Who to Listen to?: A Model of Endogenous Delegation Online Supplementary Appendix

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This Online Appendix is structured as follows. First, Section A.1 provides the proof of Theorem 1. Each subsection corresponds to each part of the theorem.

Second, Section A.2 proves Proposition 1. Specifically, Section A.2.1 formulates the Lagrangian and the first-order conditions. Then, Section A.2.2 provides the proof.

Third, Section A.3 discusses the monotonicity assumption.

Fourth, Section A.4 shows that the optimal allocations do not entail any money burning or randomization. Section A.4.1 introduces money burning. Section A.4.2 introduces randomization and shows that if money burning is not used then randomization is not used either. Section A.4.3 shows that, for the case in which the agents' type spaces do not overlap ($\overline{\theta}_1 \leq \underline{\theta}_2$), the optimal allocation does not use any money burning and consequently randomization. Section A.4.4 establishes the result for the special case in which the agents' type spaces fully overlap: $\Theta_1 = \Theta_2 = [0, 1]$.

Fifth, Section A.5 provides the proof of the fact that the communication protocol discussed in Section 4.2 is IC and yields a better social welfare.

Throughout this Online Appendix, for ease of exposition, denote by $f_i(\cdot) := \frac{1}{\overline{\theta}_i - \underline{\theta}_i}$ the probability density function of the uniform distribution on Θ_i. Likewise, denote by $F_i(\theta_i) := \frac{\theta_i - \underline{\theta}_i}{\overline{\theta}_i - \underline{\theta}_i}$ the cumulative distribution function of the uniform distribution on Θ_i.

A.1 Proof of Theorem 1

Throughout the proof of Theorem 1, we interchangeably use $\underline{\theta}_1 = 0$ and $\overline{\theta}_2 = 1$, respectively.

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A.1.1 Part (1)

Assume $\overline{\theta}_1 \leq \gamma \mathbb{E}_{\theta_1}[\theta_1] + (1-\gamma)\mathbb{E}_{\theta_2}[\theta_2] \leq \underline{\theta}_2$. We consider the relaxed problem where the monotonicity constraint is ignored, and show that the constant allocation is optimal among such a class of allocations.

The proof consists of seven steps. The first step rewrites the relaxed problem by substituting the local IC constraints into the objective function. The second step formulates the Lagrangian. Denote by Λ_i the Lagrange multiplier associated with agent i's local IC constraint. The third step examines the first-order conditions. In the fourth to seven steps, we substitute $a^*(\cdot) = \gamma \mathbb{E}_{\theta_1}[\theta_1] + (1-\gamma)\mathbb{E}_{\theta_2}[\theta_2]$ and find the Lagrange multipliers (Λ_1, Λ_2) such that the first-order conditions are met.

Step 1. Consider the relaxed problem in which the monotonicity constraint is ignored. Thus, the problem is to maximize the sum of the agents' ex-ante utilities subject to their local IC constraints. For agent 1, let the "reference" type of the local IC constraint be $\overline{\theta}_1$. For agent 2, let the "reference" type be $\underline{\theta}_2$. Then, we show that the relaxed problem can be rewritten as follows:

$$\max_{a(\cdot)} \gamma \left(U_1(\overline{\theta}_1) - 2\mathbb{E}_\theta \left[(a(\theta) - \theta_1)(\theta_1 - \underline{\theta}_1) \right] \right) + (1-\gamma) \left(U_2(\underline{\theta}_2) + 2\mathbb{E}_\theta \left[(a(\theta) - \theta_2)(\overline{\theta}_2 - \theta_2) \right] \right)$$

subject to $U_1(\theta_1) = U_1(\overline{\theta}_1) - 2\int_{\theta_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\tau_1, \theta_2) - \tau_1 \right] d\tau_1$ for each $\theta_1 \in [\underline{\theta}_1, \overline{\theta}_1]$ and

$$U_2(\theta_2) = U_2(\underline{\theta}_2) + 2\int_{\underline{\theta}_2}^{\theta_2} \mathbb{E}_{\theta_1}\left[a(\theta_1, \tau_2) - \tau_2 \right] d\tau_2 \text{ for each } \theta_2 \in [\underline{\theta}_2, \overline{\theta}_2].$$

Since we ignore the monotonicity constraint, the local IC constraints are sufficient. Thus, it suffices to rewrite the objective function as above. Using the local IC constraint, we rewrite agent i's ex-ante expected utility as follows. For agent 1,

$$\begin{aligned}
\mathbb{E}_{\theta_1}\left[U_1(\theta_1) \right] &= \frac{1}{\overline{\theta}_1 - \underline{\theta}_1} \int_{\underline{\theta}_1}^{\overline{\theta}_1} U_1(\theta_1) d\theta_1 = U_1(\overline{\theta}_1) - \frac{2}{\overline{\theta}_1 - \underline{\theta}_1} \int_{\underline{\theta}_1}^{\overline{\theta}_1} \int_{\theta_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\tau_1, \theta_2) - \tau_1 \right] d\tau_1 d\theta_1 \\
&= U_1(\overline{\theta}_1) - \frac{2}{\overline{\theta}_1 - \underline{\theta}_1} \int_{\underline{\theta}_1}^{\overline{\theta}_1} \int_{\underline{\theta}_1}^{\tau_1} \mathbb{E}_{\theta_2}\left[a(\tau_1, \theta_2) - \tau_1 \right] d\theta_1 d\tau_1 \\
&= U_1(\overline{\theta}_1) - \frac{2}{\overline{\theta}_1 - \underline{\theta}_1} \int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\tau_1, \theta_2) - \tau_1 \right] (\tau_1 - \underline{\theta}_1) d\tau_1 \\
&= U_1(\overline{\theta}_1) - 2\mathbb{E}_\theta \left[(a(\theta) - \theta_1)(\theta_1 - \underline{\theta}_1) \right].
\end{aligned}$$

Similarly for agent 2, we have

$$\mathbb{E}_{\theta_2}\left[U_2(\theta_2)\right] = U_2(\underline{\theta}_2) + 2\mathbb{E}_\theta\left[(a(\theta)-\theta_2)(\overline{\theta}_2-\theta_2)\right].$$

Then, the reformulation follows because the objective function is written as $\gamma\mathbb{E}_{\theta_1}\left[U_1(\theta_1)\right] + (1-\gamma)\mathbb{E}_{\theta_2}\left[U_2(\theta_2)\right]$.

Step 2. To formulate the Lagrangian of the problem formulated in Step 1, we denote by Λ_i the Lagrange multiplier associated with agent i's (local) IC constraint. Theoretically, the Lagrange multiplier Λ_i is a function of bounded variation. Without loss of generality, we normalize Λ_i by setting $\Lambda_1(\underline{\theta}_1)=0$ and $\Lambda_2(\overline{\theta}_2)=0$.

In each of Steps 4 to 7, we conjecture and verify a specific functional form of Λ_i, from the first-order condition to be found in Step 3.[1] In particular, in every step, the specific Λ_i is shown to have a density function λ_i on $[\underline{\theta}_i, \overline{\theta}_i]$. Thus, this justifies, as we assume in step, that each Λ_i has density λ_i. Then, each Λ_i can be written as follows:

$$\Lambda_1(\theta_1) = \int_{\underline{\theta}_1}^{\theta_1} \lambda_1(\tau_1)d\tau_1 \text{ for each } \theta_1 \in [\underline{\theta}_1, \overline{\theta}_1], \text{ and}$$
$$\Lambda_2(\theta_2) = -\int_{\theta_2}^{\overline{\theta}_2} \lambda_2(\tau_2)d\tau_2 \text{ for each } \theta_2 \in [\underline{\theta}_2, \overline{\theta}_2].$$

With this in mind, we define the Lagrangian as

$$\begin{aligned}\mathcal{L} :=& \gamma\left(U_1(\overline{\theta}_1) - 2\mathbb{E}_\theta\left[(a(\theta)-\theta_1)(\theta_1-\underline{\theta}_1)\right]\right) + (1-\gamma)\left(U_2(\underline{\theta}_2) + 2\mathbb{E}_\theta\left[(a(\theta)-\theta_2)(\overline{\theta}_2-\theta_2)\right]\right)\\ &+ \int_{\underline{\theta}_1}^{\overline{\theta}_1}\left(U_1(\theta_1) - U_1(\overline{\theta}_1) + 2\int_{\theta_1}^{\overline{\theta}_1}\mathbb{E}_{\theta_2}\left[a(\tau_1,\theta_2)-\tau_1\right]d\tau_1\right)d\Lambda_1(\theta_1)\\ &+ \int_{\underline{\theta}_2}^{\overline{\theta}_2}\left(U_2(\theta_2) - U_2(\underline{\theta}_2) - 2\int_{\underline{\theta}_2}^{\theta_2}\mathbb{E}_{\theta_1}\left[a(\theta_1,\tau_2)-\tau_2\right]d\tau_2\right)d\Lambda_2(\theta_2).\end{aligned}$$

[1] For instance, in Step 4, we conjecture and verify the following: for agent 1, Λ_1 is given by (A.5) and consequently λ_1 is given by (A.6); and for agent 2, Λ_2 is given by (A.7) and consequently λ_2 is given by (A.8).

We show that the Lagrangian can be rewritten as:

$$
\begin{aligned}
\mathcal{L} =& -\mathbb{E}_{\theta_2}\left[(a(\bar{\theta}_1,\theta_2)-\bar{\theta}_1)^2\right](\gamma-\Lambda_1(\bar{\theta}_1)) - 2\gamma\mathbb{E}_\theta\left[(a(\theta)-\theta_1)(\theta_1-\underline{\theta}_1)\right] \\
& -\int_{\underline{\theta}_1}^{\bar{\theta}_1}\mathbb{E}_{\theta_2}\left[(a(\theta)-\theta_1)^2\right]\lambda_1(\theta_1)d\theta_1 + 2\int_{\underline{\theta}_1}^{\bar{\theta}_1}\mathbb{E}_{\theta_2}\left[a(\theta)-\theta_1\right]\Lambda_1(\theta_1)d\theta_1 \\
& -\mathbb{E}_{\theta_1}\left[(a(\theta_1,\underline{\theta}_2)-\underline{\theta}_2)^2\right](1-\gamma+\Lambda_2(\underline{\theta}_2)) + 2(1-\gamma)\mathbb{E}_\theta\left[(a(\theta)-\theta_2)(\bar{\theta}_2-\theta_2)\right] \\
& -\int_{\underline{\theta}_2}^{\bar{\theta}_2}\mathbb{E}_{\theta_1}\left[(a(\theta)-\theta_2)^2\right]\lambda_2(\theta_2)d\theta_2 + 2\int_{\underline{\theta}_2}^{\bar{\theta}_2}\mathbb{E}_{\theta_1}\left[a(\theta)-\theta_2\right]\Lambda_2(\theta_2)d\theta_2. \qquad \text{(A.1)}
\end{aligned}
$$

To see this, the part of the Lagrangian that corresponds to agent 1's local IC constraint is rewritten as:

$$
\begin{aligned}
&\int_{\underline{\theta}_1}^{\bar{\theta}_1}\left(U_1(\theta_1)-U_1(\bar{\theta}_1)+2\int_{\theta_1}^{\bar{\theta}_1}\mathbb{E}_{\theta_2}\left[a(\tau_1,\theta_2)-\tau_1\right]d\tau_1\right)\lambda_1(\theta_1)d\theta_1 \\
=& -U_1(\bar{\theta}_1)\Lambda_1(\bar{\theta}_1) + \int_{\underline{\theta}_1}^{\bar{\theta}_1}U_1(\theta_1)\lambda_1(\theta_1)d\theta_1 + 2\int_{\underline{\theta}_1}^{\bar{\theta}_1}\int_{\theta_1}^{\bar{\theta}_1}\mathbb{E}_{\theta_2}\left[a(\tau_1,\theta_2)-\tau_1\right]d\tau_1\lambda_1(\theta_1)d\theta_1 \\
=& -U_1(\bar{\theta}_1)\Lambda_1(\bar{\theta}_1) + \int_{\underline{\theta}_1}^{\bar{\theta}_1}U_1(\theta_1)\lambda_1(\theta_1)d\theta_1 + 2\int_{\underline{\theta}_1}^{\bar{\theta}_1}\mathbb{E}_{\theta_2}\left[a(\tau_1,\theta_2)-\tau_1\right]\int_{\underline{\theta}_1}^{\tau_1}\lambda_1(\theta_1)d\theta_1d\tau_1 \\
=& -U_1(\bar{\theta}_1)\Lambda_1(\bar{\theta}_1) + \int_{\underline{\theta}_1}^{\bar{\theta}_1}U_1(\theta_1)\lambda_1(\theta_1)d\theta_1 + 2\int_{\underline{\theta}_1}^{\bar{\theta}_1}\mathbb{E}_{\theta_2}\left[a(\theta)-\theta_1\right]\Lambda_1(\theta_1)d\theta_1.
\end{aligned}
$$

Similarly for agent 2, we have:

$$
\begin{aligned}
&\int_{\underline{\theta}_2}^{\bar{\theta}_2}\left(U_2(\theta_2)-U_2(\underline{\theta}_2)-2\int_{\underline{\theta}_2}^{\theta_2}\mathbb{E}_{\theta_1}\left[a(\theta_1,\tau_2)-\tau_2\right]d\tau_2\right)\lambda_2(\theta_2)d\theta_2 \\
=&U_2(\underline{\theta}_2)\Lambda_2(\underline{\theta}_2) + \int_{\underline{\theta}_2}^{\bar{\theta}_2}U_2(\theta_2)\lambda_2(\theta_2)d\theta_2 + 2\int_{\underline{\theta}_2}^{\bar{\theta}_2}\mathbb{E}_{\theta_1}\left[a(\theta)-\theta_2\right]\Lambda_2(\theta_2)d\theta_2.
\end{aligned}
$$

Substituting $U_i(\theta_i) = -\mathbb{E}_{\theta_{-i}}\left[(a(\theta_i,\theta_{-i})-\theta_i)^2\right]$, the Lagrangian reduces to Expression (A.1).

Step 3. We take the point-wise first-order condition for each $a(\theta)$. For any θ, the first-order condition is

$$\frac{1}{\overline{\theta}_2-\underline{\theta}_2}\left\{\gamma\frac{\theta_1-\underline{\theta}_1}{\overline{\theta}_1-\underline{\theta}_1}+\lambda_1(\theta_1)(a(\theta)-\theta_1)-\Lambda_1(\theta_1)\right\}$$
$$+\frac{1}{\overline{\theta}_1-\underline{\theta}_1}\left\{-(1-\gamma)\frac{\overline{\theta}_2-\theta_2}{\overline{\theta}_2-\underline{\theta}_2}+\lambda_2(\theta_2)(a(\theta)-\theta_2)-\Lambda_2(\theta_2)\right\}$$
$$+\frac{(a(\overline{\theta}_1,\theta_2)-\overline{\theta}_1)(\gamma-\Lambda_1(\overline{\theta}_1))}{(\overline{\theta}_1-\underline{\theta}_1)(\overline{\theta}_2-\underline{\theta}_2)}\mathbb{I}(\theta_1=\overline{\theta}_1)+\frac{(a(\theta_1,\underline{\theta}_2)-\underline{\theta}_2)(1-\gamma+\Lambda_2(\underline{\theta}_2))}{(\overline{\theta}_1-\underline{\theta}_1)(\overline{\theta}_2-\underline{\theta}_2)}\mathbb{I}(\theta_2=\underline{\theta}_2)=0. \tag{A.2}$$

From now on, we find (Λ_1, Λ_2) such that the first-order conditions are satisfied at

$$a^*=\gamma\mathbb{E}_{\theta_1}[\theta_1]+(1-\gamma)\mathbb{E}_{\theta_2}[\theta_2]=\frac{\gamma\overline{\theta}_1+(1-\gamma)(1+\underline{\theta}_2)}{2}.$$

For the rest of the proof, we consider the following four cases: (i) $\overline{\theta}_1 < a^* < \underline{\theta}_2$; (ii) $\overline{\theta}_1 = a^* < \underline{\theta}_2$; (iii) $\overline{\theta}_1 < a^* = \underline{\theta}_2$; and (iv) $\overline{\theta}_1 = a^* = \underline{\theta}_2 (= 1-\gamma)$.

Step 4. First, suppose $\overline{\theta}_1 < a^* < \underline{\theta}_2$. We conjecture and verify $\Lambda_1(\overline{\theta}_1) = \gamma$ and $\Lambda_2(\underline{\theta}_2) = -(1-\gamma)$. Then, by the first-order conditions, there exist constants α_1 and α_2 such that

$$\alpha_1 =\gamma\frac{\theta_1-\underline{\theta}_1}{\overline{\theta}_1-\underline{\theta}_1}+\lambda_1(\theta_1)(a^*-\theta_1)-\Lambda_1(\theta_1), \tag{A.3}$$

$$\alpha_2 =-(1-\gamma)\frac{\overline{\theta}_2-\theta_2}{\overline{\theta}_2-\underline{\theta}_2}+\lambda_2(\theta_2)(a^*-\theta_2)-\Lambda_2(\theta_2), \text{ and} \tag{A.4}$$

$$0 =\frac{\alpha_1}{\overline{\theta}_2-\underline{\theta}_2}+\frac{\alpha_2}{\overline{\theta}_1-\underline{\theta}_1}.$$

For agent 1, we show that

$$\Lambda_1(\theta_1)=\frac{\theta_1-\underline{\theta}_1}{a^*-\theta_1}\left(-\frac{\gamma}{2}\frac{\theta_1-\underline{\theta}_1}{\overline{\theta}_1-\underline{\theta}_1}+\alpha_1\right), \text{ where } \alpha_1=\gamma\left(\frac{1}{2}+\frac{a^*-\overline{\theta}_1}{\overline{\theta}_1-\underline{\theta}_1}\right).$$

That is, we show:

$$\Lambda_1(\theta_1)=\gamma\frac{(\theta_1-\underline{\theta}_1)(2a^*-\overline{\theta}_1-\theta_1)}{2(a^*-\theta_1)(\overline{\theta}_1-\underline{\theta}_1)}. \tag{A.5}$$

It can be seen that $\Lambda_1(\underline{\theta}_1)=0$, $\Lambda_1(\overline{\theta}_1)=\gamma$, and that the function Λ_1 has its density:

$$\lambda_1(\theta_1)=\gamma\frac{(a^*-\theta_1)^2+(a^*-\underline{\theta}_1)(a^*-\overline{\theta}_1)}{2(\overline{\theta}_1-\underline{\theta}_1)(a^*-\theta_1)^2}\geq 0. \tag{A.6}$$

For agent 2, we show that

$$\Lambda_2(\theta_2) = -\frac{\overline{\theta}_2 - \theta_2}{a^* - \theta_2}\left(\frac{(1-\gamma)(\overline{\theta}_2 - \theta_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)} + \alpha_2\right), \text{ where } \alpha_2 = -(1-\gamma)\left(\frac{1}{2} - \frac{a^* - \underline{\theta}_2}{\overline{\theta}_2 - \underline{\theta}_2}\right).$$

That is, we show:

$$\Lambda_2(\theta_2) = (1-\gamma)\frac{(\overline{\theta}_2 - \theta_2)(\theta_2 + \underline{\theta}_2 - 2a^*)}{2(a^* - \theta_2)(\overline{\theta}_2 - \underline{\theta}_2)}. \tag{A.7}$$

It can be seen that $\Lambda_2(\underline{\theta}_2) = -(1-\gamma)$, $\Lambda_2(\overline{\theta}_2) = 0$, and that the function Λ_2 has its density:

$$\lambda_2(\theta_2) = (1-\gamma)\frac{(a^* - \theta_2)^2 + (a^* - \underline{\theta}_2)(a^* - \overline{\theta}_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)(a^* - \theta_2)^2} \geq 0. \tag{A.8}$$

Moreover, we have

$$\frac{\alpha_1}{\overline{\theta}_2 - \underline{\theta}_2} + \frac{\alpha_2}{\overline{\theta}_1 - \underline{\theta}_1} = \frac{2a^* - (\gamma\underline{\theta}_1 + (1-\gamma)(1 + \underline{\theta}_2))}{2(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)} = 0.$$

We start with agent 1. To prove Expression (A.5), observe that Expression (A.3) is a linear first-order differential equation. Since we have

$$\frac{d}{d\theta_1}\Lambda_1(\theta_1)(a^* - \theta_1) = \lambda_1(\theta_1)(a^* - \theta_1) - \Lambda_1(\theta_1) = \alpha_1 - \gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1},$$

integrating both sides from $\underline{\theta}_1$ to θ_1 yields

$$\Lambda_1(\theta_1)(a^* - \theta_1) = \alpha_1(\theta_1 - \underline{\theta}_1) - \frac{\gamma}{2}\frac{(\theta_1 - \underline{\theta}_1)^2}{\overline{\theta}_1 - \underline{\theta}_1}.$$

Hence, we have

$$\Lambda_1(\theta_1) = -\frac{\gamma(\theta_1 - \underline{\theta}_1)^2}{2(\overline{\theta}_1 - \underline{\theta}_1)(a^* - \theta_1)} + \alpha_1\frac{\theta_1 - \underline{\theta}_1}{a^* - \theta_1} = \frac{\theta_1 - \underline{\theta}_1}{a^* - \theta_1}\left(-\frac{\gamma}{2}\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \alpha_1\right).$$

To get $\Lambda_1(\overline{\theta}_1) = \gamma$, we must have

$$\gamma = \frac{\overline{\theta}_1 - \underline{\theta}_1}{a^* - \overline{\theta}_1}\left(-\frac{\gamma}{2} + \alpha_1\right), \text{ that is, } \alpha_1 = \gamma\left(\frac{1}{2} + \frac{a^* - \overline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1}\right).$$

Then, we obtain:

$$\Lambda_1(\theta_1) = \gamma\frac{\theta_1 - \underline{\theta}_1}{a^* - \theta_1}\left(-\frac{1}{2}\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \frac{1}{2} + \frac{a^* - \overline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1}\right) = \gamma\frac{(\theta_1 - \underline{\theta}_1)(2a^* - \overline{\theta}_1 - \theta_1)}{2(a^* - \theta_1)(\overline{\theta}_1 - \underline{\theta}_1)},$$

as desired. We can obtain Expression (A.6) by taking the derivative of Λ_1.

Next, we move on to agent 2. To prove Expression (A.7), observe that Expression (A.4) is a linear first-order differential equation. Similarly to the case of agent 1, we have

$$\frac{d}{d\theta_2}\Lambda_2(\theta_2)(a^* - \theta_2) = \lambda_2(\theta_2)(a^* - \theta_2) - \Lambda_2(\theta_2) = \alpha_2 + (1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2}.$$

Integrating both sides from θ_2 to $\overline{\theta}_2$ yields

$$-\Lambda_2(\theta_2)(a^* - \theta_2) = \alpha_2(\overline{\theta}_2 - \theta_2) + (1-\gamma)\frac{(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)}.$$

Hence, we have

$$\Lambda_2(\theta_2) = -(1-\gamma)\frac{(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)(a^* - \theta_2)} - \alpha_2\frac{\overline{\theta}_2 - \theta_2}{a^* - \theta_2} = -\frac{\overline{\theta}_2 - \theta_2}{a^* - \theta_2}\left(\frac{(1-\gamma)(\overline{\theta}_2 - \theta_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)} + \alpha_2\right).$$

To get $\Lambda_2(\underline{\theta}_2) = -(1-\gamma)$, we have to have

$$-(1-\gamma) = -\frac{\overline{\theta}_2 - \underline{\theta}_2}{a^* - \underline{\theta}_2}\left(\frac{1-\gamma}{2} + \alpha_2\right), \text{ that is, } \alpha_2 = -(1-\gamma)\left(\frac{1}{2} - \frac{a^* - \underline{\theta}_2}{\overline{\theta}_2 - \underline{\theta}_2}\right).$$

Then, we obtain:

$$\Lambda_2(\theta_2) = -(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{a^* - \theta_2}\left(\frac{\overline{\theta}_2 - \theta_2}{2(\overline{\theta}_2 - \underline{\theta}_2)} - \frac{1}{2} + \frac{a^* - \underline{\theta}_2}{\overline{\theta}_2 - \underline{\theta}_2}\right) = (1-\gamma)\frac{(\overline{\theta}_2 - \theta_2)(\theta_2 + \underline{\theta}_2 - 2a^*)}{2(a^* - \theta_2)(\overline{\theta}_2 - \underline{\theta}_2)},$$

as desired. We obtain Expression (A.8) by taking the derivative of Λ_2.

The Lagrangian reduces to:

$$\begin{aligned}\mathcal{L} =& \mathbb{E}_\theta\left[-\gamma(a(\theta) - \theta_1)^2\left(\frac{1}{2} + \frac{(a^* - \underline{\theta}_1)(a^* - \overline{\theta}_1)}{2(a^* - \theta_1)^2}\right) - \gamma(a(\theta) - \theta_1)\frac{(\theta_1 - \underline{\theta}_1)(\overline{\theta}_1 - \theta_1)}{(a^* - \theta_1)}\right] \\ &+ \mathbb{E}_\theta\left[-(1-\gamma)(a(\theta) - \theta_2)^2\left(\frac{1}{2} + \frac{(a^* - \underline{\theta}_2)(a^* - \overline{\theta}_2)}{2(a^* - \theta_2)^2}\right) - (1-\gamma)(a(\theta) - \theta_2)\frac{(\theta_2 - \underline{\theta}_2)(\overline{\theta}_2 - \theta_2)}{(a^* - \theta_2)}\right].\end{aligned}$$

The Lagrangian is a concave function in a. By construction, the first-order conditions are satisfied at a^*.

Step 5. Second, suppose $\overline{\theta}_1 = a^* < \underline{\theta}_2$. By Expression (A.2), the first-order condition reduces to:

$$
\begin{aligned}
&\frac{1}{\overline{\theta}_2-\underline{\theta}_2}\left\{\gamma\frac{\theta_1-\underline{\theta}_1}{\overline{\theta}_1-\underline{\theta}_1}+\lambda_1(\theta_1)(a^*-\theta_1)-\Lambda_1(\theta_1)\right\} \\
&+\frac{1}{\overline{\theta}_1-\underline{\theta}_1}\left\{-(1-\gamma)\frac{\overline{\theta}_2-\theta_2}{\overline{\theta}_2-\underline{\theta}_2}+\lambda_2(\theta_2)(a^*-\theta_2)-\Lambda_2(\theta_2)\right\} \\
&+\frac{(a^*-\underline{\theta}_2)}{(\overline{\theta}_1-\underline{\theta}_1)(\overline{\theta}_2-\underline{\theta}_2)}(1-\gamma+\Lambda_2(\underline{\theta}_2))\mathbb{I}(\theta_2=\underline{\theta}_2)=0.
\end{aligned}
$$

We conjecture and verify $\Lambda_2(\underline{\theta}_2)=-(1-\gamma)$. Then, there exist constants α_1 and α_2 with:

$$
\begin{aligned}
\alpha_1 &= \gamma\frac{\theta_1-\underline{\theta}_1}{\overline{\theta}_1-\underline{\theta}_1}+\lambda_1(\theta_1)(a^*-\theta_1)-\Lambda_1(\theta_1) \text{ for each } \theta_1\in[\underline{\theta}_1,\overline{\theta}_1], \\
\alpha_2 &= -(1-\gamma)\frac{\overline{\theta}_2-\theta_2}{\overline{\theta}_2-\underline{\theta}_2}+\lambda_2(\theta_2)(a^*-\theta_2)-\Lambda_2(\theta_2), \text{ and} \\
0 &= \frac{\alpha_1}{\overline{\theta}_2-\underline{\theta}_2}+\frac{\alpha_2}{\overline{\theta}_1-\underline{\theta}_1}.
\end{aligned}
$$

For agent 2, we have the same differential equation as in Step 4 (except that $a^*=\frac{\gamma\underline{\theta}_1+(1-\gamma)(1+\underline{\theta}_2)}{2}$ coincides exactly with $\overline{\theta}_1$). Thus,

$$
\Lambda_2(\theta_2)=-\frac{\overline{\theta}_2-\theta_2}{a^*-\theta_2}\left(\frac{(1-\gamma)(\overline{\theta}_2-\theta_2)}{2(\overline{\theta}_2-\underline{\theta}_2)}+\alpha_2\right), \text{ where } \alpha_2=-(1-\gamma)\left(\frac{1}{2}-\frac{a^*-\underline{\theta}_2}{\overline{\theta}_2-\underline{\theta}_2}\right).
$$

That is,

$$
\Lambda_2(\theta_2)=\frac{(1-\gamma)(\overline{\theta}_2-\theta_2)(\theta_2+\underline{\theta}_2-2a^*)}{2(a^*-\theta_2)(\overline{\theta}_2-\underline{\theta}_2)}. \tag{A.9}
$$

It can be seen that $\Lambda_2(\underline{\theta}_2)=-(1-\gamma)$, $\Lambda_2(\overline{\theta}_2)=0$, and that the function Λ_2 has density:

$$
\lambda_2(\theta_2)=(1-\gamma)\frac{(a^*-\theta_2)^2+(a^*-\underline{\theta}_2)(a^*-\overline{\theta}_2)}{2(\overline{\theta}_2-\underline{\theta}_2)(a^*-\theta_2)^2}\geq 0. \tag{A.10}
$$

As in Step 4, we have:

$$
\alpha_1=\gamma\left(\frac{1}{2}+\frac{a^*-\overline{\theta}_1}{\overline{\theta}_1-\underline{\theta}_1}\right).
$$

However, since $a^*=\overline{\theta}_1$, it reduces to

$$
\alpha_1=\frac{\gamma}{2}.
$$

Then, the differential equation for Λ_1 is:

$$
\frac{\gamma}{2}=\gamma\frac{\theta_1-\underline{\theta}_1}{\overline{\theta}_1-\underline{\theta}_1}+\lambda_1(\theta_1)(\overline{\theta}_1-\theta_1)-\Lambda_1(\theta_1) \text{ for each } \theta_1\in[\underline{\theta}_1,\overline{\theta}_1].
$$

We show that there exists a differentiable function Λ_1 with $\Lambda_1(\underline{\theta}_1) = 0$ and $\Lambda_1(\overline{\theta}_1) = \frac{\gamma}{2}$. Indeed, since this first-order linear differential equation is rewritten as

$$\frac{d}{d\theta_1}\left[\Lambda_1(\theta_1)(\overline{\theta}_1 - \theta_1)\right] = \gamma\left(\frac{1}{2} - \frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1}\right),$$

integrating both sides from $\underline{\theta}_1$ to θ_1 yields

$$\Lambda_1(\theta_1) = \frac{\gamma(\theta_1 - \underline{\theta}_1)}{2(\overline{\theta}_1 - \underline{\theta}_1)}. \tag{A.11}$$

Thus, Λ_1 is a linear function. Then, we have:

$$\lambda_1(\theta_1) = \frac{\gamma}{2}\frac{1}{\overline{\theta}_1 - \underline{\theta}_1} \geq 0. \tag{A.12}$$

The Lagrangian is:

$$\begin{aligned}\mathcal{L} =& \mathbb{E}_\theta\left[-\frac{\gamma}{2}(a(\theta) - \theta_1)^2 - \gamma(a(\theta) - \theta_1)(\theta_1 - \underline{\theta}_1)\right] \\ &+ \mathbb{E}_\theta\left[-(1-\gamma)(a(\theta) - \theta_2)^2\left(\frac{1}{2} + \frac{(a^* - \underline{\theta}_2)(a^* - \overline{\theta}_2)}{2(a^* - \theta_2)^2}\right) - (1-\gamma)(a(\theta) - \theta_2)\frac{(\overline{\theta}_2 - \theta_2)(\theta_2 - \underline{\theta}_2)}{(a^* - \theta_2)}\right].\end{aligned} \tag{A.13}$$

The Lagrangian is a concave function in a. By construction, the first-order conditions are satisfied at a^*.

Step 6. Third, suppose $\overline{\theta}_1 < a^* = \underline{\theta}_2$. By Expression (A.2), the first-order condition reduces to:

$$\begin{aligned}&\frac{1}{\overline{\theta}_2 - \underline{\theta}_2}\left\{\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)(a^* - \theta_1) - \Lambda_1(\theta_1)\right\} \\ &+ \frac{1}{\overline{\theta}_1 - \underline{\theta}_1}\left\{-(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} + \lambda_2(\theta_2)(a^* - \theta_2) - \Lambda_2(\theta_2)\right\} \\ &+ \frac{(a^* - \overline{\theta}_1)}{(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)}(\gamma - \Lambda_1(\overline{\theta}_1))\mathbb{I}(\theta_1 = \overline{\theta}_1) = 0.\end{aligned}$$

We conjecture and verify $\Lambda_1(\overline{\theta}_1) = \gamma$. Then, there exist constants α_1 and α_2 such that:

$$\alpha_1 = \gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)(a^* - \theta_1) - \Lambda_1(\theta_1),$$
$$\alpha_2 = -(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} + \lambda_2(\theta_2)(a^* - \theta_2) - \Lambda_2(\theta_2), \text{ and}$$
$$0 = \frac{\alpha_1}{\overline{\theta}_2 - \underline{\theta}_2} + \frac{\alpha_2}{\overline{\theta}_1 - \underline{\theta}_1}.$$

For agent 1, we have the same differential equation as in Step 4 (except that $a^* = \frac{\gamma\overline{\theta}_1 + (1-\gamma)(1+\underline{\theta}_2)}{2}$ coincides exactly with $\underline{\theta}_2$). Thus,

$$\Lambda_1(\theta_1) = \frac{\theta_1 - \underline{\theta}_1}{a^* - \theta_1}\left(-\frac{\gamma}{2}\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \alpha_1\right), \text{ where } \alpha_1 = \gamma\left(\frac{1}{2} + \frac{a^* - \overline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1}\right).$$

That is,

$$\Lambda_1(\theta_1) = \gamma\frac{(\theta_1 - \underline{\theta}_1)(2a^* - \overline{\theta}_1 - \theta_1)}{2(a^* - \theta_1)(\overline{\theta}_1 - \underline{\theta}_1)}. \tag{A.14}$$

It can be seen that $\Lambda_1(\underline{\theta}_1) = 0$, $\Lambda_1(\overline{\theta}_1) = \gamma$, and that the function Λ_1 has density:

$$\lambda_1(\theta_1) = \gamma\frac{(a^* - \theta_1)^2 + (a^* - \underline{\theta}_1)(a^* - \overline{\theta}_1)}{2(\overline{\theta}_1 - \underline{\theta}_1)(a^* - \theta_1)^2} \geq 0. \tag{A.15}$$

As in Step 4, we have:

$$\alpha_2 = -(1-\gamma)\left(\frac{1}{2} - \frac{a^* - \underline{\theta}_2}{\overline{\theta}_2 - \underline{\theta}_2}\right).$$

However, since $a^* = \underline{\theta}_2$, it reduces to

$$\alpha_2 = -\frac{1-\gamma}{2}.$$

Then, the differential equation for Λ_2 is:

$$-\frac{1-\gamma}{2} = -(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} + \lambda_2(\theta_2)(\underline{\theta}_2 - \theta_2) - \Lambda_2(\theta_2).$$

We show that there exists a differentiable function Λ_2 with $\Lambda_2(\underline{\theta}_2) = -\frac{1-\gamma}{2}$ and $\Lambda_2(\overline{\theta}_2) = 0$. Indeed, since this first-order linear differential equation is rewritten as

$$\frac{d}{d\theta_2}\left[\Lambda_2(\theta_2)(\underline{\theta}_2 - \theta_2)\right] = (1-\gamma)\left(\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} - \frac{1}{2}\right),$$

integrating both sides from θ_2 to $\overline{\theta}_2$ yields

$$\Lambda_2(\theta_2) = -(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{2(\overline{\theta}_2 - \underline{\theta}_2)}. \tag{A.16}$$

Thus, Λ_2 is a linear function. Then, we have:

$$\lambda_2(\theta_2) = \frac{1-\gamma}{2(\overline{\theta}_2 - \underline{\theta}_2)} \geq 0. \tag{A.17}$$

The Lagrangian is:

$$\begin{aligned}\mathcal{L} =& \mathbb{E}_\theta\left[-\gamma(a(\theta)-\theta_1)^2\left(\frac{1}{2} + \frac{(a^*-\underline{\theta}_1)(a^*-\overline{\theta}_1)}{2(a^*-\theta_1)^2}\right) - \gamma(a(\theta)-\theta_1)\frac{(\theta_1-\underline{\theta}_1)(\overline{\theta}_1-\theta_1)}{(a^*-\theta_1)}\right] \\ &+\mathbb{E}_\theta\left[-\frac{1-\gamma}{2}(a(\theta)-\theta_2)^2 + (1-\gamma)(a(\theta)-\theta_2)(\overline{\theta}_2-\theta_2)\right].\end{aligned}$$

The Lagrangian is a concave function in a. By construction, the first-order conditions are satisfied at a^*.

Step 7. Fourth, suppose $\overline{\theta}_1 = a^* = \underline{\theta}_2 (= 1-\gamma)$. By Expression (A.2), the first-order condition reduces to:

$$\begin{aligned}&\frac{1}{\overline{\theta}_2 - \underline{\theta}_2}\left\{\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)(a^*-\theta_1) - \Lambda_1(\theta_1)\right\} \\ +&\frac{1}{\overline{\theta}_1 - \underline{\theta}_1}\left\{-(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} + \lambda_2(\theta_2)(a^*-\theta_2) - \Lambda_2(\theta_2)\right\} = 0.\end{aligned}$$

Thus, the first-order conditions imply that there exist constants α_1 and α_2 such that

$$\begin{aligned}\alpha_1 =& \gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)(a^*-\theta_1) - \Lambda_1(\theta_1), \\ \alpha_2 =& -(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} + \lambda_2(\theta_2)(a^*-\theta_2) - \Lambda_2(\theta_2), \text{ and} \\ 0 =& \frac{\alpha_1}{\overline{\theta}_2 - \underline{\theta}_2} + \frac{\alpha_2}{\overline{\theta}_1 - \underline{\theta}_1}.\end{aligned}$$

Thus, the analyses in Steps 5 and 6 apply. We also see that the solutions Λ_1 and Λ_2 are obtained as the limit case of Step 4. For agent 1, we have:

$$\Lambda_1(\theta_1) = \frac{\theta_1 - \underline{\theta}_1}{a^* - \theta_1}\left(-\frac{\gamma}{2}\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \alpha_1\right), \text{ where } \alpha_1 = \gamma\left(\frac{1}{2} + \frac{a^* - \overline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1}\right) = \frac{\gamma}{2}.$$

That is,

$$\Lambda_1(\theta_1) = \gamma \frac{(\theta_1 - \underline{\theta}_1)(2a^* - \overline{\theta}_1 - \theta_1)}{2(a^* - \theta_1)(\overline{\theta}_1 - \underline{\theta}_1)} = \gamma \frac{\theta_1 - \underline{\theta}_1}{2(\overline{\theta}_1 - \underline{\theta}_1)}. \tag{A.18}$$

It can be seen that $\Lambda_1(\underline{\theta}_1) = 0$, $\Lambda_1(\overline{\theta}_1) = \frac{\gamma}{2}$, and that the function Λ_1 has its density:

$$\lambda_1(\theta_1) = \gamma \frac{(a^* - \theta_1)^2 + (a^* - \underline{\theta}_1)(a^* - \overline{\theta}_1)}{2(\overline{\theta}_1 - \underline{\theta}_1)(a^* - \theta_1)^2} = \frac{\gamma}{2(\overline{\theta}_1 - \underline{\theta}_1)} \geq 0. \tag{A.19}$$

For agent 2, we have:

$$\Lambda_2(\theta_2) = -\frac{\overline{\theta}_2 - \theta_2}{a^* - \theta_2}\left(\frac{(1-\gamma)(\overline{\theta}_2 - \theta_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)} + \alpha_2\right), \text{ where } \alpha_2 = -(1-\gamma)\left(\frac{1}{2} - \frac{a^* - \underline{\theta}_2}{\overline{\theta}_2 - \underline{\theta}_2}\right) = -\frac{1-\gamma}{2}.$$

That is,

$$\Lambda_2(\theta_2) = (1-\gamma)\frac{(\overline{\theta}_2 - \theta_2)(\theta_2 + \underline{\theta}_2 - 2a^*)}{2(a^* - \theta_2)(\overline{\theta}_2 - \underline{\theta}_2)} = -(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{2(\overline{\theta}_2 - \underline{\theta}_2)}. \tag{A.20}$$

It can be seen that $\Lambda_2(\underline{\theta}_2) = -\frac{1-\gamma}{2}$, $\Lambda_2(\overline{\theta}_2) = 0$, and the function Λ_2 has its density:

$$\lambda_2(\theta_2) = (1-\gamma)\frac{(a^* - \theta_2)^2 + (a^* - \underline{\theta}_2)(a^* - \overline{\theta}_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)(a^* - \theta_2)^2} = \frac{1-\gamma}{2(\overline{\theta}_2 - \underline{\theta}_2)} \geq 0. \tag{A.21}$$

Moreover, we have

$$\frac{\alpha_1}{\overline{\theta}_2 - \underline{\theta}_2} + \frac{\alpha_2}{\overline{\theta}_1 - \underline{\theta}_1} = \frac{\gamma(\overline{\theta}_1 - \underline{\theta}_1) - (1-\gamma)(\overline{\theta}_2 - \underline{\theta}_2)}{2(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)} = \frac{\gamma(1-\gamma-0) - (1-\gamma)(1-(1-\gamma))}{2(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)} = 0.$$

The Lagrangian reduces to:

$$\mathcal{L} = -\mathbb{E}_\theta\left[\frac{\gamma}{2}(a(\theta) - \theta_1)^2 + \frac{1-\gamma}{2}(a(\theta) - \theta_2)^2 + \gamma(a(\theta) - \theta_1)(\theta_1 - \underline{\theta}_1) - (1-\gamma)(a(\theta) - \theta_2)(\overline{\theta}_2 - \theta_2)\right].$$

The Lagrangian is a concave function in a. By construction, the first-order conditions are satisfied at $a = a^*$. The proof is complete.

A.1.2 Part (2)

Suppose $\overline{\theta}_1 \leq \underline{\theta}_2$ and $\underline{\theta}_2 \leq \frac{2-\gamma}{1-\gamma}\overline{\theta}_1 - 1$. The latter condition is equivalent to $\overline{\theta}_1 \geq \frac{\gamma\overline{\theta}_1 + (1-\gamma)(\underline{\theta}_2 + 1)}{2}$. Note that we interchangeably use $\underline{\theta}_1 = 0$ and $\overline{\theta}_2 = 1$, respectively. Denote by a^* the delegation solution for agent 1. Also, for ease of notation, we denote by $a^*(\theta_1) = a^*(\theta_1, \theta_2)$

as it does not depend on θ_2. Specifically, denoting by $k_1 = \frac{1-\gamma}{2-\gamma}(\underline{\theta}_2 + 1)$ agent 1's cap,

$$a^*(\theta_1) = \begin{cases} k_1 & \text{if } \theta_2 \in [0, k_1] \\ \theta_1 & \text{if } \theta_2 \in \left[k_1, \overline{\theta}_1\right] \end{cases}.$$

If $\overline{\theta}_1 = k_1$ then the delegation allocation $a^*(\cdot) = k_1$ reduces to the optimal (constant) allocation $a^*(\cdot) = \frac{\gamma\overline{\theta}_1 + (1-\gamma)(\underline{\theta}_2+1)}{2}$ found in Theorem 1 (1). Thus, without loss of generality, we can assume $\overline{\theta}_1 > \frac{\gamma\overline{\theta}_1 + (1-\gamma)(\underline{\theta}_2+1)}{2}$. Especially, $\overline{\theta}_1 > 1 - \gamma$, as $\overline{\theta}_1 \leq \underline{\theta}_2$.

The proof consists of four steps. In the first step, we consider the following relaxed problem: for each agent i, the local IC constraint is imposed; and the allocation is required to be monotonic in agent 2's types given agent 1's types. We also set up the Lagrangian which incorporates agents' IC constraints. Denote by Λ_i the Lagrange multiplier associated with agent i's local IC constraint. Thus, the problem is to maximize the Lagrangian subject to the (relaxed) monotonicity constraint. In the second step, we explicitly incorporate the (relaxed) monotonicity constraint using the Lagrangian approach again. Denote by B the Lagrange multiplier associated with the monotonicity constraint on agent 2's types θ_2 (for any given θ_1). The third step formulates the first-order conditions. The fourth step finds the multipliers $(\Lambda_1, \Lambda_2, B)$ under which the first-order conditions are met for the delegation allocation. Note that we have fewer cases than Part (1), which has dealt with the boundary cases.

Step 1. Consider the relaxed problem in which the monotonicity constraint is replaced by the one that an allocation a is monotonic in θ_2 for any given θ_1:

$$a(\theta_1, \theta_2) \text{ is non-decreasing in } \theta_2 \text{ for any given } \theta_1.$$

More precisely, we impose the following (relaxed) monotonicity constraint: for any given $\theta_1 \in [\underline{\theta}_1, \overline{\theta}_1]$,

$$\frac{\partial a(\theta_1, \theta_2)}{\partial \theta_2} \geq 0 \text{ for almost all } \theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]. \tag{A.22}$$

The problem is then to maximize the Lagrangian given by Expression (A.1) subject to the above (relaxed) monotonicity constraint (A.22). Denote by $\mathcal{L}_0$ Expression (A.1), the Lagrangian which incorporates agents' local IC constraints. Note that, as in the proof of Theorem 1 (1), we can assume, without loss of generality, $\Lambda_1(\underline{\theta}_1) = 0$ and $\Lambda_2(\overline{\theta}_2) = 0$.

Step 2. We now incorporate the monotonicity constraint (A.22) into the objective function

$\mathcal{L}_0$. Thus, the Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = \mathcal{L}_0 + \int_{\underline{\theta}_1}^{\overline{\theta}_1} \int_{\underline{\theta}_2}^{\overline{\theta}_2} B(\theta_1, \theta_2) \frac{\partial a(\theta_1, \theta_2)}{\partial \theta_2} d\theta_2 d\theta_1,$$

where, for each $\theta_1 \in [0, \overline{\theta}_1]$, the function $B(\theta_1, \cdot)$ is the Lagrange multiplier associated with the monotonicity constraint $\frac{\partial a}{\partial \theta_2}(\theta_1, \cdot) \geq 0$. Hence, the Lagrangian is rewritten as:

$$\begin{aligned}
\mathcal{L} = & - \mathbb{E}_{\theta_2}\left[(a(\overline{\theta}_1, \theta_2) - \overline{\theta}_1)^2\right](\gamma - \Lambda_1(\overline{\theta}_1)) - 2\gamma \mathbb{E}_\theta\left[(a(\theta) - \theta_1)(\theta_1 - \underline{\theta}_1)\right] \\
& - \int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[(a(\theta) - \theta_1)^2\right]\lambda_1(\theta_1) d\theta_1 + 2\int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\theta) - \theta_1\right]\Lambda_1(\theta_1) d\theta_1 \\
& - \mathbb{E}_{\theta_1}\left[(a(\theta_1, \underline{\theta}_2) - \underline{\theta}_2)^2\right](1 - \gamma + \Lambda_2(\underline{\theta}_2)) + 2(1-\gamma)\mathbb{E}_\theta\left[(a(\theta) - \theta_2)(\overline{\theta}_2 - \theta_2)\right] \\
& - \int_{\underline{\theta}_2}^{\overline{\theta}_2} \mathbb{E}_{\theta_1}\left[(a(\theta) - \theta_2)^2\right]\lambda_2(\theta_2) d\theta_2 + 2\int_{\underline{\theta}_2}^{\overline{\theta}_2} \mathbb{E}_{\theta_1}\left[a(\theta) - \theta_2\right]\Lambda_2(\theta_2) d\theta_2 \\
& + \int_{\underline{\theta}_1}^{\overline{\theta}_1} B(\theta_1, \overline{\theta}_2) a(\theta_1, \overline{\theta}_2) d\theta_1 - \int_{\underline{\theta}_1}^{\overline{\theta}_1} B(\theta_1, \underline{\theta}_2) a(\theta_1, \underline{\theta}_2) d\theta_1 - \int_{\underline{\theta}_1}^{\overline{\theta}_1} \int_{\underline{\theta}_2}^{\overline{\theta}_2} \frac{\partial B(\theta_1, \theta_2)}{\partial \theta_2} a(\theta_1, \theta_2) d\theta_2 d\theta_1.
\end{aligned} \tag{A.23}$$

Step 3. For each (θ_1, θ_2), the first-order condition at $a = a^*$ is:

$$\begin{aligned}
\frac{\partial B(\theta_1, \theta_2)}{\partial \theta_2} = & \frac{2}{\overline{\theta}_2 - \underline{\theta}_2}\left\{-\gamma \frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} - \lambda_1(\theta_1)(a^*(\theta_1) - \theta_1) + \Lambda_1(\theta_1)\right\} \\
& + \frac{2}{\overline{\theta}_1 - \underline{\theta}_1}\left\{(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} - \lambda_2(\theta_2)(a^*(\theta_1) - \theta_2) + \Lambda_2(\theta_2)\right\} \\
& - \frac{1}{\overline{\theta}_2 - \underline{\theta}_2}\left\{B(\theta_1, \underline{\theta}_2) + \frac{2}{\overline{\theta}_1 - \underline{\theta}_1}(a^*(\theta_1) - \underline{\theta}_2)(1 - \gamma + \Lambda_2(\underline{\theta}_2))\right\}\mathbb{I}(\theta_2 = \underline{\theta}_2) \\
& + \frac{B(\theta_1, \overline{\theta}_2)}{\overline{\theta}_2 - \underline{\theta}_2}\mathbb{I}(\theta_2 = \overline{\theta}_2).
\end{aligned} \tag{A.24}$$

Note that we have used $a^*(\overline{\theta}_1) = \overline{\theta}_1$.

Step 4. From now on, we will construct $(\Lambda_1, \Lambda_2, B)$ which satisfy the first-order conditions at a^*. We conjecture and verify the following:

$$B(\theta_1, \overline{\theta}_2) = 0 \text{ for all } \theta_1 \in [\underline{\theta}_1, \overline{\theta}_1], \text{ and} \tag{A.25}$$

$$\begin{aligned} B(\theta_1, \underline{\theta}_2) &= -\frac{2}{\overline{\theta}_1 - \underline{\theta}_1}(a^*(\theta_1) - \underline{\theta}_2)(1 - \gamma + \Lambda_2(\underline{\theta}_2)) \\ &= \begin{cases} -\frac{2}{\overline{\theta}_1 - \underline{\theta}_1}(k_1 - \underline{\theta}_2)(1 - \gamma + \Lambda_2(\underline{\theta}_2)) & \text{if } \theta_1 \in [0, k_1] \\ -\frac{2}{\overline{\theta}_1 - \underline{\theta}_1}(\theta_1 - \underline{\theta}_2)(1 - \gamma + \Lambda_2(\underline{\theta}_2)) & \text{if } \theta_1 \in \left[k_1, \overline{\theta}_1\right] \end{cases}. \end{aligned} \tag{A.26}$$

We substitute a^* into Expression (A.24) to get the following two forms of first-order conditions. First, for $\theta_1 \in [0, k_1]$,

$$\begin{aligned} \frac{\partial B(\theta_1, \theta_2)}{\partial \theta_2} &= \frac{2}{\overline{\theta}_2 - \underline{\theta}_2} \left\{ -\gamma \frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} - \lambda_1(\theta_1)\,(k_1 - \theta_1) + \Lambda_1(\theta_1) \right\} \\ &\quad + \frac{2}{\overline{\theta}_1 - \underline{\theta}_1} \left\{ (1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} - \frac{d}{d\theta_2}\left[\Lambda_2(\theta_2)\,(k_1 - \theta_2)\right] \right\}. \end{aligned}$$

Second, for $\theta_1 \in \left[k_1, \overline{\theta}_1\right]$,

$$\begin{aligned} \frac{\partial B(\theta_1, \theta_2)}{\partial \theta_2} &= \frac{2}{\overline{\theta}_2 - \underline{\theta}_2} \left\{ -\gamma \frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \Lambda_1(\theta_1) \right\} - \frac{2\theta_1}{\overline{\theta}_1 - \underline{\theta}_1}\lambda_2(\theta_2) \\ &\quad + \frac{2}{\overline{\theta}_1 - \underline{\theta}_1} \left\{ (1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} + \frac{d}{d\theta_2}\left[\Lambda_2(\theta_2)\theta_2\right] \right\}. \end{aligned}$$

We integrate each of the above equations with respect to θ_2 from θ_2 to $\overline{\theta}_2$. For $\theta_1 \in [0, k_1]$,

$$\begin{aligned} B(\theta_1, \theta_2) &= 2\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} \left\{ \gamma \frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)\,(k_1 - \theta_1) - \Lambda_1(\theta_1) \right\} \\ &\quad - \frac{2}{\overline{\theta}_1 - \underline{\theta}_1} \left\{ (1-\gamma)\frac{(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)} + (k_1 - \theta_2)\,\Lambda_2(\theta_2) \right\}. \end{aligned} \tag{A.27}$$

For $\theta_1 \in \left[k_1, \overline{\theta}_1\right]$,

$$\begin{aligned} B(\theta_1, \theta_2) &= 2\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} \left\{ \gamma \frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} - \Lambda_1(\theta_1) \right\} - \frac{2\theta_1}{\overline{\theta}_1 - \underline{\theta}_1}\Lambda_2(\theta_2) \\ &\quad + \frac{2}{\overline{\theta}_1 - \underline{\theta}_1} \left\{ -(1-\gamma)\frac{(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)} + \Lambda_2(\theta_2)\theta_2 \right\}. \end{aligned} \tag{A.28}$$

We substitute $\theta_2 = \underline{\theta}_2$ into the above equations. For $\theta_1 \in [0, k_1]$,

$$B(\theta_1, \underline{\theta}_2) = 2\left\{\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)(k_1 - \theta_1) - \Lambda_1(\theta_1)\right\} - \frac{2}{\overline{\theta}_1 - \underline{\theta}_1}\left\{(1-\gamma)\frac{\overline{\theta}_2 - \underline{\theta}_2}{2} + (k_1 - \underline{\theta}_2)\Lambda_2(\underline{\theta}_2)\right\}. \tag{A.29}$$

For $\theta_1 \in \left[k_1, \overline{\theta}_1\right]$,

$$B(\theta_1, \underline{\theta}_2) = 2\left\{\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} - \Lambda_1(\theta_1)\right\} - 2\frac{\theta_1 - \underline{\theta}_2}{\overline{\theta}_1 - \underline{\theta}_1}\Lambda_2(\underline{\theta}_2) - (1-\gamma)\frac{\overline{\theta}_2 - \underline{\theta}_2}{\overline{\theta}_1 - \underline{\theta}_1}. \tag{A.30}$$

Now, we solve for Λ_1. First, since $\Lambda_1(\underline{\theta}_1) = 0$, we start with the case when $\theta_1 \in [0, k_1]$. For $\theta_1 \in [0, k_1]$, it follows from Expressions (A.26) and (A.29) that

$$\lambda_1(\theta_1)(k_1 - \theta_1) - \Lambda_1(\theta_1) + \gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + (1-\gamma)\frac{2k_1 - \underline{\theta}_2 - \overline{\theta}_2}{2(\overline{\theta}_1 - \underline{\theta}_1)} = 0.$$

This is a linear first-order differential equation. In fact, since we have

$$\frac{d}{d\theta_1}\left[\Lambda_1(\theta_1)(k_1 - \theta_1)\right] = -\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} - (1-\gamma)\frac{2k_1 - \underline{\theta}_2 - \overline{\theta}_2}{2(\overline{\theta}_1 - \underline{\theta}_1)},$$

integrating both sides from $\underline{\theta}_1$ to θ_1 yields

$$\begin{aligned}\Lambda_1(\theta_1)(k_1 - \theta_1) &= -\gamma\frac{(\theta_1 - \underline{\theta}_1)^2}{2(\overline{\theta}_1 - \underline{\theta}_1)} - (1-\gamma)\frac{(2k_1 - \underline{\theta}_2 - \overline{\theta}_2)(\theta_1 - \underline{\theta}_1)}{2(\overline{\theta}_1 - \underline{\theta}_1)} \\ &= -\frac{(\theta_1 - \underline{\theta}_1)(\gamma(\theta_1 - \underline{\theta}_1) + (1-\gamma)(2k_1 - \underline{\theta}_2 - \overline{\theta}_2))}{2(\overline{\theta}_1 - \underline{\theta}_1)}.\end{aligned}$$

Hence, for $\theta_1 \in [0, k_1]$, we have:

$$\Lambda_1(\theta_1) = -\frac{(\theta_1 - \underline{\theta}_1)(\gamma(\theta_1 - \underline{\theta}_1) + (1-\gamma)(2k_1 - \underline{\theta}_2 - \overline{\theta}_2))}{2(\overline{\theta}_1 - \underline{\theta}_1)(k_1 - \theta_1)}.$$

In order for $\Lambda_1(k_1)$ to be well-defined, we have to have

$$(1-\gamma)(2k_1 - \underline{\theta}_2 - \overline{\theta}_2) = -\gamma k_1, \text{ that is, } k_1 = \frac{1-\gamma}{2-\gamma}(1 + \underline{\theta}_2),$$

provided that $k_1 \neq 0$. Then, we obtain

$$\Lambda_1(\theta_1) = \frac{\gamma}{2}\frac{(\theta_1 - \underline{\theta}_1)}{(\overline{\theta}_1 - \underline{\theta}_1)}.$$

When $\theta_1 = k_1$, we have

$$\Lambda_1\left(k_1\right) = \frac{\gamma(k_1 - \underline{\theta}_1)}{2(\overline{\theta}_1 - \underline{\theta}_1)} = \frac{\gamma(1-\gamma)}{2(2-\gamma)} \frac{1+\underline{\theta}_2}{\overline{\theta}_1 - \underline{\theta}_1}.$$

Second, when $\theta_1 \in \left[k_1, \overline{\theta}_1\right]$, it follows from Expressions (A.26) and (A.30) that

$$\Lambda_1(\theta_1) = \frac{2\gamma(\theta_1 - \underline{\theta}_1) + (1-\gamma)(\overline{\theta}_2 - \underline{\theta}_2)}{2(\overline{\theta}_1 - \underline{\theta}_1)} = \frac{\theta_1}{\overline{\theta}_1 - \underline{\theta}_1} - \frac{1-\gamma}{2} \frac{1+\underline{\theta}_2}{\overline{\theta}_1 - \underline{\theta}_1} = \frac{2\theta_1 - (1-\gamma)(1+\underline{\theta}_2)}{2(\overline{\theta}_1 - \underline{\theta}_1)}.$$

It can be seen that

$$\Lambda_1(k_1) = \frac{\gamma(1-\gamma)(1+\underline{\theta}_2)}{2(2-\gamma)(\overline{\theta}_1 - \underline{\theta}_1)},$$

as desired. In sum, Λ_1 is a piece-wise-linear and continuous function given by

$$\Lambda_1(\theta_1) = \begin{cases} \dfrac{\gamma}{2} \dfrac{(\theta_1 - \underline{\theta}_1)}{(\overline{\theta}_1 - \underline{\theta}_1)} & \text{if } \theta_1 \in [0, k_1] \\ \dfrac{2\theta_1 - (1-\gamma)(1+\underline{\theta}_2)}{2(\overline{\theta}_1 - \underline{\theta}_1)} & \text{if } \theta_1 \in \left[k_1, \overline{\theta}_1\right] \end{cases}. \tag{A.31}$$

Also, we have

$$\lambda_1(\theta_1) = \begin{cases} \dfrac{\gamma}{2(\overline{\theta}_1 - \underline{\theta}_1)} \geq 0 & \text{if } \theta_1 \in [0, k_1) \\ \dfrac{1}{\overline{\theta}_1 - \underline{\theta}_1} \geq 0 & \text{if } \theta_1 \in \left(k_1, \overline{\theta}_1\right] \end{cases}. \tag{A.32}$$

Now, we substitute Λ_1 and λ_1 into Expressions (A.27) and (A.28) to rewrite B:

$$B(\theta_1, \theta_2) = \begin{cases} -\dfrac{2}{\overline{\theta}_1 - \underline{\theta}_1} \left\{ (1-\gamma) \dfrac{(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)} + (k_1 - \theta_2)\, \Lambda_2(\theta_2) \right\} + \dfrac{\gamma(k_1 - \underline{\theta}_1)(\overline{\theta}_2 - \theta_2)}{(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)} & \text{if } \theta_1 \in [0, k_1] \\ -\dfrac{2}{\overline{\theta}_1 - \underline{\theta}_1} \left\{ (1-\gamma) \dfrac{(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)} + (\theta_1 - \theta_2)\, \Lambda_2(\theta_2) \right\} + \dfrac{(1-\gamma)\,(1 + \underline{\theta}_2 - 2\theta_1)\,(\overline{\theta}_2 - \theta_2)}{(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)} & \text{if } \theta_1 \in \left[k_1, \overline{\theta}_1\right] \end{cases}. \tag{A.33}$$

It can be seen that B satisfies Expressions (A.25) and (A.26).

Next, we conjecture and verify:

$$B(\theta_1, \theta_2) = 0 \text{ if } \theta_1 \in [0, k_1]\,. \tag{A.34}$$

In fact, it follows from Expression (A.33) that Expression (A.34) holds if and only if

$$\Lambda_2(\theta_2) = \frac{\gamma}{2} \frac{k_1 - \underline{\theta}_1}{k_1 - \theta_2} \frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} - \frac{(1-\gamma)(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)(k_1 - \theta_2)} = -\frac{(1-\gamma)(\overline{\theta}_2 - \theta_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)} \left(1 + \frac{k_1 - \underline{\theta}_2}{k_1 - \theta_2}\right). \tag{A.35}$$

Note that we especially have

$$\Lambda_2(\underline{\theta}_2) = -(1-\gamma).$$

Also, we have

$$\lambda_2(\theta_2) = \frac{1-\gamma}{2(\overline{\theta}_2 - \underline{\theta}_2)} + \frac{(1-\gamma)(\underline{\theta}_2 - k_1)(\overline{\theta}_2 - k_1)}{2(\overline{\theta}_2 - \underline{\theta}_2)(k_1 - \theta_2)^2} \geq 0. \tag{A.36}$$

Since we have found Λ_1 and Λ_2, we remark on the limit case in which $\overline{\theta}_1 = k_1$ holds, which is covered in the proof of Theorem 1 (1). In the limit case, $a^* = k_1$. For agent 2, Expressions (A.9) and (A.35) coincide, as it can be seen that

$$\frac{(1-\gamma)(\overline{\theta}_2 - \theta_2)(\theta_2 + \underline{\theta}_2 - 2k_1)}{2(k_1 - \theta_2)(\overline{\theta}_2 - \underline{\theta}_2)} = \frac{(1-\theta_2)(\gamma k_1 - (1-\gamma)(1-\theta_2))}{2(k_1 - \theta_2)(1 - \underline{\theta}_2)}.$$

Generally, for agent 1, Expressions (A.11) and (A.31) coincide for $\theta_1 \in [0, k_1]$. Thus, in the particular limit case $\overline{\theta}_1 = k_1$, they coincide on Θ_1.

Substituting Expression (A.35) into Expression (A.33), we obtain B as:

$$B(\theta_1, \theta_2) = \begin{cases} 0 & \text{if } \theta_1 \in [0, k_1] \\ (1-\gamma)\dfrac{(\theta_2 - \underline{\theta}_2)(\theta_2 - 1)((1-\gamma)(\underline{\theta}_2 + 1) - (2-\gamma)\theta_1)}{(\overline{\theta}_1 - \underline{\theta}_1)(1 - \underline{\theta}_2)((1-\gamma)(\underline{\theta}_2 + 1) - (2-\gamma)\theta_2)} & \text{if } \theta_1 \in \left[k_1, \overline{\theta}_1\right] \end{cases}. \tag{A.37}$$

where we have used $\overline{\theta}_2 = 1$. The Lagrange multiplier on the monotonicity constraint is zero when $\theta_1 \in [0, k_1]$.

Since the Lagrangian reduces to

$$\begin{aligned} \mathcal{L} = &- 2\gamma \mathbb{E}_\theta \left[(a(\theta) - \theta_1)(\theta_1 - \underline{\theta}_1)\right] - \int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[(a(\theta) - \theta_1)^2\right] \lambda_1(\theta_1) d\theta_1 + 2\int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\theta) - \theta_1\right] \Lambda_1(\theta_1) d\theta_1 \\ &+ 2(1-\gamma)\mathbb{E}_\theta \left[(a(\theta) - \theta_2)(\overline{\theta}_2 - \theta_2)\right] - \int_{\underline{\theta}_2}^{\overline{\theta}_2} \mathbb{E}_{\theta_1}\left[(a(\theta) - \theta_2)^2\right] \lambda_2(\theta_2) d\theta_2 + 2\int_{\underline{\theta}_2}^{\overline{\theta}_2} \mathbb{E}_{\theta_1}\left[a(\theta) - \theta_2\right] \Lambda_2(\theta_2) d\theta_2 \\ &- \int_{k_1}^{\overline{\theta}_1} \int_{\underline{\theta}_2}^{\overline{\theta}_2} \frac{\partial B(\theta_1, \theta_2)}{\partial \theta_2} a(\theta_1, \theta_2) d\theta_2 d\theta_1, \end{aligned}$$

the Lagrangian $\mathcal{L}$ is a concave function in a. Thus, the first-order conditions are sufficient.

A.1.3 Part (3)

The proof is similar to that of Part (2), exchanging the role of agents 1 and 2. Especially, we consider the relaxed monotonicity constraint which requires the allocation to be monotonic in agent 1's types given agent 2's types. Here, we only report the Lagrange multipliers. To that end, denote by k_2 agent 2's cap:

$$k_2 = \frac{\gamma\overline{\theta}_1 + 1 - \gamma}{1+\gamma}.$$

Now, for agent 1's local IC constraint, the multiplier Λ_1 is:

$$\Lambda_1(\theta_1) = \frac{\gamma}{2}\frac{\theta_1 - \underline{\theta}_1}{(\overline{\theta}_1 - \underline{\theta}_1)}\left\{1 + \frac{k_2 - \overline{\theta}_1}{k_2 - \theta_1}\right\}. \tag{A.38}$$

Especially, its density is:

$$\lambda_1(\theta_1) = \frac{\gamma}{2(\overline{\theta}_1 - \underline{\theta}_1)} + \frac{\gamma}{2}\frac{k_2 - \overline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1}\frac{k_2 - \underline{\theta}_1}{(k_2 - \theta_1)^2} \geq 0. \tag{A.39}$$

For agent 2's local IC constraint, the multiplier Λ_2 is:

$$\Lambda_2(\theta_2) = \begin{cases} \dfrac{2\theta_2 - \gamma\overline{\theta}_1 - 2(1-\gamma)}{2(\overline{\theta}_2 - \underline{\theta}_2)} & \text{if } \theta_2 \in [\underline{\theta}_2, k_2] \\ -\dfrac{(1-\gamma)(\overline{\theta}_2 - \theta_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)} & \text{if } \theta_2 \in [k_2, 1] \end{cases}. \tag{A.40}$$

Consequently, its density is:

$$\lambda_2(\theta_2) = \begin{cases} \dfrac{1}{\overline{\theta}_2 - \underline{\theta}_2} \geq 0 & \text{if } \theta_2 \in [\underline{\theta}_2, k_2) \\ \dfrac{1-\gamma}{2(\overline{\theta}_2 - \underline{\theta}_2)} \geq 0 & \text{if } \theta_2 \in (k_2, 1] \end{cases}. \tag{A.41}$$

For the Monotonicity constraint, denote by B the Lagrange multiplier associated with the monotonicity constraint on agent 1's types θ_1 (for any given θ_2). Then,

$$B(\theta_1, \theta_2) = \begin{cases} \gamma\dfrac{(\theta_1 - \underline{\theta}_1)(\theta_1 - \overline{\theta}_1)(1 - \gamma + \gamma\overline{\theta}_1 - (1+\gamma)\theta_2)}{(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)(1 - \gamma + \gamma\overline{\theta}_1 - (1+\gamma)\theta_1)} & \text{if } \theta_2 \in [\underline{\theta}_2, k_2] \\ 0 & \text{if } \theta_2 \in [k_2, 1] \end{cases}. \tag{A.42}$$

Thus, the Lagrange multiplier on the monotonicity constraint is zero when $\theta_2 \in [k_2, 1]$.

Finally, we remark on the limit case in which $\underline{\theta}_2 = k_2$ holds, which is covered in the proof of Theorem 1 (1). For agent 1, Expressions (A.14) and (A.38) coincide. Generally, for agent 2, Expressions (A.16) and (A.40) coincide for $\theta_2 \in [k_2, 1]$. Thus, in the particular limit case $\underline{\theta}_2 = k_2$, they coincide on Θ_2.

A.1.4 Part (4)

The proof of Theorem 1 (4) is in two steps. First, if an optimal allocation depends on at most one agent's information then the optimal allocation satisfies ex-post IC constraints, including Monotonicity. In particular, the optimal allocation which depends on at most one agent's information is a constrained delegation solution. Note that the constrained delegation solution is continuous. Second, we find the unique optimal continuous ex-post IC allocation, which can depend on both agents' information. We show that this allocation depends on both agents' information and is better than the best allocation which depends on at most one agent's information.

Step 1. Consider an allocation a which depends on (at most) agent i's information. Denote by $a(\theta) = a(\theta_i)$. Denote by γ_i the Pareto weight on agent i's utility. Since the objective function is

$$-\int_{\underline{\theta}_i}^{\overline{\theta}_i}\int_{\underline{\theta}_{-i}}^{\overline{\theta}_{-i}} \left\{\gamma_i(a(\theta_i)-\theta_i)^2 + \gamma_{-i}(a(\theta_i)-\theta_{-i})^2\right\} dF_i(\theta_i)dF_{-i}(\theta_{-i}),$$

the problem reduces to:

$$\max_a -\int_{\underline{\theta}_i}^{\overline{\theta}_i} (a(\theta_i) - (\gamma_i\theta_i + \gamma_{-i}\mathbb{E}_{\theta_{-i}}[\theta_{-i}]))^2 f_i(\theta_i)d\theta_i$$
$$\text{subject to } -(a(\theta_i)-\theta_i)^2 \geq -(a(\hat{\theta}_i)-\theta_i)^2.$$

Thus, without loss, we can restrict attention to ex-post IC constraints. In particular, since ex-post IC constraints imply Monotonicity, if an optimal allocation depends on at most one agent's types then it satisfies Monotonicity naturally without imposing it.

For this problem, similarly to Melumad and Shibano (1991), one can show that the optimal allocation a^i is a constrained delegation allocation with possibly two caps:

$$a^i(\theta_i) = \min\left(k_i^h, \max\left(\theta_i, k_i^\ell\right)\right) = \begin{cases} k_i^\ell & \text{if } \theta_i \in [\underline{\theta}_i, k_i^\ell] \\ \theta_i & \text{if } \theta_i \in [k_i^\ell, k_i^h] \\ k_i^h & \text{if } \theta_i \in [k_i^h, \overline{\theta}_i] \end{cases},$$

where $\underline{\theta}_i \leq k_i^\ell \leq k_i^h \leq \overline{\theta}_i$ (note that if $k_i^\ell = k_i^h$ then the allocation is constant). Also, note that a^i is continuous. Substituting the constrained delegation allocation a^i into the social welfare, we can find the optimal cutoffs (k_i^ℓ, k_i^h) by taking the first-order conditions. See Remark A.1 at the end of this subsection for the optimal allocations a^1 and a^2.

Step 2. The second step finds the unique optimal continuous ex-post IC allocation, which indeed depends on both agents' information and which yields strictly better social welfare than the best allocation in the first step. The proof of this step is in four sub-steps.

In the first sub-step, as Martimort and Semenov (2008) characterize continuous and ex-post IC allocations in the uniform-quadratic setting (see also Moulin, 1980), also in our setting of Θ, an allocation a is ex-post IC and continuous if and only if

$$a(\theta_1, \theta_2) = \min\left(x, \max(\theta_1, y_1), \max(\theta_2, y_2), \max(\theta_1, \theta_2, z)\right)$$

for some (x, y_1, y_2, z) with $z \leq y_1, y_2 \leq x$.

In the second sub-step, we show that, for an optimal continuous ex-post IC allocation, x and z can be dropped. Intuitively, if $x < \overline{\theta}_1$, then the allocation a does not use agent 1's information even when both agents' types lie in the common set $[x, \overline{\theta}_1]$, which is inefficient. Likewise, if $z > \underline{\theta}_2$, then the allocation a does not use agent 2's information even when both agents' types lie in the common set $[\underline{\theta}_2, z]$, which is inefficient.

Lemma A.1. *Suppose $\underline{\theta}_2 \leq \overline{\theta}_1$. An optimal ex-post incentive-compatible and continuous allocation a satisfies*

$$a(\theta_1, \theta_2) = \min\left(\max(\theta_1, y_1), \max(\theta_2, y_2), \max(\theta_1, \theta_2)\right) \text{ for some } (y_1, y_2).$$

Proof of Lemma A.1. Suppose that x and z are essential in an optimal allocation, i.e., $\underline{\theta}_2 \leq z \leq y_1, y_2 \leq x \leq \overline{\theta}_1$. We show that it is optimal to set $x = \overline{\theta}_1$ and $z = \underline{\theta}_2$, in which case, it is without loss to drop x and z from the expression for a.

First, we determine the optimal constant x. The part that depends on x is

$$\begin{aligned}
&-\frac{1}{\overline{\theta}_1(1-\underline{\theta}_2)}\int_x^{\overline{\theta}_1}\int_x^1 \left\{\gamma(x-\theta_1)^2 + (1-\gamma)(x-\theta_2)^2\right\} d\theta_2 d\theta_1 \\
&= -\frac{(\overline{\theta}_1 - x)(1-x)\left((x-(1-\gamma+\gamma\overline{\theta}_1))^2 + \gamma(1-\gamma)(1-\overline{\theta}_1)^2\right)}{3\overline{\theta}_1(1-\underline{\theta}_2)} \leq 0,
\end{aligned}$$

and the unique maximum (in $x \leq \overline{\theta}_1$) is obtained at $x = \overline{\theta}_1$.

Second, to determine the optimal constant z, the part that depends on z is

$$\begin{aligned}
&-\frac{1}{\overline{\theta}_1(1-\underline{\theta}_2)}\int_0^z\int_{\underline{\theta}_2}^z \left\{\gamma(z-\theta_1)^2 + (1-\gamma)(z-\theta_2)^2\right\} d\theta_2 d\theta_1 \\
&= -\frac{(z-(1-\gamma)\underline{\theta}_2)^2 + \gamma(1-\gamma)\underline{\theta}_2^2}{3\overline{\theta}_1(1-\underline{\theta}_2)}(z-\underline{\theta}_2)z \leq 0.
\end{aligned}$$

The unique maximum (in $z \geq \underline{\theta}_2$) is attained at $z = \underline{\theta}_2$. □

In the third sub-step, we find the optimal continuous ex-post IC allocation.

Lemma A.2. *Suppose $\underline{\theta}_2 \leq \overline{\theta}_1$. The optimal ex-post incentive-compatible and continuous allocation a is*

$$a(\theta_1, \theta_2) = \min\left(\max(\theta_1, 1-\gamma), \max(\theta_2, \gamma\overline{\theta}_1 + (1-\gamma)\underline{\theta}_2), \max(\theta_1, \theta_2)\right).$$

Figures 1 and 2 illustrate the optimal continuous ex-post IC allocation (when $\underline{\theta}_2 < 1-\gamma < \overline{\theta}_1$). It can be seen that a indeed depends on both agents' information.

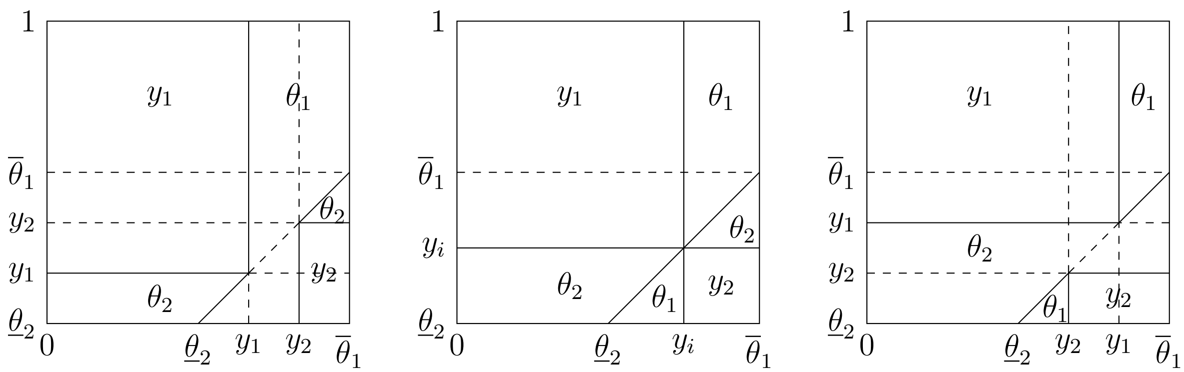

Figure 1: Illustration of the Optimal Continuous Ex-Post IC Allocation (when $\underline{\theta}_2 < 1-\gamma < \overline{\theta}_1$). The optimal continuous ex-post IC allocation has constants $(y_1, y_2) = (1-\gamma, \gamma\overline{\theta}_1 + (1-\gamma)\underline{\theta}_2)$. The left panel depicts the optimal allocation when $\frac{1-\underline{\theta}_2}{1+\overline{\theta}_1-\underline{\theta}_2} < \gamma$, in which case $y_1 < y_2$. The central panel depicts the optimal allocation when $\frac{1-\underline{\theta}_2}{1+\overline{\theta}_1-\underline{\theta}_2} = \gamma$, in which case $y_1 = y_2$. The right panel depicts the optimal allocation when $\frac{1-\underline{\theta}_2}{1+\overline{\theta}_1-\underline{\theta}_2} > \gamma$, in which case $y_1 > y_2$.

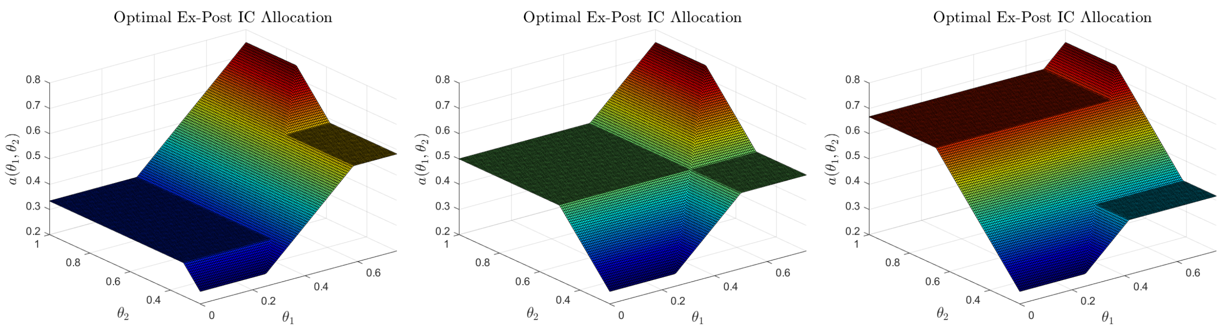

Figure 2: Illustration of the Optimal Continuous Ex-Post IC Allocation: $\overline{\theta}_1 = \frac{3}{4}$ and $\underline{\theta}_2 = \frac{1}{4}$. The left panel depicts the optimal allocation when $\gamma = \frac{2}{3}$. The central panel depicts the optimal allocation when $\gamma = \frac{1}{2}$. The right panel depicts the optimal allocation when $\gamma = \frac{1}{3}$.

Proof of Lemma A.2. It follows from Lemma A.1 that the allocation reduces to

$$a(\theta_1,\theta_2) = \begin{cases} \min(\theta_2, \max(\theta_1, y_1)) = \text{med}(\theta_1,\theta_2,y_1) & \text{if } \theta_1 \le \theta_2 \\ \min(\theta_1, \max(\theta_2, y_2)) = \text{med}(\theta_1,\theta_2,y_2) & \text{if } \theta_2 \le \theta_1 \end{cases}.$$

First, the part of the social welfare that depends on y_1 is:

$$\begin{aligned} W_1 :=& -\frac{\gamma}{\overline{\theta}_1(1-\underline{\theta}_2)} \int_{\underline{\theta}_2}^{y_1} \int_0^{\underline{\theta}_2} (\theta_2-\theta_1)^2 d\theta_1 d\theta_2 \\ & - \frac{1}{\overline{\theta}_1(1-\underline{\theta}_2)} \int_0^{y_1} \int_{y_1}^{1} \left\{\gamma(y_1-\theta_1)^2 + (1-\gamma)(y_1-\theta_2)^2\right\} d\theta_2 d\theta_1 \\ & - \frac{1-\gamma}{\overline{\theta}_1(1-\underline{\theta}_2)} \int_{y_1}^{\overline{\theta}_1} \int_{\theta_1}^{1} (\theta_1-\theta_2)^2 d\theta_2 d\theta_1. \end{aligned}$$

It can be seen that W_1 is a quartic equation with a positive coefficient on y_1^4. Differentiating, its derivative is

$$\frac{\partial W_1}{\partial y_1} = \frac{y_1(y_1-(1-\gamma))(y_1-1)}{\overline{\theta}_1(1-\underline{\theta}_2)}.$$

Thus, the first-order condition yields three distinct roots which are ordered by $0 < 1-\gamma < 1$. Hence, we obtain

$$y_1 = 1-\gamma.$$

Note that if we restrict attention to $y_1 \in [\underline{\theta}_2, \overline{\theta}_1]$, then

$$y_1 = \text{med}(\underline{\theta}_2, 1-\gamma, \overline{\theta}_1) = \begin{cases} \overline{\theta}_1 & \text{if } \overline{\theta}_1 \le 1-\gamma \\ 1-\gamma & \text{if } \underline{\theta}_2 \le 1-\gamma \le \overline{\theta}_1 \\ \underline{\theta}_2 & 1-\gamma \le \underline{\theta}_2 \end{cases}.$$

However, since the value of $\text{med}(\theta_1,\theta_2,y_1)$ does not change, we can simply take $y_1 = 1-\gamma$.

Second, the part of the social welfare that depends on y_2 is:

$$\begin{aligned} W_2 :=& -\frac{1-\gamma}{\overline{\theta}_1(1-\underline{\theta}_2)} \int_{\underline{\theta}_2}^{y_2} \int_{\underline{\theta}_2}^{\theta_1} (\theta_1-\theta_2)^2 d\theta_2 d\theta_1 \\ & - \frac{1}{\overline{\theta}_1(1-\underline{\theta}_2)} \int_{y_2}^{\overline{\theta}_1} \int_{\underline{\theta}_2}^{y_2} \{\gamma(y_2-\theta_1)^2 + (1-\gamma)(y_2-\theta_2)^2\} d\theta_2 d\theta_1 \\ & - \frac{\gamma}{\overline{\theta}_1(1-\underline{\theta}_2)} \int_{y_2}^{\overline{\theta}_1} \int_{y_2}^{\theta_1} (\theta_2-\theta_1)^2 d\theta_2 d\theta_1. \end{aligned}$$

It can be seen that W_2 is a quartic function in y_2 with a positive coefficient on y_2^4, and its derivative is

$$\frac{\partial W_2}{\partial y_2} = \frac{(y_2 - \underline{\theta}_2)(y_2 - \overline{\theta}_1)(y_2 - (\gamma\overline{\theta}_1 + (1-\gamma)\underline{\theta}_2))}{\overline{\theta}_1(1-\underline{\theta}_2)}.$$

Since the first-order condition yields the three distinct roots which are ordered as

$$\underline{\theta}_2 < \gamma\overline{\theta}_1 + (1-\gamma)\underline{\theta}_2 < \overline{\theta}_1,$$

we obtain the best constant y_2 as

$$y_2 = \gamma\overline{\theta}_1 + (1-\gamma)\underline{\theta}_2.$$

The proof is complete. Note that the optimal allocation is unique. □

Now, by construction, the optimal continuous ex-post IC allocation a yields strictly better social welfare than a^1 and a^2. The proof of Part (4) is complete.

To conclude this subsection, we make two remarks on the comparisons of social welfare. First, notice that the best allocation which depends on at most one agent's information is written as

$$\begin{aligned} a^i(\theta_i) &= \min(k_i^h, \max(\theta_i, k_i^\ell)) \\ &= \min\left(k_i^h, \max(\theta_i, k_i^\ell), \max(\theta_{-i}, k_i^h), \max(\theta_1, \theta_2, k_i^\ell)\right). \end{aligned}$$

Note that the last term $\max(\theta_1, \theta_2, k_i^\ell)$ is redundant and a^i depends only on θ_i. Now, even the allocation

$$a(\theta_1, \theta_2) = \min\left(\max(\theta_i, k_i^\ell), \max(\theta_{-i}, k_i^h), \max(\theta_1, \theta_2)\right)$$

turns out to be a strict improvement, although it is cumbersome to consider many cases in which the cutoffs and the end-points of agents' type spaces are ordered and also to prove the improvement for both agents (comparing the social welfare associated with a^1 and a^2 is quite cumbersome when γ may not be $\frac{1}{2}$). Intuitively, if both agents' types are above the higher cap k_1^h of agent 1 then it is better to make the allocation the minimum of the agents' types instead of the constant; and similarly, if both agents' types are below the lower cap k_2^ℓ of agent 2 then it is better to make the allocation the maximum of the types. The optimal continuous ex-post IC allocation is indeed of this form with different constants $1-\gamma$ and $\gamma\overline{\theta}_1 + (1-\gamma)(1+\underline{\theta}_2)$.

Second, to conclude the proof of Part (4) of Theorem 1, we provide the optimal cutoffs (k_i^ℓ, k_i^h) for the delegation allocations a^1 and a^2. Figure 3 illustrates them.

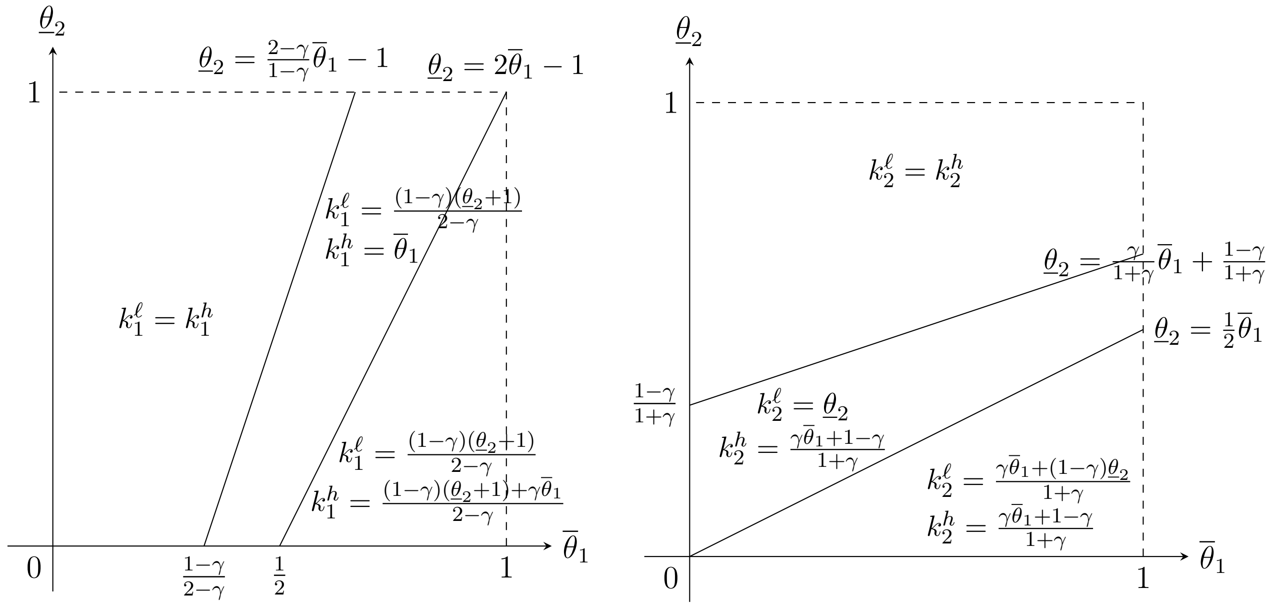

Figure 3: Cutoffs for Delegation Allocations a^1 (Left) and $a^2(\cdot)$ (Right). In the left panel, if $k_1^\ell = k_1^h$ then the best allocation (which depends on at most agent 1's information) is the best constant allocation. Likewise, in the right panel, if $k_2^\ell = k_2^h$ then the best allocation (which depends on at most agent 2's information) is the best constant allocation.

Remark A.1. 1. Suppose that a depends on at most agent 1's types. Then, the optimal allocation a^1 is either the best constant allocation or the delegation allocation for agent 1, given as follows. If $\underline{\theta}_2 \geq \frac{2-\gamma}{1-\gamma}\overline{\theta}_1 - 1$, then the optimal allocation is the best constant allocation (i.e., the optimal allocation does not even incorporate agent 1's information). If $\frac{2-\gamma}{1-\gamma}\overline{\theta}_1 - 1 \geq \underline{\theta}_2 \geq 2\overline{\theta}_1 - 1$, then the optimal allocation is the delegation allocation for agent 1 of the following form:

$$a^1(\theta_1) = \begin{cases} \frac{(1-\gamma)(\underline{\theta}_2+1)}{2-\gamma} & \text{if } \theta_1 \in \left[0, \frac{(1-\gamma)(\underline{\theta}_2+1)}{2-\gamma}\right] \\ \theta_1 & \text{if } \theta_1 \in \left[\frac{(1-\gamma)(\underline{\theta}_2+1)}{2-\gamma}, \overline{\theta}_1\right] \end{cases}.$$

If $2\overline{\theta}_1 - 1 \geq \underline{\theta}_2$, then the optimal allocation is the delegation allocation for agent 1 of the following form:

$$a^1(\theta_1) = \begin{cases} \frac{(1-\gamma)(\underline{\theta}_2+1)}{2-\gamma} & \text{if } \theta_1 \in \left[0, \frac{(1-\gamma)(\underline{\theta}_2+1)}{2-\gamma}\right] \\ \theta_1 & \text{if } \theta_1 \in \left[\frac{(1-\gamma)(\underline{\theta}_2+1)}{2-\gamma}, \frac{(1-\gamma)(\underline{\theta}_2+1)+\gamma\overline{\theta}_1}{2-\gamma}\right] \\ \frac{(1-\gamma)(\underline{\theta}_2+1)+\gamma\overline{\theta}_1}{2-\gamma} & \text{if } \theta_1 \in \left[\frac{(1-\gamma)(\underline{\theta}_2+1)+\gamma\overline{\theta}_1}{2-\gamma}, \overline{\theta}_1\right] \end{cases}.$$

2. Suppose that a depends on at most agent 2's types. Then, the optimal allocation a^2 is either the best constant allocation or the delegation allocation for agent 2, given as follows. If $\underline{\theta}_2 \geq \frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma}$, then the optimal allocation is the best constant allocation (i.e., the optimal allocation does not even incorporate agent 2's information). If $\frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma} \geq \underline{\theta}_2 \geq \frac{\overline{\theta}_1}{2}$, then the optimal allocation is the delegation allocation for agent 2 of the following form:

$$a^2(\theta_2) = \begin{cases} \theta_2 & \text{if } \theta_2 \in \left[\underline{\theta}_2, \frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma}\right] \\ \frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma} & \text{if } \theta_2 \in \left[\frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma}, 1\right] \end{cases}.$$

If $\frac{\overline{\theta}_1}{2} \geq \underline{\theta}_2$, then the optimal allocation is the delegation allocation for agent 2 of the following form:

$$a^2(\theta_2) = \begin{cases} \frac{\gamma\overline{\theta}_1+(1-\gamma)\underline{\theta}_2}{1+\gamma} & \text{if } \theta_2 \in \left[\underline{\theta}_2, \frac{\gamma\overline{\theta}_1+(1-\gamma)\underline{\theta}_2}{1+\gamma}\right] \\ \theta_2 & \text{if } \theta_2 \in \left[\frac{\gamma\overline{\theta}_1+(1-\gamma)\underline{\theta}_2}{1+\gamma}, \frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma}\right] \\ \frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma} & \text{if } \theta_2 \in \left[\frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma}, 1\right] \end{cases}.$$

A.2 Proof of Proposition 1

Section A.2.1 sets up the Lagrangian and derives the first-order conditions. Section A.2.2 provides the proof of Proposition 1.

A.2.1 The Lagrangian and the First-Order Conditions

To set up the Lagrangian, recall that each agent i's expected utility of allocation is denoted by $U_i(\theta_i) = -\mathbb{E}_{\theta_{-i}}\left[(a(\theta_1,\theta_2)-\theta_i)^2\right]$. Letting γ_i be agent i's Pareto weight, the social welfare is $\sum_{i=1}^2 \gamma_i \mathbb{E}_{\theta_i}\left[U_i(\theta_i)\right]$. We consider the relaxed problem only with the local IC constraints:

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + 2\int_{\underline{\theta}_i}^{\theta_i} \left(\mathbb{E}_{\theta_{-i}}\left[a(\tau,\theta_{-i})\right] - \tau\right) d\tau \text{ for all } \theta_i \in \Theta_i.$$

Assume that each Lagrange multiplier Λ_i associated with agent i's local IC constraint is absolutely continuous. Denote by λ_i its density. Then, we set up the Lagrangian as:

$$\mathcal{L} = \sum_{i=1}^2 \gamma_i \mathbb{E}_{\theta_i}\left[U_i\left(\theta_i\right)\right] - \sum_{i=1}^2 \int_{\underline{\theta}_i}^{\overline{\theta}_i} \left\{ U_i(\theta_i) - U_i(\underline{\theta}_i) - 2\int_{\underline{\theta}_i}^{\theta_i} \left(\mathbb{E}_{\theta_{-i}}\left[a(\tau,\theta_{-i})\right] - \tau\right) d\tau \right\} \lambda_i(\theta_i) d\theta_i. \tag{A.43}$$

Without loss of generality, we normalize Λ_1 and Λ_2 by $\Lambda_1(\underline{\theta}_1) = \Lambda_2(\overline{\theta}_2) = 0$.

We show that the Lagrangian can be written as the (weighted) sum of virtual utilities:

$$\mathcal{L} = \sum_{i=1}^{2} \gamma_i \mathbb{E}_\theta \left[-\left(a(\theta_i,\theta_{-i}) - \theta_i\right)^2 \left(1 - \frac{\lambda_i(\theta_i)}{\gamma_i f_i(\theta_i)}\right) + 2(a(\theta_i,\theta_{-i}) - \theta_i) \left(\frac{\Lambda_i(\overline{\theta}_i) - \Lambda_i(\theta_i)}{\gamma_i f_i(\theta_i)}\right)\right]. \tag{A.44}$$

To obtain this expression, for the terms of the Lagrangian (A.43) that correspond to agents' local IC constraints, first, we have

$$\begin{aligned}\sum_{i=1}^{2} \int_{\underline{\theta}_i}^{\overline{\theta}_i} \{U_i(\theta_i) - U_i(\underline{\theta}_i)\} \lambda_i(\theta_i) d\theta_i &= \sum_{i=1}^{2} \int_{\underline{\theta}_i}^{\overline{\theta}_i} U_i(\theta_i) \frac{\lambda_i(\theta_i)}{f_i(\theta_i)} dF_i(\theta_i) - \sum_{i=1}^{n} U_i(\underline{\theta}_i) \int_{\underline{\theta}_i}^{\overline{\theta}_i} \lambda_i(\theta_i) d\theta_i \\ &= \sum_{i=1}^{2} \mathbb{E}_{\theta_i}\left[U_i(\theta_i)\frac{\lambda_i(\theta_i)}{f_i(\theta_i)}\right] - \sum_{i=1}^{2} U_i(\underline{\theta}_i)\left(\Lambda_i(\overline{\theta}_i) - \Lambda_i(\underline{\theta}_i)\right).\end{aligned}$$

Second, we have:

$$\begin{aligned}\sum_{i=1}^{2} \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\underline{\theta}_i}^{\theta_i} \mathbb{E}_{\theta_{-i}}\left[a(\tau,\theta_{-i}) - \tau\right] d\tau \lambda_i(\theta_i) d\theta_i &= \sum_{i=1}^{2} \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\tau}^{\overline{\theta}_i} \mathbb{E}_{\theta_{-i}}\left[a(\tau,\theta_{-i}) - \tau\right] \lambda_i(\theta_i) d\theta_i d\tau \\ &= \sum_{i=1}^{2} \int_{\underline{\theta}_i}^{\overline{\theta}_i} \mathbb{E}_{\theta_{-i}}\left[a(\tau,\theta_{-i}) - \tau\right] \left(\Lambda_i(\overline{\theta}_i) - \Lambda_i(\tau)\right) d\tau \\ &= \sum_{i=1}^{2} \mathbb{E}_{\theta_i}\left[\mathbb{E}_{\theta_{-i}}\left[a(\theta_i,\theta_{-i}) - \theta_i\right]\left(\frac{\Lambda_i(\overline{\theta}_i) - \Lambda_i(\theta_i)}{f_i(\theta_i)}\right)\right].\end{aligned}$$

Substituting these expressions into the Lagrangian and taking the first-order condition with respect to the action $a(\theta_1,\theta_2)$ yields:

$$\begin{aligned}\sum_{i=1}^{2} \Bigg[(a(\theta) - \theta_i) \left(\gamma_i - \frac{\lambda_i(\theta_i)}{f_i(\theta_i)}\right) - \left(\frac{\Lambda_i(\overline{\theta}_i) - \Lambda_i(\theta_i)}{f_i(\theta_i)}\right) \\ + \mathbb{I}(\theta_i = \underline{\theta}_i)(a(\underline{\theta}_i,\theta_{-i}) - \underline{\theta}_i) \left(\frac{\Lambda_i(\overline{\theta}_i) - \Lambda_i(\underline{\theta}_i)}{f_i(\theta_i)}\right)\Bigg] = 0.\end{aligned}$$

Now, we show that

$$\Lambda_i(\underline{\theta}_i) = \Lambda_i(\overline{\theta}_i) = 0,$$

using the assumption that $\gamma = \frac{1}{2}$ and $\overline{\theta}_1 = 1 - \varepsilon$ and $\underline{\theta}_2 = \varepsilon$ for some $\varepsilon \in [0, \frac{1}{2})$.[2] We prove the statement for $i = 1$ (the case with $i = 2$ is similar).

[2]The statement holds as long as $a(\underline{\theta}_i, \overline{\theta}_{-i}) \neq \underline{\theta}_i$ (which is guaranteed by the assumption as $a(\underline{\theta}_i, \overline{\theta}_{-i}) = \frac{1}{2}$).

On the one hand, the first-order condition with respect to $a(\underline{\theta}_1, \overline{\theta}_2)$ is:

$$\begin{aligned}
&a(\underline{\theta}_1, \overline{\theta}_2)\left(1 - \frac{\lambda_1(\underline{\theta}_1)}{f_1(\underline{\theta}_1)} - \frac{\lambda_2(\overline{\theta}_2)}{f_2(\overline{\theta}_2)} + \frac{\Lambda_1(\overline{\theta}_1) - \Lambda_1(\underline{\theta}_1)}{f_1(\underline{\theta}_1)}\right) \\
=&\underline{\theta}_1\left(\gamma_1 - \frac{\lambda_1(\underline{\theta}_1)}{f_1(\underline{\theta}_1)}\right) + \overline{\theta}_2\left(\gamma_2 - \frac{\lambda_2(\overline{\theta}_2)}{f_2(\overline{\theta}_2)}\right) + \frac{\Lambda_1(\overline{\theta}_1) - \Lambda_1(\underline{\theta}_1)}{f_1(\underline{\theta}_1)}(1 + \underline{\theta}_1).
\end{aligned}$$

On the other hand, the first-order condition with respect to $a(\theta_1, \overline{\theta}_2)$ with $\theta_1 \neq \underline{\theta}_1$ is:

$$\begin{aligned}
&a(\theta_1, \overline{\theta}_2)\left(1 - \frac{\lambda_1(\theta_1)}{f_1(\theta_1)} - \frac{\lambda_2(\overline{\theta}_2)}{f_2(\overline{\theta}_2)}\right) \\
=&\theta_1\left(\gamma_1 - \frac{\lambda_1(\theta_1)}{f_1(\theta_1)}\right) + \overline{\theta}_2\left(\gamma_2 - \frac{\lambda_2(\overline{\theta}_2)}{f_2(\overline{\theta}_2)}\right) + \frac{\Lambda_1(\overline{\theta}_1) - \Lambda_1(\theta_1)}{f_1(\theta_1)}.
\end{aligned}$$

As $a(\theta_1, \overline{\theta}_2)$ is (assumed to be) monotonic (or assuming $a(\theta_1, \overline{\theta}_2)$ is continuous), letting $\theta_1 \downarrow \underline{\theta}_1$ implies

$$\left(a(\underline{\theta}_1, \overline{\theta}_2) - \underline{\theta}_1\right)\frac{\Lambda_1(\overline{\theta}_1) - \Lambda_1(\underline{\theta}_1)}{f_1(\underline{\theta}_1)} = 0.$$

By symmetry, $a(\underline{\theta}_1, \overline{\theta}_2) = \frac{1}{2} \neq 0$.[3] Thus, it follows that $\Lambda_1(\overline{\theta}_1) = \Lambda_1(\underline{\theta}_1)$, as desired.

Then, the Lagrangian reduces to Expression (A.44), as desired. Now, the first-order condition with respect to $a(\theta_1, \theta_2)$ can be rewritten as:

$$\sum_{i=1}^{2}\gamma_i\left[(a(\theta) - \theta_i)\left(1 - \frac{\lambda_i(\theta_i)}{\gamma_i f_i(\theta_i)}\right) + \frac{\Lambda_i(\theta_i)}{\gamma_i f_i(\theta_i)}\right] = 0. \tag{A.45}$$

We numerically verify the second-order condition, under which the Lagrangian is concave in a:

$$1 - \frac{\lambda_1(\theta_1)}{f_1(\theta_1)} - \frac{\lambda_2(\theta_2)}{f_2(\theta_2)} > 0 \text{ for all } (\theta_1, \theta_2) \in \Theta. \tag{A.46}$$

Generally, the optimal action is:

$$a(\theta_1, \theta_2) = \frac{(\gamma_1 f_1(\theta_1) - \lambda_1(\theta_1))f_2(\theta_2)\theta_1 + (\gamma_2 f_2(\theta_2) - \lambda_2(\theta_2))f_1(\theta_1)\theta_2 - f_2(\theta_2)\Lambda_1(\theta_1) - f_1(\theta_1)\Lambda_2(\theta_2)}{(\gamma_1 f_1(\theta_1) - \lambda_1(\theta_1))f_2(\theta_2) + (\gamma_2 f_2(\theta_2) - \lambda_2(\theta_2))f_1(\theta_1)}.$$

Particularly in our setting in which f_1 and f_2 are a constant and in which $\gamma_1 = \gamma_2 = \frac{1}{2}$,

[3] Consider the problem in which each agent i's type is rescaled as $1 - \theta_i$ and in which the role of agents 1 and 2 are exchanged. Thus, we have $a(\theta_1, \theta_2) = 1 - a(1 - \theta_2, 1 - \theta_1)$.

denoting by $f = f_i$, the optimal action is:

$$a(\theta_1, \theta_2) = \frac{(\frac{f}{2} - \lambda_1(\theta_1))\theta_1 + (\frac{f}{2} - \lambda_2(\theta_2))\theta_2 - \Lambda_1(\theta_1) - \Lambda_2(\theta_2)}{(\frac{f}{2} - \lambda_1(\theta_1)) + (\frac{f}{2} - \lambda_2(\theta_2))}. \tag{A.47}$$

Moreover, we show that

$$\Lambda_1(\theta_1) = -\Lambda_2(1 - \theta_1) \tag{A.48}$$

and consequently that

$$\lambda_1(\theta_1) = \lambda_2(1 - \theta_1).$$

To see this, we decompose agent i's local IC constraint into the downward and upward ones:

$$U_i(\theta_i) \geq U_i(\underline{\theta}_i) + 2\int_{\underline{\theta}_i}^{\theta_i} \mathbb{E}_{\theta_{-i}}\left[a(\tau, \theta_{-i}) - \tau\right] d\tau; \text{ and} \tag{LICi-DOWN}$$

$$U_i(\theta_i) \leq U_i(\underline{\theta}_i) + 2\int_{\underline{\theta}_i}^{\theta_i} \mathbb{E}_{\theta_{-i}}\left[a(\tau, \theta_{-i}) - \tau\right] d\tau. \tag{LICi-UP}$$

Next, denote by Λ_i^- and Λ_i^+ the Lagrange multiplier associated with Expressions (LICi-DOWN) and (LICi-UP), respectively. Then, we have

$$\Lambda_i = \Lambda_i^- - \Lambda_i^+. \tag{A.49}$$

The downward (resp. upward) local IC constraint that type $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$ of agent i faces has to be the same as the upward (resp. downward) local IC constraint that type $1-\theta_{-i} \in [\underline{\theta}_{-i}, \overline{\theta}_{-i}]$ of agent $-i$ faces. Thus, we have:

$$\Lambda_1^-(\theta_1) = \Lambda_2^+(1 - \theta_1) \text{ for each } \theta_1 \in [\underline{\theta}_1, \overline{\theta}_1] \text{ and}$$
$$\Lambda_2^-(\theta_2) = \Lambda_1^+(1 - \theta_2) \text{ for each } \theta_2 \in [\underline{\theta}_2, \overline{\theta}_2].$$

Thus, for each $\theta_1 \in [\underline{\theta}_1, \overline{\theta}_1]$, we have

$$\Lambda_1(\theta_1) = \Lambda_1^-(\theta_1) - \Lambda_1^+(\theta_1) = -\Lambda_2^-(1 - \theta_1) + \Lambda_2^+(1 - \theta_1) = -\Lambda_2(1 - \theta_1),$$

establishing expression (A.48).

Moreover, in the fully-overlapping case $\Theta_1 = \Theta_2 = [0, 1]$ (i.e., $\varepsilon = 0$), we have $\Lambda_1 = \Lambda_2$, which additionally implies $\Lambda_i(\frac{1}{2}) = 0$. This would correspond to the fact that, for type $\theta_i \in [\frac{1}{2}, 1]$, the relevant IC constraint is an upward one while, for type $\theta_i \in [0, \frac{1}{2}]$, the relevant IC constraint is a downward one.

Then, we can rewrite the action $a(\theta_1, \theta_2)$ as

$$a(\theta_1,\theta_2) = \frac{(\frac{f}{2} - \lambda_1(\theta_1))\theta_1 + (\frac{f}{2} - \lambda_1(1-\theta_2))\theta_2 - \Lambda_1(\theta_1) + \Lambda_1(1-\theta_2)}{(\frac{f}{2} - \lambda_1(\theta_1)) + (\frac{f}{2} - \lambda_1(1-\theta_2))}. \tag{A.50}$$

A.2.2 Ex-Post Inefficiency

Now, we move on to the proof of Proposition 1. For $\theta_1 \in [\underline{\theta}_2, \overline{\theta}_1]$, it follows from Expressions (A.47) and (A.50) that

$$a(\theta_1,\theta_1) = \theta_1 - \frac{\Lambda_1(\theta_1) + \Lambda_2(\theta_1)}{\frac{1}{1-\varepsilon} - \lambda_1(\theta_1) - \lambda_2(\theta_1)} = \theta_1 - \frac{\Lambda_1(\theta_1) - \Lambda_1(1-\theta_1)}{\frac{1}{1-\varepsilon} - \lambda_1(\theta_1) - \lambda_1(1-\theta_1)}. \tag{A.51}$$

By the second-order condition (A.46), the denominator of the second term is always positive. Since $\Lambda_1 = \Lambda_1^- - \Lambda_1^+$, it must be the case that

$$\Lambda_1(\theta_1) - \Lambda_1(1-\theta_1) \begin{cases} > 0 & \text{if } \theta_1 \in (\overline{\theta}_2, \frac{1}{2}) \\ = 0 & \text{if } \theta_1 = \frac{1}{2} \\ < 0 & \text{if } \theta_1 \in (\frac{1}{2}, \overline{\theta}_1) \end{cases}.$$

That is, when $\theta_1 \in (\overline{\theta}_2, \frac{1}{2})$, the downward local IC constraint associated with $\Lambda_1(\theta_1) > 0$ and the upward local IC constraint associated with $\Lambda_1(1-\theta_1) < 0$ yields $\Lambda_1(\theta_1) - \Lambda_1(1-\theta_1) > 0$. Similarly, when $\theta_1 \in (\frac{1}{2}, \overline{\theta}_1)$, the upward local IC constraint associated with $\Lambda_1(\theta_1) < 0$ and the downward local IC constraint associated with $\Lambda_1(1-\theta_1) > 0$ yields $\Lambda_1(\theta_1) - \Lambda_1(1-\theta_1) < 0$. The proof is complete.

To conclude the proof of Proposition 1, we remark on the role of Monotonicity. To that end, observe that agent i's local IC constraint would be written as

$$\mathbb{E}_{\theta_{-i}}\left[\frac{\partial a(\theta_i,\theta_{-i})}{\partial \theta_i}\left(a(\theta_i,\theta_{-i}) - \theta_i\right)\right] = 0.$$

The fact that Monotonicity plays a role in the delegation solutions comes from the fact that the delegation allocation, say, for agent 2 is always above agent 1's type. Thus, a slight decrease in an allocation (the violation of Monotonicity, which is of the second-order inefficiency loss) has a larger impact on relaxing the Bayesian IC constraint (which is of the first-order efficiency gain). In contrast, when agents' type spaces may overlap, the allocation $a(\theta_1, \theta_1) - \theta_1$ changes the sign at $\theta_1 = \frac{1}{2}$. In this case, the Bayesian IC constraint can be relaxed not by violating Monotonicity (i.e., the first term of the expectation) but by making the allocation further away from the reported type (i.e., the second term of the expectation). Thus, even when Monotonicity is explicitly imposed, the optimal allocation would still exhibit ex-post inefficiency.

A.3 Monotonicity

In the main text, we impose (Mon) because the use of (Mon) rather than (Ex-Mon$i$) (for each $i \in N$) significantly reduces the difficulty of characterizing certain properties of the optimal allocation. Naturally, the reader might question what happens when one does not impose it. We have investigated this question numerically. For the overlapping case ($\underline{\theta}_2 < \overline{\theta}_1$), our simulations (such as the ones illustrated in Figures 3 and 5) only imposing (ICi) show that the violation of Monotonicity (in terms of the slope of the allocation) is typically of the order of 10^{-3} in the region in which the optimal allocation is flat. There, one agent's type is high while the other agent's is low. Thus we conjecture that it is without loss. In contrast, for the non-overlapping case, we can analytically show that the monotonicity constraint is binding for a delegation allocation. While we have found the violation of Monotonicity extensively in the region in which the delegation allocation is flat, we have also found the violation to be typically of the order of 10^{-3}.

A.4 Money Burning and Stochastic Allocations

In the main text, we restrict the allocation to be deterministic and furthermore we rule out the use of money burning. Within the delegation literature, Kováč and Mylovanov (2009), Amador and Bagwell (2013, 2020), and Ambrus and Egorov (2017) among others, have studied the use of stochastic allocations or money burning. In particular, they provide conditions under which money burning or stochastic allocations might be useful and when they are not. Given our use of quadratic preferences (which is common in this literature), agents essentially care about the expected allocation and conditional variance (see, for instance, Goltsman, Hörner, Pavlov, and Squintani, 2009). In particular, the variance term enters just like money burning into the agent's utility function. Thus, if we cannot achieve a better outcome by allowing money burning, stochastic allocations would not help either.

When we have a complete characterization of the allocation as in the main results in Theorem 1, in Section A.4.3, we solve the relaxed problem allowing for money burning and verify analytically that it is not used. Thus, our main result about the endogenous delegation is robust to this extension.[4]

[4]For the special case in which one of the agents' type spaces is degenerate we can also use the results of Amador and Bagwell (2013) to establish that money burning is not used.

For the special case in which the agents' type spaces coincide, even though we do not have a complete characterization of the optimal allocation, using the Lagrange multipliers, in Section A.4.4, we still analytically show that money burning is not used. For the case in which the agents' type spaces partially overlap, our numerical simulations suggest that money burning is never used.

Note that, even though money burning is not used, Proposition 1 shows that certain actions exhibit ex-post inefficiency.

Below, we show that the optimal allocation does not entail money burning or randomization. To that end, Sections A.4.1 and A.4.2 state the problems allowing for money burning and randomization, respectively.

A.4.1 Problem with Money Burning

Here, we allow the possibility that each type θ_i of each agent i can burn money $t_i(\theta_i)$. Thus, the problem is to find an allocation $(a,(t_i)_{i\in N})$ which maximizes the sum of agents' utilities subject to the incentive-compatibility and monotonicity constraints and the money burning constraint $t_i(\cdot)\geq 0$. Namely:

$$\max_{(a,(t_i)_i)} \mathbb{E}_\theta \left[-\gamma(a(\theta)-\theta_1)^2 - t_1(\theta_1) - (1-\gamma)(a(\theta)-\theta_2)^2 - t_2(\theta_2)\right]$$

subject to $\mathbb{E}_{\theta_{-i}}\left[-(a(\theta_i,\theta_{-i})-\theta_i)^2\right] - t_i(\theta_i) \geq \mathbb{E}_{\theta_{-i}}\left[-(a(\hat{\theta}_i,\theta_{-i})-\theta_i)^2\right] - t_i(\hat{\theta}_i)$ for each $i,\theta_i,\hat{\theta}_i$;

$a(\cdot)$ is non-decreasining in each $\theta_i\in\Theta_i$; and

$t_i(\cdot)\geq 0$ for each $i\in N$.

As is well known in mechanism design theory, the above IC constraint is decomposed into (i) the local IC constraint: for all $\theta_i\in\Theta_i$,

$$-\mathbb{E}_{\theta_{-i}}\left[(a(\theta_i,\theta_{-i})-\theta_i)^2\right] - t_i(\theta_i) = -\mathbb{E}_{\theta_{-i}}\left[(a(\underline{\theta}_i,\theta_{-i})-\underline{\theta}_i)^2\right] - t_i(\underline{\theta}_i) + 2\int_{\underline{\theta}_i}^{\theta_i}\left(\mathbb{E}_{\theta_{-i}}\left[a(\tau,\theta_{-i})\right]-\tau\right)d\tau; \qquad \text{(LIC}i\text{-MB)}$$

and (ii) the expected monotonicity constraint: for any $\theta_i,\hat{\theta}_i\in\Theta_i$ with $\theta_i\geq\hat{\theta}_i$,

$$\mathbb{E}_{\theta_{-i}}\left[a(\theta_i,\theta_{-i})\right] \geq \mathbb{E}_{\theta_{-i}}\left[a(\hat{\theta}_i,\theta_{-i})\right].$$

A.4.2 Stochastic Allocations

Next, we consider stochastic allocations and show that if an optimal allocation does not entail money burning then it is a deterministic allocation. We first define a stochastic allocation as a function $P : \Theta \to \Delta(\mathcal{A})$. We define the conditional expected allocation and the variance of the conditional expected allocation associated with P, respectively, as follows:

$$\overline{a}_P(\theta) := \int_{\mathcal{A}} a dP\left(a|\theta\right) \text{ and } \sigma_P^2(\theta) := \int_{\mathcal{A}} \left(a - \overline{a}\left(\theta\right)\right)^2 dP\left(a|\theta\right).$$

We often denote $\overline{a}(\cdot) := \overline{a}_P(\cdot)$ and $\sigma^2(\cdot) := \sigma_P^2(\cdot)$.

We say that the stochastic allocation P satisfies Monotonicity if

$$\overline{a} \text{ is non-decreasing in each } \theta_i \in \Theta_i. \tag{Mon-S}$$

The stochastic allocation P satisfies agent i's (Bayesian) incentive-compatibility (IC) constraint if, for all $\theta_i, \hat{\theta}_i \in \Theta_i$,

$$-\mathbb{E}_{\theta_{-i}}\left[\int_{\mathcal{A}} \left(a - \theta_i\right)^2 dP(a|\theta_i, \theta_{-i})\right] \geq -\mathbb{E}_{\theta_{-i}}\left[\int_{\mathcal{A}} \left(a - \theta_i\right)^2 dP(a|\hat{\theta}_i, \theta_{-i})\right]. \tag{ICi-S}$$

The stochastic allocation P is incentive-compatible (IC) if it satisfies (ICi-S) for each $i \in N$.

The problem is to maximize the sum of (ex-ante) expected utilities subject to the incentive-compatibility and monotonicity constraints:

$$\begin{aligned} &\max_P -\mathbb{E}_\theta\left[\int_{\mathcal{A}} \left\{\gamma(a-\theta_1)^2 + (1-\gamma)(a-\theta_2)^2\right\} dP(a \mid \theta)\right] \\ &\text{subject to (IC1-S), (IC2-S), and (Mon-S).} \end{aligned}$$

To characterize the (Bayesian) IC constraint, with some abuse of notation, we define the expected utility of agent i from the stochastic allocation P when her type is θ_i and she announces $\hat{\theta}_i$:

$$U_i(\theta_i, \hat{\theta}_i) := -\mathbb{E}_{\theta_{-i}}\left[\int_{\mathcal{A}} \left(a - \theta_i\right)^2 dP(a|\hat{\theta}_i, \theta_{-i})\right].$$

Denote by $U_i(\theta_i) := U_i(\theta_i, \theta_i)$ the (interim) expected utility of type θ_i of agent i from the stochastic allocation P when she announces her type truthfully. A stochastic allocation P is IC if and only if (i) it satisfies the local IC constraint: for all $\theta_i \in \Theta_i$,

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + 2\int_{\underline{\theta}_i}^{\theta_i} \left(\mathbb{E}_{\theta_{-i}}\left[\overline{a}(\tau, \theta_{-i})\right] - \tau\right) d\tau;$$

and (ii) it satisfies the expected monotonicity constraint: for any $\theta_i, \hat{\theta}_i \in \Theta_i$ with $\theta_i \geq \hat{\theta}_i$,

$$\mathbb{E}_{\theta_{-i}}\left[\overline{a}(\theta_i, \theta_{-i})\right] \geq \mathbb{E}_{\theta_{-i}}\left[\overline{a}(\hat{\theta}_i, \theta_{-i})\right].$$

Now, we show that, since the agents have quadratic preferences, their payoffs depend only on the conditional expectation and variance of a stochastic allocation (see, for instance, Goltsman, Hörner, Pavlov, and Squintani, 2009). In fact, each agent i's expected utility from a stochastic allocation is rewritten as:

$$U_i(\theta_i, \hat{\theta}_i) = -\mathbb{E}_{\theta_{-i}}\left[\left(\overline{a}(\hat{\theta}_i, \theta_{-i}) - \theta_i\right)^2\right] - \mathbb{E}_{\theta_{-i}}\left[\sigma^2(\hat{\theta}_i, \theta_{-i})\right].$$

Especially,

$$U_i\left(\theta_i\right) = -\mathbb{E}_{\theta_{-i}}\left[\left(\overline{a}(\theta_i, \theta_{-i}) - \theta_i\right)^2\right] - \mathbb{E}_{\theta_{-i}}\left[\sigma^2(\theta_i, \theta_{-i})\right].$$

This immediately implies that if an allocation $(a, (t_i)_i)$ does not entail any money burning then the optimal allocation is deterministic: $a = \overline{a}$ and $\sigma = 0$. Below, we show that the optimal allocation does not entail any money burning.

For the rest of this section, again with some abuse of notation, denote agent i's expected utility from an allocation $(a, (t_i)_i)$ by

$$U_i(\theta_i, \hat{\theta}_i) := \mathbb{E}_{\theta_{-i}}\left[-(a(\hat{\theta}_i, \theta_{-i}) - \theta_i)^2\right] - t_i(\hat{\theta}_i).$$

Denote also $U_i(\theta_i) := U_i(\theta_i, \theta_i)$. Now, the problem is:

$$\begin{aligned}
&\max_{(a,(t_i)_i)} \mathbb{E}_\theta\left[-\gamma(a(\theta) - \theta_1)^2 - t_1(\theta_1) - (1-\gamma)(a(\theta) - \theta_2)^2 - t_2(\theta_2)\right] \\
&\quad \text{subject to (Mon), (LIC}i\text{-MB) for each } i \in N, \text{ and} \\
&\qquad\qquad t_i(\cdot) \geq 0 \text{ for each } i \in N.
\end{aligned}$$

A.4.3 The Non-Overlapping Case

Here, we show that the optimal allocation does not entail any money burning for the non-overlapping case $\overline{\theta}_1 \leq \underline{\theta}_2$. Section A.4.3.1 sets up the Lagrangian of the problem that allows for money burning. Section A.4.3.2 proves the result.

A.4.3.1 Lagrangian

We formulate the Lagrangian which incorporates the agents' local IC and money-burning constraints (subject to the monotonicity constraint). Letting Ψ_i be the Lagrange multiplier associated with the money burning constraint $t_i \geq 0$, we guess and verify that Ψ_i has its density $\psi_i \geq 0$. To that end, as in the proof of Theorem 1, we take the "reference type" of agent 1 to be $\overline{\theta}_1$, while we take the "reference type" of agent 2 to be $\underline{\theta}_2$.

As in the proof of Theorem 1, we start with rewriting the objective function. Since

$$\begin{aligned}\mathbb{E}_{\theta_1}[U_1(\theta_1)] &= U_1(\overline{\theta}_1) - 2\mathbb{E}_\theta[(a(\theta)-\theta_1)(\theta_1-\underline{\theta}_1)] \text{ and}\\ \mathbb{E}_{\theta_2}[U_2(\theta_2)] &= U_2(\underline{\theta}_2) + 2\mathbb{E}_\theta[(a(\theta)-\theta_2)(\overline{\theta}_2-\theta_2)],\end{aligned}$$

the objective function is rewritten as

$$\begin{aligned}&\mathbb{E}_{\theta_1}[\gamma U_1(\theta_1) - (1-\gamma)t_1(\theta_1)] + \mathbb{E}_{\theta_2}[(1-\gamma)U_2(\theta_2) - \gamma t_2(\theta_2)]\\ =&\gamma U_1(\overline{\theta}_1) + (1-\gamma)U_2(\underline{\theta}_2) - 2\gamma\mathbb{E}_\theta[(a(\theta)-\theta_1)(\theta_1-\underline{\theta}_1)] + 2(1-\gamma)\mathbb{E}_\theta[(a(\theta)-\theta_2)(\overline{\theta}_2-\theta_2)]\\ &-(1-\gamma)\mathbb{E}_{\theta_1}[t_1(\theta_1)] - \gamma\mathbb{E}_{\theta_2}[t_2(\theta_2)].\end{aligned}$$

Next, for agent 1's local IC constraint, as in the proof of Theorem 1, we have:

$$\begin{aligned}&\int_{\underline{\theta}_1}^{\overline{\theta}_1}\left(U_1(\theta_1) - U_1(\overline{\theta}_1) + 2\int_{\theta_1}^{\overline{\theta}_1}\mathbb{E}_{\theta_2}\left[a(\tau_1,\theta_2)-\tau_1\right]d\tau_1\right)\lambda_1(\theta_1)d\theta_1\\ =&-U_1(\overline{\theta}_1)\Lambda_1(\overline{\theta}_1) + \int_{\underline{\theta}_1}^{\overline{\theta}_1}U_1(\theta_1)\lambda_1(\theta_1)d\theta_1 + 2\int_{\underline{\theta}_1}^{\overline{\theta}_1}\mathbb{E}_{\theta_2}\left[a(\theta)-\theta_1\right]\Lambda_1(\theta_1)d\theta_1.\end{aligned}$$

For agent 2's local IC constraint, as in the proof of Theorem 1, we have:

$$\begin{aligned}&\int_{\underline{\theta}_2}^{\overline{\theta}_2}\left(U_2(\theta_2) - U_2(\underline{\theta}_2) - 2\int_{\underline{\theta}_2}^{\theta_2}\mathbb{E}_{\theta_1}\left[a(\theta_1,\tau_2)-\tau_2\right]d\tau_2\right)\lambda_2(\theta_2)d\theta_2\\ =&U_2(\underline{\theta}_2)\Lambda_2(\underline{\theta}_2) + \int_{\underline{\theta}_2}^{\overline{\theta}_2}U_2(\theta_2)\lambda_2(\theta_2)d\theta_2 + 2\int_{\underline{\theta}_2}^{\overline{\theta}_2}\mathbb{E}_{\theta_1}\left[a(\theta)-\theta_2\right]\Lambda_2(\theta_2)d\theta_2.\end{aligned}$$

As in the proof of Theorem 1, the Lagrangian can be written as:

$$\begin{aligned}\mathcal{L} = \mathcal{L}_0 &+ \int_{\underline{\theta}_1}^{\overline{\theta}_1}t_1(\theta_1)\left(\psi_1(\theta_1) - (1-\gamma+\lambda_1(\theta_1))\right)d\theta_1 + \int_{\underline{\theta}_2}^{\overline{\theta}_2}t_2(\theta_2)\left(\psi_2(\theta_2) - (\gamma+\lambda_2(\theta_2))\right)d\theta_2\\ &- t_1(\overline{\theta}_1)\left(\gamma - \Lambda_1(\overline{\theta}_1)\right) - t_2(\underline{\theta}_2)\left(1-\gamma+\Lambda_2(\underline{\theta}_2)\right),\end{aligned}$$

where $\mathcal{L}_0$ is given by Expression (A.1), i.e., the part of the Lagrangian that correspond to the agents' local IC constraints.

A.4.3.2 No Money Burning

Since each optimal allocation a^* found in Parts (1)-(3) of Theorem 1 satisfies the first-order conditions of the Lagrangian $\mathcal{L}_0$ under the respective Lagrange multipliers (Λ_1, Λ_2) (possibly under the monotonicity constraint), if $\Lambda_1(\overline{\theta}_1) \leq \gamma$, $\Lambda_2(\underline{\theta}_2) \geq -(1-\gamma)$, and if

$$\psi_1 = 1 - \gamma + \lambda_1 \geq 0 \text{ and } \psi_2 = \gamma + \lambda_2 \geq 0,$$

then the allocation $(a^*, (t_i)_i)$ with $t_1 = t_2 = 0$ is the optimal allocation, as desired. In fact, in each of Parts (1)-(3) of Theorem 1, we have $\Lambda_1(\overline{\theta}_1) \leq \gamma$, $\Lambda_2(\underline{\theta}_2) \geq -(1-\gamma)$, and $\lambda_i \geq 0$ for each $i \in N$. Hence, the optimal allocation does not entail any money burning.

A.4.4 The Fully-Overlapping Case

Here, we show that the optimal allocation does not entail any money burning for the symmetric fully-overlapping case (i.e., $\gamma = \frac{1}{2}$ and $\Theta_1 = \Theta_2 = [0,1]$). Section A.4.4.1 sets up the Lagrangian of the problem that allows for money burning. Section A.4.4.2 proves the result.

A.4.4.1 Lagrangian

We formulate the Lagrangian which incorporates the agents' local IC and money-burning constraints (subject to the monotonicity constraint). Letting Ψ_i be the Lagrange multiplier associated with the money burning constraint $t_i \geq 0$, we guess and verify that Ψ_i has its density $\psi_i \geq 0$. To that end, as in the proof of Proposition 1, we take the "reference type" of agent i to be $\underline{\theta}_i$.

Denote by γ_i agent i's Pareto weight. The objective function is

$$\sum_{i=1}^{2} \left(\gamma_i \mathbb{E}_{\theta_i}\left[U_i(\theta_i)\right] - (1-\gamma_i)\mathbb{E}_{\theta_i}[t_i(\theta_i)]\right).$$

For a part of the terms of the Lagrangian that correspond to the agents' local IC constraints,

$$\begin{aligned}
&\sum_{i=1}^{2} \int_{\underline{\theta}_i}^{\overline{\theta}_i} \{U_i(\theta_i) - U_i(\underline{\theta}_i)\} \lambda_i(\theta_i) d\theta_i = \sum_{i=1}^{2} \mathbb{E}_{\theta_i}\left[U_i(\theta_i)\frac{\lambda_i(\theta_i)}{f_i(\theta_i)}\right] - \sum_{i=1}^{2} U_i(\underline{\theta}_i)\left(\Lambda_i(\overline{\theta}_i) - \Lambda_i(\underline{\theta}_i)\right) \\
&= -\sum_{i=1}^{2} \mathbb{E}_{\theta_i}\left[\mathbb{E}_{\theta_{-i}}\left[(a(\theta_i,\theta_{-i}) - \theta_i)^2\right]\frac{\lambda_i(\theta_i)}{f_i(\theta_i)}\right] - \sum_{i=1}^{2} \mathbb{E}_{\theta_i}\left[t_i(\theta_i)\frac{\lambda_i(\theta_i)}{f_i(\theta_i)}\right] \\
&\quad + \sum_{i=1}^{2} \mathbb{E}_{\theta_{-i}}\left[(a(\underline{\theta}_i,\theta_{-i}) - \underline{\theta}_i)^2\right]\left(\Lambda_i(\overline{\theta}_i) - \Lambda_i(\underline{\theta}_i)\right) + \sum_{i=1}^{2} t(\underline{\theta}_i)\left(\Lambda_i(\overline{\theta}_i) - \Lambda_i(\underline{\theta}_i)\right).
\end{aligned}$$

As in the proof of Proposition 1, the Lagrangian can be written as:

$$\mathcal{L} = \mathcal{L}_0 + \sum_{i=1}^{2} \left\{ \int_{\underline{\theta}_i}^{\overline{\theta}_i} t_i(\theta_i) \left(\psi_i(\theta_i) - (1 - \gamma_i - \lambda_i(\theta_i))\right) d\theta_i - t(\underline{\theta}_i) \left(\Lambda_i(\overline{\theta}_i) - \Lambda_i(\underline{\theta}_i)\right) \right\},$$

where $\mathcal{L}_0$ is given by Expression (A.44), i.e., the part of the Lagrangian that corresponds to the agents' local IC constraints.

A.4.4.2 No Money Burning

Since the optimal allocation a^* satisfies the first-order conditions of the Lagrangian $\mathcal{L}_0$ under the corresponding Lagrange multipliers (Λ_1, Λ_2), if

$$\psi_1 = 1 - \gamma - \lambda_1 \geq 0 \text{ and } \psi_2 = \gamma - \lambda_2 \geq 0,$$

then the allocation $(a^*, (t_i)_i)$ with $t_1 = t_2 = 0$ is the optimal allocation, as desired. Also, note that if these inequalities hold with strict inequality then (i) the first-order condition of the Lagrangian of the original problem (that does not allow for money burning) is also sufficient[5] and also (ii) the optimal allocation (in the problem in which money burning is allowed) does not entail any money burning.

Here, we show that money burning is not used (i.e., $1 - \gamma_i \geq \lambda_i$) for the symmetric fully-overlapping case. Noting that $\Lambda_1 = \Lambda_2$, we drop the subscript from Λ_i and λ_i. We define

$$\mu(\theta_i) = 1 - 2\lambda(\theta_i) \text{ and } M(\theta_i) = \int_{\theta_i}^{1} \mu(\tau_i) d\tau_i \left(= 1 - \theta_i + 2\Lambda(\theta_i)\right).$$

Then, observe that the action $a(\theta_1, \theta_1)$ is written as

$$a(\theta_1, \theta_1) = \theta_1 + \frac{1 - \theta_1 - M(\theta_1)}{\mu(\theta_1)}.$$

For any point θ_1 with $\mu(\theta_1) = 0$, one should have $0 = 1 - \theta_1 - M(\theta_1)$. Also, $1 - \theta_1 - M(\theta_1) = 0$ for $\theta \in \{0, \frac{1}{2}, 1\}$. We define a golden point as a point in $\mathbf{cl}(\{0, \frac{1}{2}, 1\} \cup \mu^{-1}(\{0\}))$ where $\mathbf{cl}$ is the closure operator. By construction, the set of non-golden points is open, and as a result the non-golden points are written as a union of open sets. Hence, any non-golden point $\theta \in [0, 1]$ is between two adjacent golden points $\theta^1 < \theta^2$ (i.e. the ones which have no other golden points in between). One has $0 = 1 - \theta^1 - M(\theta^1) = 1 - \theta^2 - M(\theta^2)$ then one has

[5] In fact, each agent's virtual utility in Expression (A.44) is strictly concave.

$\theta^2 - \theta^1 = \int_{\theta^1}^{\theta^2} \mu(\theta) d\theta$. Hence one should have $\mu(\theta) > 0$ on (θ^1, θ^2) because μ does not change sign between two golden points. As a result, $\mu(\theta_1) \geq 0$ for all $\theta_1 \in [0, 1]$. Thus, $\frac{1}{2} \geq \lambda_i$. The proof is complete.

A.5 The Better Communication Protocol

Here we consider the allocation with two rounds of communication without commitment that is discussed in the main text. To show that the allocation is IC, as in Alonso, Dessein, and Matouschek (2008), we invoke symmetry so that it suffices to show that type $\frac{1}{2} + c$ of agent 1 is indifferent between lying downward and upward. For ease of exposition, we denote the partitions by $[c_{-2}, c_{-1}]$, $[c_{-1}, c_0]$, $[c_0, c_1]$, and $[c_1, c_2]$. That is,

$$c_{-2} = 0, c_{-1} = \frac{1}{2} - c, c_0 = \frac{1}{2}, c_1 = \frac{1}{2} + c, \text{ and } c_2 = 1.$$

When agent 1 lies downward, her expected utility from the allocation is

$$\begin{aligned}
&- \left(\frac{c_{-2} + c_{-1} + c_0 + c_1}{4} - c_1 \right)^2 (c_{-1} - c_{-2}) - \left(\frac{c_{-1} + c_0 + c_0 + c_1}{4} - c_1 \right)^2 (c_0 - c_{-1}) \\
&- \frac{1}{2} \left(\frac{c_0 + c_1}{2} - c_1 \right)^2 (c_1 - c_0) - \frac{1}{2} \left(\frac{\frac{c_0 + c_1}{2} + c_1}{2} - c_1 \right)^2 (c_1 - c_0) \\
&- \left(\frac{c_1 + c_2 + c_0 + c_1}{4} - c_1 \right)^2 (c_2 - c_1).
\end{aligned}$$

When agent 1 lies upward, her expected utility from the allocation is

$$\begin{aligned}
&- \left(\frac{c_{-2} + c_{-1} + c_1 + c_2}{4} - c_1 \right)^2 (c_{-1} - c_{-2}) \\
&- \left(\frac{c_{-1} + c_0 + c_1 + c_2}{4} - c_1 \right)^2 (c_0 - c_{-1}) - \left(\frac{c_0 + c_1 + c_1 + c_2}{4} - c_1 \right)^2 (c_1 - c_0) \\
&- \frac{1}{2} \left(\frac{c_1 + c_2}{2} - c_1 \right)^2 (c_2 - c_1) - \frac{1}{2} \left(\frac{c_1 + \frac{c_1 + c_2}{2}}{2} - c_1 \right)^2 (c_2 - c_1).
\end{aligned}$$

Equating these two equations, we obtain the desired cubic equation:

$$48c^3 - 36c^2 - 30c + 1 = 0.$$

It can be seen that the equation has a unique root in $(0, \frac{1}{2})$. After some algebra, one can also show that the (ex-ante) expected utility from this allocation is slightly better than $-\frac{1}{21}$.

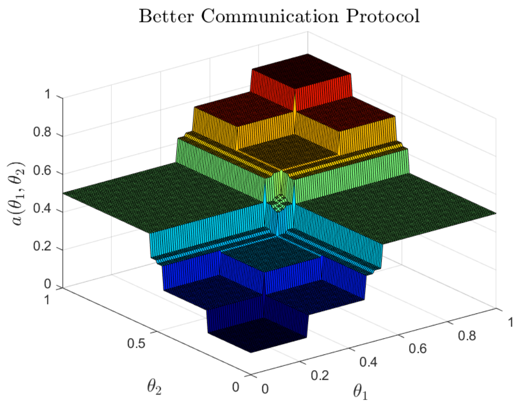

Figure 4: The Better Communication Protocol

Figure 4 depicts the allocation. Note that, the principal's IC is satisfied because the principal always chooses the average of expected value of the two agents' types given any information which is the ex-post optimal action.

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