Topology-free Constructions of a Universal Type Space as Coherent Belief Hierarchies*

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Abstract

This paper constructs a universal type space on an arbitrary measurable space of nature states as the set of coherent belief hierarchies, proposing the right notion of coherent belief hierarchies. Since any type space induces belief hierarchies of countable depths, coherency in this paper requires that a belief hierarchy (consisting of all finite levels of beliefs) extend to any subsequent countable levels in a way such that all countable levels of beliefs do not conflict with one another. The paper shows that the space of such coherent belief hierarchies is a universal type space without any topological assumption on nature states. Such universal type space coincides exactly with the topology-free universal type space constructed as the set of belief hierarchies that are induced by some type of some type space. Hence, this paper shows that, under the coherency condition that all countable levels of beliefs do not conflict with one another, the previous approaches yield the same universal space in the most general measurable environment without any topological assumption. Moreover, the need for keeping track of all countable levels of beliefs in constructing the universal type space without a topological assumption has a game-theoretic counterpart: the need for transfinite levels of reasoning (e.g., eliminations of strictly dominated actions) in solving infinite games with general measurable action spaces employing rationalizability solution concepts.

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1 Introduction

Suppose players are uncertain about some exogenously-given values such as their strategies or payoff functions. Call such exogenously-given values states of nature. In such a situation, the players' behaviors, such as their strategy choices, would depend on their belief hierarchies over nature states: the players' beliefs over nature states, their beliefs over their opponents' beliefs over nature states, and so forth.

A Harsanyi (1967-1968) type space is a self-referential representation of players' belief hierarchies. Each player has a set of types, and each type induces a probability measure over nature states and the opponents' types. The idea of capturing the players' belief hierarchies using a type space hinges on the question of whether there exists a "universal" type space that contains all possible belief hierarchies. Pioneering papers such as Armbruster and Böge (1979), Böge and Eisele (1979), and Mertens and Zamir (1985) have established the existence of the universal type space when the underlying set of states of nature is equipped with a topological structure.

Broadly, there are two approaches, which I call "explicit" and "implicit," to establishing the universal type space. The two approaches, however, may lead to different type spaces without an appropriate topological assumption on nature states. The purpose of this paper is to provide new explicit-approach constructions of a universal type space so that the two approaches lead to the same universal type space without any topological assumption. This paper unifies the previously-known constructions at the most general level in which no topological assumptions are imposed. Moreover, the construction of the universal type space in this paper has a game-theoretic implication in rationalizability solution concepts.

The "explicit" approach by Brandenburger and Dekel (1993) and Mertens and Zamir (1985) constructs the universal type space explicitly as the set of belief hierarchies that satisfies a certain coherency condition (to be mentioned below) under some topological assumption on nature states.[1] In contrast, the "implicit" approach of Heifetz and Samet (1998, Theorem 5.5) constructs a universal (precisely, terminal) type space by collecting all possible belief hierarchies that can be induced by some type spaces without any topological assumption.[2] Thus, any type of any particular type space is associated with the corresponding belief hierarchy in the type space of Heifetz and Samet (1998).

As shown by Heifetz and Samet (1999), however, absent any topological assumption, the space of coherent belief hierarchies (in which all the finite levels of beliefs

[1] See, for example, Heifetz (1993), Mertens, Sorin, and Zamir (2015), and Pintér (2005, 2018) for generalizations of topological assumptions. For a survey, see Siniscalchi (2010).

[2] The terminologies, "explicit" and "implicit," as a means to construct a universal type space are for the classification purpose. Note that Heifetz and Samet (1999) use the terminologies "explicit" and "implicit," as a means to represent interactive probabilistic beliefs, in a different context. The explicit approach refers to explicitly describing players' beliefs about nature states, their beliefs about their beliefs about nature states, and so on, while the implicit approach is to represent interactive beliefs through a type space.

do not contradict each other) may not admit an extension to a countable level of beliefs and may be strictly larger than the terminal type space of Heifetz and Samet (1998). Hence, it has been unclear what conditions on belief hierarchies make the explicit-approach construction possible without a topological assumption. The explicit approach of Brandenburger and Dekel (1993) and Mertens and Zamir (1985) may not lead to a terminal type space on an arbitrary measurable space of nature states. The terminal type space of Heifetz and Samet (1998) may not be characterized merely as the space of coherent belief hierarchies.

This paper provides explicit-approach constructions of a universal type space without any topological assumption on nature states by extending a coherency condition on belief hierarchies to all countable (not necessarily restricted to finite) levels of beliefs (i.e., by proposing a stronger yet conceptually natural notion of coherency). On the one hand, this paper elucidates the implicit assumptions on belief hierarchies in the implicit approach. Roughly, the terminal type space of Heifetz and Samet (1998) is characterized as the space of belief hierarchies that can coherently extend to transfinite levels of beliefs. On the other hand, this paper shows that the constructions of Brandenburger and Dekel (1993) and Mertens and Zamir (1985) can be extended to any measurable space of nature. Any applications that hinge on the explicit-approach universal type space in strands of literature such as robustness or solution concepts in games and mechanisms can be extended to any measurable space of nature states.[3] As will be discussed, the stronger coherency condition on countable levels of belief hierarchies without a topological assumption naturally corresponds to transfinite levels of interactive reasoning in rationalizability solution concepts in a game in which the players' action spaces are not compact or their payoff functions are discontinuous or in which the players' beliefs are not countably additive—in such a game, the predictions based on rationalizability may require transfinite elimination of, say, strictly dominated actions.[4]

[3]In the context of robust mechanism design, see, for instance, Bergemann and Morris (2005, Section 2.5) for their discussions on the role that topological restrictions plays in implementation on universal type spaces. For solution concepts such as interim correlated rationalizability, see, for example, Dekel, Fudenberg, and Morris (2007). They restrict attention to a finite set of nature states, and interchangeably use the implicit-approach (topology-free) universal type space of Heifetz and Samet (1998) and the explicit-approach (topological) universal type spaces of Brandenburger and Dekel (1993) and Mertens and Zamir (1985). One of their open questions is whether the transfinite induction is needed to equate the iterative and fixed-point definitions of rationalizability, and this paper suggests that the transfinite induction would be needed.

[4]For (rationalizability) solution concepts in games, see, for example, Arieli (2010), Chen, Long, and Luo (2007), Chen, Luo, and Qu (2016), Dufwenberg and Stegeman (2002), Lipman (1994), and Samet (2015). In the context of robust mechanism design, see, for instance, Bergemann and Morris (2011), Bergemann, Morris, and Tercieux (2011), and Kunimoto and Serrano (2011). The literature on the formalizations of common belief (and on an implication of common belief in rationality as a solution concept) also examines transfinite iterations of mutual beliefs; see, for instance, Barwise (1989), Fukuda (2020, 2021), Heifetz (1999), and Lismont and Mongin (1995). Lipman (1991) considers a transfinite regress in a decision procedure to choose a procedure to ... to choose an

To examine the right coherency condition, recall the coherency condition of Brandenburger and Dekel (1993): the universal type space is the set of coherent belief hierarchies that satisfy common certainty of coherency. A belief hierarchy is coherent in their sense if all its (finite) levels of beliefs do not contradict one another. They show that, under a certain topological assumption, the coherent belief hierarchies induce the beliefs over nature states and the opponents' (not-necessarily-coherent) belief hierarchies. The universal type space is carved out by the common-certainty-of-coherency condition (i.e., everybody is certain that beliefs are coherent, everybody is certain that everybody is certain that beliefs are coherent, and so forth *ad infinitum*).[5]

The key conceptual observation in this paper is the following: when the players' beliefs are represented by a type space, they are always able to reason at least about their countable levels of beliefs beyond finite levels. For two players, if each player has a (first-order) belief over nature states, a (second-order) belief over nature states and the (first-order) beliefs of the opponent, and so on, then each player i is still uncertain about the opponent j's beliefs over i's own finite-level belief hierarchies. Hence, first, I define a belief hierarchy consisting of all the countable levels of beliefs. Call it a transfinite belief hierarchy; in contrast, when I refer to a belief hierarchy without the adjective, "transfinite," it means that the hierarchy consists of all the finite level beliefs. Second, I define a notion of coherency for a transfinite belief hierarchy: simply, any levels of beliefs in the transfinite belief hierarchy do not contradict one another. Then, I call a belief hierarchy *coherent* if it extends to some coherent transfinite belief hierarchy. I show that the construction of the universal type space imposing the stronger coherency condition generalizes Brandenburger and Dekel (1993) without any topological assumption. The stronger coherency condition reduces to the standard one under an appropriate topological assumption. I show that this stronger coherency condition is a natural generalization in the sense that the same idea of capturing all countable levels of beliefs also generalizes the explicit-approach construction of Mertens and Zamir (1985) without any topological assumption.

Section 4 provides the explicit-approach construction of a universal type space by generalizing coherency and common certainty of coherency. I show that a coherent belief hierarchy (in the stronger sense) induces a belief over nature states and the opponents' (not-necessarily-coherent) belief hierarchies without the Kolgomorov extension theorem, which is the key topological tool in the previous literature (Proposition 1). Then, the Brandenburger and Dekel (1993) type space is extracted under common certainty of coherency (Theorem 1). However, unlike the topological case, common certainty of coherency may require transfinite iterations of mutual certainty.

action.

[5]Also, Brandenburger and Dekel (1993) informally argue that the common-certainty-of-coherency condition captures the idea that the players are commonly certain of the model. This paper shows that their argument does not depend on a topological assumption, which is indeed orthogonal to the conceptual question, when the analysts allow for transfinite levels of interactive reasoning.

I show that the resulting space coincides with the implicit-approach terminal type space of Heifetz and Samet (1998) (Theorem 2).

Section 5 provides another explicit-approach construction by generalizing the "recursive coherency" condition of Mertens and Zamir (1985): each level of a coherent belief hierarchy is a belief about all the lower-level coherent beliefs.[6] The explicit and implicit approaches again yield the same universal type space respecting the recursive coherency condition. Hence, this paper shows that the idea of capturing transfinite levels of beliefs yields the universal type space irrespective of a particular explicit-approach construction. Connecting the two approaches of Brandenburger and Dekel (1993) and Heifetz and Samet (1998) and those of Mertens and Zamir (1985) and Heifetz and Samet (1998) without any topological assumption is new.[7] Moreover, this paper also connects the two different explicit-approach constructions in the most general measurable environment without any topological assumption.[8] This paper shows not only that the universal type spaces of Brandenburger and Dekel (1993) and Mertens and Zamir (1985) are isomorphic but also that the universal type space of Mertens and Zamir (1985) is also characterized as the space of recursive coherent belief hierarchies satisfying common certainty of recursive coherency (Theorem 4).

In sum, this paper shows that the explicit and implicit approaches yield the same universal type space when a coherency condition on belief hierarchies is extended to all countable levels. As summarized in Table 1, I consider coherency and common certainty of coherency (Section 4) and recursive coherency (Section 5). For the second or the third row of the table, this paper generalizes the implicit and explicit approaches by extending the coherency condition on belief hierarchies to all countable levels, to show that both approaches reduce to the same universal type space. This paper makes it explicit how the implicit-approach universal type space is constructed: the belief hierarchies induced by type spaces coincide with the belief hierarchies that extend to transfinite belief hierarchies respecting a coherency condition. Moreover, this paper shows that, within the explicit approach, the universal type spaces of Brandenburger and Dekel (1993) and Mertens and Zamir (1985) are isomorphic without any topological assumption when the coherency conditions are extended to countable

[6] I use the term, recursive coherency, to distinguish the coherency condition of Mertens and Zamir (1985) from the coherency condition of Brandenburger and Dekel (1993). Battigalli, Friedenberg, and Siniscalchi (In Preparation) distinguish the universal-type-space constructions of Brandenburger and Dekel (1993) and Mertens and Zamir (1985) by referring to the former as "top-down" and the latter as "bottom-up" (private communication).

[7] Technically, I slightly modify the implicit-approach construction of a universal type space by Heifetz and Samet (1998) for each of two different coherency conditions. Thus, the implicit-approach construction that turns out to be exactly the same as Brandenburger and Dekel (1993) and the one that turns out to be exactly the same as Mertens and Zamir (1985) are not exactly the same but are isomorphic, given the difference in the coherency conditions. See also footnotes 9 and 13 for how I modify the implicit-approach constructions for each coherency condition.

[8] De Vito (2010a) shows that, under a weak topological assumption (namely, an underlying space of nature states is Hausdorff), the explicit-approach universal type space of Brandenburger and Dekel (1993) is isomorphic to that of Mertens and Zamir (1985).

Table 1: Constructions of a Universal Type Space

	Explicit Approach	Implicit Approach
Coherency and Common Certainty of Coherency (Section 4)	Brandenburger and Dekel (1993) (Theorems 1 and 2)	Heifetz and Samet (1998)
Recursive Coherency (Section 5)	Mertens and Zamir (1985) (Theorems 3 and 4)	Heifetz and Samet (1998)

belief hierarchies, i.e., the constructions in the second and the third rows of the table are isomorphic.

The stronger coherency condition on transfinite levels of belief hierarchies has a natural counterpart in rationalizability solution concepts in games and mechanisms: in a game without an appropriate topological condition, predictions under common certainty of rationality under such solution concepts may require transfinite levels of interactive reasoning based on mutual beliefs in rationality (recall footnote 4). Section 6 briefly discusses transfinite levels of interactive reasoning. Specifically, it provides a simple example of a strategic game in which a unique prediction under common certainty of rationality requires transfinite levels of elimination of strictly dominated actions. That is, the unique prediction under common certainty of rationality requires not only all finite levels of mutual beliefs in rationality (e.g., the players believe that they are rational, they believe that they believe that they are rational, and so forth, for any finite levels) but also the mutual belief in the fact that they believe their rationality for all finite levels. This is analogous to the fact that common certainty of coherency may require transfinite iterations of mutual certainty, in employing the explicit approach to constructing a universal type space. In fact, this is not a mere coincidence in the sense that the notion of common certainty requires that common certainty be belief-closed (Mertens and Zamir, 1985): whenever the players are commonly certain of their rationality, everybody is certain that they are commonly certain of their rationality; and similarly, whenever the players are commonly certain that their belief hierarchies are coherent, everybody is certain that they are commonly certain that their belief hierarchies are coherent. In order for common certainty to be belief-closed, transfinite iterations of mutual beliefs may be required in a general environment.

The rest of the paper is structured as follows. Section 2 provides technical preliminaries. Section 3 defines type spaces and belief hierarchies. Section 4 provides the explicit-approach topology-free construction of a universal type space by generalizing the concept of coherency and common certainty of coherency. It also shows that the explicit and implicit approaches yield the same universal type space. Section 5 extends the results to the recursive coherency condition. Section 6 discusses the main results. Specifically, Section 6.1 provides the simple strategic game in which the prediction under common certainty of rationality requires transfinite levels of mutual beliefs in rationality. Section 6.2 discusses the related literature. Section 6.3 provides concluding remarks. Proofs are relegated to Appendix A.

2 Technical Preliminaries

Denote by $\mathcal{P}(\cdot)$ the operation of taking the power set. Denote by $\sigma(\cdot)$ the operation of taking the generated σ-algebra. For any measurable space $(M, \mathcal{M})$, let $\Delta(M)$ be the set of probability measures on $(M, \mathcal{M})$, endowed with the σ-algebra $\mathcal{M}_\Delta := \sigma(\{\{\mu \in \Delta(M) \mid \mu(E) \geq p\} \in \mathcal{P}(\Delta(M)) \mid (E, p) \in \mathcal{M} \times [0,1]\})$. If $\varphi : (M, \mathcal{M}) \to (N, \mathcal{N})$ is a measurable map, then the map $\Delta(\varphi) : \Delta(M) \ni \mu \mapsto \mu \circ \varphi^{-1} \in \Delta(N)$ is measurable with respect to $(\mathcal{M}_\Delta, \mathcal{N}_\Delta)$. Call a measurable map a (measurable) isomorphism if it is bijective and its inverse is measurable. Denote by id_M the identity map on $(M, \mathcal{M})$.

For a probability measure μ on a measurable space $(M, \mathcal{M})$, denote by $\mu^* : \mathcal{P}(M) \to [0,1]$ the outer measure induced by μ. That is, $\mu^*(F) := \inf\{\mu(E) \in [0,1] \mid F \subseteq E \in \mathcal{M}\}$ for each $F \in \mathcal{P}(M)$.

For any measurable space $(M, \mathcal{M})$ and for any subset N of M, $\mathcal{M} \cap N := \{E \cap N \in \mathcal{P}(N) \mid E \in \mathcal{M}\}$ is a σ-algebra on N. Also, define $\tilde{\Delta}(N) := \{\mu \in \Delta(M) \mid \mu(E) = 1 \text{ for any } E \in \mathcal{M} \text{ with } N \subseteq E\}$. There is an isomorphism $(\Delta(N), (\mathcal{M} \cap N)_\Delta) \ni \mu \mapsto \mu(\cdot \cap N) \in (\tilde{\Delta}(N), \mathcal{M}_\Delta \cap \tilde{\Delta}(N))$. The inverse of such isomorphism maps $\nu \in \tilde{\Delta}(N)$ to the outer measure $\nu^*|_{\mathcal{M} \cap N} \in \Delta(N)$ of ν restricted on $\mathcal{M} \cap N$ (Heifetz and Samet, 1999, Lemma 2.2). That is, for any $\nu \in \tilde{\Delta}(N)$, $\nu(\cdot) = \nu^*(\cdot \cap N)$.

I consider the product measurable space for any product of measurable spaces. For example, $(M_1 \times M_2, \mathcal{M}_1 \times \mathcal{M}_2)$ denotes the product measurable space of $(M_1, \mathcal{M}_1)$ and $(M_2, \mathcal{M}_2)$. For a collection of measurable spaces $((M_\lambda, \mathcal{M}_\lambda))_{\lambda \in \Lambda}$ (henceforth, abbreviated as $(M_\lambda, \mathcal{M}_\lambda)_{\lambda \in \Lambda}$), the product measurable space is written as $(\prod_{\lambda \in \Lambda} M_\lambda, \prod_{\lambda \in \Lambda} \mathcal{M}_\lambda)$. If $f_\lambda : (M_\lambda, \mathcal{M}_\lambda) \to (N_\lambda, \mathcal{N}_\lambda)$ is a measurable map for each $\lambda \in \Lambda$, then $f : (\prod_{\lambda \in \Lambda} M_\lambda, \prod_{\lambda \in \Lambda} \mathcal{M}_\lambda) \to (\prod_{\lambda \in \Lambda} N_\lambda, \prod_{\lambda \in \Lambda} \mathcal{N}_\lambda)$ defined by $f((m_\lambda)_{\lambda \in \Lambda}) := (f_\lambda(m_\lambda))_{\lambda \in \Lambda}$ is measurable. Denote by π^M_N the projection from $(M, \mathcal{M})$ to $(N, \mathcal{N})$ (e.g., $(M_1 \times M_2, \mathcal{M}_1 \times \mathcal{M}_2)$ to $(M_1, \mathcal{M}_1)$). While the operation of taking the product (measurable) space is not necessarily associative, I often re-order and identify product spaces. For example, while $M_1 \times (M_2 \times M_3)$ and $(M_1 \times M_2) \times M_3$ are (generally) different spaces, I identify these. Denote by marg the operation of taking the marginal: letting π^M_N be the projection, $\mathrm{marg}_N \, \mu := \mu \circ (\pi^M_N)^{-1} \in \Delta(N)$ for any $\mu \in \Delta(M)$.

Denote by ω the least infinite ordinal. Specifically, $\omega := \{0, 1, 2, \dots\}$. Likewise, denote by Ω the least uncountable ordinal. Denote by $[\alpha, \beta)$ the set of ordinals γ with $\alpha \leq \gamma < \beta$. For ease of notation, I often identify $\mathbb{N} = [1, \omega)$. Define $[\alpha, \beta]$ as the set of ordinals γ with $\alpha \leq \gamma \leq \beta$.

Let M be a set, and let $(\mathcal{M}_\alpha)_{\alpha \in [1, \Omega)}$ be a sequence of σ-algebras on M such that $\alpha \leq \beta$ implies $\mathcal{M}_\alpha \subseteq \mathcal{M}_\beta$. Then, $\bigcup_{\alpha \in [1, \Omega)} \mathcal{M}_\alpha$ is a σ-algebra.

3 Type Spaces and Belief Hierarchies

For simplicity, throughout the paper, suppose two players $I := \{1, 2\}$ face a non-empty measurable space of nature states $(S, \mathcal{S})$. Let $I_0 := I \cup \{0\}$. For each player $i \in I$, denote by $j (= 3 - i)$ her opponent.

3.1 Type Spaces

I define a type space and a universal type space which is formally defined to be a terminal type space. A *type space* is a tuple $\overrightarrow{T} = \langle (T_i, \mathcal{T}_i)_{i \in I_0}, (m_i)_{i \in I} \rangle$ such that: (i) each $(T_i, \mathcal{T}_i)$ is a measurable space with $(T_0, \mathcal{T}_0) = (S, \mathcal{S})$; and that (ii) the *type mapping* $m_i : (T_i, \mathcal{T}_i) \to (\Delta(S \times T_j), (\mathcal{S} \times \mathcal{T}_j)_\Delta)$ of player i is a measurable map. The type mapping m_i associates, with each *type* $t_i \in T_i$, a probability measure over the nature states S and the opponent's types T_j. It is a self-referential representation of her belief hierarchies.

Call a profile of measurable maps $\varphi = (\varphi_i)_{i \in I_0}$ from $\overrightarrow{T}$ to $\overrightarrow{T'}$ a *(type) morphism* if $\varphi_0 = \mathrm{id}_S$ and if $\varphi_i : (T_i, \mathcal{T}_i) \to (T'_i, \mathcal{T}'_i)$ satisfies $m_i(\cdot) \circ \varphi_{-i}^{-1} = m'_i(\varphi_i(\cdot))$ for each $i \in I$, where $\varphi_{-i} = (\mathrm{id}_S, \varphi_j) : (S \times T_j, \mathcal{S} \times \mathcal{T}_j) \to (S \times T'_j, \mathcal{S} \times \mathcal{T}'_j)$. A morphism φ is a *(type) isomorphism* if each φ_i is bijective and the inverses form a morphism.

Now, call a type space $\overrightarrow{T^*} = \langle (T^*_i, \mathcal{T}^*_i)_{i \in I_0}, (m^*_i)_{i \in I} \rangle$ *terminal* if, for any type space $\overrightarrow{T}$, there is a unique morphism $\varphi : \overrightarrow{T} \to \overrightarrow{T^*}$. Two remarks are in order. First, a terminal type space is unique up to isomorphism. Second, a type space in this framework forms a "coalgbera" in category theory (Moss and Viglizzo, 2004, 2006). This means that, by Lambek (1968)'s Lemma in category theory, the measurable maps $(\mathrm{id}_S, (m^*_i)_{i \in I})$ of a terminal type space $\overrightarrow{T^*}$ form a type isomorphism. Especially, $m^*_i : T^*_i \to \Delta(S \times T^*_j)$ is a measurable isomorphism.

Call a type space $\overrightarrow{T}$ *complete* (e.g., Brandenburger, 2003) if each m_i is surjective. The above remark implies that, in this framework in which each type of a player is associated with a probability measure over nature states and the types of the opponent(s), a terminal type space is complete.

As discussed in the Introduction, there are two approaches to establishing a terminal type space. The explicit approach constructs the terminal type space explicitly as the set of belief hierarchies that satisfy a coherency condition. The implicit approach constructs the terminal type space implicitly by collecting all possible belief hierarchies that can be induced by some type spaces. This paper demonstrates that the two approaches yield the same terminal type space without any topological assumption on nature states $(S, \mathcal{S})$ by reformulating coherency conditions of Brandenburger and Dekel (1993) (in Section 4) and Mertens and Zamir (1985) (in Section 5).

3.2 Belief Hierarchies

The following two sections construct a terminal type space as the set of belief hierarchies that satisfy a certain coherency condition: coherency and common certainty of coherency (Section 4) and recursive coherency (Section 5). To accommodate two different coherency conditions within the same framework, this subsection defines belief hierarchies (Definition 1) and belief hierarchies induced by a type space (Definition 2), separately from these coherency conditions. Moreover, I define two kinds of belief hierarchies. One is a belief hierarchy that consists of all finite-order beliefs. The other is a transfinite belief hierarchy that consists of all countable-order beliefs.

To that end, Definition 1 (1) inductively defines a hierarchy space $H^{<\alpha}$ of order less than $\alpha \in [2, \Omega]$, which is a set of belief hierarchies of order less than α. Specifically, the hierarchy space of order less than 2 is the space of first-order beliefs $\Delta(S)$. A hierarchy space of order less than ω is a set of finite-order beliefs. A hierarchy space of order less than Ω is a space of countable-order beliefs.

Each element h of $H^{<\alpha}$ is a belief hierarchy of order less than α. For example, a belief hierarchy h of order less than 3 is a sequence of probability measures $h = (h^1, h^2)$ such that $h^1 \in \Delta(S)$ is a first-order belief on nature states S and that $h^2 \in \Delta(S \times \Delta(S))$ is a second-order belief on nature states S and the first-order beliefs (of the other player).

To express a belief hierarchy $h \in H^{<\alpha}$ of order less than $\alpha \in [2, \Omega]$ explicitly, Definition 1 (2) introduces the space $Z^{<\beta}$ with $\beta \in [1, \alpha]$, starting from $Z^{<1} = S$ and then $Z^{<\beta} = S \times H^{<\beta}$, so that $\Delta(Z^{<\beta})$ represents the beliefs on the nature states S and all the lower-level beliefs of order less than β. Then, in Definition 1 (3), the belief hierarchy $h \in H^{<\alpha}$ of order less than $\alpha \in [2, \Omega]$ is a sequence of probability measures $h = (h^\beta)_{\beta \in [1,\alpha)} \in \prod_{\beta \in [1,\alpha)} \Delta(Z^{<\beta})$ on nature states and lower-level beliefs.

Especially, Definition 1 (3) defines a belief hierarchy consisting of finite-order beliefs (when $\alpha = \omega$) and a transfinite belief hierarchy consisting of countable-order beliefs (when $\alpha = \Omega$).

Definition 1. 1. Call a measurable space $(H^{<\alpha}, \mathcal{H}^{<\alpha})$ a *hierarchy space of order less than* $\alpha \in [2, \Omega]$ if, starting from $(H^{<2}, \mathcal{H}^{<2}) := (\Delta(S), \mathcal{S}_\Delta)$, it is inductively constructed from $(H^{<\beta}, \mathcal{H}^{<\beta})_{\beta \in [2,\alpha]}$ as follows:

(a) For a successor ordinal $\beta = \gamma + 1 \in [3, \alpha]$, $H^{<\beta} \subseteq H^{<\gamma} \times \Delta(S \times H^{<\gamma})$ and $\mathcal{H}^{<\beta} := (\mathcal{H}^{<\gamma} \times (\mathcal{S} \times \mathcal{H}^{<\gamma})_\Delta) \cap H^{<\beta}$ is the product σ-algebra on $H^{<\beta}$.

(b) For a limit ordinal $\beta \in [2, \alpha]$, $H^{<\beta}$ is the set of all $(h^\gamma)_{\gamma \in [1,\beta)}$ such that $(h^\delta)_{\delta \in [1,\gamma)} \in H^{<\gamma}$ for all $\gamma \in [2, \beta)$. For each $\gamma \in [2, \beta)$, denote by $\pi^\beta_\gamma : H^{<\beta} \to H^{<\gamma}$ the projection, and define the product σ-algebra $\mathcal{H}^{<\beta}$ on $H^{<\beta}$ as

$$\mathcal{H}^{<\beta} := \sigma\left(\{(\pi^\beta_\gamma)^{-1}(E) \in \mathcal{P}(H^{<\beta}) \mid E \in \mathcal{H}^{<\gamma} \text{ for some } \gamma \in [2, \beta)\}\right).$$

For each $\beta \in [2, \alpha)$, call the lower-level space $(H^{<\beta}, \mathcal{H}^{<\beta})$ the hierarchy space *associated with* $(H^{<\alpha}, \mathcal{H}^{<\alpha})$. By construction, the projection $\pi^{\alpha}_{\beta} : (H^{<\alpha}, \mathcal{H}^{<\alpha}) \to (H^{<\beta}, \mathcal{H}^{<\beta})$ is a well-defined measurable map.

2. Let $(H^{<\alpha}, \mathcal{H}^{<\alpha})$ be a hierarchy space of order less than $\alpha \in [2, \Omega]$, and let $(H^{<\beta}, \mathcal{H}^{<\beta})_{\beta \in [2,\alpha)}$ be the hierarchy spaces associated with $(H^{<\alpha}, \mathcal{H}^{<\alpha})$. Then, for each $\beta \in [1, \alpha)$, define

$$(Z^{<\beta}, \mathcal{Z}^{<\beta}) := \begin{cases} (S, \mathcal{S}) & \text{if } \beta = 1 \\ (S \times H^{<\beta}, \mathcal{S} \times \mathcal{H}^{<\beta}) & \text{if } \beta \in [2, \alpha) \end{cases}.$$

For each $\beta \in [2, \alpha)$, the space $\Delta(Z^{<\beta})$ represents the beliefs on the nature states and all the lower-level beliefs of order less than β.

3. (a) Call each $h \in H^{<\alpha}$ a *belief hierarchy of order less than* α. By construction, h is a sequence of probability measures $h = (h^{\beta})_{\beta \in [1,\alpha)} \in \prod_{\beta \in [1,\alpha)} \Delta(Z^{<\beta})$ on nature states and the opponent's lower-level beliefs.

   (b) Especially, call $h = (h^k)_{k \in \mathbb{N}} \in H^{<\omega}$ a *belief hierarchy*, and call $\overline{h} = (\overline{h}^{\beta})_{\beta \in [1,\Omega)} \in H^{<\Omega}$ a *transfinite belief hierarchy*.

When I will construct a terminal type space as a (sub-)set of belief hierarchies $(H^{<\omega}, \mathcal{H}^{<\omega})$, I impose a specific coherency condition on the set of belief hierarchies using the set of transfinite belief hierarchies $(H^{<\Omega}, \mathcal{H}^{<\Omega})$ (thus, take $\alpha = \Omega$ in Definition 1). While in Definition 1 (1a), the hierarchy space $H^{<\gamma+1}$ of order less than $\gamma + 1$ is defined merely as a subset of $H^{<\gamma} \times \Delta(S \times H^{<\gamma})$, I will consider two different notions of coherency in Sections 4 and 5,

Specifically, first, at each step $\beta \in [3, \Omega]$ in Definition 1 (1a), Section 4 considers the largest set $H^{<\beta}$, i.e., take $H^{<\gamma+1} = H^{<\gamma} \times \Delta(S \times H^{<\gamma})$, separately from a coherency condition to construct $(H^{<\Omega}, \mathcal{H}^{<\Omega})$. Then, impose an appropriate coherency condition on $(H^{<\omega}, \mathcal{H}^{<\omega})$ using $(H^{<\Omega}, \mathcal{H}^{<\Omega})$: namely, coherency and common certainty of coherency in Definitions 3 and 4.

Second, at each step $\beta \in [3, \Omega]$ in Definition 1 (1a), Section 5 restricts attention to belief hierarchies $(H^{<\omega}, \mathcal{H}^{<\omega})$ that satisfy an appropriate coherency condition using $(H^{<\Omega}, \mathcal{H}^{<\Omega})$: namely, recursive coherency in Definition 6. The hierarchy space $(H^{<\Omega}, \mathcal{H}^{<\Omega})$ is inductively constructed from the recursive coherency condition, and thus in Definition 1 (1a), $H^{<\gamma+1}$ is in fact defined as a subset of $H^{<\gamma} \times \Delta(S \times H^{<\gamma})$ that satisfies the recursive coherency condition.

In either way, this paper considers transfinite belief hierarchies $(H^{<\Omega}, \mathcal{H}^{<\Omega})$ because a type space can always capture all countable levels of beliefs beyond the finite ones: player i has a belief over her opponent j's belief hierarchies, over j's belief over i's belief hierarchies, and so on. Thus, while the resulting terminal type space (in each section) is constructed as a subset of belief hierarchies, I use an auxiliary notion of a transfinite belief hierarchy in a way such that the terminal space (consisting of belief hierarchies) induces transfinite belief hierarchies in a coherent manner.

In constructing a terminal type space as a subset of belief hierarchies $H^{<\omega}$, since every player faces the same space of nature states S, the hierarchy space $H^{<\omega}$ does not depend on the identity of the players, i.e., it is not necessary to define the hierarchy space $(H_i^{<\alpha}, \mathcal{H}_i^{<\alpha})$ of a particular player $i \in I$.

Since the terminal type space to be constructed consists of belief hierarchies, before the actual constructions, here I define a belief hierarchy induced by a type mapping of a type space. Then, I define what it means by the statement that a hierarchy space contains belief hierarchies induced by a type space.

Given a type space $\overrightarrow{T} = \langle (T_i, \mathcal{T}_i)_{i \in I_0}, (m_i)_{i \in I} \rangle$, the type mapping m_i associates, with each type t_i, a transfinite belief hierarchy starting from the first-order belief $m_i(t_i) \circ (\pi_S^{S \times T_j})^{-1} \in \Delta(S)$. Since the set of types in the terminal space does not depend on the identity of the players, I simply define the condition when a hierarchy space $(H^{<\alpha}, \mathcal{H}^{<\alpha})$ contains the belief hierarchies induced by type mappings of both players (instead of defining the condition when $(H_i^{<\alpha}, \mathcal{H}_i^{<\alpha})$ contains the belief hierarchies induced by a type mapping m_i for each $i \in I$).

Definition 2. A hierarchy space $(H^{<\alpha}, \mathcal{H}^{<\alpha})$ (of order less than $\alpha \in [2, \Omega]$) *contains belief hierarchies induced by a type space* $\overrightarrow{T} = \langle (T_i, \mathcal{T}_i)_{i \in I_0}, (m_i)_{i \in I} \rangle$ if the following measurable maps $\eta_i^{<\beta} : (T_i, \mathcal{T}_i) \to (H^{<\beta}, \mathcal{H}^{<\beta})$ are well-defined for all $(i, \beta) \in I \times [2, \alpha]$.

1. For each $i \in I$, define a measurable map $\eta_i^{<2} : (T_i, \mathcal{T}_i) \to (H^{<2}, \mathcal{H}^{<2})$ by
$$\eta_i^{<2}(\cdot) := m_i(\cdot) \circ (\pi_S^{S \times T_j})^{-1}.$$
2. For a successor ordinal $\beta = \gamma + 1 \geq 3$, assume that a measurable map $\eta_i^{<\gamma} : (T_i, \mathcal{T}_i) \to (H^{<\gamma}, \mathcal{H}^{<\gamma})$ is well-defined for each $i \in I$. Let
$$\eta_i^{<\beta}(\cdot) := (\eta_i^{<\gamma}(\cdot), m_i(\cdot) \circ (\eta_{-i}^{<\gamma})^{-1})$$
for each $i \in I$, where $\eta_{-i}^{<\gamma} = (\mathrm{id}_S, \eta_j^{<\gamma}) : (S \times T_j, \mathcal{S} \times \mathcal{T}_j) \to (S \times H^{<\gamma}, \mathcal{S} \times \mathcal{H}^{<\gamma})$. The map $\eta_i^{<\beta} : (T_i, \mathcal{T}_i) \to (H^{<\beta}, \mathcal{H}^{<\beta})$ is a well-defined measurable map if $H^{<\beta}$ contains every $\eta_i^{<\beta}(t_i)$.
3. For a limit ordinal β, assume that a measurable map $\eta_i^{<\gamma} : (T_i, \mathcal{T}_i) \to (H^{<\gamma}, \mathcal{H}^{<\gamma})$ is well-defined for all $(i, \gamma) \in I \times [2, \beta)$. Then, for each $i \in I$, let $\eta_i^{<\beta} : (T_i, \mathcal{T}_i) \to (H^{<\beta}, \mathcal{H}^{<\beta})$ be the unique measurable map such that
$$\eta_i^{<\gamma} = \pi_\gamma^\beta \circ \eta_i^{<\beta} \text{ for every } \gamma \in [2, \beta).$$

Suppose that $(H^{<\alpha}, \mathcal{H}^{<\alpha})$ contains the belief hierarchies induced by the type space $\overrightarrow{T}$. For each $\beta \in [2, \alpha]$, call $\eta_i^{<\beta} : (T_i, \mathcal{T}_i) \to (H^{<\beta}, \mathcal{H}^{<\beta})$ *the hierarchy map of player*

$i \in I$ *(of order less than* β*)*. Also, for each $\beta \in [2, \alpha]$, call $\eta_i^{<\beta}(t_i) \in H^{<\beta}$ *the belief hierarchy of player* $i \in I$ *(of order less than* β*) at* $t_i \in T_i$. By construction,

$$\eta_i^{<\beta}(t_i) = (m_i(t_i) \circ (\pi_S^{S \times T_j})^{-1}, (m_i(t_i) \circ (\eta_{-i}^{<\gamma})^{-1})_{\gamma \in [2,\beta)}) \text{ for each } (i, \beta) \in I \times [2, \alpha]. \quad (1)$$

For ease of notation, write $\eta_i := \eta_i^{<\omega}$ and $\overline{\eta}_i := \eta_i^{<\Omega}$.

In order for a hierarchy space $H^{<\alpha}$ to contain the belief hierarchies induced by a type space $\overrightarrow{T}$, it is necessary and sufficient that, for each successor ordinal $\beta = \gamma + 1 \in [3, \alpha]$, the associated hierarchy space $H^{<\beta}$ contains each $\eta_i^{<\beta}(t_i)$. This is because in Definition 1 (1a), the space $H^{<\beta}$ is defined as a subset of $H^{<\gamma} \times \Delta(S \times H^{<\gamma})$.

In Definition 2 (1), each $\eta_i^{<2}(t_i) \in \Delta(S)$ is player i's first-order belief on nature states S. In Definition 2 (2), each $\eta_i^{<\beta}(t_i) \in H^{<\gamma} \times \Delta(S \times H^{<\gamma})$ consists of player i's belief hierarchy of order less than γ and her beliefs on nature states S and belief hierarchies of order less than γ. Again, since $H^{<\beta}$ is a subset of $H^{<\gamma} \times \Delta(S \times H^{<\gamma})$, the hierarchy map $\eta_i^{<\beta}$ is well-defined when the hierarchy space $H^{<\beta}$ contains each belief hierarchy $\eta_i^{<\beta}(t_i)$. If $\eta_i^{<\beta} : T_i \to H^{<\beta}$ is well-defined, then $\eta_i^{<\beta} : (T_i, \mathcal{T}_i) \to (H^{<\beta}, \mathcal{H}^{<\beta})$ is, by construction, measurable. In Definition 2 (3), each belief hierarchy $\eta_i^{<\beta}(t_i) = (h^\delta)_{\delta \in [1,\beta)}$ is defined in such a way that any lower-level belief hierarchy $\eta_i^{<\gamma}(t_i)$, with $\gamma < \beta$, satisfies $\eta_i^{<\gamma}(t_i) = (h^\delta)_{\delta \in [1,\gamma)}$.

The following two properties can be proved.

Remark 1. Let $\varphi : \overrightarrow{T} \to \overrightarrow{T'}$ be a morphism between type spaces $\overrightarrow{T}$ and $\overrightarrow{T'}$. Suppose that a hierarchy space $(H^{<\alpha}, \mathcal{H}^{<\alpha})$ of order less than $\alpha \in [2, \Omega]$ contains the belief hierarchies induced by the type spaces $\overrightarrow{T}$ and $\overrightarrow{T'}$.

1. For each $i \in I$, $\eta_i^{<\alpha} = \eta_i'^{<\alpha} \circ \varphi_i$.
2. If $\eta_i'^{<\alpha} : (T_i', \mathcal{T}_i') \to (H^{<\alpha}, \mathcal{H}^{<\alpha})$ is injective for each $i \in I$, then φ is a unique morphism from $\overrightarrow{T}$ to $\overrightarrow{T'}$.

The first part states that if $\varphi : \overrightarrow{T} \to \overrightarrow{T'}$ is a morphism, then the belief hierarchy (of order less than α) at $t_i \in T_i$ is identical with the one at $\varphi_i(t_i) \in T_i'$. The proof is similar to that of Heifetz and Samet (1998, Proposition 5.1) for the case when $\alpha = \omega$. The second part immediately follows from the first. The second part for the case of $\alpha = \omega$ will be used to show that there is a unique type morphism from a given type space to each terminal type space to be constructed in Sections 4 and 5. There, the terminal type space turns out to be a subset of belief hierarchies $(H^{<\omega}, \mathcal{H}^{<\omega})$ so that each hierarchy map η_i turns out to be an inclusion map, which is injective. Call a type space $\overrightarrow{T}$ *non-redundant* (e.g., Mertens and Zamir, 1985) if each η_i is injective. The fact that there is a unique morphism from a given type space to the terminal type space comes from the fact that the terminal space to be constructed is non-redundant.

To conclude this section, I briefly explain the idea behind the explicit-approach constructions of a terminal type space, in relation to Definitions 1 and 2. In either section, a terminal type space is a measurable subspace of belief hierarchies $(H^{<\omega}, \mathcal{H}^{<\omega})$

that is defined using the measurable space $(H^{<\Omega}, \mathcal{H}^{<\Omega})$ of transfinite belief hierarchies. Especially, the terminal space is constructed in a way such that it satisfies a certain coherency condition. For each type space $\overrightarrow{T}$, the hierarchy maps η_i associated with the type space $\overrightarrow{T}$, defined according to Definition 2, turn out to be a unique morphism from the given type space to the terminal space.

4 The Explicit Approach with Coherency and Common Certainty of Coherency

This section constructs a terminal type space by generalizing the coherency condition of Brandenburger and Dekel (1993) in three steps: the terminal type space consists of coherent belief hierarchies that satisfy common certainty of coherency. The first step defines the set of transfinite belief hierarchies using Definition 1. The second step defines coherent belief hierarchies using transfinite belief hierarchies. The set of coherent belief hierarchies is isomorphic to the set of probability measures on the nature states and the set of belief hierarchies, with no topological assumptions (Proposition 1). The third step defines the set of coherent belief hierarchies that satisfy common certainty of coherency, without the aid of a topological assumption. It turns out to be a type space (Theorem 1). The type space is terminal and coincides with the implicit-approach terminal type space of Heifetz and Samet (1998) (Theorem 2), which I denote by $\overrightarrow{T^{\mathrm{HS}}} = \langle (T_i^{\mathrm{HS}}, \mathcal{T}_i^{\mathrm{HS}})_{i \in I_0}, (m_i^{\mathrm{HS}})_{i \in I} \rangle$.[9] The resulting space is also complete and non-redundant.

The first step defines the measurable space $(H^{<\Omega}, \mathcal{H}^{<\Omega})$ of transfinite belief hierarchies without any restriction on each step β in Definition 1 (1a), i.e., for a successor ordinal $\beta = \gamma + 1 \in [3, \Omega]$, let

$$H^{<\gamma+1} := H^{<\gamma} \times \Delta(S \times H^{<\gamma}).$$

Then, denote by $(\overline{T}^0, \overline{\mathcal{T}}^0) := (H^{<\Omega}, \mathcal{H}^{<\Omega})$ the space of transfinite belief hierarchies, and denote by $(T^0, \mathcal{T}^0) := (H^{<\omega}, \mathcal{H}^{<\omega})$ the space of belief hierarchies. I also remark that, for any type space $\overrightarrow{T}$, each hierarchy map $\eta_i : (T_i, \mathcal{T}_i) \to (T^0, \mathcal{T}^0)$ in Definition 2 is well-defined.

The second step defines coherency. First, a transfinite belief hierarchy is coherent if different levels of beliefs do not contradict one another. Second, a belief hierarchy is coherent if it extends to a coherent transfinite belief hierarchy.

[9] Appendix A.1.5 directly reconstructs the terminal type space $\overrightarrow{T^{\mathrm{HS}}}$. This is because, in the original type space of Heifetz and Samet (1998), each type induces a belief over the type of all the players (whose marginal on her own types puts probability one to her own realized type). By the reconstruction, Theorem 2 establishes that the explicit and implicit approaches yield exactly the same type space. The by-product of this re-construction is that the resulting implicit-approach terminal type space of Heifetz and Samet (1998) is complete (Meier (2012) shows that the original terminal type space of Heifetz and Samet (1998) is complete in the syntactic framework).

Definition 3. 1. (a) A transfinite belief hierarchy $\overline{h} = (\overline{h}^{\alpha})_{\alpha\in[1,\Omega)} \in \overline{T}^0$ is *coherent* if $\overline{h}^{\beta} = \mathrm{marg}_{Z^{<\beta}} \overline{h}^{\alpha}$ for any $\alpha, \beta \in [1, \Omega)$ with $\beta < \alpha$.

(b) Denote by $\overline{T}^1$ the set of coherent transfinite belief hierarchies. Let $\overline{\mathcal{T}}^1 := \overline{\mathcal{T}}^0 \cap \overline{T}^1$ be the σ-algebra on $\overline{T}^1$.

2. (a) A belief hierarchy $h \in T^0$ is *coherent* if there is a coherent transfinite belief hierarchy $\overline{h} \in \overline{T}^1$ such that $h = (\overline{h}^k)_{k\in\mathbb{N}}$.

(b) Denote by T^1 the set of coherent belief hierarchies. Let $\mathcal{T}^1 := \mathcal{T}^0 \cap T^1$ be the σ-algebra on T^1.

Since a type mapping can unpack beliefs of countable orders, I define coherency for a belief hierarchy $h = (h^k)_{k\in\mathbb{N}}$ in a way such that the belief hierarchy h can extend to a coherent transfinite belief hierarchy $\overline{h}$. To rephrase, since a type mapping induces a transfinite belief hierarchy that accounts for countable levels of interactive reasoning, the truncation to a belief hierarchy consisting of finite-level beliefs is deemed coherent only when the belief hierarchy is indeed obtained from the coherent transfinite belief hierarchy that represents interactive beliefs of all countable orders.

In fact, for any type space $\overrightarrow{T}$ and the hierarchy map $\eta_i : (T_i, \mathcal{T}_i) \to (T^0, \mathcal{T}^0)$ of each player i, every belief hierarchy $\eta_i(t_i)$ is, by construction, coherent.[10] Consequently, $\eta_i : (T_i, \mathcal{T}_i) \to (T^1, \mathcal{T}^1)$ is a well-defined measurable map. Also, I remark that T^1 is technically the image of $\overline{T}^1$ under the projection π^{Ω}_{ω}.

Now, I show that a coherent belief hierarchy induces a belief over the nature states S and the belief hierarchies of the opponent T^0 without any topological assumption.

Proposition 1. *There is an isomorphism* $\psi : (T^1, \mathcal{T}^1) \to (\Delta(S \times T^0), (\mathcal{S} \times \mathcal{T}^0)_{\Delta})$.

With each $h \in T^1$, the measurable isomorphism ψ associates a unique probability measure $\psi(h) \in \Delta(S \times T^0) = \Delta(Z^{<\omega})$ such that $\psi(h) \circ \pi_k^{-1} = h^k \in \Delta(Z^{<k})$ for each $k \in \mathbb{N}$, where $\pi_k : Z^{<\omega} \to Z^{<k}$ is the projection. The probability measure $\psi(h)$ on $\mathcal{Z}^{<\omega}$ is uniquely determined from $h = (h^k)_{k\in\mathbb{N}}$ because the collection $\bigcup_{k\in\mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})$ is an algebra on $Z^{<\omega}$ which generates $\mathcal{Z}^{<\omega}$.

However, the belief hierarchy determines $\psi(h)$ only on the algebra. The previous literature (e.g., Brandenburger and Dekel, 1993) makes an appropriate topological assumption on S (e.g., S is a Polish space) which allows for the extension of $\psi(h)$ to the generated σ-algebra $\mathcal{Z}^{<\omega}$ through the use of the Kolmogorov extension theorem, when h satisfies the "standard" coherency condition ($h^k = \mathrm{marg}_{Z^{<k}} h^{k+1}$ for all $k \in \mathbb{N}$).

In contrast, in the proof of Proposition 1, I establish an auxiliary isomorphism $\overline{\psi} : \overline{T}^1 \to \Delta(S \times \overline{T}^0)$ which associates, with each $\overline{h} \in \overline{T}^1$, the probability measure $\overline{\psi}(\overline{h}) \in \Delta(S \times \overline{T}^0)$ such that $\overline{\psi}(\overline{h}) \circ \overline{\pi}_{\alpha}^{-1} = \overline{h}^{\alpha} \in \Delta(Z^{<\alpha})$ for each $\alpha \in [1, \Omega)$, where

[10] The belief hierarchy $\eta_i(t_i)$ extends to a coherent transfinite belief hierarchy $\overline{\eta}_i(t_i)$. The coherency of $\overline{\eta}_i(t_i)$ follows from Expression (1).

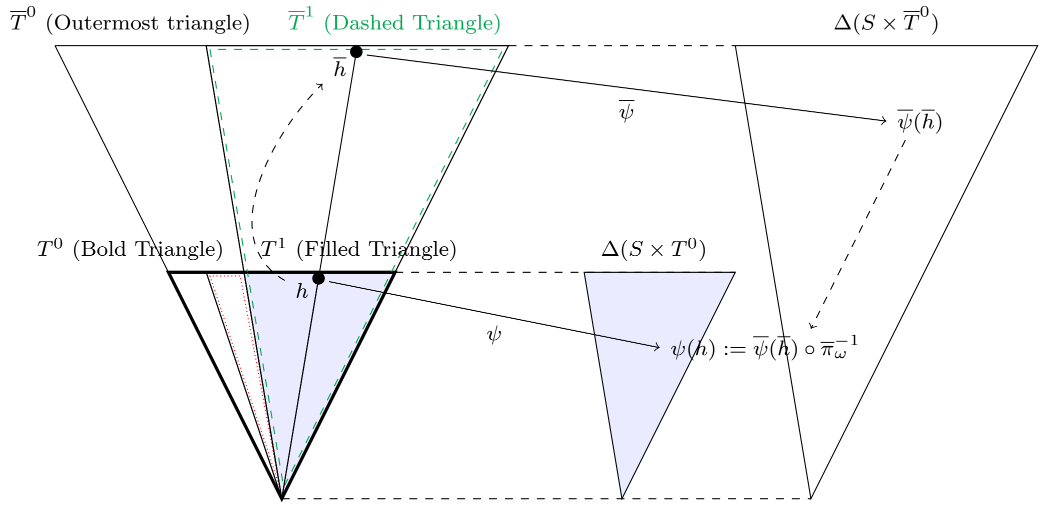

Figure 1: Illustration of Proposition 1

$\overline{\pi}_\alpha : Z^{<\Omega} \to Z^{<\alpha}$ is the projection. That is, for each transfinite belief hierarchy $\overline{h}$ and each countable level $\alpha \in [1, \Omega)$, the marginal of $\overline{\psi}(\overline{h})$ on $Z^{<\alpha}$ is exactly $\overline{h}^\alpha$. Here, the collection $\bigcup_{\alpha\in[1,\Omega)} \overline{\pi}_\alpha^{-1}(\mathcal{Z}^{<\alpha})$ itself is a σ-algebra (recall the last part of Section 2), unlike the algebra $\bigcup_{k\in\mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})$. Hence, $\overline{\psi}$ is well-defined without any topological assumption. Using this auxiliary $\overline{\psi}$, I construct the isomorphism ψ. The smallest uncountable ordinal Ω is the smallest ordinal β such that $\bigcup_{\alpha\in[1,\beta)} \overline{\pi}_\alpha^{-1}(\mathcal{Z}^{<\alpha})$ is always a σ-algebra for any measurable space $(S, \mathcal{S})$ of states of nature. This is the reason that I require a coherent belief hierarchy $h = (h^k)_{k\in\mathbb{N}}$ to extend to $\overline{h} = (\overline{h}^\alpha)_{\alpha\in[1,\Omega)}$ which takes into account all the countable-level beliefs, not just the first countably-infinite level of beliefs $\overline{h}^\omega$.

Figure 1 illustrates the construction of ψ. First, for any $h \in T^1$, there is $\overline{h} \in \overline{T}^1$ with $h = (\overline{h}^k)_{k\in\mathbb{N}}$. This corresponds to the dashed arrow from h to $\overline{h}$. Second, I define the unique probability measure $\psi(h)$ on the entire domain $\mathcal{Z}^{<\omega}$ using the auxiliary probability measure $\overline{\psi}(\overline{h})$: namely,

$$\psi(h) := \overline{\psi}(\overline{h}) \circ \overline{\pi}_\omega^{-1} (= \overline{h}^\omega).$$

Coherency guarantees that $\psi(h) = \overline{\psi}(\overline{h}) \circ \overline{\pi}_\omega^{-1}$ is uniquely defined irrespective of the particular choice of $\overline{h}$. This corresponds to the solid arrow from $\overline{h}$ to $\overline{\psi}(\overline{h})$ and then the dashed arrow from $\overline{\psi}(\overline{h})$ to $\psi(h)$. Then, the mapping $\psi : T_1 \ni h \mapsto \psi(h)(= \overline{\psi}(\overline{h}) \circ \overline{\pi}_\omega^{-1}) \in \Delta(S \times T^0)$ is well-defined. That is, the solid arrow from h to $\psi(h)$ is well-defined.

In fact, the probability measure $\psi(h)$ is defined on the σ-algebra generated by $\bigcup_{k\in\mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})$ because

$$\psi(h) \circ \pi_k^{-1} = (\overline{\psi}(\overline{h}) \circ \overline{\pi}_\omega^{-1}) \circ \pi_k^{-1} = \overline{\psi}(\overline{h}) \circ \overline{\pi}_k^{-1}$$

and because the domain of $\overline{\psi}(\overline{h})$ includes the σ-algebra generated by $\bigcup_{k\in\mathbb{N}} \overline{\pi}_k^{-1}(\mathcal{Z}^{<k})$. Then, since $\psi(h) \circ \pi_k^{-1} = \overline{h}^{\omega} \circ \pi_k^{-1} = h^k$ for each $k \in \mathbb{N}$, $\psi(h)$ does not depend on the particular choice of $\overline{h}$. This is how the extension $\overline{\psi}(\overline{h})$ makes it possible to define $\psi(h)$ on the σ-algebra generated by $\bigcup_{k\in\mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})$. Moreover, the Appendix shows that ψ is a measurable isomorphism, using the property that

$$\psi(h) \circ \pi_k^{-1} = h^k \text{ for each } k \in \mathbb{N}.$$

Thus, the filled triangles T^1 and $\Delta(S \times T^0)$ are isomorphic.

I discuss how Proposition 1 generalizes the standard topological case. On the one hand, with an appropriate topological assumption, there exists an isomorphism f between the set $D := \{h \in T^0 \mid \operatorname{marg}_{Z^{<k}} h^{k+1} = h^k \text{ for all } k \in \mathbb{N}\}$ satisfying the standard "coherency" condition and $\Delta(S \times T^0)$; see Brandenburger and Dekel (1993, Lemma 1). With each $h \in D$, the isomorphism f associates $f(h)$ such that $f(h) \circ \pi_k^{-1} = h^k$ for all $k \in \mathbb{N}$. On the other hand, consider the inverse ψ^{-1} of $\psi : T^1 \to \Delta(S \times T^0)$ in Proposition 1 which does not require a topological assumption. The inverse ψ^{-1} maps $f(h)$ to $\psi^{-1}f(h) \in T^1$ such that $\psi^{-1}f(h) = (f(h) \circ \pi_k^{-1})_{k\in\mathbb{N}} = h$. Thus, the mapping $\psi^{-1} \circ f : D \to T^1$ is an inclusion map. Under the topological assumption, it follows from $T^1 \subseteq D$ that $D = T^1$ and also that $f = \psi$. However, here I consider a general case in which an appropriate topological condition may not be satisfied. As illustrated in Figure 1, the set D of belief hierarchies that satisfy the "standard" coherency condition may be larger than T^1 so that D and $\Delta(S \times T^0)$ may not necessarily be isomorphic. In the figure, the dotted area depicts the set of belief hierarchies that satisfy the "standard" coherency condition but do not satisfy coherency in Definition 3, i.e., the set $D \setminus T^1$.

The third step defines the set of coherent belief hierarchies that satisfy common certainty of coherency. To that end, for each $h \in T^1$, denote by $\psi^*(h)$ the outer measure induced by $\psi(h)$. For each ordinal $\alpha \geq 2$, define

$$T^{<\alpha} := \bigcap_{\beta\in\alpha} T^{\beta} \text{ and } T^{\alpha} := T^{<\alpha} \cap \{h \in T^1 \mid \psi^*(h)(S \times T^{<\alpha}) = 1\}. \tag{2}$$

The set T^{α} is the set of belief hierarchies h in the intersection $T^{<\alpha} = \bigcap_{\beta\in\alpha} T^{\beta}$ of all the previous sets $(T^{\beta})_{\beta\in\alpha}$ such that the outer measure $\psi^*(h)$ assigns probability-one belief in $S \times T^{<\alpha}$. Appendix A.1.3 shows that $T^{<\alpha}$ and T^{α} are not empty for each $\alpha \geq 2$. Since $(T^{\alpha})_{\alpha}$ is a decreasing sequence, there exists the smallest ordinal α with $T^{\alpha} = T^{\alpha+1}$. Define $T^{\mathrm{BD}} := T^{\alpha}$ and $\mathcal{T}^{\mathrm{BD}} := \mathcal{T}^1 \cap T^{\mathrm{BD}}$ for this α.

Definition 4. Call T^{BD} the set of coherent belief hierarchies satisfying *common certainty of coherency.*

Starting with $\alpha = 2$, a belief hierarchy $h \in T^2$ of a player is certain that the opponent's belief hierarchies are coherent, a belief hierarchy $h \in T^3$ of a player is, in addition, certain that the opponent's belief hierarchies are certain that her own belief hierarchies are coherent, and so on. Under an appropriate topological assumption, T^1 is a closed set so that each belief hierarchy $h \in T^{<\omega} = \bigcap_{k\in\omega} T^k$ satisfies coherency and common certainty of coherency. With no topological assumptions, however, T^1 may not be an object of beliefs (i.e., may not be measurable) in the space of belief hierarchies $(T^0, \mathcal{T}^0)$. Hence, I extend each probability measure $\psi(h) \in \Delta(S \times T^0)$ to the outer measure $\psi^*(h)$ to accommodate T^1 (and consequently every T^α). The difference between the probability measure $\psi(h)$ and the outer measure $\psi^*(h)$ is that the notion of certainty induced by the latter may not be continuous from above: for a decreasing sequence of events of which $\psi^*(h)$ is certain, $\psi^*(h)$ may not be certain of the limit. Hence, common certainty of coherency may require transfinite iterations of certainty.[11]

The common-certainty-of-coherency condition endows the subset of coherent belief hierarchies T^{BD} with a type structure because T^{BD} is "belief-closed" (e.g., Mertens and Zamir, 1985) in that $T^{\mathrm{BD}} = \{h \in T^{\mathrm{BD}} \mid \psi^*(h)(S \times T^{\mathrm{BD}}) = 1\}$. Generally, I call a set of coherent belief hierarchies T to be belief-closed if, for each belief hierarchy $h \in T$, the outer measure $\psi^*(h)$ assigns probability-one belief to $S \times T$. Put differently, a subset T of T^1 is belief-closed if ψ^* induces a type mapping (and thus a type space) identified as a mapping from T to $\Delta(S \times T)$. Formally:

Definition 5. Let T be a subset of T^1 endowed with the σ-algebra $\mathcal{T} := \mathcal{T}^0 \cap T$.

1. For any $h \in T$, define $\psi^*_T(h) := \psi^*(h)|_{\mathcal{S}\times\mathcal{T}}$, i.e., the outer measure $\psi^*(h)$ restricted on $\mathcal{S} \times \mathcal{T}$.

2. Call the measurable subspace $(T, \mathcal{T})$ *belief-closed* if $\psi^*_T : (T, \mathcal{T}) \to (\Delta(S \times T), (\mathcal{S} \times \mathcal{T})_\Delta)$ is a measurable map.

It is shown in the Appendix that a subset T of T^1 is belief-closed if and only if (hereafter, iff)

$$T = \{h \in T \mid \psi^*(h)(S \times T) = 1\}.$$

Thus, T^{BD} is belief-closed. This means that, for any coherent belief hierarchy h in the set T^{BD} satisfying common certainty of coherency, the belief hierarchy h itself is a belief over the nature states S and the set of coherent belief hierarchies T^{BD} satisfying common certainty of coherency.

In fact, the first main result, Theorem 1 below, states that $(T^{\mathrm{BD}}, \mathcal{T}^{\mathrm{BD}})$ is the largest belief-closed subspace (or that it contains any belief-closed subspace). Thus, $\langle (T^{\mathrm{BD}}, \mathcal{T}^{\mathrm{BD}}), \psi^*_{T^{\mathrm{BD}}} \rangle$ induces the type space of Brandenburger and Dekel (1993). Each "type" $h \in T^{\mathrm{BD}}$ of player i induces a belief (probability measure) $\psi^*_{T^{\mathrm{BD}}}(h) \in \Delta(S \times$

[11]See footnote 4 for the literature on the characterization of common certainty (or common belief). There, if mutual certainty (mutual beliefs) may not be continuous from above, that is, if, for a decreasing sequence of events for which everybody is certain, it may not be the case that everybody is certain of the limit (i.e., the conjunction of events), then common certainty may require transfinite iterations of mutual certainty.

T^{BD}) over the nature states and the "types" of the other player. Moreover, $\psi^*_{T^{\mathrm{BD}}}$ is a measurable isomorphism. The theorem also implies that a set of coherent belief hierarchies satisfies common certainty of coherency iff it is a belief-closed subspace, analogously to the fixed-point characterization of common certainty.[12]

Theorem 1. *The space* $(T^{\mathrm{BD}}, \mathcal{T}^{\mathrm{BD}})$ *is the largest belief-closed subspace of* $(T^1, \mathcal{T}^1)$*: (i)* $(T^{\mathrm{BD}}, \mathcal{T}^{\mathrm{BD}})$ *is belief-closed; and (ii) if* $(T, \mathcal{T})$ *is belief-closed, then its hierarchy map* η *associated with* $\overrightarrow{T}$ *is an inclusion map* $\eta : (T, \mathcal{T}) \to (T^{\mathrm{BD}}, \mathcal{T}^{\mathrm{BD}})$ *so that* $T \subseteq T^{\mathrm{BD}}$*. Moreover,* $\psi^*_{T^{\mathrm{BD}}} : (T^{\mathrm{BD}}, \mathcal{T}^{\mathrm{BD}}) \to (\Delta(S \times T^{\mathrm{BD}}), (\mathcal{S} \times \mathcal{T}^{\mathrm{BD}})_\Delta)$ *is a measurable isomorphism.*

Formally, denote by $\overrightarrow{T^{\mathrm{BD}}} = \langle (T_i^{\mathrm{BD}}, \mathcal{T}_i^{\mathrm{BD}})_{i \in I_0}, (m_i^{\mathrm{BD}})_{i \in I} \rangle$ the type space induced by $\langle (T_i^{\mathrm{BD}}, \mathcal{T}_i^{\mathrm{BD}}), m_i^{\mathrm{BD}} \rangle := \langle (T^{\mathrm{BD}}, \mathcal{T}^{\mathrm{BD}}), \psi^*_{T^{\mathrm{BD}}} \rangle$ for each $i \in I$. Then, $\overrightarrow{T^{\mathrm{BD}}}$ is a terminal type space on the arbitrary measurable space of nature $(S, \mathcal{S})$. For any type space $\overrightarrow{T}$ and any type $t_i \in T_i$, the induced belief hierarchy $\eta_i(t_i)$ satisfies coherency and common certainty of coherency, i.e., $\eta_i(t_i) \in T^{\mathrm{BD}}$ (see Appendix A.1.3). Then, the hierarchy map $\eta_i : (T_i, \mathcal{T}_i) \to (T^{\mathrm{BD}}, \mathcal{T}^{\mathrm{BD}})$ associated with $\overrightarrow{T}$ is measurable. Since the hierarchy map $\eta_i^{\mathrm{BD}} : T^{\mathrm{BD}} \to T^{\mathrm{BD}}$ is shown to be the identity, it follows that $(\mathrm{id}_S, \eta_1, \eta_2) : \overrightarrow{T} \to \overrightarrow{T^{\mathrm{BD}}}$ is the unique morphism. In sum:

Remark 2. The type space $\overrightarrow{T^{\mathrm{BD}}}$ is terminal: for any type space $\overrightarrow{T}$, the unique type morphism is $(\mathrm{id}_S, \eta_1, \eta_2) : \overrightarrow{T} \to \overrightarrow{T^{\mathrm{BD}}}$, where (η_1, η_2) is the profile of the hierarchy maps in $\overrightarrow{T}$. Note also that $\overrightarrow{T^{\mathrm{BD}}}$ is is non-redundant and complete.

What is more, I show that the explicit and implicit approaches yield exactly the same type space: $\overrightarrow{T^{\mathrm{BD}}} = \overrightarrow{T^{\mathrm{HS}}}$ (Remark 2 suggests only that these terminal spaces are isomorphic).

Theorem 2. *For any measurable space of nature states* $(S, \mathcal{S})$*,* $\overrightarrow{T^{\mathrm{BD}}} = \overrightarrow{T^{\mathrm{HS}}}$*.*

To prove Theorem 2, Appendix A.1.5 directly reconstructs the terminal type space $\overrightarrow{T^{\mathrm{HS}}}$ (recall footnote 9). Then, Appendix A.1.6 proves the theorem in three steps. The first step shows that $T_i^{\mathrm{BD}} \subseteq T_i^{\mathrm{HS}}$ by proving that the unique morphism from the type space $\overrightarrow{T^{\mathrm{BD}}}$ to the terminal type space $\overrightarrow{T^{\mathrm{HS}}}$, which is the hierarchy space associated with $\overrightarrow{T^{\mathrm{BD}}}$, is an inclusion map. The second step shows that $T_i^{\mathrm{HS}} \subseteq T_i^{\mathrm{BD}}$ by proving that each belief hierarchy in T_i^{HS} satisfies coherency and common certainty of coherency and then by invoking Theorem 1. The third step finishes the proof by showing $\mathcal{T}_i^{\mathrm{BD}} = \mathcal{T}_i^{\mathrm{HS}}$ and $m_i^{\mathrm{BD}} = m_i^{\mathrm{HS}}$.

[12] For the characterization of common certainty of an event E as the largest belief-closed event implying the mutual certainty of E, see, for instance, Barwise (1989), Chen, Long, and Luo (2007), Fukuda (2020), Heifetz (1999), Lismont and Mongin (1995), Monderer and Samet (1989), and Samet (2015).

Theorem 2 provides a surprisingly simple answer to the question of what conditions on belief hierarchies make the explicit approach (with coherency and common certainty of coherency) possible and hence connect the explicit and implicit approaches on a general measurable space of nature. The set of belief hierarchies that can be induced by some type of some type space (i.e., the implicit-approach terminal type space) is equal to the set of belief hierarchies which are coherent, that is, can extend to all countable levels in a way such that any lower levels of beliefs do not conflict with each other, and which satisfies common certainty of coherency (i.e., the explicit-approach terminal type space).

If S has an appropriate topological assumption, the construction of the terminal type space $\overrightarrow{T^{\text{BD}}}$ in this paper coincides exactly with that of the terminal type space of Brandenburger and Dekel (1993). Hence, under (or irrespective of) the topological assumption, Theorem 2 shows also that the terminal type spaces of Brandenburger and Dekel (1993) and Heifetz and Samet (1998) are exactly the same.

5 The Explicit Approach with Recursive Coherency

In the last section, the key idea of the explicit approach is to extend coherency to all possible countable-level beliefs. To demonstrate that the extension of coherency to countable levels of beliefs is natural, this section shows that the explicit-approach type space satisfying the recursive coherency condition of Mertens and Zamir (1985) is terminal when the coherency condition is extended to all countable-level beliefs.[13]

I reformulate the coherency condition of Mertens and Zamir (1985): a belief hierarchy of order less than $\alpha \in [2, \Omega]$ is recursively coherent if lower-level beliefs do not conflict with each other and if it is a belief about the nature states and all the lower-level recursively coherent beliefs of the opponent. I construct a terminal type space as the set of recursively coherent belief hierarchies that can extend to recursively coherent transfinite belief hierarchies. Formally:

Definition 6. Let $(H^{<\alpha}, \mathcal{H}^{<\alpha})$ be a hierarchy space of order less than α, and let $(H^{<\beta}, \mathcal{H}^{<\beta})_{\beta\in[2,\alpha)}$ be the associated lower-level hierarchy spaces. The hierarchy space $(H^{<\alpha}, \mathcal{H}^{<\alpha})$ is *recursively coherent* if the spaces $(H^{<\beta}, \mathcal{H}^{<\beta})_{\beta\in[2,\alpha]}$ satisfy the following: for any $\beta \in [2, \alpha]$, $(h^\gamma)_{\gamma\in[1,\beta)} \in H^{<\beta}$ iff $\text{marg}_{Z^{<\delta}} h^\gamma = h^\delta$ for any (γ, δ) with $1 \leq \delta < \gamma < \beta$.

I construct the terminal type space $\overrightarrow{T^{\text{MZ}}} = \langle (T_i^{\text{MZ}}, \mathcal{T}_i^{\text{MZ}})_{i\in I_0}, (m_i^{\text{MZ}})_{i\in I}\rangle$ in three steps. The first step defines the measurable space $(H^{<\Omega}, \mathcal{H}^{<\Omega})$ of recursively coherent

[13] Technically, in Mertens and Zamir (1985), the set of states of the world Ω (which may not necessarily consist of a profile of a nature state and the players' types) is a primitive: for each player i, each state (not a type of player i) $\omega \in \Omega$ induces a probability measure over the set of states Ω (not the nature states and the types of the other players). Thus, I reformulate the coherency condition of Mertens and Zamir (1985) in the type-space context of Section 2 as in Heifetz and Samet (1999).

transfinite belief hierarchies in Definition 1 (1a): for a successor ordinal $\beta = \gamma + 1$, define

$$H^{<\gamma+1} := \{(h^\delta)_{\delta\in[1,\gamma+1)} \in H^{<\gamma} \times \Delta(S \times H^{<\gamma}) \mid \operatorname{marg}_{Z^{<\delta}} h^\gamma = h^\delta \text{ for all } \delta \in [1, \gamma)\}.$$

Then, the space $(H^{<\Omega}, \mathcal{H}^{<\Omega})$ is written as

$$H^{<\Omega} = \left\{ (h^\beta)_{\beta\in[1,\Omega)} \in \prod_{\beta\in[1,\Omega)} \Delta(Z^{<\beta}) \;\middle|\; (h^\beta)_{\beta\in[1,\alpha)} \in H^{<\alpha} \text{ for all } \alpha \in [2, \Omega) \right\} \text{ and}$$
$$\mathcal{H}^{<\Omega} = \{(\pi^\Omega_\alpha)^{-1}(E_\alpha) \in \mathcal{P}(H^{<\Omega}) \mid E_\alpha \in \mathcal{H}^{<\alpha} \text{ for some } \alpha \in [2, \Omega)\}.$$

Note that the equality for $\mathcal{H}^{<\Omega}$ holds because the right-hand side itself is a σ-algebra (recall Section 2 and the discussion after Proposition 1 in Section 4).

The second step defines T^{MZ} as the set of recursively coherent belief hierarchies $h \in H^{<\omega}$ that extend to recursively coherent transfinite belief hierarchies, i.e., the projection of $H^{<\Omega}$ onto $H^{<\omega}$:

$$T^{\mathrm{MZ}} := \{h \in H^{<\omega} \mid h = (\overline{h}^\alpha)_{\alpha\in[1,\omega)} \text{ for some } \overline{h} = (\overline{h}^\alpha)_{\alpha\in[1,\Omega)} \in H^{<\Omega}\}.$$

Define $\mathcal{T}^{\mathrm{MZ}} := \mathcal{H}^{<\omega} \cap T^{\mathrm{MZ}}$. For each $i \in I$, let $(T^{\mathrm{MZ}}_i, \mathcal{T}^{\mathrm{MZ}}_i) := (T^{\mathrm{MZ}}, \mathcal{T}^{\mathrm{MZ}})$.

Before moving on to the third step, I remark on the fact that the hierarchy map of any type space is well-defined. Namely, for any type space $\overrightarrow{T}$, the hierarchy map $\eta^{<\alpha}_i : (T_i, \mathcal{T}_i) \to (H^{<\alpha}, \mathcal{H}^{<\alpha})$ of player i is, by construction, a well-defined measurable map. Moreover, since the type mapping induces $\overline{\eta}_i(\cdot) \in H^{<\Omega}$ which is an extension of $\eta_i(\cdot) \in H^{<\omega}$, it follows that the hierarchy map is a well-defined measurable map $\eta^{<\alpha}_i : (T_i, \mathcal{T}_i) \to (T^{\mathrm{MZ}}, \mathcal{T}^{\mathrm{MZ}})$. Recall Expression (1)

The third step defines $m^{\mathrm{MZ}} : T^{\mathrm{MZ}} \to \Delta(S \times T^{\mathrm{MZ}})$ (then, let $m^{\mathrm{MZ}}_i = m^{\mathrm{MZ}} : T^{\mathrm{MZ}}_i \to \Delta(S \times T^{\mathrm{MZ}}_j)$ for each $i \in I$) in three sub-steps. As in Section 4, I use transfinite belief hierarchies to extend the domain of each $m^{\mathrm{MZ}}(h)$ to $\mathcal{S} \times \mathcal{T}^{\mathrm{MZ}}$. Unlike Section 4, however, this section directly constructs the type mapping $m^{\mathrm{MZ}} : T^{\mathrm{MZ}} \to \Delta(S \times T^{\mathrm{MZ}})$ without defining the auxiliary mapping "ψ."

In the first sub-step, for any $\overline{h} \in H^{<\Omega}$, define $\overline{m}(\overline{h}) \in \Delta(S \times H^{<\Omega})$ as

$$\overline{m}(\overline{h}) \circ \overline{\pi}^{-1}_\alpha := \overline{h}^\alpha \in \Delta(Z^{<\alpha}) \text{ for all } \alpha \in [1, \Omega),$$

where $\overline{\pi}_\alpha : Z^{<\Omega} \to Z^{<\alpha}$ is the projection. Since the σ-algebra $\mathcal{Z}^{<\Omega}$ on $Z^{<\Omega} = S \times H^{<\Omega}$ is exactly $\bigcup_{\alpha\in[1,\Omega)} \overline{\pi}^{-1}_\alpha(\mathcal{Z}^{<\alpha})$, it follows that $\overline{m}(\overline{h}) : \mathcal{Z}^{<\Omega} \to [0, 1]$ is well-defined. Appendix A.2.2 shows that the mapping

$$\overline{m} : (H^{<\Omega}, \mathcal{H}^{<\Omega}) \to (\Delta(S \times H^{<\Omega}), (\mathcal{S} \times \mathcal{H}^{<\Omega})_\Delta)$$

is a well-defined measurable map.[14] In Figure 2, this corresponds to the solid arrow from $\overline{h}$ to $\overline{m}(\overline{h})$.

[14]Moreover, as $\overline{\psi}$ in Section 4 is an isomorphism (see Remark 3 in Appendix A.1.1 for the proof), the similar proof shows that $\overline{m}$ is an isomorphism.

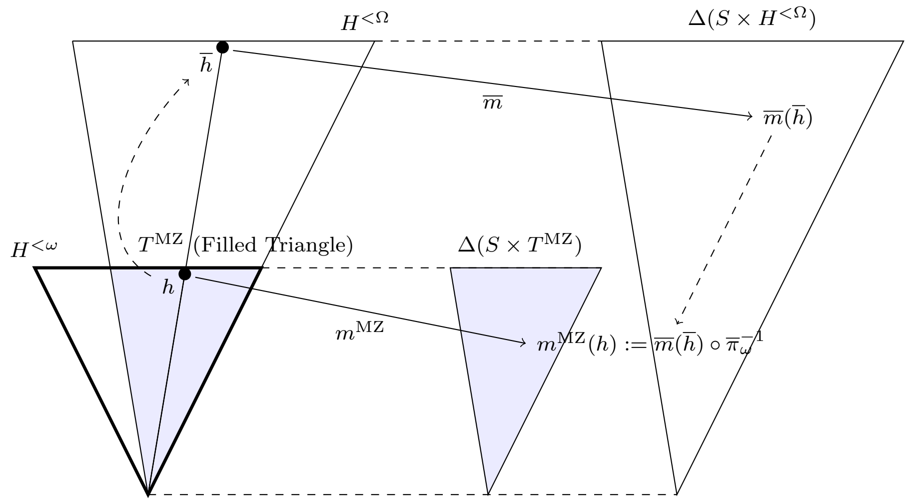

Figure 2: Illustration of m^{MZ}

In the second sub-step, for any $h \in T^{\mathrm{MZ}}$, take some $\overline{h} = (\overline{h}^\alpha)_{\alpha \in [1,\Omega)} \in H^{<\Omega}$ with $h = (\overline{h}^\alpha)_{\alpha \in \mathbb{N}}$. In Figure 2, this corresponds to the dashed arrow from h to $\overline{h}$.

In the third sub-step, for h and $\overline{h}$ chosen in the second sub-step, as illustrated in Figure 2, define

$$m^{\mathrm{MZ}}(h) := \overline{m}(\overline{h}) \circ \overline{\pi}_\omega^{-1},$$

where $\overline{\pi}_\omega$ is identified as $\overline{\pi}_\omega : S \times H^{<\Omega} \to S \times T^{\mathrm{MZ}}$ because, by construction, T^{MZ} is the projection of $H^{<\Omega}$. This corresponds to the dashed line from $\overline{m}(\overline{h})$ to $m^{\mathrm{MZ}}(h)$.

Since $H^{<\Omega}$ is recursively coherent, $m^{\mathrm{MZ}}(h)$ does not depend on the choice of $\overline{h}$. Since the domain of $\overline{m}(\overline{h})$ includes the σ-algebra generated by $\bigcup_{k \in \mathbb{N}} \overline{\pi}_k^{-1}(\mathcal{Z}^{<k})$, it follows that $m^{\mathrm{MZ}}(h)$ is defined on the σ-algebra $\sigma\left(\bigcup_{k \in \mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})\right)$ on $S \times T^{\mathrm{MZ}}$, where, by slightly abusing the notation, $\pi_k : S \times T^{\mathrm{MZ}} \to Z^{<k}$ is the projection.[15] As in Section 4, this is where the extension of belief hierarchies to transfinite levels plays a role. This means that $m^{\mathrm{MZ}}(h) : \mathcal{S} \times \mathcal{T}^{\mathrm{MZ}} \to [0,1]$ is well-defined. In fact, Appendix A.2.2 shows that the mapping $m^{\mathrm{MZ}} : (T^{\mathrm{MZ}}, \mathcal{T}^{\mathrm{MZ}}) \to (\Delta(S \times T^{\mathrm{MZ}}), (S \times T^{\mathrm{MZ}})_\Delta)$ is measurable.

In sum, each belief hierarchy $h = (h^k)_{k \in \mathbb{N}} \in T^{\mathrm{MZ}}$ induces a belief over $S \times T^{\mathrm{MZ}}$ in the sense that, for any $k \in \mathbb{N}$, the belief hierarchy h induces a belief

$$m^{\mathrm{MZ}}(h) \circ \pi_k^{-1} = h^k$$

on $\mathcal{Z}^{<k}$. Recursive coherency guarantees that $m^{\mathrm{MZ}}(h)$ is well-defined on the algebra $\bigcup_{k \in \mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})$ that generates $\mathcal{S} \times \mathcal{T}^{\mathrm{MZ}}$. As discussed above, $m^{\mathrm{MZ}}(h)$ is indeed well-defined on the σ-algebra $\mathcal{S} \times \mathcal{T}^{\mathrm{MZ}}$ by using $\overline{m}(\overline{h})$. Thus, as in Figure 2, this section

[15] In the notation of Section 4, this projection is the restriction $\pi_k|_{S \times T^{\mathrm{MZ}}}$ of the projection $\pi_k : Z^{<\omega} \to Z^{<k}$. Here, for ease of notation, I denote by π_k the restriction itself.

demonstrates that the idea of capturing all countable levels of beliefs also extends the construction of Mertens and Zamir (1985), in addition to that of Brandenburger and Dekel (1993).

With an appropriate topological assumption on the nature states S, there exists a type mapping from $H^{<\omega}$ to $\Delta(S \times H^{<\omega})$. The type mapping, in turn, maps each belief hierarchy $h \in H^{<\omega}$ to a recursively coherent transfinite belief hierarchy $\overline{h} \in H^{<\Omega}$. Hence, $T^{\mathrm{MZ}} = H^{<\omega}$ under the appropriate topological assumption. With no topological assumptions, however, I restrict attention to recursively coherent belief hierarchies that can extend to recursively coherent transfinite belief hierarchies. In Figure 2, the filled triangle T^{MZ} may be a strict subset of the triangle $H^{<\omega}$ without a topological assumption.

Once the type space $\overrightarrow{T^{\mathrm{MZ}}}$ has been constructed, for any type space $\overrightarrow{T}$ and any type $t_i \in T_i$, the induced belief hierarchy $\eta_i(t_i)$ satisfies recursive coherency, i.e., $\eta_i(t_i) \in T^{\mathrm{MZ}}$. Then, $\eta_i : (T_i, \mathcal{T}_i) \to (T^{\mathrm{MZ}}, \mathcal{T}^{\mathrm{MZ}})$ is measurable. Since the hierarchy map $\eta_i^{\mathrm{MZ}} : T^{\mathrm{MZ}} \to T^{\mathrm{MZ}}$ associated with $\overrightarrow{T^{\mathrm{MZ}}}$ is shown to be the identity, the hierarchy map $(\mathrm{id}_S, \eta_1, \eta_2) : \overrightarrow{T} \to \overrightarrow{T^{\mathrm{MZ}}}$ associated with $\overrightarrow{T}$ is the unique morphism from $\overrightarrow{T}$ to $\overrightarrow{T^{\mathrm{MZ}}}$. This means that $\overrightarrow{T^{\mathrm{MZ}}}$ is terminal, non-redundant, and complete. Especially, as discussed in Section 3.1, $m_i^{\mathrm{MZ}} : (T_i^{\mathrm{MZ}}, \mathcal{T}_i^{\mathrm{MZ}}) \to (\Delta(S \times T_j^{\mathrm{MZ}}), (\mathcal{S} \times \mathcal{T}_j^{\mathrm{MZ}})_\Delta)$ is a measurable isomorphism.

Theorem 3 below shows that a terminal type space $\overrightarrow{T^{\mathrm{MZ}}}$ coincides with the implicit-approach construction that satisfies recursive coherency. Denote by $\overrightarrow{T^{\mathrm{IM}}} = \langle (T_i^{\mathrm{IM}}, \mathcal{T}_i^{\mathrm{IM}})_{i \in I_0}, (m_i^{\mathrm{IM}})_{i \in I} \rangle$ the implicit-approach terminal type space constructed in Appendix A.2.1 that respects recursive coherency. The construction of $\overrightarrow{T^{\mathrm{IM}}}$ is similar to that of the Heifetz and Samet (1998) terminal type space $\overrightarrow{T^{\mathrm{HS}}}$ in Section 4 (precisely, Appendix A.1.5) that respects coherency and common certainty of coherency. Each T_i^{IM} consists of recursively coherent belief hierarchies that can be induced by some type of some type space. Now:

Theorem 3. *For any measurable space of nature states $(S, \mathcal{S})$, $\overrightarrow{T^{\mathrm{MZ}}} = \overrightarrow{T^{\mathrm{IM}}}$ is terminal.*

For the rest of this section, I characterize T^{MZ} as the set of recursively coherent belief hierarchies satisfying common certainty of recursive coherency and as the largest belief-closed subset of $H^{<\omega}$ as in Section 4. Thus, the recursively coherent belief hierarchies $h \in H^{<\omega}$ that extend to some recursively coherent transfinite belief hierarchy are exactly the ones that satisfy common certainty of recursive coherency.

First, I characterize T^{MZ} as the set of recursively coherent belief hierarchies satisfying common certainty of recursive coherency. Starting from $(T^0, \mathcal{T}^0) := (H^{<\omega}, \mathcal{H}^{<\omega})$, I introduce an auxiliary mapping ψ on T^0 and define a set T^α of belief hierarchies for each ordinal α. To that end, with each $h \in H^{<\omega}$, define the set function $\psi(h)$ on the algebra $\bigcup_{k\in\mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})$ as

$$\psi(h) \circ \pi_k^{-1} = h^k \in \Delta(Z^{<k}) \text{ for each } k \in \mathbb{N}.$$

Define T^1 as the set of belief hierarchies h in T^0 such that $\psi(h)$ is a probability measure on the σ-algebra $\sigma\left(\bigcup_{k\in\mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})\right)$:

$$T^1 := \{h \in T^0 \mid \psi(h) \in \Delta(S \times T^0)\} \text{ and } \mathcal{T}^1 = \mathcal{T}^0 \cap T^1.$$

In the Appendix, it is shown that $T^{\mathrm{MZ}} \subseteq T^1$. Especially, T^1 is not empty. By construction,

$$\psi|_{T^1} : (T^1, \mathcal{T}^1) \to (\Delta(S \times T^0), (S \times T^0)_\Delta)$$

is a well-defined measurable mapping.

Restating the previous discussion in this section, under an appropriate topological assumption, $T^1 = T^0 = T^{\mathrm{MZ}}$ (and also $\psi = m^{\mathrm{MZ}}$). Without the topological assumption, however, T^1 may be a strict subset of T^0.

For each $h \in T^1$, denote by $\psi^*(h)$ the outer measure induced by $\psi(h)$. As in Section 4, for each ordinal $\alpha \geq 2$, define

$$T^{<\alpha} := \bigcap_{\beta\in\alpha} T^\beta \text{ and } T^\alpha := T^{<\alpha} \cap \{h \in T^1 \mid \psi^*(h)(S \times T^{<\alpha}) = 1\}.$$

As in Section 4 (precisely, Appendix A.1.3), it can be shown that $T^{<\alpha}$ and T^α are not empty for each $\alpha \geq 2$. Since $(T^\alpha)_\alpha$ is a decreasing sequence, there exists the smallest ordinal α with $T^\alpha = T^{\alpha+1}$. Define $T^{\mathrm{RC}} := T^\alpha$ and $\mathcal{T}^{\mathrm{RC}} := \mathcal{T}^1 \cap T^{\mathrm{RC}}$ for this α. Call T^{RC} the set of recursively coherent belief hierarchies satisfying *common certainty of recursive coherency.*

The set T^{RC} is also characterized as the largest belief-closed subset of T^1. To that end, as in Definition 5, let T be a subset of T^1 endowed with the σ-algebra $\mathcal{T} := \mathcal{T}^0 \cap T$. For any $h \in T$, define $\psi_T^*(h) := \psi^*|_{\mathcal{S}\times\mathcal{T}}(h)$. Then, call the measurable subspace $(T, \mathcal{T})$ *belief-closed* if $\psi_T^* : (T, \mathcal{T}) \to (\Delta(S \times T), (\mathcal{S} \times \mathcal{T})_\Delta)$ is a measurable map. The definition parallels Definition 5. Now, I establish:

Theorem 4. *1.* $T^{\mathrm{MZ}} = T^{\mathrm{RC}}$.

2. The space $(T^{\mathrm{MZ}}, \mathcal{T}^{\mathrm{MZ}})$ *is the largest belief-closed subspace of* $(T^1, \mathcal{T}^1)$*: (i)* $(T^{\mathrm{MZ}}, \mathcal{T}^{\mathrm{MZ}})$ *is belief-closed; and (ii) if* $(T, \mathcal{T})$ *is belief-closed, then its hierarchy map* η *associated with* $\overrightarrow{T}$ *is an inclusion map* $\eta : (T, \mathcal{T}) \to (T^{\mathrm{MZ}}, \mathcal{T}^{\mathrm{MZ}})$ *so that* $T \subseteq T^{\mathrm{MZ}}$*. Moreover,* $\psi_{T^{\mathrm{MZ}}}^* = m^{\mathrm{MZ}}$ *is a measurable isomorphism.*

Part (1) states that the set T^{MZ} is exactly the set of recursively coherent belief hierarchies satisfying common certainty of recursive coherency. Part (2) states that the set T^{MZ} is the largest belief-closed set.

6 Discussion

This section discusses the main results. Section 6.1 provides a simple example of a strategic game in which the unique prediction under common certainty of rationality requires transfinite elimination of strictly dominated actions, which would come from predictions under transfinite levels of mutual beliefs in rationality. Section 6.2 discusses the related literature. Section 6.3 provides concluding remarks.

6.1 A Strategic-Game Example

The key idea in constructing a terminal type space without any topological assumption as the space of coherent belief hierarchies that satisfy common certainty of coherency is the need to keep track of transfinite levels of interactive beliefs (Sections 4 and 5). This idea is important not only in the construction of the terminal type space itself but also in the applications of the terminal type space to characterizations of solution concepts of games, as it is indeed well-known that predictions under common certainty of rationality may require transfinite levels of mutual beliefs in rationality (see footnote 4). Here, I briefly introduce one such game, in which the unique prediction under elimination of strictly dominated actions may require transfinite levels of elimination.[16] Thus, the prediction under common certainty of rationality is never pinned down by finite levels of mutual beliefs in rationality, as common certainty of coherency may never be achieved through finite iterations of mutual certainty of coherency.

Consider the following two-player strategic game $\langle (A_i)_{i\in I}, (u_i)_{i\in I}\rangle$. The set A_i of actions available to player $i \in I := \{1,2\}$ is $A_i = \omega \cup \{a,b\} = \{0,1,2,\dots,n,\dots\} \cup \{a,b\}$. The players reason about their action profiles. Thus, the set of states of nature is $(S,\mathcal{S}) = (A_1 \times A_2, \mathcal{P}(S))$. The set S is countable as each A_i is countable. Thus, $(S,\mathcal{S})$ is a measurable space.

Assume that the actions are ordered as $0 < 1 < 2 < \dots < n < \dots < a < b$. Player i's payoff function $u_i : A_i \times A_j \to \mathbb{R}$ is given by

$$u_i(a_i,a_j) = \begin{cases} 0 & \text{if } a_i \neq b \text{ and } a_i \leq a_j \\ 1 & \text{if } a_i = b \\ 2 & \text{if } a_i \neq b \text{ and } a_i > a_j \end{cases}.$$

[16] For this example, see also Arieli (2010), Chen, Long, and Luo (2007), Fukuda (2020, 2021), and Samet (2015) who consider related strategic games in which the players need to engage in interactive reasoning of transfinite levels. Especially, the game is a particular case of Fukuda (2021), with a different focus on qualitative beliefs and knowledge. Lipman (1994) is the first who have considered a strategic game which requires transfinite levels of interactive reasoning to pin down the prediction(s) under common certainty of rationality. Generally, Arieli (2010) shows that, for a game in which every player's strategy set is a Polish space and in which every player's payoff function is bounded and continuous, a process of elimination of "never-best-replies" terminates before or at the first uncountable ordinal and that the bound is tight.

Table 2 depicts the payoff matrix.

	0	1	2	⋯⋯	$n-1$	n	$n+1$	⋯⋯	a	b
0	0, 0	0, 2	0, 2	⋯⋯	0, 2	0, 2	0, 2	⋯⋯	0, 2	0, 1
1	2, 0	0, 0	0, 2	⋯⋯	0, 2	0, 2	0, 2	⋯⋯	0, 2	0, 1
2	2, 0	2, 0	0, 0	⋯⋯	0, 2	0, 2	0, 2	⋯⋯	0, 2	0, 1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
n	2, 0	2, 0	2, 0	⋯⋯	2, 0	0, 0	0, 2	⋯⋯	0, 2	0, 1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
a	2, 0	2, 0	2, 0	⋯⋯	2, 0	2, 0	2, 0	⋯⋯	0, 0	0, 1
b	1, 0	1, 0	1, 0	⋯⋯	1, 0	1, 0	1, 0	⋯⋯	1, 0	1, 1

Table 2: The Payoff Matrix $(u_1(a_1, a_2), u_2(a_1, a_2))_{(a_1,a_2)\in A_1\times A_2}$.

As is well-known, an action $a_i \in A_i$ of player i is *strictly dominated* if there exists a mixed strategy $\alpha_i \in \Delta(A_i)$ such that

$$\sum_{\tilde{a}_i\in A_i} \alpha_i(\tilde{a}_i)u_i(\tilde{a}_i, a_j) > u_i(a_i, a_j) \text{ for all } a_j \in A_j.$$

To define a process of iterated elimination of strictly dominated actions (IESDA), identify any subset $A' = \bigtimes_{i\in I} A'_i$ of $A = \bigtimes_{i\in I} A_i$ with a (sub-)game. A process of *iterated elimination of strictly dominated actions (IESDA)* is an ordinal sequence of $A^\alpha = \bigtimes_{i\in I} A^\alpha_i$, with $|\alpha| \leq |A|$, defined as follows: (i) $A^0 = A$; (ii) for an ordinal $\alpha > 0$, if the subgame $\bigcap_{\beta<\alpha} A^\beta$ has strictly dominated actions then A^α is obtained by eliminating *at least one* such action from $\bigcap_{\beta<\alpha} A^\beta$; and (iii) if the subgame $\bigcap_{\beta<\alpha} A^\beta$ does not contain any strictly dominated action then let $A^\alpha = \bigcap_{\beta<\alpha} A^\beta$. Since $(A^\alpha)_\alpha$ is a decreasing sequence, there exists the least ordinal α (with $|\alpha| \leq |A|$) with $A^\alpha = \bigcap_{\beta<\alpha} A^\beta$. Define $A^{\text{IESDA}} := A^\alpha$. An action profile $a \in A$ *survives* the process of IESDA if $a \in A^{\text{IESDA}}$.

In this example, consider the following process of IESDA. At the initial step, $a_i = 0$ is strictly dominated and thus eliminated. Next, once actions $\{0, 1, \ldots, n\}$ have been eliminated from A_i, the unique action $a_i = n + 1$ is a strictly dominated action, and it is thus eliminated. Then, once actions $\{0, 1, 2, \ldots, n, \ldots\}(= \omega)$ have been eliminated, i.e., intuitively, the players are rational, they believe that they are rational, they believe that they believe that they are rational, and so on for any finite levels, the players would face the subgame in which their action spaces are $\{a, b\}$. In this subgame (precisely, at step ω of the process), action a is strictly dominated. Hence, the unique prediction under the process of IESDA is $A^{\text{IESDA}} = \{b\}$, which is attained by $\omega + 1$ steps of mutual beliefs in rationality.[17]

It is not a mere coincidence that, in both common certainty of coherency in the constructions of a terminal type space and common certainty of rationality in the epistemic characterization of IESDA, common certainty is achieved when transfinite iterations of mutual certainty are made. In both contexts, the notion of belief closure that common certainty possesses plays an important role. In the construction of the terminal type space, T^{BD} (or also T^{MZ}) is the largest belief-closed space of coherent belief hierarchies. In IESDA, when rationality and common certainty of rationality are formalized as events in a type space, the event that the players are commonly certain of their rationality is formulated as the largest belief-closed event which implies mutual belief in rationality. The literature on the formalization of common certainty (see footnotes 4 and 12) has also shown that, when mutual beliefs do not satisfy a certain continuity property (e.g., everybody believes each of a decreasing sequence of events yet it may not be the case that everybody believes its intersection), common certainty may be characterized as transfinite iterations of mutual beliefs. In constructing T^{BD} (or also characterizing T^{MZ}) as the space of coherent belief hierarchies satisfying common certainty of coherency, the outer measure ψ^* may not be continuous so that transfinite iterations of mutual certainty of coherency may be required.

6.2 Related Literature

This subsection discusses the related literature on constructions of a terminal type space and on the role of transfinite belief hierarchies in constructing a terminal type space.[18]

I discuss three related papers on constructions of a terminal type space. First, Theorem 2 of this paper is related to De Vito (2010b). Both papers provide a stronger coherency notion of belief hierarchies under which there is a set of coherent (in a stronger sense) belief hierarchies T which is isomorphic to the space $\Delta(S \times T)$ of beliefs over S and the coherent belief hierarchies T of the opponent. Under an appropriate topological assumption, the coherency notions in both papers reduce to the standard notion of coherency. To the best of my knowledge, De Vito (2010b) is the first paper to demonstrate that the explicit approach by Brandenburger and Dekel (1993) and the implicit approach by Heifetz and Samet (1998) can be reconciled under an appropriate (strong) notion of coherency.

As the stronger coherency notions considered by De Vito (2010b) and this paper are quite different, the resulting constructions are different. Specifically, in De Vito

[17]This example can be generalized to the case in which the players need to engage in one more step after an arbitrarily high countable ordinal step α (i.e., this example corresponds to $\alpha = \omega$).

[18]For the role of transfinite levels of reasoning in certain game-theoretic solution concepts that are implications of common certainty of rationality and in the formalization of common certainty, see footnotes 4 and 12.

(2010b), a belief hierarchy is “strongly coherent” if lower-level beliefs do not conflict with one another when a player is able to reason about any subset (including a non-measurable event), i.e., when, for each level of beliefs, its outer measure is considered. This paper, in contrast, considers all countable levels of beliefs beyond finite levels: a belief hierarchy is coherent (in the sense of this paper) if it admits an extension to a coherent transfinite belief hierarchy such that all lower-level beliefs do not conflict with one another. This is because a type space can in fact unpack countable levels of interactive beliefs. Informally, the extension of coherency is “vertical” in the contents of beliefs (for each finite level of beliefs) in De Vito (2010b), while the extension in this paper is “horizontal” in all countable levels of beliefs.

With this extension, first, this paper shows that the explicit and implicit approaches by Brandenburger and Dekel (1993) and Heifetz and Samet (1998) exactly coincide with each other and that the ways in which a specific type space is associated with the terminal spaces of Brandenburger and Dekel (1993) and Heifetz and Samet (1998) (i.e., the unique morphisms from the given space to the terminal spaces) are also the same. In contrast, De Vito (2010b, Remark 4) may not possess the latter feature. Second, this paper provides an explicit procedure, paralleling and generalizing Brandenburger and Dekel (1993), to carve out the space T^{BD} of coherent belief hierarchies satisfying common certainty of coherency without a topological assumption, when the procedure is allowed to be transfinite. While seemingly complex, such transfinite procedure to reach common certainty of coherency has analogues to the formalization of common certainty and its game-theoretic applications. Third, the idea to extend belief hierarchies to countable levels also generalizes the construction of a terminal type space by Mertens and Zamir (1985). Thus, this paper shows that the terminal spaces of Brandenburger and Dekel (1993) and Mertens and Zamir (1985) are isomorphic regardless of topological assumptions, when the notion of coherent belief hierarchies are refined. The paper also shows that the terminal type space of Mertens and Zamir (1985) is characterized as the space of recursively coherent belief hierarchies that satisfy common certainty of recursive coherency.

Second, Theorem 3 is related to Heifetz and Samet (1999, Section 5), which connects the explicit and implicit approaches of Mertens and Zamir (1985) and Heifetz and Samet (1998). They give a procedure to extract their terminal type space from the set of recursively coherent belief hierarchies (i.e., the space $H^{<\omega}$ in Section 5) in a transfinite process. Additional contributions of this paper are as follows. First, this paper shows that their transfinite procedure corresponds to the property that the Mertens and Zamir (1985) terminal space satisfies common certainty of recursive coherency. On a related point, this paper explicitly shows the parallel between the terminal type spaces of Brandenburger and Dekel (1993) and Mertens and Zamir (1985) in terms of common certainty of coherency and belief closure. Second, while the construction (extraction) of the terminal type space by Heifetz and Samet (1999) as a certain subset of recursively coherent belief hierarchies is implicit, this paper provides an exact characterization of the terminal space as the set of recursively coherent belief hierarchies that extend to recursively coherent transfinite belief hierarchies. Informally, this paper shows that the Mertens and Zamir (1985) terminal type space is characterized as the set of (recursively) coherent belief hierarchies and that the explicit and implicit approaches lead to the same terminal type space when the notion of coherency is strengthened, contrary to Heifetz and Samet (1999). They show that the set of coherent (in the standard sense) belief hierarchies may be a strict super-set of the implicit-approach terminal type space when coherency is taken as the standard notion but also that the implicit-approach terminal type space is carved out from the set of coherent (in the standard sense) belief hierarchies in a transfinite procedure.

Third, Pintér (2018) shows that the terminal belief space of Pintér (2005) consists of coherent belief hierarchies. Pintér (2005) constructs a terminal belief space on a measurable space of nature states by introducing a topological structure not on nature states but on players' beliefs. A (generalized) Kolgomorov existence theorem of his plays a crucial role in his construction. Since the notions of coherency are different between his paper and this one, Pintér (2018) does not show that the set of "coherent" belief hierarchies satisfying common certainty of coherency is a terminal type space or that the terminal type spaces of Brandenburger and Dekel (1993) and Heifetz and Samet (1998) turn out to be the same space under a stronger notion of coherency.

Moving on to the role of transfinite belief hierarchies, when players are interactively reasoning about a measurable space of nature states, a terminal type space may consist of transfinite-belief hierarchies if beliefs may not be countably additive. Meier (2006) constructs a terminal finitely-additive type space in a different framework involving epistemic logic, in which players in the terminal space would be able to reason about their interactive beliefs of transfinite levels. His terminal type space would consist of (coherent) transfinite belief hierarchies when belief hierarchies are explicitly introduced as in the framework of type spaces. Also, Fukuda (2021) studies coherency of beliefs for transfinite levels when a notion of beliefs is more general (either qualitative, including knowledge, or non-/finitely-/countably-additive). Thus, the notion of coherency is formalized by syntactic formulas representing players' beliefs as opposed to coherency on belief hierarchies in this paper. The terminal (qualitative) belief space consists of all feasible transfinite levels of coherent beliefs.

The major difference of this paper is to restrict attention to all the finite levels of coherent beliefs that extend to coherent transfinite levels of beliefs. Hence, the terminal type space consists of a (sub-)set of belief hierarchies consisting only of finite levels of beliefs. The transfinite levels of beliefs are used only as a conceptual auxiliary idea in this paper. This is possible because types are countably additive. In contrast, when beliefs are not countably additive, one would need to keep track of all the feasible levels of a coherent belief hierarchy. Thus, each belief hierarchy in such a terminal type space would consist of transfinite levels of beliefs.

6.3 Concluding Remarks

Theorems 1 and 2 show that, absent any topological condition on the nature states $(S, \mathcal{S})$, there exists an isomorphism m^{BD} between the space T^{BD} and the beliefs $\Delta(S \times T^{\mathrm{BD}})$ over $S \times T^{\mathrm{BD}}$. The space T^{BD} is carved out from the set T^1 of coherent belief hierarchies (where coherency is defined as in Definition 3) by requiring common certainty of coherency (Definition 4). The theorems ensure that each belief hierarchy $h \in T^{\mathrm{BD}}$ is a *type*. The isomorphism m^{BD} maps each type to a probability measure on the nature states S and the types of the opponent. By Theorem 2, the explicit and implicit approaches of Brandenburger and Dekel (1993) and Heifetz and Samet (1998) yield the same terminal type space.

Previously, Heifetz and Samet (1999, Section 4) show that the set of "coherent" belief hierarchies (coherency in this context only requires that finite levels of beliefs do not conflict with one another) may be strictly larger than their terminal type space (Heifetz and Samet, 1998) by proving that a coherent type may not admit a countably-additive extension. On the one hand, this paper technically circumvents Heifetz and Samet (1999, Section 4) by proposing a stronger notion of coherency. On the other hand, conceptually, since a type induces countable levels of beliefs, I have incorporated into the definition of coherency the requirement that all possible countable levels of beliefs do not contradict one another. This enables one to establish the measurable isomorphism in Proposition 1 and consequently in Theorem 1 without any topological apparatus (i.e., the Kolgomorov extension theorem). The idea to capture transfinite levels of coherent beliefs also generalizes the construction of a terminal type space by Mertens and Zamir (1985) (Theorem 3). Moreover, the resulting terminal type space $\overrightarrow{T^{\mathrm{MZ}}}$ can also be characterized as the set of recursively coherent belief hierarchies that satisfy common certainty of recursive coherency and as the largest belief-closed subset of recursive coherent belief hierarchies (Theorem 4).

This paper leaves some avenues of future research, on both applications and technical sides. On the applications side, it would be interesting to extend solution concepts on a terminal type space with topological assumptions to those on the terminal type space constructed here. For example, Dekel, Fudenberg, and Morris (2007) base their analysis of interim correlated rationalizability on Heifetz and Samet (1998)'s implicit-approach terminal type space and also use the properties of Mertens and Zamir (1985)'s explicit-approach terminal type space when a set of nature states is finite. Given that their terminal spaces are the same for any measurable space of nature states, it would be interesting to extend the solution concept of interim correlated rationalizability on a purely measurable environment, although this brings difficulties of additional measurability issues such as that of best responses.[19]

On the technical side, it is interesting to extend the explicit approach to construct-

[19] Weinstein and Yildiz (2017) consider the case in which players' type spaces are complete and separable metric space, their action spaces are compact metric spaces, and in which their payoff functions are continuous, beyond Dekel, Fudenberg, and Morris (2007).

ing a terminal type space to richer forms of beliefs and preferences without imposing a topological assumption.[20] For example, the following papers provide the explicit approach to constructing a terminal type space where types represent a richer form of beliefs: ambiguous types (Ahn, 2007), conditional types (Battigalli and Siniscalchi, 1999), possibility types (Mariotti, Meier, and Piccione, 2005), and preference types (Chen, 2010; Epstein and Wang, 1996).[21] An additional difficulty is that these concepts of beliefs may require additional topological assumptions.[22] This is in contrast to the fact that a countably-additive measure can be defined on a purely measurable space. Although the construction of a terminal type space for such richer forms of beliefs is beyond the scope of this paper, the paper would demonstrate that the implicit and explicit approaches yield the same terminal type space when the coherency condition is refined, even with the presence of topological assumptions. More generally, the analysis of this paper sheds light on the following question: for what conditions on beliefs, do the belief hierarchies up to level ω determine all subsequent beliefs (see also Brandenburger and Keisler, 2006; Fagin et al., 1999; Friedenberg, 2010), beyond standard countably-additive beliefs? When each player's type is a single probability measure, countable additivity grantees this property. Finding such conditions in a general environment makes it possible to apply the idea of this paper to richer forms of beliefs.

A Proofs

A.1 Section 4

A.1.1 Proof of Proposition 1

The proof consists of two steps. The first step defines an auxiliary measurable map $\overline{\psi} : \overline{T}^1 \to \Delta(S \times \overline{T}^0)$. Then, the second step establishes a measurable isomorphism $\psi : T^1 \to \Delta(S \times T^0)$. The first step (i.e., the mapping $\overline{\psi}$) enables one to define the isomorphism ψ without invoking the Kolgomorov extension theorem.[23]

Step 1. The first step constructs a measurable map $\overline{\psi} : (\overline{T}^1, \overline{\mathcal{T}}^1) \to (\Delta(S \times \overline{T}^0), (\mathcal{S} \times$

[20]For implicit-approach topology-free constructions of conditional type spaces, see, for instance, Fukuda (2021) and Guarino (2017).

[21]Also, Brandenburger, Friedenberg, and Keisler (2008) construct a complete type space for lexicographic beliefs.

[22]An ambiguous type of player i may be associated with a compact subset of probability distributions over nature states S and the types of the other player T_j. For a conditional type, the set of conditioning events form a collection of non-empty closed-and-open sets. A possibility type may be associated with a compact subset of nature states S and the types of the other player T_j. A preference type may be associated with a "regular" preference relation over the measurable acts on $S \times T_j$.

[23]In fact, the mapping $\overline{\psi}$ is also an isomorphism. See Remark 3 below.

$\overline{\mathcal{T}}^0)_\Delta)$ in the following three sub-steps.

First, $S \times \overline{T}^0$ is identified with $Z^{<\Omega}$:

$$S \times \overline{T}^0 = S \times \prod_{\alpha \in [1,\Omega)} \Delta\left(Z^{<\alpha}\right) = Z^{<\Omega}.$$

For each $\alpha \in [1, \Omega)$, denote by $\overline{\pi}_\alpha : Z^{<\Omega} \to Z^{<\alpha}$ the projection. The σ-algebra $\mathcal{Z}^{<\Omega}$ on $Z^{<\Omega}$ satisfies

$$\mathcal{Z}^{<\Omega} = \bigcup_{\alpha \in [1,\Omega)} \{\overline{\pi}_\alpha^{-1}(E) \in \mathcal{P}\left(Z^{<\Omega}\right) \mid E \in \mathcal{Z}^{<\alpha}\}, \tag{3}$$

because the right-hand side (the generator of $\mathcal{Z}^{<\Omega}$) itself forms a σ-algebra. For the fact that the right-hand side is a σ-algebra, recall Section 2; also, footnote 24 below sketches a direct proof of this fact.

Second, define $\overline{\psi} : \overline{T}^1 \to \Delta(Z^{<\Omega})$ as follows. Take $\overline{h} = (\overline{h}^\alpha)_\alpha \in \overline{T}^1$, and let

$$\overline{\psi}(\overline{h})(\overline{\pi}_\alpha^{-1}(E)) := \overline{h}^\alpha(E) \text{ for any } E \in \mathcal{Z}^{<\alpha} \text{ with } \alpha \in [1, \Omega).$$

Since $\overline{h}$ is coherent, if $\overline{\pi}_\alpha^{-1}(E) = \overline{\pi}_\beta^{-1}(F)$ then $\overline{\psi}(\overline{h})(\overline{\pi}_\alpha^{-1}(E)) = \overline{\psi}(\overline{h})(\overline{\pi}_\beta^{-1}(F))$. Hence, $\overline{\psi}(\overline{h})$ is a well-defined set function from $\mathcal{Z}^{<\Omega}$ to $[0, 1]$.

Now, I show that $\overline{\psi}(\overline{h})$ is a probability measure on $(Z^{<\Omega}, \mathcal{Z}^{<\Omega})$. Regardless of the choice of α, $\overline{\psi}(\overline{h})(Z^{<\Omega}) = \overline{\psi}(\overline{h})(\overline{\pi}_\alpha^{-1}(Z^{<\alpha})) = \overline{h}^\alpha(Z^{<\alpha}) = 1$ and $\overline{\psi}(\overline{h})(\emptyset) = \overline{\psi}(\overline{h})(\overline{\pi}_\alpha^{-1}(\emptyset)) = \overline{h}^\alpha(\emptyset) = 0$. Take a disjoint countable collection $(\overline{\pi}_{\alpha_\lambda}^{-1}(E_\lambda))_{\lambda \in \Lambda}$ from $\mathcal{Z}^{<\Omega}$, where $(\alpha_\lambda)_{\lambda \in \Lambda}$ is a countable collection of non-zero countable ordinals (i.e., Λ is a countable set and $\alpha_\lambda \in [1, \Omega)$ for each $\lambda \in \Lambda$). Then, $\alpha := \sup_{\lambda \in \Lambda} \alpha_\lambda < \Omega$. Now, rewrite $\overline{\pi}_{\alpha_\lambda}^{-1}(E_\lambda) = \overline{\pi}_\alpha^{-1}(E_\lambda^\alpha)$, where $E_\lambda^\alpha := (\pi_{Z^{<\alpha_\lambda}}^{Z^{<\alpha}})^{-1}(E_\lambda)$. Then, $\bigcup_{\lambda \in \Lambda} \overline{\pi}_{\alpha_\lambda}^{-1}(E_\lambda) = \overline{\pi}_\alpha^{-1}\left(\bigcup_{\lambda \in \Lambda} E_\lambda^\alpha\right)$, and thus

$$\overline{\psi}(\overline{h})\left(\bigcup_{\lambda \in \Lambda} \overline{\pi}_{\alpha_\lambda}^{-1}(E_\lambda)\right) = \overline{h}^\alpha\left(\bigcup_{\lambda \in \Lambda} E_\lambda^\alpha\right) = \sum_{\lambda \in \Lambda} \overline{h}^\alpha(E_\lambda^\alpha) = \sum_{\lambda \in \Lambda} \overline{\psi}(\overline{h})(\overline{\pi}_{\alpha_\lambda}^{-1}(E_\lambda)).$$

Hence, $\overline{\psi}(\overline{h})$ is a probability measure.[24]

Third, I show that $\overline{\psi} : (\overline{T}^1, \overline{\mathcal{T}}^1) \to (\Delta(Z^{<\Omega}), (\mathcal{Z}^{<\Omega})_\Delta)$ is measurable. For any $\alpha \in [1, \Omega)$ and for any $(E, p) \in \mathcal{Z}^{<\alpha} \times [0, 1]$,

$$\overline{\psi}^{-1}(\{\mu \in \Delta(S \times \overline{T}^0) \mid \mu(\overline{\pi}_\alpha^{-1}(E)) \geq p\}) = \{\overline{h} \in \overline{T}^1 \mid \overline{h}^\alpha(E) \geq p\} \in \overline{\mathcal{T}}^1.$$

[24]Incidentally, the previous argument has shown that the right-hand side of Expression (3) contains $\emptyset$ and $Z^{<\Omega}$, and is closed under countable union. Since $(\overline{\pi}_\alpha^{-1}(E))^c = \overline{\pi}_\alpha^{-1}(E^c)$, the right-hand side of Expression (3) is also closed under complementation. Hence, it is indeed a σ-algebra.

Step 2. The second step consists of three sub-steps. First, noticing that $S \times T^0$ is identified with $Z^{<\omega}$, I define $\psi : T^1 \to \Delta(Z^{<\omega})$ as follows. For any $h \in T^1$, take $\overline{h} \in \overline{T}^1$ with $h = (\overline{h}^k)_{k\in\mathbb{N}}$. Define

$$\psi(h) := \overline{\psi}(\overline{h}) \circ \overline{\pi}_\omega^{-1} = \overline{h}^\omega.$$

Especially, for any $E \in \mathcal{Z}^{<k}$ with $k \in \mathbb{N}$, denoting by $\pi_k : Z^{<\omega} \to Z^{<k}$ the projection,

$$\psi(h)(\pi_k^{-1}(E)) = h^k(E).$$

For the σ-algebra $\mathcal{Z}^{<\omega}$ on $Z^{<\omega}$, the generator $\mathcal{G} := \bigcup_{k\in\mathbb{N}}\{\pi_k^{-1}(E) \in \mathcal{P}(Z^{<\omega}) \mid E \in \mathcal{Z}^{<k}\}$ is an algebra. If $\pi_k^{-1}(E) = \pi_\ell^{-1}(F)$, then it follows from the coherency of h that

$$\psi(h)(\pi_k^{-1}(E)) = h^k(E) = h^\ell(F) = \psi(h)(\pi_\ell^{-1}(F)).$$

Hence, $\psi(h)$ is a well-defined set function from $\mathcal{G}$ into $[0, 1]$. The set function $\psi(h)$ is defined on the σ-algebra $\sigma(\mathcal{G})$ because

$$\psi(h) \circ \pi_k^{-1} = (\overline{\psi}(\overline{h}) \circ \overline{\pi}_\omega^{-1}) \circ \pi_k^{-1} = \overline{\psi}(\overline{h}) \circ \overline{\pi}_k^{-1}$$

and because the domain of $\overline{\psi}(\overline{h})$ includes the σ-algebra $\overline{\pi}_\omega^{-1}(\sigma(\mathcal{G}))$, which is equal to the σ-algebra $\sigma(\overline{\pi}_\omega^{-1}(\mathcal{G}))$ generated by $\overline{\pi}_\omega^{-1}(\mathcal{G}) = \bigcup_{k\in\mathbb{N}} \overline{\pi}_k^{-1}(\mathcal{Z}^{<k})$. Hence, $\psi(h) : \sigma(\mathcal{G}) \to [0, 1]$ is a well-defined probability measure on $\sigma(\mathcal{G})$, i.e., it does not depend on the choice of $\overline{h}$. Hence, $\psi : T^1 \to \Delta(S \times T^0)$ is well-defined.

Second, I show that $\psi : (T^1, \mathcal{T}^1) \to (\Delta(S \times T^0), (\mathcal{S} \times \mathcal{T}^0)_\Delta)$ is measurable. This is because, for any $k \in \mathbb{N}$, $\pi_k^{-1}(E)$ with $E \in \mathcal{Z}^{<k}$, and for any $p \in [0, 1]$,

$$\begin{aligned}\psi^{-1}(\{\mu \in \Delta(S \times T^0) \mid \mu(\pi_k^{-1}(E)) \geq p\}) &= \{h \in T^1 \mid \psi(h)(\pi_k^{-1}(E)) \geq p\} \\ &= \{h \in T^1 \mid h^k(E) \geq p\} \in \mathcal{T}^1.\end{aligned}$$

Third, I show that ψ is injective. For any $h, h' \in \Delta(S \times T^0)$ with $h \neq h'$, there is $k \in \mathbb{N}$ with $\operatorname{marg}_{Z^{<k}} h^k \neq \operatorname{marg}_{Z^{<k}} h'^k$. Thus, there is $E \in \mathcal{Z}^{<k}$ with $h^k(E) \neq h'^k(E)$. Then,

$$\psi(h)(\pi_k^{-1}(E)) = h^k(E_k) \neq h'^k(E_k) = \psi(h')(\pi_k^{-1}(E)).$$

Hence, ψ is injective.

Fourth, I show that ψ is surjective. For any $\mu \in \Delta(S \times T^0) = \Delta(Z^{<\omega})$, let $h = (\operatorname{marg}_{Z^{<k}} \mu)_{k\in\mathbb{N}} \in T^0$. Observe that the set T^0 of belief hierarchies (without imposing coherency) is the projection of the set of transfinite belief hierarchies: $T^0 = \pi_\omega^\Omega(\overline{T}^0)$, where $\pi_\omega^\Omega : H^{<\Omega} \to H^{<\omega}$ is the projection. Thus, there is some $\overline{h} \in \overline{T}^0$ with $h = (\overline{h}^k)_{k\in\mathbb{N}}$. Then, $\overline{\mu} := \overline{\psi}(\overline{h}) \in \Delta(S \times \overline{T}^0) = \Delta(Z^{<\Omega})$. Now, $(\operatorname{marg}_{Z^{<\alpha}} \overline{\mu})_{\alpha\in[1,\Omega)} \in \overline{T}^1$ and $h = (\operatorname{marg}_{Z^{<k}} \overline{\mu})_{k\in\mathbb{N}}$. Hence, $h \in T^1$. Since $\psi(h) = \overline{\psi}(\overline{h}) \circ \overline{\pi}_\omega^{-1} = \overline{\mu} \circ \overline{\pi}_\omega^{-1}$ and μ coincide on the generator of $\mathcal{Z}^{<\omega}$, it follows that $\psi(h) = \mu$.

Fifth, I show that the inverse $\psi^{-1} : (\Delta(S\times T^0), (\mathcal{S}\times\mathcal{T}^0)_\Delta) \ni \mu \mapsto (\operatorname{marg}_{Z^{<k}} \mu)_{k\in\mathbb{N}} \in (T^1, \mathcal{T}^1)$ is measurable. For any $k \in \mathbb{N}$ and for any $(E, p) \in \mathcal{Z}^{<k} \times [0, 1]$,

$$\begin{aligned}
&(\psi^{-1})^{-1}\left(\left(\pi^{T^1}_{\Delta(Z^{<k})}\right)^{-1}(\{\nu \in \Delta(Z^{<k}) \mid \nu(E) \geq p\})\right)\\
=&\{\mu \in \Delta(S \times T^0) \mid (\operatorname{marg}_{Z^{<k}} \mu)(E) \geq p\}\\
=&\{\mu \in \Delta(S \times T^0) \mid \mu(\pi_k^{-1}(E)) \geq p\} \in (\mathcal{S} \times \mathcal{T}^0)_\Delta.
\end{aligned}$$

□

Remark 3 (Remark on $\overline{\psi}$ in the proof of Proposition 1)**.** The measurable map $\overline{\psi} : (\overline{T}^1, \overline{\mathcal{T}}^1) \to (\Delta(Z^{<\Omega}), (\mathcal{Z}^{<\Omega})_\Delta)$ is an isomorphism. The proof is similar to that of Proposition 1. First, I show $\overline{\psi}$ is bijective. For any $\mu \in \Delta(S \times \overline{T}^0)$, $\overline{\psi}\left((\operatorname{marg}_{Z^{<\alpha}} \mu)_{\alpha\in[1,\Omega)}\right) = \mu$. Thus, $\overline{\psi}$ is surjective. Take any $\overline{h}, \overline{h}' \in \overline{T}^1$ with $\overline{h} \neq \overline{h}'$. Then there is $\alpha \in [1, \Omega)$ with $\overline{h}^\alpha \neq \overline{h}'^\alpha$. Thus, there is $E \in \mathcal{Z}^{<\alpha}$ with $\overline{h}^\alpha(E) \neq \overline{h}'^\alpha(E)$. Then,

$$\overline{\psi}(\overline{h})(\overline{\pi}_\alpha^{-1}(E)) = \overline{h}^\alpha(E) \neq \overline{h}'^\alpha(E) = \overline{\psi}(\overline{h}')(\overline{\pi}_\alpha^{-1}(E)).$$

Hence, $\overline{\psi}$ is injective.

Second, I show $\overline{\psi}^{-1} : (\Delta(S\times\overline{T}^0), (\mathcal{S}\times\overline{\mathcal{T}}^0)_\Delta) \ni \mu \mapsto (\operatorname{marg}_{Z^{<\alpha}} \mu)_{\alpha\in[1,\Omega)} \in (\overline{T}^1, \overline{\mathcal{T}}^1)$ is measurable. For any $\alpha \in [1, \Omega)$, $E_\alpha \in \mathcal{Z}^{<\alpha}$, and for any $p \in [0, 1]$,

$$\begin{aligned}
&(\overline{\psi}^{-1})^{-1}\left(\left(\pi^{\overline{T}^1}_{\Delta(Z^{<\alpha})}\right)^{-1}(\{\nu \in \Delta(Z^{<\alpha}) \mid \nu(E_\alpha) \geq p\})\right)\\
=&\{\mu \in \Delta(S \times \overline{T}^0) \mid (\operatorname{marg}_{Z^{<\alpha}} \mu)(E_\alpha) \geq p\}\\
=&\{\mu \in \Delta(S \times \overline{T}^0) \mid \mu(\overline{\pi}_\alpha^{-1}(E_\alpha)) \geq p\} \in (\mathcal{S} \times \overline{\mathcal{T}}^0)_\Delta.
\end{aligned}$$

A.1.2 Proof of Theorem 1

The proof consists of three parts. The first part shows that $(T^{\mathrm{BD}}, \mathcal{T}^{\mathrm{BD}})$ is belief-closed. By construction, $T^{\mathrm{BD}} \subseteq T^1$ and $\mathcal{T}^{\mathrm{BD}} = \mathcal{T}^1 \cap T^{\mathrm{BD}}$. Hence, it suffices to show that $\psi^*_{T^{\mathrm{BD}}} : (T^{\mathrm{BD}}, \mathcal{T}^{\mathrm{BD}}) \to (\Delta(S \times T^{\mathrm{BD}}), (\mathcal{S} \times \mathcal{T}^{\mathrm{BD}})_\Delta)$ is a well-defined measurable map. Recall from the main text that $T^{\mathrm{BD}} = \{h \in T^1 \mid \psi^*(h)(S \times T^{\mathrm{BD}}) = 1\}$ holds by definition.

Part 1. To establish the first part, I prove the following lemma: any subset T of T^1 is belief-closed iff $T = \{h \in T \mid \psi^*(h)(S \times T) = 1\}$. The lemma implies that T^{BD} is belief-closed.

For the "only if" part of the lemma, suppose $\psi^*_T : (T, \mathcal{T}) \to (\Delta(S\times T), (\mathcal{S}\times\mathcal{T})_\Delta)$. Then, for any $h \in T$, $\psi^*(h)(S \times T) = \psi^*_T(h)(S \times T) = 1$. Hence, $T = \{h \in T \mid \psi^*(h)(S \times T) = 1\}$.

For the "if" part, suppose $T = \{h \in T \mid \psi^*(h)(S \times T) = 1\}$. Note $T \subseteq T^1 \subseteq T^0$. Take any $h \in T$. Then, $\psi(h) \in \tilde{\Delta}(S \times T)$ because, for any $E \in \mathcal{S} \times \mathcal{T}^0$ with $S \times T \subseteq E$,

$$1 = \psi^*(h)(S \times T) = \inf\{\psi(h)(F) \mid S \times T \subseteq F \in \mathcal{S} \times \mathcal{T}^0\} \leq \psi(h)(E) \leq 1.$$

Then, since $\psi : (T^1, \mathcal{T}^1) \to (\Delta(S \times T^0), (\mathcal{S} \times \mathcal{T}^0)_\Delta)$ is measurable, so is $\psi|_T : (T, \mathcal{T}) \to (\tilde{\Delta}(S \times T^0), (\mathcal{S} \times \mathcal{T}^0)_\Delta \cap \tilde{\Delta}(S \times T^0))$. Since $(\tilde{\Delta}(S \times T^0), (\mathcal{S} \times \mathcal{T}^0)_\Delta \cap \tilde{\Delta}(S \times T^0)) \ni \nu \mapsto \nu^*|_{\mathcal{S} \times \mathcal{T}} \in (\Delta(S \times T), (\mathcal{S} \times \mathcal{T})_\Delta)$ is an isomorphism by invoking Heifetz and Samet (1999, Lemma 2.2), $\psi^*_T : (T, \mathcal{T}) \to (\Delta(S \times T), (\mathcal{S} \times \mathcal{T})_\Delta)$ is measurable.

Part 2. The second part establishes that if $(T, \mathcal{T})$ is belief-closed then $T \subseteq T^{\text{BD}}$ in the following three sub-steps. Let $(T, \mathcal{T})$ be belief-closed. The first sub-step defines a type space $\overrightarrow{T} = \langle (T_i, \mathcal{T}_i)_{i \in I_0}, (m_i)_{i \in I} \rangle$ as follows: $(T_i, \mathcal{T}_i) = (T, \mathcal{T})$ for each $i \in I$; and $m_i = \psi^*_T : (T_i, \mathcal{T}_i) \to (\Delta(S \times T_j), (\mathcal{S} \times \mathcal{T}_j)_\Delta)$ for each $i \in I$. The second sub-step shows that, for each $h_i \in T_i$, the coherent transfinite belief hierarchy $\overline{\eta}_i(h_i)$ satisfies $\eta_i(h_i) = h_i$, implying that $\eta_i : (T_i, \mathcal{T}_i) \to (T^1, \mathcal{T}^1)$ is an inclusion map. The third sub-step, by induction, establishes $\eta_i(\cdot) \in T^\beta$ for any ordinal $\beta \geq 1$, i.e., $\eta_i(\cdot) \in T^{\text{BD}}$. By construction, $\eta_i : (T_i, \mathcal{T}_i) \to (T^{\text{BD}}, \mathcal{T}^{\text{BD}})$ is an inclusion map. Hence, $T \subseteq T^{\text{BD}}$.

To prove the second sub-step, it suffices to show by induction that the hierarchy map $\eta_i^{<\beta} : (T_i, \mathcal{T}_i) \to (H^{<\beta}, \mathcal{H}^{<\beta})$ satisfies $\eta_i^{<\beta}(h_i) = h_i^{<\beta} := (h_i^\gamma)_{\gamma \in [1,\beta)}$ for each $(\beta, i) \in [2, \omega] \times I$. Let $\beta = 2$. For each $i \in I$,

$$\begin{aligned} \eta_i^{<2}(h_i)(\cdot) &= m_i(h_i) \circ (\pi_S^{S \times T_j})^{-1}(\cdot) = \psi^*_T(h_i) \circ (\pi_S^{S \times T_j})^{-1}(\cdot) \\ &= \psi^*_T(h_i)((\cdot) \times T^0 \cap S \times T_j) = \psi(h_i) \circ \pi_1^{-1}(\cdot) = h_i^{<2}(\cdot). \end{aligned}$$

The first equality follows from Definition 2. The second equality follows because $m_i = \psi^*|_T$. The third and fourth equalities follow from properties of projections. The last equality follows from the construction of the isomorphism ψ in Proposition 1.

For a successor ordinal $\beta = \gamma + 1 \in [3, \omega)$, assume $\eta_i^{<\gamma}(h_i) = h_i^{<\gamma}$ for each $i \in I$. Fix $i \in I$. Since $(\text{id}_S, \eta_j^{<\gamma}) = \pi_\gamma|_{S \times T_j} : (S \times T_j, \mathcal{S} \times \mathcal{T}_j) \to (Z^{<\gamma}, \mathcal{Z}^{<\gamma})$,

$$\begin{aligned} \eta_i^{<\beta}(h_i) &= (\eta_i^{<\gamma}(h_i), m_i(h_i) \circ (\pi_\gamma|_{S \times T_j})^{-1}) = (h_i^{<\gamma}, m_i(h_i) \circ (\pi_\gamma|_{S \times T_j})^{-1}) \\ &= (h_i^{<\gamma}, \psi^*_T(h_i) \circ (\pi_\gamma|_{S \times T_j})^{-1}) = (h_i^{<\gamma}, \psi(h_i) \circ \pi_\gamma^{-1}) = (h_i^{<\gamma}, h_i^\gamma) = h_i^{<\beta}. \end{aligned}$$

The first equality follows from Definition 2. The second equality follows from the induction hypothesis. The third equality follows because $m_i = \psi^*|_T$. The fourth equality follows from properties of a projection and an outer measure. The fifth equality follows from the construction of the isomorphism ψ in Proposition 1. The last equality follows by definition.

For the limit ordinal $\beta = \omega$, assume $\eta_i^{<\gamma}(h_i) = h_i^{<\gamma}$ for all $(\gamma, i) \in [2, \beta) \times I$. Fix $i \in I$. On the one hand, $\eta_i^{<\beta}$ is a unique map from $(T_i, \mathcal{T}_i)$ to $(H^{<\beta}, \mathcal{H}^{<\beta})$ satisfying $\pi_\gamma^\beta \circ \eta_i^{<\beta} = \eta_i^{<\gamma}$ for all $\gamma \in [2, \beta)$. Thus, $\pi_\gamma^\beta \circ \eta_i^{<\beta}(h_i) = \eta_i^{<\gamma}(h_i)$. On the other hand,

$h_i^{<\beta}$ is a unique element of $H^{<\beta}$ satisfying $\pi_\gamma^\beta \circ h_i^{<\beta} = h_i^{<\gamma}$ for all $\gamma \in [2, \beta)$. Thus, for any h_i, $\eta_i^{<\beta}(h_i) = h_i^{<\beta}$.

For the third sub-step, fix $i \in I$. By construction, each $\eta_i(h_i)$ is coherent, i.e., $\eta_i(\cdot) \in T^1 = T^{<2}$. For a successor ordinal $\beta = \gamma + 1$ with $\gamma \geq 2$, assume that $\eta_i(\cdot) \in T^{<\gamma}$. Since T_i is belief-closed, $\psi^*(\eta_i(\cdot))(S \times T^{<\gamma}) = 1$, i.e., $\eta_i(\cdot) \in T^{<\beta}$. For a limit ordinal β, if $\eta_i(\cdot) \in T^{<\gamma}$ for any $\gamma < \beta$, then $\eta_i(\cdot) \in T^{<\beta}$. This also shows that $\eta_i(\cdot) \in T^\beta$ for all β. Summing up, for any $h_i = h_i^{<\omega} \in T_i$, $\eta_i(h_i) = h_i^{<\omega} = h_i \in T^{\mathrm{BD}}$.

Part 3. The third part shows that $m^{\mathrm{BD}} = \psi^*_{T^{\mathrm{BD}}}$ is an isomorphism. By the definition of T^{BD}, $\psi|_{T^{\mathrm{BD}}} : (T^{\mathrm{BD}}, \mathcal{T}^{\mathrm{BD}}) \to (\tilde{\Delta}(S \times T^{\mathrm{BD}}), (\mathcal{S} \times \mathcal{T}^0)_\Delta \cap \tilde{\Delta}(S \times T^{\mathrm{BD}}))$ is an isomorphism. Next, denote by $\chi : (\tilde{\Delta}(S \times T^{\mathrm{BD}}), (\mathcal{S} \times \mathcal{T}^0)_\Delta \cap \tilde{\Delta}(S \times T^{\mathrm{BD}})) \to (\Delta(S \times T^{\mathrm{BD}}), (\mathcal{S} \times \mathcal{T}^{\mathrm{BD}})_\Delta)$ the isomorphism that associates, with each $\mu \in \tilde{\Delta}(S \times T^{\mathrm{BD}})$, the induced outer measure restricted on $S \times T^{\mathrm{BD}} = S \times (T^0 \cap T^{\mathrm{BD}})$ (Heifetz and Samet, 1999, Lemma 2.2; see also Section 2). By construction, $\psi^*_{T^{\mathrm{BD}}} = \chi \circ \psi|_{T^{\mathrm{BD}}}$. Since $\psi^*_{T^{\mathrm{BD}}}$ is the composite of two isomorphisms, it is an isomorphism. □

A.1.3 Remark on Expression (2)

It follows from the proof of Theorem 1 that if T is a belief-closed subset of T^1 then $T \subseteq T^\beta$ for all β. Hence, Expression (2) is well-defined (indeed, each T^β is not empty) if there exists a non-empty belief-closed subset T.

I show that a non-empty belief-closed subset exists given $S \neq \emptyset$. In doing so, I show: (i) when S is not empty, there exists a type space; and (ii) the belief hierarchy induced by a type in any type space constitutes a belief-closed subset.

First, since $S \neq \emptyset$, take $s \in S$. I show that there exists a type space $\overrightarrow{T^s} = \langle (T_i^s, \mathcal{T}_i^s)_{i \in I_0}, (m_i^s)_{i \in I} \rangle$. Namely, let $T_i^s = \{s\}$ and $\mathcal{T}_i^s = \{\emptyset, \{s\}\}$ for each $i \in I$. Then, define $m_i^s : T_i \to \Delta(S \times T_j)$ as $m_i^s(s)(E \times \{s\}) = \delta_s(E)$ for each $E \in \mathcal{S}$, where δ_s is the Dirac measure on $(S, \mathcal{S})$ concentrated at $s \in S$. Note that $\mathcal{S} \times \mathcal{T}_j = \{E \times \{s\} \mid E \in \mathcal{S}\}$.

Second, let $\overrightarrow{T}$ be any type space (by the previous argument, a type space exists). Choose some $i \in I$ and $t_i \in T_i$. Letting $\eta_i : (T_i, \mathcal{T}_i) \to (H^{<\omega}, \mathcal{H}^{<\omega})$ be the hierarchy map associated with the type space $\overrightarrow{T}$, let $\eta_i^{t_i} := \eta_i(t_i) \in T^1$ be the coherent belief hierarchy. Then, define $T_i' = \{\eta_i^{t_i}\}$ and $\mathcal{T}_i' = \mathcal{T}^1 \cap T_i' = \mathcal{P}(T_i')$ for each $i \in I$. Now, $T_i' = \{h \in T_i' \mid \psi^*(h)(S \times T_i') = 1\}$ follows because $\psi^*(\eta_i^{t_i})(S \times \{\eta_i^{t_i}\}) = 1$. Hence, $(T_i', \mathcal{T}_i')$ is non-empty and belief-closed. Also, this means that, for any type t_i of any player i in any type space $\overrightarrow{T}$, the associated belief hierarchy $\eta_i(t_i)$ belongs to T^{TB}, as Theorem 1 implies that T^{TB} is the largest belief-closed subset of T^1.

A.1.4 Proof of Remark 2

First, I show that $(\mathrm{id}_S, \eta_1, \eta_2) : \overrightarrow{T} \to \overrightarrow{T^{\mathrm{BD}}}$ is a type morphism: $m_i(\cdot) \circ (\mathrm{id}_S, \eta_j)^{-1} = m_i^{\mathrm{BD}}(\eta_i(\cdot))$ for each $i \in I$. It is enough to show the equality for a generator of $\mathcal{S} \times \mathcal{T}^{\mathrm{BD}}$.

For any $t_i \in T_i$ and $\pi_k^{-1}(E)$ with $E \in \mathcal{Z}^k$,

$$m_i(t_i)((\mathrm{id}_S, \eta_j)^{-1}(\pi_k^{-1}(E))) = m_i(t_i)((\pi_k \circ (\mathrm{id}_S, \eta_j))^{-1}(E)) = m_i(t_i) \circ (\mathrm{id}_S, \eta_j^{<k})^{-1}(E)$$
$$= (\eta_i(t_i))^k(E) = m_i^{\mathrm{BD}}(\eta_i(t_i))(\pi_k^{-1}(E)).$$

The first equality follows from a property of inverses. The second equality follows from a property of a projection. The third equality follows from Definition 2. The last equality follows from the construction of m^{BD}.

Second, the uniqueness follows from Remark 1 and Theorem 1. Theorem 1 implies that T^{BD} is belief-closed and that the hierarchy map associated with $\overrightarrow{T^{\mathrm{BD}}}$ is $\eta_i = \mathrm{id}_{T^{\mathrm{BD}}}$. Then, Remark 1 implies that there is a unique type morphism from $\overrightarrow{T}$ into $\overrightarrow{T^{\mathrm{BD}}}$. Finally, note that Remark 2 also implies that $m^{\mathrm{BD}} : T^{\mathrm{BD}} \to \Delta(S \times T^{\mathrm{BD}})$ is an isomorphism. □

A.1.5 The Implicit Approach with Coherency and Common Certainty of Coherency

To prove Theorem 2 in Section A.1.6, here I re-construct the implicit-approach terminal space $\overrightarrow{T^{\mathrm{HS}}}$ of Heifetz and Samet (1998) that satisfies coherency and common certainty of coherency in three steps (see footnote 9).

The first step defines $(T_i^{\mathrm{HS}}, \mathcal{T}_i^{\mathrm{HS}})_{i \in I_0}$, where $(T_0^{\mathrm{HS}}, \mathcal{T}_0^{\mathrm{HS}}) := (S, \mathcal{S})$. Fix $i \in I$. Recalling that the hierarchy map $\eta_i : (T_i, \mathcal{T}_i) \to (T^1, \mathcal{T}^1)$ is a well-defined measurable map for any type space $\overrightarrow{T} = \langle (T_i, \mathcal{T}_i)_{i \in I_0}, (m_i)_{i \in I} \rangle$, define

$$T_i^{\mathrm{HS}} := \{h_i \in T^1 \mid h_i = \eta_i(t_i) \text{ for some type space } \overrightarrow{T} \text{ and } t_i \in T_i\}.$$

By taking the type space $\overrightarrow{T^s}$ defined in Appendix A.1.3, $\eta_i^s \in T_i^{\mathrm{HS}}$ and thus $T_i^{\mathrm{HS}} \neq \emptyset$. Let $\mathcal{T}_i^{\mathrm{HS}} := \mathcal{T}^1 \cap T_i^{\mathrm{HS}}$ be the σ-algebra on T_i^{HS}. Note that $(T_i^{\mathrm{HS}}, \mathcal{T}_i^{\mathrm{HS}})$ does not depend on the identity of the players: $(T_1^{\mathrm{HS}}, \mathcal{T}_1^{\mathrm{HS}}) = (T_2^{\mathrm{HS}}, \mathcal{T}_2^{\mathrm{HS}})$. Hereafter in this proof, for any type space $\overrightarrow{T}$, we identify the hierarchy map η_i as $\eta_i : (T_i, \mathcal{T}_i) \to (T_i^{\mathrm{HS}}, \mathcal{T}_i^{\mathrm{HS}})$.

The second step defines $m_i^{\mathrm{HS}} : T_i^{\mathrm{HS}} \to \Delta(S \times T_j^{\mathrm{HS}})$ for each $i \in I$. Fix $i \in I$. For each $t_i^{\mathrm{HS}} \in T_i^{\mathrm{HS}}$, there are a type space $\overrightarrow{T}$ and $t_i \in T_i$ such that $t_i^{\mathrm{HS}} = \eta_i(t_i)$. Define $m_i^{\mathrm{HS}}(t_i^{\mathrm{HS}}) := m_i(t_i) \circ \eta_{-i}^{-1}$.

I show that $m_i^{\mathrm{HS}} : (T_i^{\mathrm{HS}}, \mathcal{T}_i^{\mathrm{HS}}) \to (\Delta(S \times T_j^{\mathrm{HS}}), (\mathcal{S} \times \mathcal{T}_j^{\mathrm{HS}})_\Delta)$ is a well-defined measurable map. For each $k \in \mathbb{N}$, for ease of notation, let $\pi_k : S \times T_i^{\mathrm{HS}} \to Z^{<k}$ be the projection (precisely, the restriction of $\pi_{Z^{<k}}^{Z^{<\omega}}$ on $S \times T_i^{\mathrm{HS}}$). For any $k \in \mathbb{N}$ and any choice of $\overrightarrow{T}$ and $t_i \in T_i$ with $t_i^{\mathrm{HS}} = \eta_i(t_i)$,

$$m_i^{\mathrm{HS}}(t_i^{\mathrm{HS}}) \circ \pi_k^{-1} = m_i(t_i) \circ (\mathrm{id}_S, \eta_j^{<k})^{-1} = (\eta_i(t_i))^k = (t_i^{\mathrm{HS}})^k.$$

Thus, $m_i^{\mathrm{HS}}(t_i^{\mathrm{HS}})$ is well-defined on $\sigma\left(\bigcup_{k \in \mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})\right)$ irrespective of the specific choice of $\overrightarrow{T}$ and $t_i \in T_i$ with $t_i^{\mathrm{HS}} = \eta_i(t_i)$. The map m_i^{HS} is measurable because, for any

$k \in \mathbb{N}$ and $(E, p) \in \mathscr{Z}^{<k} \times [0,1]$,

$$(m_i^{\text{HS}})^{-1}(\{\mu \in \Delta(S \times T_j^{\text{HS}}) \mid \mu(\pi_k^{-1}(E)) \geq p\}) = \{t_i^{\text{HS}} \in T_i^{\text{HS}} \mid (t_i^{\text{HS}})^k(E) \geq p\}.$$

So far, I have established that $\overrightarrow{T^{\text{HS}}}$ is a type space. Moreover, for any type space $\overrightarrow{T}$, the measurable maps $(\text{id}_S, \eta_1, \eta_2)$ form a morphism. This is because $(m_i^{\text{HS}} \circ \eta_i)(\cdot) = m_i(\cdot) \circ \eta_{-i}^{-1}$ for all $i \in I$.

The third step shows that $\overrightarrow{T^{\text{HS}}}$ is terminal. By Remark 1, it suffices to show that the hierarchy map $\eta_i^{\text{HS}} : T_i^{\text{HS}} \to T_i^{\text{HS}}$ associated with $\overrightarrow{T^{\text{HS}}}$ is injective for each $i \in I$. I show it is the identity by proving that $(\eta_i^{\text{HS}})^{<k} : T_i^{\text{HS}} \to H_i^{<k}$ is the projection for each $(i,k) \in I \times \mathbb{N} \backslash \{1\}$. For $k = 2$, $(\eta_i^{\text{HS}})^{<2}(t_i^{\text{HS}}) = m_i^{\text{HS}}(t_i^{\text{HS}}) \circ (\pi_S^{S \times T_j^{\text{HS}}})^{-1} = (t_i^{\text{HS}})^1$. Assume the induction hypothesis for k. For $k+1$, $(\eta_i^{\text{HS}})^{<k+1}(t_i^{\text{HS}}) = ((\eta_i^{\text{HS}})^{<k}(t_i^{\text{HS}}), m_i^{\text{HS}}(t_i^{\text{HS}}) \circ ((\eta_{-i}^{\text{HS}})^{<k})^{-1}) = (t_i^{\text{HS}})^{<k+1}$.

A.1.6 Proof of Theorem 2

As discussed in the main text, the proof of Theorem 2 is in three steps.

Step 1. Fix $i \in I$. First, $T_i^{\text{BD}} \subseteq T_i^{\text{HS}}$ follows from the fact that, for the unique morphism $\eta = (\text{id}_S, \eta_1, \eta_2)$ from a type space $\overrightarrow{T^{\text{BD}}}$ to a terminal type space $\overrightarrow{T^{\text{HS}}}$, the hierarchy map $\eta_i : T_i^{\text{BD}} \to T_i^{\text{HS}}$ associated with $\overrightarrow{T^{\text{BD}}}$ is an inclusion map.

Step 2. Fix $i \in I$. I show $T_i^{\text{HS}} \subseteq T_i^{\text{BD}}$. For any $h_i \in T_i^{\text{HS}}$, there are $\overrightarrow{T}$ and $t_i \in T_i$ such that $h_i = \eta_i^{\text{HS}}(t_i) \in T_i^{\text{BD}}$, where the last set belonging follows from the argument in Appendix A.1.3.

Step 3. Fix $i \in I$. So far, I have shown that $T_i^{\text{BD}} = T_i^{\text{HS}}$. Then, $\mathcal{T}_i^{\text{BD}} = \mathcal{T}^0 \cap T_i^{\text{BD}} = \mathcal{T}^0 \cap T_i^{\text{HS}} = \mathcal{T}_i^{\text{HS}}$. Also, $m_i^{\text{HS}} = \psi^*_{T_i^{\text{HS}}} = \psi^*_{T_i^{\text{BD}}} = m_i^{\text{BD}}$ (one can also show $m_i^{\text{HS}} = m_i^{\text{BD}}$ from the fact that the unique morphism from from $\overrightarrow{T^{\text{HS}}}$ to $\overrightarrow{T^{\text{BD}}}$ is the identity morphism). Hence, $\overrightarrow{T^{\text{BD}}} = \overrightarrow{T^{\text{HS}}}$. □

A.2 Section 5

A.2.1 The Implicit Approach with Recursive Coherency

Here, I (re-)construct the implicit-approach terminal type space that satisfies recursive coherency in three steps. The construction is similar to Appendix A.1.5. However, to distinguish it from the implicit-approach terminal type space $\overrightarrow{T^{\text{HS}}}$ in Appendix A.1.5, denote it by $\overrightarrow{T^{\text{IM}}}$. Technically, while $\overrightarrow{T^{\text{IM}}}$ and $\overrightarrow{T^{\text{HS}}}$ are different, they are isomorphic.

The first step is to define $(T_i^{\text{IM}}, \mathcal{T}_i^{\text{IM}})_{i \in I_0}$, where $(T_0^{\text{IM}}, \mathcal{T}_0^{\text{IM}}) := (S, \mathcal{S})$. Fix $i \in I$. Recalling that the hierarchy map $\eta_i : (T_i, \mathcal{T}_i) \to (H^{<\omega}, \mathcal{H}^{<\omega})$ is a well-defined measurable map for any type space $\overrightarrow{T} = \langle (T_i, \mathcal{T}_i)_{i \in I_0}, (m_i)_{i \in I} \rangle$, define

$$T_i^{\mathrm{IM}} = \{h_i \in H^{<\omega} \mid h_i = \eta_i(t_i) \text{ for some type space } \overrightarrow{T} \text{ and } t_i \in T_i\}.$$

As in Appendix A.1.5, since $S \neq \emptyset$, there exist a type space $\overrightarrow{T}$ and $t_i \in T_i$ such that $\eta_i(t_i) \in T_i^{\mathrm{IM}} \neq \emptyset$. Let $\mathcal{T}_i^{\mathrm{IM}} := \mathcal{H}^{<\omega} \cap T_i^{\mathrm{IM}}$. Note that $(T_i^{\mathrm{IM}}, \mathcal{T}_i^{\mathrm{IM}})$ does not depend on the identity of the players: $(T_1^{\mathrm{IM}}, \mathcal{T}_1^{\mathrm{IM}}) = (T_2^{\mathrm{IM}}, \mathcal{T}_2^{\mathrm{IM}})$. Hereafter in this proof, for any type space $\overrightarrow{T}$, we identify the hierarchy map η_i as $\eta_i : (T_i, \mathcal{T}_i) \to (T_i^{\mathrm{IM}}, \mathcal{T}_i^{\mathrm{IM}})$.

The second step defines $m_i^{\mathrm{IM}} : T_i^{\mathrm{IM}} \to \Delta(S \times T_j^{\mathrm{IM}})$ for each $i \in I$ (let $m_0^{\mathrm{IM}} := \mathrm{id}_S$). Fix $i \in I$. For each $t_i^{\mathrm{IM}} \in T_i^{\mathrm{IM}}$, there are a type space $\overrightarrow{T}$ and $t_i \in T_i$ such that $t_i^{\mathrm{IM}} = \eta_i(t_i)$. Define $m_i^{\mathrm{IM}}(t_i^{\mathrm{IM}}) := m_i(t_i) \circ \eta_{-i}^{-1}$.

I show that $m_i^{\mathrm{IM}} : (T_i^{\mathrm{IM}}, \mathcal{T}_i^{\mathrm{IM}}) \to (\Delta(S \times T_j^{\mathrm{IM}}), (\mathcal{S} \times \mathcal{T}_j^{\mathrm{IM}})_{\Delta(S \times T_j^{\mathrm{IM}})})$ is a well-defined measurable map. For each $k \in \mathbb{N}$, for ease of notation, let $\pi_k : S \times T_i^{\mathrm{IM}} \to Z^{<k}$ be the projection (precisely, the restriction of $\pi_{Z^{<k}}^{Z^{<\omega}}$ on $S \times T_i^{\mathrm{IM}}$).

For any $k \in \mathbb{N}$ and for any choice of $\overrightarrow{T}$ and $t_i \in T_i$ with $t_i^{\mathrm{IM}} = \eta_i(t_i)$,

$$\begin{aligned} m_i^{\mathrm{IM}}(t_i^{\mathrm{IM}}) \circ \pi_k^{-1} &= (m_i(t_i) \circ \eta_{-i}^{-1}) \circ \pi_k^{-1} = m_i(t_i) \circ (\pi_k \circ \eta_{-i})^{-1} \\ &= m_i(t_i) \circ (\eta_{-i}^{<k})^{-1} = (\eta_i(t_i))^{k+1} = (t_i^{\mathrm{IM}})^{k+1}. \end{aligned}$$

The first equality follows from the definition of m_i^{IM}. The second equality follows from a property of inverses. The third equality follows from a property of a projection. The fourth equality follows from Definition 2. The last equality follows because $t_i^{\mathrm{IM}} = \eta_i(t_i)$.

Thus, $m_i^{\mathrm{IM}}(t_i^{\mathrm{IM}})$ is well-defined for the generator $\bigcup_{k \in \mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})$ irrespective of the specific choice of $\overrightarrow{T}$ and $t_i \in T_i$ with $t_i^{\mathrm{IM}} = \eta_i(t_i)$. Since $m_i^{\mathrm{IM}}(t_i^{\mathrm{IM}})$ on the algebra uniquely extends to the generated σ-algebra, $m_i^{\mathrm{IM}}(t_i^{\mathrm{IM}}) \in \Delta(S \times T_j^{\mathrm{IM}})$ is well-defined irrespective of the specific choice of a type space and a type.

To show that the map m_i^{IM} is measurable, observe that, for any $k \in \mathbb{N}$ and $(E, p) \in \mathcal{Z}^{<k} \times [0, 1]$,

$$(m_i^{\mathrm{IM}})^{-1}(\{\mu \in \Delta(S \times T_j^{\mathrm{IM}}) \mid \mu(\pi_k^{-1}(E)) \geq p\}) = \{t_i^{\mathrm{IM}} \in T_i^{\mathrm{IM}} \mid (t_i^{\mathrm{IM}})^{<k+1}(E) \geq p\}.$$

Since $\{(t_i^{\mathrm{IM}})^{<k+1} \in \Delta(S \times H_j^{<k}) \mid (t_i^{\mathrm{IM}})^{k+1}(E) \geq p\} \in (\mathcal{S} \times \mathcal{H}_j^{<k})_{\Delta(S \times H_j^{<k})}$, it follows that m_i^{IM} is measurable.

So far, I have established that $\overrightarrow{T^{\mathrm{IM}}}$ is a type space. Moreover, for any type space $\overrightarrow{T}$, the measurable maps $\eta_i : T_i \to T_i^{\mathrm{IM}}$ form a morphism. This is because $(m_i^{\mathrm{IM}} \circ \eta_i)(\cdot) = m_i(\cdot) \circ \eta_{-i}^{-1}$ for all $i \in I$.

The third step shows that $\overrightarrow{T^{\mathrm{IM}}}$ is terminal. By Remark 1, it suffices to show that the hierarchy map $\eta_i^{\mathrm{IM}} : T_i^{\mathrm{IM}} \to T_i^{\mathrm{IM}}$ associated with $\overrightarrow{T^{\mathrm{IM}}}$ is injective for each $i \in I$. Indeed, I show that η_i^{IM} is the identity for each $i \in I$ (as well as $\eta_0 = \mathrm{id}_S$) by proving that $(\eta_i^{\mathrm{IM}})^{<k} : T_i^{\mathrm{IM}} \to H_i^{<k}$ is the projection for each $(i, k) \in I \times \mathbb{N} \setminus \{1\}$. For $k = 2$, $(\eta_i^{\mathrm{IM}})^{<2}(t_i^{\mathrm{IM}}) = m_i^{\mathrm{IM}}(t_i^{\mathrm{IM}}) \circ (\pi_S^{S \times T_j^{\mathrm{IM}}})^{-1} = (t_i^{\mathrm{IM}})^1$. Assume the induction hypothesis for k. For $k+1$, $(\eta_i^{\mathrm{IM}})^{<k+1}(t_i^{\mathrm{IM}}) = ((\eta_i^{\mathrm{IM}})^{<k}(t_i^{\mathrm{IM}}), m_i^{\mathrm{IM}}(t_i^{\mathrm{IM}}) \circ \pi_k^{-1}) = (t_i^{\mathrm{IM}})^{<k+1}$.

A.2.2 Remark on the Mappings $\overline{m}$ and m^{MZ}

First, I show that

$$\overline{m} : (H^{<\Omega}, \mathcal{H}^{<\Omega}) \to (\Delta(S \times H^{<\Omega}), (\mathcal{S} \times \mathcal{H}^{<\Omega})_\Delta)$$

is a well-defined measurable map. The proof of the fact that $\overline{m}(\overline{h})$ is a probability measure on $(Z^{<\Omega}, \mathcal{Z}^{<\Omega})$ is the same as the proof of the fact that $\overline{\psi}(\overline{h})$ is a probability measure on $(Z^{<\Omega}, \mathcal{Z}^{<\Omega})$ in the proof of Proposition 1, basically by replacing $\overline{\psi}$ with $\overline{m}$. Thus, $\overline{m} : H^{<\Omega} \to \Delta(S \times H^{<\Omega})$ is well-defined. The proof of the fact that $\overline{m}$ is measurable is the same as the proof of the fact that $\overline{\psi}$ is measurable in the proof of Proposition 1.

Second, I show that $m^{\mathrm{MZ}} : (T^{\mathrm{MZ}}, \mathcal{T}^{\mathrm{MZ}}) \to (\Delta(S \times T^{\mathrm{MZ}}), (S \times T^{\mathrm{MZ}})_\Delta)$ is a well-defined measurable map in two sub-steps. The proof is similar to the proof of the fact that $\psi : (T^1, \mathcal{T}^1) \to (\Delta(S \times T^0), (\mathcal{S} \times \mathcal{T}^0)_\Delta)$ in the proof of Proposition 1.

The first sub-step shows that each $m^{\mathrm{MZ}}(h)$ is a probability measure on $S \times T^{\mathrm{MZ}}$. For any $h \in T^{\mathrm{MZ}}$, there exists $\overline{h} \in H^{<\Omega}$ such that, irrespective of the choice of $\overline{h}$,

$$m^{\mathrm{MZ}}(h) \circ \pi_k^{-1} = (\overline{m}(\overline{h}) \circ \overline{\pi}_\omega^{-1}) \circ \pi_k^{-1} = \overline{m}(\overline{h}) \circ (\overline{\pi}_\omega^{-1} \circ \pi_k^{-1}) \text{ for all } k \in \mathbb{N}.$$

The set function $m^{\mathrm{MZ}}(h)$ is defined on $\sigma\left(\bigcup_{k\in\mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})\right)$ if $\overline{m}(\overline{h})$ is defined on $\overline{\pi}_\omega^{-1}\left(\sigma\left(\bigcup_{k\in\mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})\right)\right)$. This is the case because

$$\overline{\pi}_\omega^{-1}\left(\sigma\left(\bigcup_{k\in\mathbb{N}} \pi_k^{-1}(\mathcal{Z}^{<k})\right)\right) = \sigma\left(\bigcup_{k\in\mathbb{N}} \overline{\pi}_k^{-1}(\mathcal{Z}^{<k})\right)$$

and because $\overline{m}(\overline{h})$ is defined on the σ-algebra $\bigcup_{\alpha\in[1,\Omega]} \overline{\pi}_\alpha^{-1}(\mathcal{Z}^{<\alpha})$, which includes the algebra $\bigcup_{k\in\mathbb{N}} \overline{\pi}_k^{-1}(\mathcal{Z}^{<k})$. Then, since $m^{\mathrm{MZ}}(h) = \overline{h}^{\omega}$ on $\mathcal{S} \times \mathcal{T}^{\mathrm{MZ}}$, it is a probability measure on $S \times T^{\mathrm{MZ}}$.

The second sub-step shows that m^{MZ} is measurable. For any $k \in \mathbb{N}$, $\pi_k^{-1}(E)$ with $E \in \mathcal{Z}^{<k}$, and for any $p \in [0,1]$,

$$\begin{aligned}(m^{\mathrm{MZ}})^{-1}(\{\mu \in \Delta(S \times T^{\mathrm{MZ}}) \mid \mu(\pi_k^{-1}(E)) \geq p\}) &= \{h \in T^{\mathrm{MZ}} \mid \psi(h)(\pi_k^{-1}(E)) \geq p\} \\ &= \{h \in T^{\mathrm{MZ}} \mid h^k(E) \geq p\} \in \mathcal{T}^{\mathrm{MZ}}.\end{aligned}$$

A.2.3 Proof of Theorem 3

Take the type space $\overrightarrow{T^{\mathrm{MZ}}}$ constructed in the main text. The proof is in two steps. The first step establishes that, for any type space $\overrightarrow{T}$, the hierarchy map $\eta : \overrightarrow{T} \to \overrightarrow{T^{\mathrm{MZ}}}$ is a unique morphism. That is, $\overrightarrow{T^{\mathrm{MZ}}}$ is terminal. Then, terminal type spaces $\overrightarrow{T^{\mathrm{MZ}}}$ and $\overrightarrow{T^{\mathrm{IM}}}$ are isomorphic. The second step shows that $\overrightarrow{T^{\mathrm{MZ}}} = \overrightarrow{T^{\mathrm{IM}}}$.

Step 1. The measurable maps $\eta_i : T_i \to T_i^{\mathrm{MZ}}$ form a morphism. This is because $(m_i^{\mathrm{MZ}} \circ \eta_i)(\cdot) = m_i(\cdot) \circ \eta_{-i}^{-1}$ for all $i \in I$. Now, I show that the morphism is unique. By Remark 1, it suffices to show that the hierarchy map η^{MZ} is injective. Indeed, I show that the type morphism $\eta_i^{\mathrm{MZ}} : T_i^{\mathrm{MZ}} \to T_i^{\mathrm{MZ}}$ is the identity for each $i \in I$ (as well as $\eta_0 = \mathrm{id}_S$) by proving that $(\eta_i^{\mathrm{MZ}})^{<k} : T_i^{\mathrm{MZ}} \to H_i^{<k}$ is the projection for each $(i,k) \in I \times \mathbb{N} \setminus \{1\}$. For $k = 2$, $(\eta_i^{\mathrm{MZ}})^{<2}(t_i^{\mathrm{MZ}}) = m_i^{\mathrm{MZ}}(t_i^{\mathrm{MZ}}) \circ (\pi_S^{S \times T_j^{\mathrm{MZ}}})^{-1} = (t_i^{\mathrm{MZ}})^1$. Assume the induction hypothesis for k. For $k+1$, $(\eta_i^{\mathrm{MZ}})^{<k+1}(t_i^{\mathrm{MZ}}) = m_i^{\mathrm{MZ}}(t_i^{\mathrm{MZ}}) \circ (\pi_{-i}^k)^{-1} = (t_i^{\mathrm{MZ}})^{<k+1}$.

Step 2. The proof in this step is similar to that of Theorem 2. Fix $i \in I$. I start with showing $T_i^{\mathrm{MZ}} \subseteq T_i^{\mathrm{IM}}$. Indeed, it follows from the fact that, for the unique morphism $\eta = (\mathrm{id}_S, \eta_1, \eta_2)$ from a type space $\overrightarrow{T^{\mathrm{MZ}}}$ to a terminal type space $\overrightarrow{T^{\mathrm{IM}}}$, the hierarchy map $\eta_i : T_i^{\mathrm{MZ}} \to T_i^{\mathrm{IM}}$ of $\overrightarrow{T^{\mathrm{MZ}}}$ is an inclusion map.

Next, I show $T_i^{\mathrm{IM}} \subseteq T_i^{\mathrm{MZ}}$. For any $h_i \in T_i^{\mathrm{IM}}$, there are $\overrightarrow{T}$ and $t_i \in T_i$ such that $h_i = \eta_i(t_i) \in T_i^{\mathrm{MZ}}$, where the last set belonging follows from the argument in the main text that any belief hierarchy at some type (in some type space) is contained in T_i^{MZ}.

So far, I have shown that $T_i^{\mathrm{MZ}} = T_i^{\mathrm{IM}}$. Then, $\mathcal{T}_i^{\mathrm{MZ}} = \mathcal{T}^0 \cap T_i^{\mathrm{MZ}} = \mathcal{T}^0 \cap T_i^{\mathrm{IM}} = \mathcal{T}_i^{\mathrm{IM}}$. Also, $m_i^{\mathrm{IM}} = m_i^{\mathrm{MZ}}$ follows because the unique morphism from from $\overrightarrow{T^{\mathrm{IM}}}$ to $\overrightarrow{T^{\mathrm{MZ}}}$ is the identity morphism. Hence, $\overrightarrow{T^{\mathrm{MZ}}} = \overrightarrow{T^{\mathrm{IM}}}$. □

A.2.4 Proof of Theorem 4 (1)

The proof is in two steps. The first step establishes $T^{\mathrm{MZ}} \subseteq T^{\mathrm{RC}}$. The second does $T^{\mathrm{RC}} \subseteq T^{\mathrm{MZ}}$.

Step 1. The first step shows that $T^{\mathrm{MZ}} \subseteq T^{\mathrm{RC}}$. It suffices to show that $T^{\mathrm{MZ}} \subseteq T^{\alpha}$ for any ordinal $\alpha \geq 1$. For $\alpha = 1$, the restriction $\psi|_{T^{\mathrm{MZ}}}$ satisfies $\psi|_{T^{\mathrm{MZ}}}(h)(\cdot) = m(h)((\cdot) \cap (S \times T^{\mathrm{MZ}})) \in \Delta(S \times T^0)$. Thus, $T^{\mathrm{MZ}} \subseteq T^1$. For any ordinal (either a successor ordinal or a non-zero limit ordinal) $\alpha \geq 2$, suppose that $T^{\mathrm{MZ}} \subseteq T^{\beta}$ for all $\beta < \alpha$, i.e., $T^{\mathrm{MZ}} \subseteq T^{<\alpha}$. Then, by construction, $T^{\mathrm{MZ}} \subseteq T^{\alpha}$.

Step 2. The second step shows that $T^{\mathrm{RC}} \subseteq T^{\mathrm{MZ}}$. It suffices to show (i) that T^{RC} induces a type space $\overrightarrow{T^{\mathrm{RC}}}$ and (ii) that the hierarchy map $\eta_i : (T_i^{\mathrm{RC}}, \mathcal{T}_i^{\mathrm{RC}}) \to (T^{\mathrm{MZ}}, \mathcal{T}^{\mathrm{MZ}})$ is an inclusion map. Thus, I first construct a type space $\overrightarrow{T^{\mathrm{RC}}}$ as in Section 4 (i.e., as in the construction of $\overrightarrow{T^{\mathrm{BD}}}$). Namely, denote by $\overrightarrow{T^{\mathrm{RC}}} = \langle (T_i^{\mathrm{RC}}, \mathcal{T}_i^{\mathrm{RC}})_{i \in I_0}, (m_i^{\mathrm{RC}})_{i \in I} \rangle$ the type space induced by $\langle (T_i^{\mathrm{RC}}, \mathcal{T}_i^{\mathrm{RC}}), m_i^{\mathrm{RC}} \rangle := \langle (T^{\mathrm{RC}}, \mathcal{T}^{\mathrm{RC}}), \psi^*_{T^{\mathrm{RC}}} \rangle$ for each $i \in I$. Then, by construction, since $T^{\mathrm{RC}} \subseteq H^{<\omega}$ and since $\psi^*_{T^{\mathrm{RC}}}(h) \circ \pi_k^{-1} = h^k$ for all $k \in \mathbb{N}$, where $\pi_k : S \times T^{\mathrm{RC}} \to Z^{<k}$ is the projection, the hierarchy map $\eta_i : (T_i^{\mathrm{RC}}, \mathcal{T}_i^{\mathrm{RC}}) \to (T_i^{\mathrm{MZ}}, \mathcal{T}_i^{\mathrm{MZ}})$ is an inclusion map. □

A.2.5 Proof of Theorem 4 (2)

The proof consists of three parts. The first part shows that T^{MZ} is belief-closed. The second part establishes that $T \subseteq T^{\mathrm{MZ}}$ for any belief-closed set T by showing that there is a type space $\overrightarrow{T}$ such that its hierarchy map $\eta : \overrightarrow{T} \to \overrightarrow{T^{\mathrm{MZ}}}$ is an inclusion map. The third part shows that $\psi^*_{T^{\mathrm{MZ}}} = m^{\mathrm{MZ}}$ is a measurable isomorphism.

Part 1. The first part shows that T^{MZ} is belief-closed. Let $\pi_k : S \times T^{\mathrm{MZ}} \to Z^{<k}$ be the projection. The mapping $m^{\mathrm{MZ}} : (T^{\mathrm{MZ}}, \mathcal{T}^{\mathrm{MZ}}) \to (\Delta(S \times T^{\mathrm{MZ}}), (\mathcal{S} \times \mathcal{T}^{\mathrm{MZ}})_\Delta)$ satisfies

$$m^{\mathrm{MZ}}(h) \circ \pi_k^{-1} = h^k \text{ for all } k \in \mathbb{N}.$$

Thus, I can identify $\psi^*|_{T^{\mathrm{MZ}}} = m^{\mathrm{MZ}}$, which means that T^{MZ} is belief-closed.

Part 2. The second part shows that if $(T, \mathcal{T})$ is belief-closed then $T \subseteq T^{\mathrm{MZ}}$ in the following two sub-steps. Let $(T, \mathcal{T})$ be belief-closed. As in the proof of Theorem 1, the first sub-step defines a type space $\overrightarrow{T} = \langle (T_i, \mathcal{T}_i)_{i \in I_0}, (m_i)_{i \in I} \rangle$ as follows: $(T_i, \mathcal{T}_i) = (T, \mathcal{T})$ for each $i \in I$; and $m_i = \psi^*_T : (T_i, \mathcal{T}_i) \to (\Delta(S \times T_j), (\mathcal{S} \times \mathcal{T}_j)_\Delta)$ for each $i \in I$. The second sub-step shows that, for each $h_i \in T_i$, the recursively coherent transfinite belief hierarchy $\overline{\eta}_i(h_i)$ satisfies $\eta_i(h_i) = h_i$, implying that $(T_i, \mathcal{T}_i) \to (T^{\mathrm{MZ}}, \mathcal{T}^{\mathrm{MZ}})$ is an inclusion map.

To prove the second sub-step, it suffices to show by induction that the hierarchy map $\eta_i^{<\beta} : (T_i, \mathcal{T}_i) \to (H^{<\beta}, \mathcal{H}^{<\beta})$ satisfies $\eta_i^{<\beta}(h_i) = h_i^{<\beta} := (h_i^\gamma)_{\gamma \in [1,\beta)}$ for each $(\beta, i) \in [2, \omega] \times I$. Let $\beta = 2$. For each $i \in I$,

$$\begin{aligned}\eta_i^{<2}(h_i)(\cdot) &= m_i(h_i) \circ (\pi_S^{S \times T_j})^{-1}(\cdot) = \psi^*(h_i) \circ (\pi_S^{S \times T_j})^{-1}(\cdot) \\ &= \psi^*(h_i)((\cdot) \times T^0 \cap S \times T_j) = \psi(h_i) \circ \pi_1^{-1}(\cdot) = h_i^{<2}(\cdot).\end{aligned}$$

For a successor ordinal $\beta = \gamma + 1 \in [3, \omega)$, assume $\eta_i^{<\gamma}(h_i) = h_i^{<\gamma}$ for each $i \in I$. Fix $i \in I$. Since $(\mathrm{id}_S, \eta_j^{<\gamma}) = \pi_\gamma|_{S \times T_j} : (S \times T_j, \mathcal{S} \times \mathcal{T}_j) \to (Z^{<\gamma}, \mathcal{Z}^{<\gamma})$,

$$\begin{aligned}\eta_i^{<\beta}(h_i) &= (\eta_i^{<\gamma}(h_i), m_i(h_i) \circ (\pi_\gamma|_{S \times T_j})^{-1}) = (h_i^{<\gamma}, m_i(h_i) \circ (\pi_\gamma|_{S \times T_j})^{-1}) \\ &= (h_i^{<\gamma}, \psi^*(h_i) \circ (\pi_\gamma|_{S \times T_j})^{-1}) = (h_i^{<\gamma}, \psi(h_i) \circ \pi_\gamma^{-1}) = (h_i^{<\gamma}, h_i^\gamma) = h_i^{<\beta}.\end{aligned}$$

For the limit ordinal $\beta = \omega$, assume $\eta_i^{<\gamma}(h_i) = h_i^{<\gamma}$ for all $(\gamma, i) \in [2, \beta) \times I$. Fix $i \in I$. On the one hand, $\eta_i^{<\beta}$ is a unique map from $(T_i, \mathcal{T}_i)$ to $(H^{<\beta}, \mathcal{H}^{<\beta})$ satisfying $\pi_\gamma^\beta \circ \eta_i^{<\beta} = \eta_i^{<\gamma}$ for all $\gamma \in [2, \beta)$. Thus, $\pi_\gamma^\beta \circ \eta_i^{<\beta}(h_i) = \eta_i^{<\gamma}(h_i)$. On the other hand, $h_i^{<\beta}$ is a unique element of $H^{<\beta}$ satisfying $\pi_\gamma^\beta \circ h_i^{<\beta} = h_i^{<\gamma}$ for all $\gamma \in [2, \beta)$. Thus, for any h_i, $\eta_i^{<\beta}(h_i) = h_i^{<\beta}$.

Part 3. For the third part, it suffices to show that $m^{\mathrm{MZ}} = \psi^*_{T^{\mathrm{MZ}}}$, as it follows from the terminality of $\overrightarrow{T^{\mathrm{MZ}}}$ that $m^{\mathrm{MZ}} = \psi^*_{T^{\mathrm{MZ}}}$ is an isomorphism. However, the first step of this proof has established that $m^{\mathrm{MZ}} = \psi^*_{T^{\mathrm{MZ}}}$. □

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