathcal{L}_\lambda^I])} : \sigma([\mathcal{L}_\lambda^I]) \to \sigma([\mathcal{L}_\lambda^I])$. Applying Lemma 3 to $B_i^{*p}|_{[\mathcal{L}_\lambda^I]} : [\mathcal{L}_\lambda^I] \to [\mathcal{L}_\lambda^I]$, the operator $B_i^{*p}|_{[\mathcal{L}_\lambda^I]}$ satisfies Definition 10 (2) (with the slight modification that events are restricted to $[\mathcal{L}_\lambda^I]$). It follows from Lemma A.2 that $B_i^{*p}|_{[\mathcal{L}_\lambda^I]}$ uniquely extends to $\sigma([\mathcal{L}_\lambda^I])$, and it coincides with $B_i^{*p}|_{\sigma([\mathcal{L}_\lambda^I])}$. Then, $B_i^{*p}|_{\sigma([\mathcal{L}_\lambda^I])} : \sigma([\mathcal{L}_\lambda^I]) \to \sigma([\mathcal{L}_\lambda^I])$.

Now, I show $[e] \in \sigma([\mathcal{L}_\lambda^I])$ for all $e \in \mathcal{L}$ by induction on the construction of $\mathcal{L}$ (recall Remark 1). For $\alpha = 0$, $[E] \in \sigma([\mathcal{L}_\lambda^I])$ for all $E \in \mathcal{L}_0 = \mathcal{A}_\kappa(\mathcal{S})$. Suppose $[e] \in \sigma([\mathcal{L}_\lambda^I])$ for all $e \in \bigcup_{\beta<\alpha} \mathcal{L}_\beta$. For any $e \in \bigcup_{\beta<\alpha} \mathcal{L}_\beta$, $[\beta_i^p(e)] = B_i^{*p}|_{\sigma([\mathcal{L}_\lambda^I])}([e]) \in \sigma([\mathcal{L}_\lambda^I])$ (note that $\beta_i^p(e)$ is the (syntactic) expression for "i p-believes e"). Thus, $[e] \in \sigma([\mathcal{L}_\lambda^I])$ for all $e \in \mathcal{L}'_\alpha$. Then, $[\neg e] = \neg[e] \in \sigma([\mathcal{L}_\lambda^I])$ for any $e \in \mathcal{L}'_\alpha$. Also, $[\bigwedge \mathcal{F}] = \bigcap_{e\in\mathcal{F}} [e] \in \sigma([\mathcal{L}_\lambda^I])$ for any $\mathcal{F} \subseteq \mathcal{L}'_\alpha$ with $0 < |\mathcal{F}| < \kappa (= \aleph_1)$. □

A.4 Section 6

Proof of Corollary 3. In each belief space (with a common belief operator), identify the common belief operator as the belief operator of a hypothetical player who represents common belief. As in Section 3, construct a candidate terminal κ-belief space $\overrightarrow{\Omega^*} = \langle(\Omega^*, \mathcal{D}^*), (B_i^*)_{i\in I}, C^*, \Theta^*\rangle$. For any belief space $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, C, \Theta\rangle$, $D^{-1}C^*[\cdot] = CD^{-1}[\cdot]$. To show that $\overrightarrow{\Omega^*}$ is terminal, it suffices to show that C^* is a common belief operator: $C^*[e] = \max\{[f] \in \mathcal{J}_I^* \mid [f] \subseteq B_I^*[e]\}$ for any $[e] \in \mathcal{D}^*$.

Let $\omega^* \in C^*[e]$. There are a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $\omega^* = D(\omega)$. Thus, $\omega \in D^{-1}C^*[e] = CD^{-1}[e]$. Since $CD^{-1}[e] \subseteq B_I D^{-1}[e] = D^{-1}B_I^*[e]$, $\omega^* = D(\omega) \in B_I^*[e]$. Thus, $C^*[e] \subseteq B_I^*[e]$. To show $C^*[e] \in \mathcal{J}_I^*$, take any $[f] \in \mathcal{D}^*$ with $C^*[e] \subseteq [f]$. Take $\omega^* \in C^*[e]$. There are a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $\omega^* = D(\omega)$. Since $CD^{-1}[e] = D^{-1}C^*[e] \subseteq D^{-1}[f]$, it follows $D^{-1}C^*[e] =$

$CD^{-1}[e] \subseteq B_I D^{-1}[f] = D^{-1}B_I^*[f]$. Thus, $\omega^* = D(\omega) \in B_I^*[f]$. It follows that $C^*[e] \subseteq \max\{[f] \in \mathcal{J}_I^* \mid [f] \subseteq B_I^*[e]\}$.

To get the converse set inclusion, take any $[f] \in \mathcal{J}_I^*$ with $[f] \subseteq B_I^*[e]$. If $\omega^* \in [f]$, then there are a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $\omega \in D^{-1}[f] \subseteq D^{-1}B_I^*[e] = B_I D^{-1}[e]$. For the belief space $\overrightarrow{\Omega}$, consider $\overrightarrow{\Omega'} = \langle (\Omega, \mathcal{D}'), (B_i|_{\mathcal{D}'})_{i \in I}, C|_{\mathcal{D}'}, \Theta \rangle$ with $\mathcal{D}' = D^{-1}(\mathcal{D}^*)$. One can show that $\overrightarrow{\Omega'}$ is a belief space and that the identify map $\mathrm{id}_\Omega : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism (see also Remark 4 in Section 7.1). Then, $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \mathrm{id}_\Omega$. Since one can retake $\omega^* = D_{\overrightarrow{\Omega'}}(\omega)$, without loss, assume $\mathcal{D} = D^{-1}(\mathcal{D}^*)$.

To establish $\omega^* \in C^*[e]$, it suffices to show that $D^{-1}[f] \in \mathcal{J}_I$, because it implies $\omega \in D^{-1}[f] \subseteq C(D^{-1}[e]) = D^{-1}C^*[e]$. Take any $E' \in \mathcal{D} = D^{-1}(\mathcal{D}^*)$ with $D^{-1}[f] \subseteq E'$. Without loss, one can assume $E' = D^{-1}[e']$ for some $[e'] \in \mathcal{D}^*$ with $[f] \subseteq [e]'$, because $D^{-1}[e' \vee f] = D^{-1}([e]' \cup [f]) = D^{-1}[e'] \cup D^{-1}[f] = D^{-1}[e'] = E'$. Since $[f] \in \mathcal{J}_I^*$, it follows $[f] \subseteq B_I^*[e']$. Thus, $D^{-1}[f] \subseteq D^{-1}B_I^*[e'] = B_I D^{-1}[e'] = B_I(E')$. □

A.5 Section 7

Proof of Proposition 5. Part (1). Let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ be a morphism. In a similar way to Heifetz and Samet (1998a), I show by induction that $\mathcal{C}_\alpha = \varphi^{-1}(\mathcal{C}'_\alpha)$ for all α. For $\alpha = 0$, $\mathcal{C}_0 = \varphi^{-1}(\mathcal{C}'_0)$ follows because $(\Theta)^{-1}(E) = \varphi^{-1}((\Theta')^{-1}(E))$ for any $E \in \mathcal{A}_\kappa(\mathcal{S})$. Suppose $\mathcal{C}_\beta = \varphi^{-1}(\mathcal{C}'_\beta)$ for all $\beta < \alpha$. Then,

$$\begin{aligned}\mathcal{C}_\alpha &= \mathcal{A}_\kappa(\{\varphi^{-1}(E') \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\} \cup \bigcup_{i \in I}\{\varphi^{-1}(B'_i(E')) \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\}) \\ &= \varphi^{-1}(\mathcal{A}_\kappa(\{E' \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\} \cup \bigcup_{i \in I}\{B'_i(E') \in \mathcal{D} \mid E' \in \bigcup_{\beta<\alpha} \mathcal{C}'_\beta\})) = \varphi^{-1}(\mathcal{C}'_\alpha).\end{aligned}$$

Thus, $\mathcal{C}'_\alpha = \mathcal{C}'_{\alpha+1}$ implies $\mathcal{C}_\alpha = \mathcal{C}_{\alpha+1}$. Hence, if the κ-rank of $\overrightarrow{\Omega'}$ is α then that of $\overrightarrow{\Omega}$ is at most α.

Part (2). Fix a λ-belief space $\overrightarrow{\Omega}$ (where $\lambda \geq \kappa$). Let $\mathcal{D}_\alpha := \mathcal{A}_\kappa(\{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}_\alpha\})$ for each $\alpha \leq \overline{\kappa}$, where $(\mathcal{L}_\alpha)_{\alpha=0}^{\overline{\kappa}}$ is defined as in Remark 1 so that $\mathcal{L} = \mathcal{L}_{\overline{\kappa}}$. I show $\mathcal{D}_\alpha = \mathcal{C}_\alpha$ for all $\alpha \leq \overline{\kappa}$. For $\alpha = 0$, $\mathcal{D}_0 = \{\Theta^{-1}(E) \in \mathcal{D} \mid E \in \mathcal{A}_\kappa(\mathcal{S})\} = \mathcal{C}_0$. If $\mathcal{D}_\beta = \mathcal{C}_\beta$ for all $\beta < \alpha$, then $\mathcal{D}_\alpha = \mathcal{A}_\kappa((\bigcup_{\beta<\alpha} \mathcal{D}_\beta) \cup \bigcup_{i \in I}\{B_i([\![e]\!]) \in \mathcal{D} \mid [\![e]\!] \in \bigcup_{\beta<\alpha} \mathcal{D}_\beta\}) = \mathcal{C}_\alpha$. Hence, $\mathcal{C}_{\overline{\kappa}} = \mathcal{D}_{\overline{\kappa}} = \{[\![e]\!]_{\overrightarrow{\Omega}} \in \mathcal{D} \mid e \in \mathcal{L}\}$, implying $\mathcal{C}_{\overline{\kappa}} = \mathcal{C}_{\overline{\kappa}+1}$, i.e., the κ-rank of $\overrightarrow{\Omega}$ is at most $\overline{\kappa}$. □

Proof of Proposition 6. Part (1). Since the λ-belief space $\overrightarrow{\Omega^*_\lambda}$ is also a κ-belief space, there is a unique morphism $D_{\overrightarrow{\Omega^*_\lambda}} : \overrightarrow{\Omega^*_\lambda} \to \overrightarrow{\Omega^*_\kappa}$, which takes the following form. While each $\omega^* = \{s \in S \mid s \in_0 \omega^*\} \sqcup \{e \in \mathcal{L}^I_\lambda(\mathcal{A}_\lambda(\mathcal{S})) \mid e \in_1 \omega^*\} \in \Omega^*_\lambda$ consists of the unique nature state s and expressions $e \in \mathcal{L}^I_\lambda(\mathcal{A}_\lambda(\mathcal{S}))$ that obtain, $D_{\overrightarrow{\Omega^*_\lambda}}(\omega^*) = \{s \in S \mid s \in_0$

$\omega^*\} \sqcup \{e \in \mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S})) \mid e \in_1 \omega^*\}$ consists of the same unique nature state s and expressions $e \in \mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S}))$ (observe $\mathcal{L}^I_\kappa(\mathcal{A}_\kappa(\mathcal{S})) \subseteq \mathcal{L}^I_\lambda(\mathcal{A}_\lambda(\mathcal{S}))$) that obtain. Thus, $D_{\overrightarrow{\Omega_\lambda}}$ is a surjective morphism (that identifies two states ω^* and $\tilde{\omega}^*$ in $\overrightarrow{\Omega^*_\lambda}$ if they induce the same κ-belief hierarchies, i.e., $D_{\overrightarrow{\Omega_\lambda}}(\omega^*) = D_{\overrightarrow{\Omega_\lambda}}(\tilde{\omega}^*)$). Hence, $|\Omega^*_\kappa| \leq |\Omega^*_\lambda|$.

Part (2). To simplify the proof, I make the following assumptions. Since the proof does not depend on the cardinality of I, let $I = \{1, 2\}$. Next, by the (second) remark following Definition 7, assume all the properties of beliefs in Definition 2. Next, assume $(S, \mathcal{S}) = (\{s_0, s_1\}, \mathcal{P}(S))$ (the proof goes through by taking $s_1 \in E$ and $s_0 \in E^c$ for E in the statement of the proposition). First, the knowledge space $\overrightarrow{\Omega}$ constructed by Hart, Heifetz, and Samet (1996) is a non-redundant κ-belief space with $|\Omega| \geq 2^{\aleph_0}$. Since the morphism $D_{\overrightarrow{\Omega}}$ is injective, $2^{\aleph_0} \leq |\Omega| \leq |\Omega^*_\kappa|$. Second, Heifetz and Samet (1998a, Theorem 2.5) construct a non-redundant κ-belief (knowledge) space $\overrightarrow{\Omega'}$ with $|\Omega'| = \kappa$. Since the morphism $D_{\overrightarrow{\Omega'}}$ is injective, $\kappa = |\Omega'| \leq |\Omega^*_\kappa|$. □

Remark A.1 (Extension of the Domain)**.** Let $(\Omega, \mathcal{D})$ be a κ-algebra. Let $B_i : \mathcal{D} \to \mathcal{D}$ satisfy the Kripke property. Define $\overline{B}_i : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ by $\overline{B}_i(E) := \{\omega \in \Omega \mid b_{B_i}(\omega) \subseteq E\}$ for each $E \in \mathcal{P}(\Omega)$. By construction, $\overline{B}_i$ satisfy the Kripke property. Also, $\overline{B}_i$ inherits Consistency, Truth Axiom, Positive Introspection, and Negative Introspection from B_i. Moreover, $B_i = \overline{B}_i|_{\mathcal{D}}$ and $b_{\overline{B}_i} = b_{B_i}$.

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