The Existence of Universal Qualitative Belief Spaces

Online Supplementary Appendix

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The Supplementary Appendix provides additional discussions and results. It is organized as follows. Appendix B supplements Remark 3 in Section 3. It introduces a notion of "behavioral equivalence" between two states, which induces the same belief hierarchy. Appendix C supplements Section 5, demonstrating the existence of a terminal space for various settings in which players interactively reason. Appendix C.1 establishes a terminal conditional-belief space, and Appendix C.2 a terminal dynamic knowledge-and-belief space. Appendix C.3 discusses further possible applications: knowledge and unawareness, preferences, and expectations.

Appendix D supplements Section 6, studying correlated equilibria. It shows that the correlating device (i.e., the underlying state space) of a correlated equilibrium can be replaced with the belief hierarchies that the state space induces if the correlated equilibrium as a belief space is non-redundant and minimal. Appendix E supplements Section 7. It provides a characterization of minimality. The proofs are relegated to Appendix F.

B Section 3

Remark S.1 (Behavioral Equivalence)**.** Two states, possibly residing in different belief spaces, are identified when the descriptions (i.e., corresponding nature states and interactive beliefs) are identical. While Fagin (1994, Section 4) and Mertens and Zamir (1985) study a notion of equivalence, the notion is extensively studied as one notion of bisimulations called "behavioral equivalence" (Kurz, 2000) in theoretical computer science. For notions of bisimulations ("observational equivalence"), see, for instance, Jacobs and Rutten (2012), Kurz (2000), and Rutten (2000).

Let $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ be belief spaces in a given category. States $(\omega, \omega') \in \Omega \times \Omega'$ are *behaviorally equivalent* if there are a belief space $\overrightarrow{\Omega''}$ and morphisms $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega''}$ and $\varphi' : \overrightarrow{\Omega'} \to \overrightarrow{\Omega''}$ with $\varphi(\omega) = \varphi'(\omega')$.

I show $(\omega, \omega') \in \Omega \times \Omega'$ are behaviorally equivalent iff $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\omega')$. This means that, in order for two states to be identical in terms of nature states and belief hierarchies, it suffices to show they are behaviorally equivalent.

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The proof goes as follows. If $(\omega, \omega') \in \Omega \times \Omega'$ are behaviorally equivalent, then $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega''}}(\varphi(\omega)) = D_{\overrightarrow{\Omega''}}(\varphi'(\omega')) = D_{\overrightarrow{\Omega'}}(\omega')$. The converse holds once the description map is shown to be a morphism (Lemmas 1, 2, and 3).

If $D_{\overrightarrow{\Omega}}$ is injective, i.e., $\overrightarrow{\Omega}$ is non-redundant, then the "co-induction proof principle" (e.g., Jacobs and Rutten, 2012; Kurz, 2000; Rutten, 2000) holds: for two states ω and ω' in Ω to be identical, it suffices to show they are behaviorally equivalent.

C Section 5

The framework of this paper applies to various settings in which players interactively reason as long as their beliefs are represented by belief operators. Section 5 has established a terminal probabilistic-belief space. The Supplementary Appendix discusses further applications. Appendix C.1 discusses a terminal space for conditional probability systems (CPSs). Appendix C.2 introduces players' knowledge and qualitative beliefs indexed by time. Appendix C.3 briefly discusses further possible applications, namely, terminal knowledge-unawareness, preference, and expectation spaces.

C.1 Terminal Conditional-Belief Space

I construct a terminal space for conditional belief systems (CPSs) (Battigalli and Siniscalchi, 1999; Guarino, 2017). While this subsection focuses on CPS-based conditional beliefs, it suggests that a terminal conditional-belief space exists for a wide variety of qualitative or probabilistic beliefs, which can be used for epistemic analyses of dynamic games including dynamic psychological games.[1]

Call a triple $(\Omega, \mathcal{D}, \mathcal{C})$ a *conditional space* if: (i) $(\Omega, \mathcal{D})$ is an $\aleph_1$-algebra; (ii) $\mathcal{C}$ is a non-empty sub-collection of $\mathcal{D}$ with $\emptyset \notin \mathcal{C}$; and (iii) there exists a *conditional probability system* (*CPS*) μ on $(\Omega, \mathcal{D}, \mathcal{C})$. A function $\mu(\cdot|\cdot) : \mathcal{D} \times \mathcal{C} \to [0,1]$ is a CPS if: (i) each $\mu(\cdot|C)$ is a countably-additive probability measure; (ii) Normality: $\mu(C|C) = 1$ for each $C \in \mathcal{C}$; and (iii) Chain Rule: $\mu(E|C) = \mu(E|D)\mu(D|C)$ for any $E \in \mathcal{D}$ and $C, D \in \mathcal{C}$ with $E \subseteq D \subseteq C$. Call each $C \in \mathcal{C}$ a *conditioning event* (or a *condition*, for short). Fix a conditional space $(S, \mathcal{A}_{\aleph_1}(\mathcal{S}), \mathcal{C}_S)$, where $(S, \mathcal{S})$ is the set of nature states and $S \in \mathcal{C}_S$.

[1] First, as discussed in footnote 35 of the main text, a state space here is not restricted to a product space. The framework here does not presuppose any topological restriction on nature states or any cardinal restriction on conditioning events, either. Thus, the construction of the terminal conditional-belief space would be complementary to Battigalli and Siniscalchi (1999) and Guarino (2017). Second, it would be interesting to examine "lexicographic probability systems (LPSs)" or hypothetical knowledge (also, belief revision and counterfactual reasoning) within this framework. Tsakas (2014) defines a formal equivalence between conditional and lexicographic belief hierarchies in respective type spaces (under some topological assumptions on nature states and beliefs), and establishes the existence of a terminal lexicographic belief space from a terminal conditional-belief space. For the connection of conditional beliefs with hypothetical knowledge and counterfactual reasoning, see, for example, Di Tillio, Halpern, and Samet (2014) and the references therein.

Denote by $\Delta^{\mathcal{C}}(\Omega)$ the set of CPSs on $(\Omega, \mathcal{D}, \mathcal{C})$ endowed with the $\aleph_1$-algebra

$$\mathcal{D}_{\Delta}^{\mathcal{C}} := \mathcal{A}_{\aleph_1}(\{\{\mu \in \Delta^{\mathcal{C}}(\Omega) \mid \mu(E|C) \geq p\} \in \mathcal{P}(\Delta^{\mathcal{C}}(\Omega)) \mid (E, C, p) \in \mathcal{D} \times \mathcal{C} \times [0,1]\}).$$

A player i's *conditional-type mapping* is a measurable map $t_i : (\Omega, \mathcal{D}) \to (\Delta^{\mathcal{C}}(\Omega), \mathcal{D}_{\Delta}^{\mathcal{C}})$. I formulate a conditional-belief space using conditional p-belief operators $B_i^p(\cdot|C)$ for each player i and each condition C, so that i's conditional p-belief operators induce her conditional-type mapping.

Definition S.1 (Conditional-Belief Space)**.** *A conditional-belief space of I on $(S, \mathcal{S}, \mathcal{C}_S)$ is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}, \mathcal{C}), (B_i^p(\cdot|C))_{(i,p,C)\in I\times[0,1]\times\mathcal{C}}, \Theta \rangle$ with the following properties.*

1. *$(\Omega, \mathcal{D}, \mathcal{C})$ is a conditional space and $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_{\aleph_1}(\mathcal{S}))$ is a measurable map with $\mathcal{C} = \Theta^{-1}(\mathcal{C}_S)$.*
2. *For each $i \in I$, player i's conditional p-belief operators $B_i^p(\cdot|C) : \mathcal{D} \to \mathcal{D}$ satisfy the following.*
   (a) *For each $C \in \mathcal{C}$, $(B_i^p(\cdot|C))_{p\in[0,1]}$ satisfies Definition 10 (2a)-(2h).*
   (b) *Certainty-of-Conditional-Beliefs: If $[t_{B_i}(\omega)] \subseteq E$ then $\omega \in B_i^1(E|\Omega)$, where*
   $$[t_{B_i}(\omega)] := \Big(\bigcap_{\substack{(E,p,C)\in\mathcal{D}\times[0,1]\times\mathcal{C} \\ \omega\in B_i^p(E|C)}} B_i^p(E|C)\Big) \cap \Big(\bigcap_{\substack{(E,p,C)\in\mathcal{D}\times[0,1]\times\mathcal{C} \\ \omega\in(\neg B_i^p)(E|C)}} (\neg B_i^p)(E|C)\Big).$$
   (c) *Normality: $B_i^1(C|C) = \Omega$ for all $C \in \mathcal{C}$.*
   (d) *Chain Rule: $B_i^p(E|D) \cap B_i^q(D|C) \subseteq B_i^{pq}(E|C)$ for any $E \in \mathcal{D}$ and $C, D \in \mathcal{C}$ with $E \subseteq D \subseteq C$.*

By the assumption $\mathcal{C} = \Theta^{-1}(\mathcal{C}_S)$ in (1), denote $B_{i,C_S}^p(\cdot) := B_i^p(\cdot|\Theta^{-1}(C_S))$ for each $(i, p, C_S) \in I \times [0,1] \times \mathcal{C}_S$. This means that conditions in each conditional-belief space are exogenously given as in Battigalli and Siniscalchi (1999) and Guarino (2017). Thus, one's conditional belief may fail to be a condition (i.e., $B_i^p(E|C) \notin \mathcal{C}$). Also, since $S \in \mathcal{C}_S$, "unconditional" beliefs $B_{i,S}^p(\cdot) = B_i^p(\cdot|\Omega)$ are also considered.

Conditions (2) characterize each player's conditional-type mapping as in Di Tillio, Halpern, and Samet (2014, Theorem 1). First, by (2a), slightly abusing the notation, a measurable map $t_{B_i} : (\Omega, \mathcal{D}) \to (\Delta^{\mathcal{C}}(\Omega), \mathcal{D}_{\Delta}^{\mathcal{C}})$ is well-defined for each condition as in Section 5.1. For conditional beliefs, Condition (2b) is the introspective property stating that player i is certain of her own conditional beliefs. The set $[t_{B_i}(\omega)]$ satisfies $[t_{B_i}(\omega)] = \{\omega' \in \Omega \mid t_{B_i}(\omega') = t_{B_i}(\omega)\}$: it consists of states ω' that player i cannot distinguish from ω based on her conditional beliefs. That is, she unconditionally believes E with probability one when E is implied by $[t_{B_i}(\omega)]$. Especially, $B_{i,C_S}^p(E) \subseteq B_{i,S}^1 B_{i,C_S}^p(E)$ and $(\neg B_{i,C_S}^p)(E) \subseteq B_{i,S}^1(\neg B_{i,C_S}^p)(E)$ hold: if player i p-believes (does not p-believe) E conditional on $\Theta^{-1}(C_S)$, then she unconditionally 1-believes that she p-believes (does not p-believe) E conditional on $\Theta^{-1}(C_S)$.

By (2c), each $t_{B_i}(\omega)(\cdot|\cdot)$ satisfies Normality. Under (2a) and (2c), it can be seen that (2d) characterizes the Chain Rule.

A *(conditional-belief) morphism* from $\overrightarrow{\Omega}$ to $\overrightarrow{\Omega'}$ is a measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ satisfying: (i) $\Theta' \circ \varphi = \Theta$; and (ii) $B^p_{i,C_S}(\varphi^{-1}(\cdot)) = \varphi^{-1}(B'^p_{i,C_S}(\cdot))$ for all $(i, p, C_S) \in I \times [0,1] \times \mathcal{C}_S$. A conditional-belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S}, \mathcal{C}_S)$ is *terminal* if, for any conditional-belief space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{S}, \mathcal{C}_S)$, there is a unique morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. By considering the probabilistic-beliefs of each player for each conditioning event, as in Section 5.1, Theorem 1 implies:

Corollary S.1 (Terminal Conditional-Belief Space)**.** *There exists a terminal conditional-belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S}, \mathcal{C}_S)$.*

C.2 Terminal Dynamic Knowledge-Belief Space

Epistemic analyses of dynamic games often call for players' knowledge and beliefs. As in Battigalli and Bonanno (1997), consider players' knowledge and beliefs indexed by time. While a knowledge operator $K_{i,t} : \mathcal{D} \to \mathcal{D}$ represents player i's knowledge at time $t \in \mathbb{N}$, a belief operator $B_{i,t} : \mathcal{D} \to \mathcal{D}$ does her qualitative belief at time t.

Definition S.2 (Dynamic Knowledge-Belief Space)**.** *A dynamic κ-knowledge-belief space of I on $(S, \mathcal{S})$ is a tuple $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (K_{i,t}, B_{i,t})_{(i,t)\in I\times\mathbb{N}}, \Theta\rangle$ with the following properties.*

1. *$(\Omega, \mathcal{D})$ is a κ-algebra and the map $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_\kappa(\mathcal{S}))$ is measurable.*

2. *Knowledge operators $K_{i,t} : \mathcal{D} \to \mathcal{D}$ satisfy Truth Axiom, (Positive Introspection,) Negative Introspection, and the Kripke property. Belief operators $B_{i,t} : \mathcal{D} \to \mathcal{D}$ satisfy Consistency, Positive Introspection, Negative Introspection, and the Kripke property.*

3. *Knowledge and belief operators jointly satisfy: (i) $K_{i,t}(\cdot) \subseteq B_{i,t}(\cdot)$; (ii) $B_{i,t}(\cdot) \subseteq K_{i,t}B_{i,t}(\cdot)$; and (iii) $B_{i,t}(\cdot) = B_{i,t}B_{i,t+1}(\cdot)$.*

In (2), for ease of illustration, I have assumed (i) both knowledge and qualitative belief are fully introspective, (ii) knowledge is truthful while qualitative belief is consistent, and (iii) both knowledge and qualitative belief are represented by a possibility correspondence. The existence of a terminal space does not hinge on assumptions on properties of knowledge and qualitative belief.

In (3), the first condition means that knowledge implies belief at each time. The second states that each player knows her own belief at each time. Note that $(\neg B_{i,t})(\cdot) \subseteq K_{i,t}(\neg B_{i,t})(\cdot)$ follows from Truth Axiom and Negative Introspection of knowledge. The third captures the idea of belief persistence (Battigalli and Bonanno, 1997): player i believes E at time t iff she believes at t that she (will) believe E at

$t+1$. Player i's knowledge satisfies *perfect recall* if $K_{i,t}(\cdot) \subseteq K_{i,t+1}(\cdot)$ for all $t \in \mathbb{N}$. A dynamic knowledge-belief space *with perfect recall* is a dynamic knowledge-belief space such that each player's knowledge satisfies perfect recall.

A dynamic knowledge-belief space is mathematically a belief space of $I \times \mathbb{N} \times \{0,1\}$, where "player $(i,t,0)$'s belief operator" is $K_{i,t}$ while "player $(i,t,1)$'s operator" is $B_{i,t}$, with the specified conditions. Thus:

Corollary S.2 (Terminal Dynamic Knowledge-Belief Space)**.** *There exists a terminal dynamic κ-knowledge-belief space (with/without perfect recall) $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S})$.*

C.3 Futher Possible Extensions

This subsection briefly discusses three possible extensions in which the domain of a "belief" space has a richer structure: a terminal knowledge-unawareness space, a terminal preference space, and a terminal expectation space.

C.3.1 Terminal Knowledge-Unawareness Space

A *knowledge-unawareness space* is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (K_i, U_i)_{i \in I}, \Theta \rangle$ where $K_i : \mathcal{D} \to \mathcal{D}$ is player i's knowledge operator and $U_i : \mathcal{D} \to \mathcal{D}$ is i's unawareness operator. By Theorem 1, a terminal knowledge-unawareness space $\overrightarrow{\Omega^*} := \langle (\Omega^*, \mathcal{D}^*), (K_i^*, U_i^*)_{i \in I}, \Theta^* \rangle$ exists under various assumptions on properties of knowledge and unawareness.

Call a knowledge-unawareness space $\overrightarrow{\Omega}$ *non-trivial* if $U_i([\![e]\!]_{\overrightarrow{\Omega}}) \neq \emptyset$ for some $(i,e) \in I \times \mathcal{L}$. Then, there exists a non-trivial knowledge-unawareness space within a given category of knowledge-unawareness spaces iff the terminal knowledge-unawareness space $\overrightarrow{\Omega^*}$ is non-trivial: $U_i^*([e]) \neq \emptyset$ for some $(i,e) \in I \times \mathcal{L}$.

It would be an interesting research avenue to accommodate a generalized state space consisting of multiple state spaces as in Heifetz, Meier, and Schipper (2006, 2008) due to some limitation of possibility correspondence models (e.g., Chen, Ely, and Luo, 2012; Dekel, Lipman, and Rustichini, 1998; Fukuda, 2021; Modica and Rustichini, 1994). While the framework of this paper requires the domain (the collection of events) $\mathcal{D}$ to be a κ-algebra of sets, the domain in the generalized state space has a more general lattice structure. A (generalized) knowledge-unawareness space would refer to a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (K_i, U_i)_{i \in I}, \Theta \rangle$ with the following properties: $\Omega = \bigcup_{\alpha \in A} \Omega_\alpha$ is a state space that consists of multiple subspaces $(\Omega_\alpha)_{\alpha \in A}$; $\mathcal{D}$ forms a κ-complete lattice; $K_i, U_i : \mathcal{D} \to \mathcal{D}$ are player i's knowledge and unawareness operators; and $\Theta : \Omega \to S$ is a map that associates, with each state of the world, the corresponding nature state. I conjecture that the idea of this paper can be applied to knowledge and unawareness operators on a κ-complete lattice.

C.3.2 Terminal Preference Space

Consider a preference(-based type) space where players reason about their interactive preferences instead of beliefs (e.g., Di Tillio, 2008; Epstein and Wang, 1996; Ganguli, Hiefetz, and Lee, 2016). Each player i's preference-type mapping associates, with each state of the world $\omega \in \Omega$, her preference relation over the set of acts (i.e., bounded measurable functions) on Ω, where Ω is endowed with a κ-algebra. Call a preference space a κ-preference space if it is defined on a κ-algebra.

The analyses of this paper suggest two points on the existence and structure of a terminal preference space. First, a terminal κ-preference space would exist, irrespective of such nature of preferences as "continuity," if one considers hierarchies of interactive preferences up to the ordinality of κ. As an example, Di Tillio (2008) constructs a terminal $\aleph_0$-preference-type space consisting of finite preference hierarchies in the category of $\aleph_0$-preference-type spaces.

Second, under a regularity condition under which players' finite-level reasoning extends to countable-levels (recall Proposition 4 in Section 5.1 for the case of probabilistic beliefs), a terminal $\aleph_1$-preference space would consist of finite-level preference hierarchies in the category of $\aleph_1$-preference spaces. As examples, in Epstein and Wang (1996), preferences satisfy some continuity properties (their P3 and P4). In Ganguli, Hiefetz, and Lee (2016), preferences are represented by a countable collection of continuous real-valued functionals over acts. Hence, I conjecture that, in the context of Di Tillio (2008) in which players' preference relations are merely complete and transitive, a terminal $\aleph_1$-preference space would exist if each preference hierarchy incorporates all countable-level reasoning.[2]

Below, instead of introducing hierarchies of preferences, I analyze players' interactive expectations. At each state of the world, each player has her (numerical) expectation of an act (or a random variable, i.e., a bounded measurable function).

C.3.3 Terminal Expectation Space

I construct a terminal expectation space where players interactively reason about their expectations of random variables by formalizing the correspondence between beliefs and expectations. While I focus on expectations that come from countably-additive beliefs, a terminal expectation space would exist for weaker notions of expectations when the objects of reasoning (e.g., a class of random variables) and the properties of expectations (i.e., additivity or continuity) are modified.[3]

[2]In the context of κ-knowledge spaces, an earlier version of this paper (Fukuda, 2017, Sections 5 and 6) provides a type-space reformulation of a knowledge space (where each type takes either 0 or 1 instead of probability $p \in [0,1]$), and constructs a terminal knowledge space consisting of hierarchies of "knowledge types" up to $\overline{\kappa}$.

[3]The construction here solves an open question raised by Golub and Morris (2017, Section 3) on the construction of a terminal expectation space. Corollary S.3 constructs a terminal expectation space by transforming a terminal probabilistic-belief space into the terminal expectation space.

To define the objects of players' expectations (i.e., random variables), denote by $\mathcal{B}(\Omega)$ the set of bounded Borel measurable functions $f : \Omega \to \mathbb{R}$ on a measurable space $(\Omega, \mathcal{D})$. Next, I define the properties of expectations (that come from countably-additive beliefs). Define the space $\Gamma(\Omega)$ of expectations on a measurable space $(\Omega, \mathcal{D})$ as the subset of the space $\mathbb{R}^{\mathcal{B}(\Omega)}$ of the mappings from $\mathcal{B}(\Omega)$ into $\mathbb{R}$ respecting the following five properties of expectations. Namely, any $J \in \Gamma(\Omega)$ satisfies: (a. Non-negativity) $f(\cdot) \geq 0$ implies $J[f] \geq 0$; (b. Additivity) $J[f+g] = J[f] + J[g]$; (c. Homogeneity) $J[cf] = cJ[f]$ for all $c \in \mathbb{R}$; (d. Constancy) $J[\mathbb{I}_\Omega] = 1$; and (e. Continuity) $f_n \uparrow f$ (in $\mathcal{B}(\Omega)$) implies $J[f_n] \uparrow J[f]$. Next, let $\mathcal{D}_\Gamma$ be the $\aleph_1$-algebra on $\Gamma(\Omega)$ generated by $\{\{J \in \Gamma(\Omega) \mid J(f) \geq r\} \in \mathcal{P}(\Gamma(\Omega)) \mid (f, r) \in \mathcal{B}(\Omega) \times \mathbb{R}\}$.

An *expectation space* (of I on $(S, \mathcal{S})$) is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (\mathbb{E}_i)_{i \in I}, \Theta \rangle$ with the following properties: (i) $(\Omega, \mathcal{D})$ is a measurable space of states of the world; (ii) $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{A}_{\aleph_1}(\mathcal{S}))$ is a measurable map that associates, with each state of the world, the corresponding nature state; and (iii) each $\mathbb{E}_i$ is player i's expectation-type mapping $\mathbb{E}_i : (\Omega, \mathcal{D}) \to (\Gamma(\Omega), \mathcal{D}_\Gamma)$ satisfying the introspective property (certainty of expectation) below. For each $\omega \in \Omega$, define the set of states $[\mathbb{E}_i(\omega)] := \{\tilde{\omega} \in \Omega \mid \mathbb{E}_i(\tilde{\omega}) = \mathbb{E}_i(\omega)\}$ at which player i cannot distinguish from ω based on her introspection into her own expectations. Then, assume that each player i is *certain of her expectation*: for any $(\omega, E) \in \Omega \times \mathcal{D}$, $[\mathbb{E}_i(\omega)] \subseteq E$ implies $\mathbb{E}_i(\omega)[\mathbb{I}_E] = 1$.

I discuss two ways in which an expectation space unpacks players' higher-order expectations. The first is analogous to a type mapping in a probabilistic-belief space as in the above definition. Player i's expectation-type mapping associates, with each state, the functional that maps each random variable f to the player's expectation of f at ω.

In contrast, the second views the expectation-type mapping as a mapping that associates, with each random variable f, another random variable $\mathbb{E}_i[f]$ that represents the player's expectation of the random variable at each state. Hence, by iterating players' expectations $\mathbb{E}_i$, one can represent higher-order expectations such as j's expectation of i's expectation of a random variable f by $\mathbb{E}_j\mathbb{E}_i[f]$.

Formally, $\mathbb{E}_i : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$ associates, with each bounded Borel measurable function f, another bounded Borel measurable function $\mathbb{E}_i(\cdot)[f]$ that represents the player i's expectation of f at each state. Denote by $\mathbb{E}_i[f \mid \omega] = \mathbb{E}_i(\omega)[f]$. The five properties on the expectation-type mapping $\mathbb{E}_i$ are: (a. Non-negativity) $f(\cdot) \geq 0$ implies $\mathbb{E}_i[f \mid \cdot] \geq 0$; (b. Additivity) $\mathbb{E}_i[f+g] = \mathbb{E}_i[f] + \mathbb{E}_i[g]$; (c. Homogeneity) $\mathbb{E}_i[cf] = c\mathbb{E}_i[f]$ for all $c \in \mathbb{R}$; (d. Constancy) $\mathbb{E}_i[\mathbb{I}_\Omega] = \mathbb{I}_\Omega$; and (e. Continuity) $f_n \uparrow f$ (in $\mathcal{B}(\Omega)$) implies $\mathbb{E}_i[f_n] \uparrow \mathbb{E}_i[f]$.

To see interactive reasoning over random variables on $(S, \mathcal{A}_{\aleph_1}(\mathcal{S}))$, for any $f \in \mathcal{B}(S)$, denote $f_\Theta = f \circ \Theta \in \mathcal{B}(\Omega)$. Player i's expectation of f on the state space $(\Omega, \mathcal{D})$ is $\mathbb{E}_i[f_\Theta] \in \mathcal{B}(\Omega)$. Thus, the analysts can keep track of players' higher-order expectations by iterating their expectation-type mappings. For example, player i's

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Footnote 9 in Appendix F.1 also shows how one can explicitly construct a terminal expectation space by paralleling the construction of a terminal probabilistic-belief space.

expectation of player j's expectation of $f \in \mathcal{B}(S)$ is a bounded Borel measurable function $\mathbb{E}_i\mathbb{E}_j[f_\Theta] \in \mathcal{B}(\Omega)$.

Next, I also remark that if player i is certain of her expectations then her expectations satisfy the law of iterated expectations. Again, denoting $\mathbb{E}_i[f \mid \omega] = \mathbb{E}_i(\omega)[f]$, it can be seen that $\mathbb{E}_i[\mathbb{E}_i[f \mid \tilde{\omega}] \mid \omega] = \mathbb{E}_i[f \mid \omega]$ for all $(\omega, f) \in \Omega \times \mathcal{B}(\Omega)$.

An *(expectation) morphism* φ between expectation spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ is a measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ with the following two properties: (i) $\Theta = \Theta' \circ \varphi$; and (ii) $\mathbb{E}_i(\omega)[f' \circ \varphi] = \mathbb{E}'_i(\varphi(\omega))[f']$ for all $(\omega, f') \in \Omega \times \mathcal{B}(\Omega')$. Call a morphism φ an (expectation) *isomorphism* if φ is bijective and its inverse φ^{-1} is a morphism. The class of expectation spaces forms a category, where each object is an expectation space and each arrow is an expectation morphism.

An expectation space $\overrightarrow{\Omega^*}$ (of I on $(S, \mathcal{S})$) is *terminal* (among the class of expectation spaces of I on $(S, \mathcal{S})$) if, for any expectation space $\overrightarrow{\Omega}$ (of I on $(S, \mathcal{S})$), there is a unique morphism φ from $\overrightarrow{\Omega}$ into $\overrightarrow{\Omega^*}$. A terminal expectation space is unique up to isomorphism. Then:

Corollary S.3 (Terminal Expectation Space)**.** *There exists a terminal expectation space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S})$.*

To construct a terminal expectation space, the proof (in Appendix F.1) exploits the (category-theoretical) equivalence between expectation spaces and probabilistic-belief spaces that comes from the one-to-one correspondence between beliefs and expectations. Especially, the terminal expectation space is constructed from a terminal probabilistic-belief space.

I briefly remark on expectations that come from finitely-additive or non-additive beliefs. When players possess expectations that come from finitely-additive or non-additive beliefs on a κ-algebra, they interactively reason about their expectations of bounded measurable functions $f : (\Omega, \mathcal{D}) \to (\mathbb{R}, \mathcal{A}_\kappa(\{[a, b) \mid a < b\}))$ (or $f : \Omega \to [0, 1]$ with respect to the appropriate κ-algebras), and their expectation-type mappings would satisfy the appropriate properties of the expectation functional.

Next, I remark on the average expectation operator $\overline{\mathbb{E}} : (\Omega, \mathcal{D}) \to (\Gamma(\Omega), \mathcal{D}_\Gamma)$.[4] Consider the following two typical cases of a set I of players (for simplicity, I focus on symmetric weights on players). First, let $I = \{1, \ldots, n\}$. Then, define $\overline{\mathbb{E}}(\omega)[f] := \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_i(\omega)[f]$ for each $(\omega, f) \in \Omega \times \mathcal{B}(\Omega)$. Second, let $I = [0, 1]$, and assume that, for each $(\omega, f) \in \Omega \times \mathcal{B}(\Omega)$, the mapping $I \ni i \mapsto \mathbb{E}_i(\omega)[f] \in \mathbb{R}$ is Borel-measurable (by construction, it is bounded: $\sup_{i \in I} |\mathbb{E}_i(\omega)[f]| \leq \sup_{\tilde{\omega} \in \Omega} |f(\tilde{\omega})| < \infty$). Hence, the following mapping $\overline{\mathbb{E}} : \Omega \times \mathcal{B}(\Omega) \to \mathbb{R}$ is well-defined: $\overline{\mathbb{E}}(\omega)[f] := \int_I \mathbb{E}_i(\omega)[f] di$ for each $(\omega, f) \in \Omega \times \mathcal{B}(\Omega)$. It can be seen that $\overline{\mathbb{E}} : (\Omega, \mathcal{D}) \to (\Gamma(\Omega), \mathcal{D}_\Gamma)$ is measurable. Especially, each agent can reason about the average expectations, because $\overline{\mathbb{E}}$ maps

[4]See Golub and Morris (2017) and the references therein for applications of average expectations to such literature as network games.

$f \in \mathcal{B}(\Omega)$ to $\overline{\mathbb{E}}[f] \in \mathcal{B}(\Omega)$. Since the mapping $I \ni i \mapsto \mathbb{E}_i^*(\omega^*)[f] \in \mathbb{R}$ is Borel-measurable (observe $\mathbb{E}_i^*(\omega^*)[f] = \mathbb{E}_i(\omega)[f \circ h]$ for some expectation space $\overrightarrow{\Omega}$ and $\omega \in \Omega$) and bounded, the average expectation operator $\overline{\mathbb{E}^*} : (\Omega^*, \mathcal{D}^*) \to (\Gamma(\Omega^*), \mathcal{D}^*_\Gamma)$ is well-defined in the terminal space.

D Section 6

D.1 Correlated Equilibria

Here, I discuss the role of the existence and structure of a terminal probabilistic-belief space on the solution concept of correlated equilibria. To that end, Section D.1.1 introduces, to a belief space, the players' common prior in addition to their posteriors (i.e., type mappings). Then, Section D.1.2 shows that a correlated equilibrium is mapped to a subspace of the terminal probabilistic-belief space with a common prior if the correlated equilibrium as a belief space is non-redundant and minimal. Put differently, the underlying states in a correlated equilibrium can be replaced with the players' belief hierarchies about their play if the correlated equilibrium is non-redundant and minimal.

D.1.1 Terminal Belief Space with a Common Prior

I introduce probabilistic-belief spaces with a common prior, and show that a terminal space exists. A *probabilistic-belief space of I on $(S, \mathcal{S})$ with a common prior* is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (B_i^p)_{(i,p) \in I \times [0,1]}, \Theta, \mu \rangle$ such that: (i) $\langle (\Omega, \mathcal{D}), (B_i^p)_{(i,p) \in I \times [0,1]}, \Theta \rangle$ is a probabilistic-belief space in the sense of Definition 10; and that (ii) μ is a (countably-additive) probability measure, called the *common prior*, satisfying

$$\mu(E) = \int_\Omega t_{B_i}(\omega)(E)\mu(d\omega) \text{ for each } (i, E) \in I \times \mathcal{D}. \tag{4}$$

Equation (4) says the prior probability of E is equal to the expectation of the posterior probabilities $t_{B_i}(\omega)(E)$ with respect to μ (e.g., Mertens and Zamir, 1985).[5]

For probabilistic-belief spaces with a common prior $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$, $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a *(probabilistic-belief) morphism* if φ is a morphism between probabilistic-belief spaces

[5]In a probabilistic-belief space with a common prior $\overrightarrow{\Omega}$, Aumann (1976)'s Agreement theorem holds. If $\mu(C^p(\bigcap_{i \in I}\{\omega \in \Omega \mid t_{B_i}(\omega)(E) = r_i\})) > 0$, then $|r_i - r_j| \leq 1 - p$ for all $i, j \in I$: if the event that it is common p-belief that each player i's belief in E is r_i has positive probability according to the common prior, then the difference between any two players' beliefs is at most $1 - p$ (see footnote 36 of the main text for common p-belief). See, for instance, Fukuda (2019) for the Agreement theorem on an arbitrary measurable space.

(i.e., $\Theta' \circ \varphi = \Theta$ and $B_i^p \varphi^{-1} = \varphi^{-1} B_i'^p$) and $\mu' = \mu \circ \varphi^{-1}$.[6] The last condition states the prior probabilities are preserved. A probabilistic-belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S})$ with a common prior is *terminal* if, for any probabilistic-belief space $\overrightarrow{\Omega}$ of I on $(S, \mathcal{S})$ with a common prior, there is a unique morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. As in Section 5.1:

Corollary S.4 (Terminal Probabilistic-Belief Space with a Common Prior)**.** *There exists a terminal probabilistic-belief space $\overrightarrow{\Omega^*}$ of I on $(S, \mathcal{S})$ with a common prior.*

In the proof, I introduce a hypothetical player so that, in each probabilistic-belief space with a common prior, the beliefs of the hypothetical player are given by the common prior. Following the construction of a terminal space in Sections 3 and 5.1, one can introduce the beliefs of the hypothetical player on the candidate terminal space. Since it induces a common prior, the candidate space is indeed terminal.

D.1.2 Correlated Equilibrium and Belief Hierarchies

I introduce correlated equilibria. Since a correlated equilibrium is a particular belief space, I show that it is mapped to a subspace of the terminal probabilistic-belief space with a common prior. That is, the underlying state space of a correlated equilibrium can be replaced with the set of belief hierarchies that the state space intends to represent.

To define a correlated equilibrium, let $\langle (A_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be an underlying strategic game with the following properties. The set A_i of i's actions is endowed with an $\aleph_1$-algebra $\mathcal{S}_i$ containing singletons: $\{a_i\} \in \mathcal{S}_i$ for all $a_i \in A_i$.[7] Endow the action profiles $A := \prod_{i \in I} A_i$ with the product $\aleph_1$-algebra $\mathcal{S}$. Let $u_i : A \to \mathbb{R}$ be i's bounded Borel measurable payoff function. Denote by $\pi_i : A \to A_i$ the projection. Any measurable function Θ from an $\aleph_1$-algebra $(\Omega, \mathcal{D})$ into $(A, \mathcal{S})$ can be decomposed into $\Theta = (\Theta_i)_{i \in I}$ such that each $\Theta_i = \pi_i \circ \Theta$ is measurable.

A *correlated equilibrium* is an $\aleph_1$-belief space $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (B_i^p)_{(i,p) \in I \times [0,1]}, \Theta, \mu \rangle$ with a common prior satisfying the following two properties. The first is the certainty of actions: $\Theta_i^{-1}(\{a_i\}) \subseteq B_i^1(\Theta_i^{-1}(\{a_i\}))$ for each $a_i \in A_i$. Whenever player i takes action a_i, she believes with probability one that she takes a_i. The second is the (ex-ante) optimality condition. For any player i and for any measurable function $\tau_i : (\Omega, \mathcal{D}) \to (A_i, \mathcal{S}_i)$ with $\tau_i^{-1}(\{a_i\}) \subseteq B_i^1(\tau_i^{-1}(\{a_i\}))$ for each $a_i \in A_i$,

$$\int_\Omega u_i(\Theta_i(\omega), \Theta_{-i}(\omega)) \mu(d\omega) \geq \int_\Omega u_i(\tau_i(\omega), \Theta_{-i}(\omega)) \mu(d\omega).$$

---

[6]Let $\overrightarrow{\Omega}$ be a probabilistic-belief space with a common prior, and let $\varphi : \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, \Theta \rangle \to \langle (\Omega', \mathcal{D}'), (B_i')_{i \in I}, \Theta' \rangle$ be a morphism in the sense of Section 5.1. Letting $\overrightarrow{\Omega'}$ with $\mu' := \mu \circ \varphi^{-1}$, $\overrightarrow{\Omega'}$ is a probabilistic-belief space with a common prior and $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism.

[7]If the players reason only about whether each player takes each action, then the $\aleph_1$-algebra $\mathcal{S}_i$ is the one generated by the singleton actions. Generally, the action space A_i may have a natural measurable structure (e.g., the Borel σ-algebra) $\mathcal{S}_i$ containing each singleton actions.

For ease of exposition, I have defined a correlated equilibrium in terms of probability-one beliefs instead of knowledge (induced by a partition (or a σ-algebra)). When each player's knowledge is induced by her partition, her type mapping (equivalently, her p-belief operators) is given as the Bayes conditional probability measure from the common prior conditional on the partition. Thus, instead of specifying the players' partitions, I specify their p-belief operators (equivalently, their type mappings).

Next, I identify the set of belief hierarchies induced by some state of some correlated equilibrium. That is, I construct a terminal correlated equilibrium, i.e., a terminal $\aleph_1$-belief space with a common prior satisfying the two requirements to be a correlated equilibrium: $\overrightarrow{\Omega^*} = \langle(\Omega^*, \mathcal{D}^*), (B_i^{*p})_{(i,p)\in I\times[0,1]}, \Theta^*, \mu^*\rangle$ such that, for any correlated equilibrium $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}), (B_i^p)_{(i,p)\in I\times[0,1]}, \Theta, \mu\rangle$, there is a unique morphism $D : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$. Indeed, as long as the underlying game admits a correlated equilibrium, the terminal correlated equilibrium exists, because the candidate terminal $\aleph_1$-belief space with a common prior $\overrightarrow{\Omega^*}$ (constructed as in Sections 3 and 5.1) satisfies the two conditions to be a correlated equilibrium: (i) $(\Theta_i^*)^{-1}(\{a_i\}) \subseteq B_i^{*1}((\Theta_i^*)^{-1}(\{a_i\}))$ for each $a_i \in A_i$; and (ii) for any player i and for any measurable function $\tau_i^* : (\Omega^*, \mathcal{D}^*) \to (A_i, \mathcal{S}_i)$ with $(\tau_i^*)^{-1}(\{a_i\}) \subseteq B_i^{*1}((\tau_i^*)^{-1}(\{a_i\}))$ for each $a_i \in A_i$,

$$\int_{\Omega^*} u_i(\Theta_i^*(\omega^*), \Theta_{-i}^*(\omega^*))\mu^*(d\omega^*) \geq \int_{\Omega^*} u_i(\tau_i^*(\omega^*), \Theta_{-i}^*(\omega^*))\mu(d\omega^*).$$

Now, whenever there exists a correlated equilibrium (and thus the terminal correlated equilibrium exists), any non-redundant and minimal correlated equilibrium $\overrightarrow{\Omega}$ can be embedded into the subspace $\overrightarrow{D(\Omega)}$ of the terminal space. Put differently, the correlating device (i.e., the state space) of any non-redundant and minimal correlated equilibrium $\overrightarrow{\Omega}$ can be replicated by the space $\overrightarrow{D(\Omega)}$ of belief hierarchies: the correlated equilibria $\overrightarrow{\Omega}$ and $\overrightarrow{D(\Omega)}$ are isomorphic as a belief space (i.e., the players' beliefs and strategies and the common prior are preserved), and consequently induce the same correlated equilibrium distribution $\mu \circ \Theta^{-1} = \mu^* \circ (\Theta^*)^{-1}$.

While the analysis here shows that any non-redundant and minimal correlated equilibrium is embedded into a subspace of the terminal correlated equilibrium, the analysis is also related to an intrinsic view of correlation in non-cooperative games. Namely, Brandenburger and Friedenberg (2008) and Du (2012) provide an *intrinsic* view of correlation in non-cooperative games, in contrast to the extrinsic view of correlation as external payoff-irrelevant signals (Aumann, 1974): the correlation in players' actions comes from correlation in their beliefs and higher-order beliefs about the play of the game, which is induced by a primitive of a model.

Both papers show that a certain correlated rationalizable (or correlated equilibrium) play cannot be played under intrinsic correlation. In contrast, this subsection asks a related but different question as an implication of a terminal probabilistic-belief space with a common prior. Brandenburger and Friedenberg (2008) study a refinement of correlated rationalizability by imposing epistemic assumptions on players' (correlated) beliefs and rationality (and common belief in rationality) in a type space, and study the given strategic game as originally. In contrast, the analysis here takes an extended game (the given strategic game augmented with a correlation device—recall that a correlated equilibrium is a Nash equilibrium of the extended game), and asks when the correlation device (in a correlated equilibrium) can be replaced with belief hierarchies about play. Next, Du (2012) characterizes a refinement of correlated equilibria in which each player's strategy in a type space is constant whenever types induce the same belief hierarchy about play. In contrast, this paper asks when the correlation device (the underlying state space) can be replaced with belief hierarchies without imposing the condition that each player's strategy is determined by her belief hierarchy alone. Consequently, the analysis here shows that whenever a belief space is non-redundant (and minimal), the correlating device can be replaced with belief hierarchies.

The same analysis carries over to a belief space in which the players commonly believe Bayesian rationality (Aumann, 1987).[8] A belief space with common belief in Bayesian rationality is an $\aleph_1$-belief space $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), (B_i^p)_{(i,p)\in I\times[0,1]}, \Theta, \mu\rangle$ with a common prior satisfying the following two properties. The first is the certainty of actions: $\Theta_i^{-1}(\{a_i\}) \subseteq B_i^1(\Theta_i^{-1}(\{a_i\}))$ for each $a_i \in A_i$. The second is Bayesian rationality (at every state): for all $\omega \in \Omega$ and for all $a_i \in A_i$,

$$\int_\Omega u_i(\Theta_i(\tilde{\omega}), \Theta_{-i}(\tilde{\omega})) t_{B_i}(\omega)(d\tilde{\omega}) \geq \int_\Omega u_i(a_i, \Theta_{-i}(\tilde{\omega})) t_{B_i}(\omega)(d\tilde{\omega}).$$

The players are Bayesian rational at every state, and thus they commonly believe (at every state) that they are Bayesian rational. It can be seen that a belief space with common belief in Bayesian rationality is a correlated equilibrium.

Whenever there exists a belief space with common belief in Bayesian rationality, the class of belief spaces with common belief in Bayesian rationality admits a terminal space. Now, the correlating device (i.e., the state space) of any non-redundant and minimal belief space with common belief in Bayesian rationality $\overrightarrow{\Omega}$ can be replicated by the space $\overrightarrow{D(\Omega)}$ of belief hierarchies.

E Section 7

The following proposition characterizes minimality as in Friedenberg and Meier (2011, Theorem 5.1). Specifically, recalling Remark 3 and its discussions, any belief morphism φ preserves descriptions: the states ω and $\varphi(\omega)$ induce the same nature state and players' belief hierarchies: $D_{\overrightarrow{\Omega}}(\omega) = D_{\overrightarrow{\Omega'}}(\varphi(\omega))$. In contrast, Proposition S.1 considers a map φ with the property that the description of (i.e., the nature state and players' belief hierarchies at) a state ω is associated with the description of $\varphi(\omega)$.

[8] The similar analysis also holds for a refinement of a subjective correlated equilibrium called an a-posteriori equilibrium (Aumann, 1974; Brandenburger and Dekel, 1987).

Friedenberg and Meier (2011) call such φ a *hierarchy morphism* as the analysts often directly work with a mapping that preserves players' belief hierarchies rather than with a belief morphism. Roughly, the following proposition states that $\overrightarrow{\Omega'}$ is minimal iff a hierarchy morphism φ is a belief morphism.

Proposition S.1. *Fix a category of κ-belief spaces of I on $(S, \mathcal{S})$.*

1. *Let $\overrightarrow{\Omega}$ be a belief space, and let $\overrightarrow{\Omega'}$ be a minimal belief space. A measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ is a belief morphism iff the map $\varphi : \Omega \to \Omega'$ satisfies $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \varphi$.*

2. *Let a belief space $\overrightarrow{\Omega'}$ be such that, for any belief space $\overrightarrow{\Omega}$, a measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ is a belief morphism iff the map $\varphi : \Omega \to \Omega'$ satisfies $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \varphi$. Then, $\overrightarrow{\Omega'}$ is minimal.*

F Proofs

F.1 Appendix C

Proof of Corollary S.1. Construct Ω^*, as in the proof of Theorem 1, by viewing the set of players in each conditional-belief space as $\overline{I} := I \times [0,1] \times \mathcal{C}_S$. To see that Ω^* is not empty, take a CPS μ on $(S, \mathcal{A}_{\aleph_1}(\mathcal{S}), \mathcal{C}_S)$. Consider $\langle (S, \mathcal{A}_{\aleph_1}(\mathcal{S}), \mathcal{C}_S), (B^p_{i,C})_{(i,p,C)\in\overline{I}}, \mathrm{id}_S \rangle$, where (i) $B^p_{i,C}(E) := \emptyset$ if $\mu(E|C) < p$; and (ii) $B^p_{i,C}(E) := S$ if $\mu(E|C) \geq p$.

Next, as in the proof of Theorem 1, define $\mathcal{D}^*$, Θ^*, and an auxiliary collection of p-belief operators $(B^{*p}_{i,C_S})_{(i,p,C_S)\in\overline{I}}$ as $B^{*p}_{i,C_S}([e]) := [\beta^p_{i,C_S}(e)]$ for each $[e] \in \mathcal{D}^*$. By construction, $D^{-1}(B^{*p}_{i,C_S}([e])) = B^p_{i,C_S}(D^{-1}[e])$. Since $(\Theta^*)^{-1}(C_S) = [C_S] \in \mathcal{D}^*$, let $\mathcal{C}^* := \{[C_S] \in \mathcal{D}^* \mid C_S \in \mathcal{A}_{\aleph_1}(\mathcal{S})\}$. By construction, $\mathcal{C}^* \subseteq \mathcal{D}^*$, $(\Theta^*)^{-1}(\mathcal{C}_S) = \mathcal{C}^*$, and $\emptyset \notin \mathcal{C}^*$ (this is because Θ^* is surjective). Then, $(\Omega^*, \mathcal{D}^*, \mathcal{C}^*)$ is a conditional space, and $(B^{*p}_{i,C_S})_{(i,p,C_S)\in\overline{I}}$ is a well-defined collection of p-belief operators (observe $B^{*p}_i(\cdot|[C_S]) = B^{*p}_{i,C_S}$). As in the proof of Corollary 2, the p-belief operators satisfy the specified properties, i.e., $\overrightarrow{\Omega^*} := \langle (\Omega^*, \mathcal{D}^*, \mathcal{C}^*), (B^{*p}_i(\cdot|[C_S]))_{(i,p,[C_S])\in I\times[0,1]\times\mathcal{C}^*}, \Theta^* \rangle$ is a conditional-belief space. By construction, $\overrightarrow{\Omega^*}$ is terminal. □

Proof of Corollary S.3. The proof consists of three steps. The first step shows that any expectation space induces the corresponding (probabilistic-)belief space and that any belief space induces the corresponding expectation space. The second step shows that any expectation morphism induces the corresponding (probabilistic-)belief morphism and that any belief morphism induces the corresponding expectation morphism. These two steps establish the equivalence between an expectation space and a belief space. The third step then shows that the expectation space induced by a terminal belief space is a terminal expectation space (see Figure S.1).

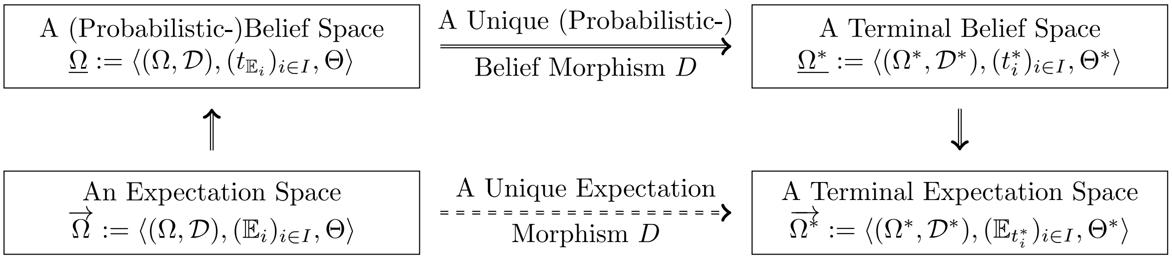

Figure S.1: The Third Step of the Proof of Corollary S.3

Step 1. The first step establishes the correspondence between expectation and belief spaces. First, let $\langle(\Omega, \mathcal{D}), (\mathbb{E}_i)_{i\in I}, \Theta\rangle$ be an expectation space. I define the corresponding probabilistic-belief space $\langle(\Omega, \mathcal{D}), (t_{\mathbb{E}_i})_{i\in I}, \Theta\rangle$: define a measurable map $t_{\mathbb{E}_i} : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ by $t_{\mathbb{E}_i}(\omega)(E) = \mathbb{E}_i(\omega)[\mathbb{I}_E]$. Note that, for ease of notation, in the proof, I formulate any probabilistic-belief space using type mappings.

Conversely, let $\langle(\Omega, \mathcal{D}), (t_i)_{i\in I}, \Theta\rangle$ be a belief space. I define the corresponding expectation space $\langle(\Omega, \mathcal{D}), (\mathbb{E}_{t_i})_{i\in I}, \Theta\rangle$. For any $f \in \mathcal{B}(\Omega)$, define $\mathbb{E}_{t_i}(\omega)[f] := \int_\Omega f(\tilde{\omega})t_i(\omega)(d\tilde{\omega})$ for each $\omega \in \Omega$. It can be seen that $\mathbb{E}_{t_i} : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$ is a well-defined map satisfying Non-negativity, Additivity, Homogeneity, Constancy, and Continuity. Moreover, $t_i = t_{\mathbb{E}_{t_i}}$ and $\mathbb{E}_i = \mathbb{E}_{t_{\mathbb{E}_i}}$ for every $i \in I$.[9]

Step 2. The second step establishes the correspondence between expectation and belief morphisms. Let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ be an expectation morphism. Thus, $\mathbb{E}_i(\omega)[f' \circ \varphi] = \mathbb{E}'_i(\varphi(\omega))[f']$ for any $f' \in \mathcal{B}(\Omega')$. Take $\mathbb{I}_{E'}$ with $E' \in \mathcal{D}$. Then, $t_{\mathbb{E}_i}(\omega)(\varphi^{-1}(E')) = t'_{\mathbb{E}'_i}(\varphi(\omega))(E')$ for each $(\omega, E') \in \Omega \times \mathcal{D}'$. Since this condition is equivalent to the one in terms of p-belief operators, the measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ is a belief morphism $\varphi : \langle(\Omega, \mathcal{D}), (t_{\mathbb{E}_i})_{i\in I}, \Theta\rangle \to \langle(\Omega', \mathcal{D}'), (t'_{\mathbb{E}'_i})_{i\in I}, \Theta'\rangle$.

Conversely, let $\varphi : \underline{\Omega} \to \underline{\Omega'}$ be a belief morphism (to distinguish expectation and belief spaces, I use the underline to indicate a belief space). For any $E' \in \mathcal{D}'$,

$$\mathbb{E}_{t_i}(\omega)[\mathbb{I}_{E'} \circ \varphi] = \mathbb{E}_{t_i}(\omega)[\mathbb{I}_{\varphi^{-1}(E')}] = t_i(\omega)(\varphi^{-1}(E')) = t'_i(\varphi(\omega))(E') = \mathbb{E}'_{t'_i}[\mathbb{I}_{E'}].$$

Using the properties of the expectation type-mapping $\mathbb{E}_i$, $\mathbb{E}_{t_i}(\omega)[f' \circ \varphi] = \mathbb{E}'_{t'_i}(\varphi(\omega))[f']$ for all $f' \in \mathcal{B}(\Omega')$. In other words, the measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ is an expectation morphism $\varphi : \langle(\Omega, \mathcal{D}), (\mathbb{E}_{t_i})_{i\in I}, \Theta\rangle \to \langle(\Omega', \mathcal{D}'), (\mathbb{E}'_{t'_i})_{i\in I}, \Theta'\rangle$.

[9] In the language of category theory, this equivalence hinges on the fact that Δ and Γ (as functors from the category of measurable spaces into itself) are isomorphic: there is a natural isomorphism η from Δ to Γ such that $\eta_{(\Omega,\mathcal{D})} : \Delta(\Omega) \ni \mu \mapsto \int_\Omega [\cdot](\omega)\mu(d\omega) \in \Gamma(\Omega)$ is a measurable isomorphism ($\eta^{-1}_{(\Omega,\mathcal{D})}(J)(E) = J(\mathbb{I}_E)$ for all $E \in \mathcal{D}$) for every measurable space $(\Omega, \mathcal{D})$. Hence, I can also construct a terminal expectation space by replacing Δ with Γ in the (explicit) construction of a terminal probabilistic type space (such as Heifetz and Samet, 1998, Section 5).

Step 3. The third step shows that the expectation space $\overrightarrow{\Omega^*} := \langle(\Omega^*, \mathcal{D}^*), (\mathbb{E}_{t_i^*})_{i\in I}, \Theta^*\rangle$ induced by a terminal probabilistic belief space $\underline{\Omega^*} := \langle(\Omega^*, \mathcal{D}^*), (t_i^*)_{i\in I}, \Theta^*\rangle$ is a terminal expectation space.

Take any expectation space $\overrightarrow{\Omega}$. Consider a corresponding belief space $\underline{\Omega} := \langle(\Omega, \mathcal{D}), (t_{\mathbb{E}_i})_{i\in I}, \Theta\rangle$. Since $\underline{\Omega^*}$ is terminal, there is a (unique) belief morphism $D : \underline{\Omega} \to \underline{\Omega^*}$. By Step 2, the measurable mapping D is an expectation morphism $\overrightarrow{\Omega}$ into $\overrightarrow{\Omega^*}$. To show that the mapping D is unique, let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ be an expectation morphism. By Step 2, $\varphi : \underline{\Omega} \to \underline{\Omega^*}$ is a belief morphism. Since $\underline{\Omega^*}$ is a terminal belief space, it follows that $\varphi = D$. □

F.2 Appendix D

Proof of Corollary S.4. The proof consists of two steps. The first step introduces a hypothetical player "0" to each probabilistic-belief space $\overrightarrow{\Omega}$ with a common prior so that it can be identified as a probabilistic belief space $\langle(\Omega, \mathcal{D}), (B_i^p)_{(i,p)\in(I\cup\{0\})\times[0,1]}, \Theta\rangle$ satisfying Expression (4). Namely, define B_0^p as follows: $B_0^p(E) = \Omega$ if $\mu(E) \geq p$; and $B_0^p(E) = \emptyset$ if $\mu(E) < p$. Hence, the player 0 has a state-independent type mapping $t_{B_0}(\omega)(\cdot) = \mu(\cdot)$ for all $\omega \in \Omega$. For any spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$, $B_0^p\varphi^{-1} = \varphi^{-1}B_0'^p$ is equivalent to $\mu \circ \varphi^{-1} = \mu'$.

The second step constructs a terminal space. Following the construction in Section 3, the p-belief operators $(B_0^{*p})_{p\in[0,1]}$ satisfy $D^{-1}B_0^{*p}([e]) = B_0^p D^{-1}([e])$ for all $[e] \in \mathcal{D}^*$. Define $\mu^* : \mathcal{D}^* \to [0,1]$ by $\mu^*([e]) := \sup\{p \in [0,1] \mid \Omega^* = B_0^{*p}([e])\}$ for each $[e] \in \mathcal{D}^*$. Since the p-belief operators $(B_0^{*p})_{p\in[0,1]}$ satisfy the properties in Definition 10 (2), μ^* is a countably-additive probability measure. Also, $\mu^* = \mu \circ D^{-1}$ follows from $D^{-1}B_0^{*p}(\cdot) = B_0^p D^{-1}(\cdot)$. Thus, μ^* satisfies Equation (4) (see footnote 6 in Appendix D.1.2). □

F.3 Appendix E

Proof of Proposition S.1. For both parts, by Remark 3, a belief morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ satisfies $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \varphi$. For Part (1), I first show that $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ is measurable. Since $D_{\overrightarrow{\Omega}} = D_{\overrightarrow{\Omega'}} \circ \varphi$, $D_{\overrightarrow{\Omega}}^{-1} = \varphi^{-1} \circ D_{\overrightarrow{\Omega'}}^{-1}$. Since $\overrightarrow{\Omega'}$ is minimal, $D_{\overrightarrow{\Omega'}}^{-1}(\mathcal{D}^*) = \mathcal{D}'$. For any $E' \in \mathcal{D}'$, there is $[e] \in \mathcal{D}^*$ with $E' = D_{\overrightarrow{\Omega'}}^{-1}([e])$. Then, $\varphi^{-1}(E') = \varphi^{-1}(D_{\overrightarrow{\Omega'}}^{-1}([e])) = D_{\overrightarrow{\Omega}}^{-1}([e]) \in \mathcal{D}$. Second, $\Theta = \Theta' \circ \varphi$. Third, I show $B_i(\varphi^{-1}(\cdot)) = \varphi^{-1}(B_i'(\cdot))$. For any $E' \in \mathcal{D}'$, take $[e] \in \mathcal{D}^*$ with $E' = D_{\overrightarrow{\Omega'}}^{-1}([e])$. Then, $B_i(\varphi^{-1}(E')) = B_i D_{\overrightarrow{\Omega}}^{-1}([e]) = D_{\overrightarrow{\Omega}}^{-1} B_i^*([e]) = \varphi^{-1} D_{\overrightarrow{\Omega'}}^{-1}(B_i^*([e]) = \varphi^{-1} B_i' D_{\overrightarrow{\Omega'}}^{-1}([e]) = \varphi^{-1} B_i'(E')$.

For Part (2), suppose that $\overrightarrow{\Omega'}$ is not minimal. By Remark 4, $\overrightarrow{\Omega'_{\overline{\kappa}}}$ is minimal and $\mathrm{id}_{\Omega'}$ satisfies $D_{\overrightarrow{\Omega'}} = D_{\overrightarrow{\Omega'_{\overline{\kappa}}}} \circ \mathrm{id}_{\Omega'}$. However, $\mathrm{id}_{\Omega'} : (\Omega', \mathcal{D}'_{\overline{\kappa}}) \to (\Omega', \mathcal{D}')$ is not measurable because $\mathcal{D}'_{\overline{\kappa}} \subsetneq \mathcal{D}'$, and hence it is not a belief morphism. □

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