Online Supplementary Appendix

Unprecedented

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B Supplementary Results

B.1 Supplementary Results for Section 4.2

Consider the synchronous model and suppose that a public randomization device is not available. We call such a model the *synchronous model without a public randomization device*. Note that the strategy $\sigma_i^{(T,1)}$ is well defined in such a model as well.

First, we show that a public randomization device is not necessary to obtain multiple equilibria when $x_i \leq 1$ for each $i = 1, 2$.

Proposition 5. *Suppose $x_i \leq 1$ for each $i = 1, 2$. For any $p_1, p_2 \in (0, 1]$, there exist $\delta' \in (0, 1)$ and $T' < \infty$ such that if $\delta_i \in (\delta', 1)$ for each $i = 1, 2$ and $T > T'$, then $\sigma^{(T,1)}$ is a PBE in the synchronous model without a public randomization device.*

The proof of this result as well as the next are relegated to Appendix B.3.1.

Second, we consider the case when $x_i > 1$ for some $i = 1, 2$ and demonstrate that there is a region of parameter values such that there are multiple equilibria in the synchronous model without a public randomization device while there is a unique PBE in our main model.

Proposition 6. *Suppose $x_i > 1$. For any $p_{-i} \in (0, 1]$, there exists $\delta' \in (0, 1)$ such that, for any $\delta_i \in (\delta', 1)$, the strategy $\sigma_i^{(T,1)}$ is a best response against $\sigma_{-i}^{(T,1)}$ if and only*

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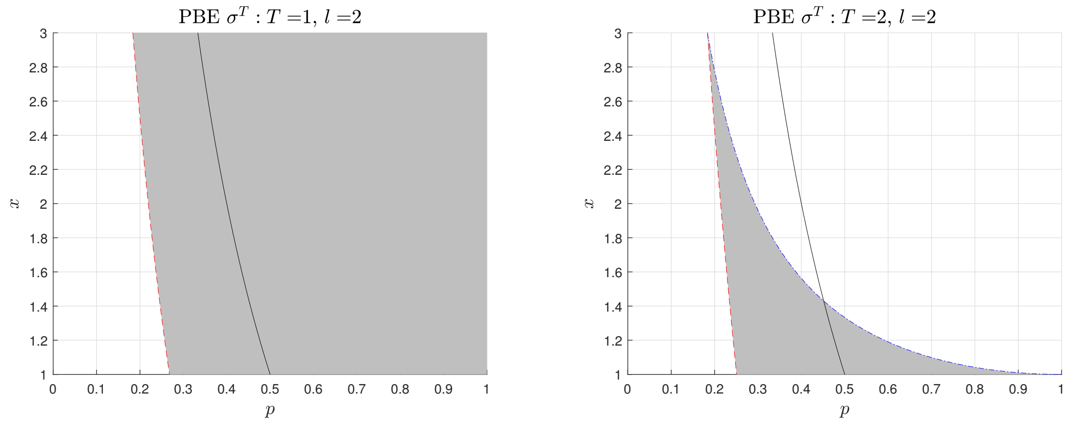


Figure 4: Illustration of Proposition 6 for different values of (p, x). The shaded region depicts the pairs of (p, x) such that $\sigma^{(T,1)}$ is a PBE for sufficiently high discount factors in the synchronous model without a public randomization device. *Left:* $T = 1$. *Right:* $T = 2$. In both panels, we set $\ell_1 = \ell_2 = 2$. The dashed and dashed-dotted curves illustrate the constraints given by (A.1) and (A.2), respectively. The solid curve illustrates the threshold below which σ^{G} is a unique PBE for sufficiently high discount factors in our main model with asynchronous moves.

if the following two conditions hold:

$$(1 - p_{-i})^T x_i \geq (1 - (1 - p_{-i})^T)(-\ell_i) + (1 - p_{-i})^T \left(T + (1 - p_{-i})^T x_i\right); \tag{A.1}$$

$$\frac{x_i - 1}{x_i} < (1 - p_{-i})^T. \tag{A.2}$$

Letting $p = p_1 = p_2$ and $x = x_1 = x_2 \geq 1$, Figure 4 uses Proposition 6 to depict the set of (p, x) such that σ^T is a PBE for sufficiently high discount factors for a fixed T. Figure 5 then depicts the set of (p, x) such that σ^T is a PBE for sufficiently high discount factors for some T. Both figures also depict the threshold probability $p = \frac{2}{1+x+l}$ below which σ^{G} is a unique PBE for sufficiently high discount factors in our main model with asynchronous moves.

B.2 Supplementary Results for Section 4.6

We provide formal statements of the alternative characterization of PBE discussed in Section 4.6 and comparative statics results based on the characterization. In our comparative statics, we show that the set of profiles of discount factors (δ_1, δ_2) under which σ^{G} is a unique PBE is weakly increasing in p_i and weakly decreasing in x_i and

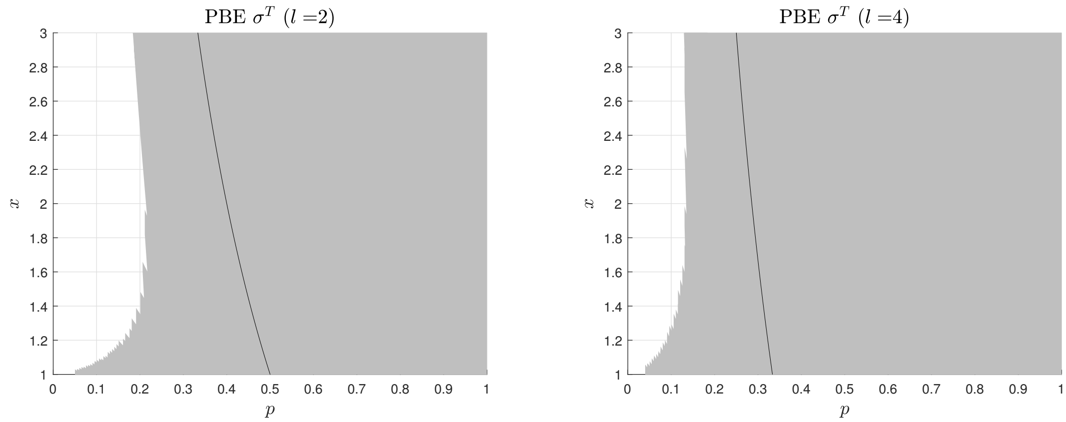


Figure 5: Illustration of Proposition 6 for different values of (p, x). The shaded region depicts the pairs of (p, x) such that $\sigma^{(T,1)}$ is a PBE for sufficiently high discount factors for some T in the synchronous model without a public randomization device. *Left:* $\ell_1 = \ell_2 = 2$. *Right:* $\ell_1 = \ell_2 = 4$. In both panels, the solid curve illustrates the threshold below which σ^{G} is a unique PBE for sufficiently high discount factors in our main model with asynchronous moves. The shaded area would occupy the entire region when $x \leq 1$.

ℓ_i (in the set-inclusion sense). To that end, we denote by

$$S = S(p_1, p_2, x_1, x_2, \ell_1, \ell_2)$$

the set of profiles of discount factors $(\delta_1, \delta_2) \in (0,1)^2$ such that σ^{G} is a unique PBE for given parameters $(p_1, p_2, x_1, x_2, \ell_1, \ell_2)$.

The results are stated for each of the following three cases that are exhaustive besides nongeneric cases: (i) $x_i > 1$ for each $i = 1, 2$; (ii) $x_i < 1$ for each $i = 1, 2$; and (iii) $x_i > 1 > x_{-i}$ for some $i = 1, 2$. The proofs are relegated to Appendix B.3.2.

First, we start with the case in which $x_i > 1$ for each $i = 1, 2$.

Proposition 7. *Fix $p_1, p_2 \in [0,1]$. Let $\delta_1, \delta_2 \in (0,1)$. Suppose $x_i > 1$ for each $i = 1, 2$. Then, $0 < \hat{\delta}_i < \overline{\delta}_i(p_{-i})$ for each $i = 1, 2$. Moreover, the following hold.*

1. *If there is $i = 1, 2$ such that $\delta_i > \hat{\delta}_i$ and $\delta_{-i} > \overline{\delta}_{-i}(p_i)$, then σ^{G} is a unique PBE.*

2. *If $\delta_i \in (\hat{\delta}_i, \overline{\delta}_i(p_{-i}))$ for each $i = 1, 2$, then both σ^{G} and σ^N are PBE.*

3. *If there is $i = 1, 2$ such that $\delta_i \in (0, \hat{\delta}_i)$ and $\delta_{-i} \in (\hat{\delta}_{-i}, \overline{\delta}_{-i}(p_i))$, then σ^N is a unique PBE.*

4. *If there is $i = 1,2$ such that $\delta_i \in (0, \hat{\delta}_i)$ and $\delta_{-i} \in (\hat{\delta}_{-i}, 1)$, then $(\sigma_i^N, \sigma_{-i}^{\mathrm{G}})$ is a unique PBE.*

This result is illustrated in the left panel of Figure 2 in Section 4.6 of the main text. Figure 6 illustrates the characterization of PBE for different values of (p_1, δ_2). For the symmetric cases, the left panel of Figure 3 in Section 4.6 of the main text illustrates the characterization.

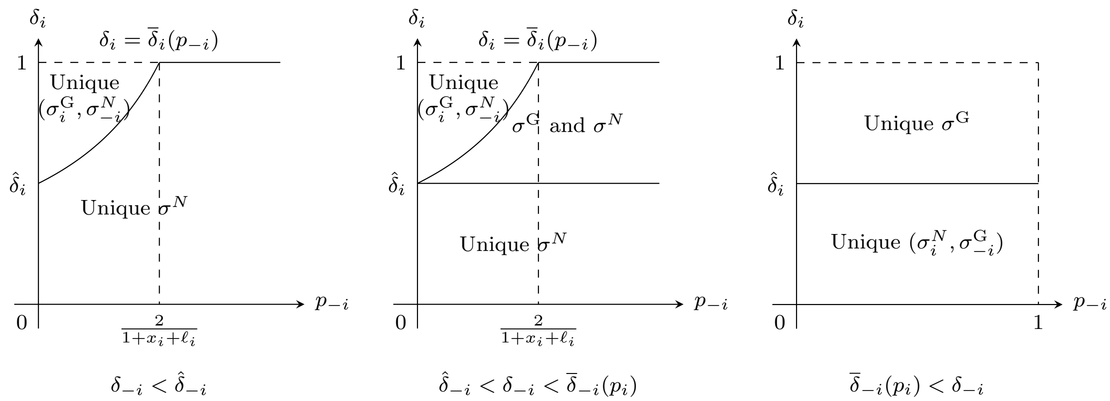


Figure 6: Illustration of Proposition 7 for different values of (p_{-i}, δ_i). *Left*: The case with $\delta_{-i} < \hat{\delta}_{-i}$. *Center*: The case with $\hat{\delta}_{-i} < \delta_{-i} < \overline{\delta}_{-i}(p_i)$. *Right*: The case with $\overline{\delta}_{-i}(p_i) < \delta_{-i}$.

Although the proposition above does not cover the cases of some knife-edge parameter values, the equilibrium characterization in such cases can be easily obtained by using upper-hemicontinuity of the set of PBE.[1]

Proposition 7 implies that, when $x_i > 1$ for each $i = 1,2$, we have

$$S = \{(\delta_1, \delta_2) \in (0,1)^2 \mid \delta_i > \hat{\delta}_i \text{ and } \delta_{-i} > \overline{\delta}_{-i}(p_i) \text{ for some } i = 1,2\}. \tag{A.3}$$

Now, we provide the comparative statics result.

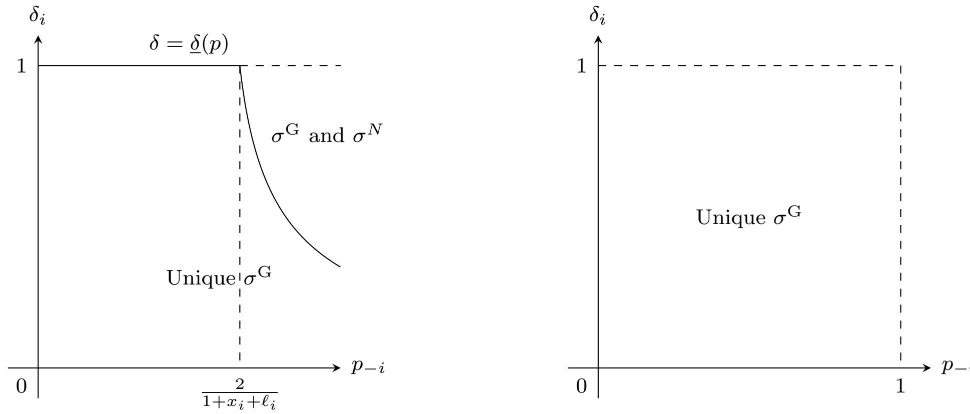
Remark 1. Suppose $x_j > 1$ for each $j = 1,2$. Fix $i = 1,2$.

1. If $p_i \geq p_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i', p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i})$.

2. If $x_i \geq x_i' > 1$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i, p_{-i}, x_i', x_{-i}, \ell_i, \ell_{-i})$.

3. If $\ell_i \geq \ell_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i, p_{-i}, x_i, x_{-i}, \ell_i', \ell_{-i})$.

[1]For example, if $\delta_1 = \frac{x_1 - 1}{x_1}$ and $\delta_2 > \overline{\delta}_2$, both σ^{G} and (σ_1^N, σ_2^*) are PBE.



$\underline{\delta}_{-i}(p_i) < \delta_{-i} < \overline{\delta}_{-i}(p_i)$ $\quad$ $\delta_{-i} < \underline{\delta}_{-i}(p_i)$ or $\overline{\delta}_{-i}(p_i) < \delta_{-i}$

Figure 7: Illustration of Proposition 8 for different values of (p_{-i}, δ_i). *Left*: The case with $\underline{\delta}_{-i}(p_i) < \delta_{-i} < \overline{\delta}_{-i}(p_i)$. *Right*: The case with $\delta_{-i} < \underline{\delta}_{-i}(p_i)$ or $\overline{\delta}_{-i}(p_i) < \delta_{-i}$.

Second, we consider the case in which $x_i < 1$ for each $i = 1, 2$.

Proposition 8. *Fix $p_1, p_2 \in [0, 1]$. Let $\delta_1, \delta_2 \in (0, 1)$. Suppose $x_i < 1$ for each $i = 1, 2$.*

1. *If $\delta_i \in (\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i}))$ for each $i = 1, 2$, then both σ^{G} and σ^N are PBE.*

2. *If $\delta_i \notin [\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})]$ for some $i = 1, 2$, then σ^{G} is a unique PBE.*

The central panel of Figure 2 in Section 4.6 of the main text depicts this result when p_i satisfies $\underline{\delta}_{-i}(p_i) \leq \overline{\delta}_{-i}(p_i)$ for each $i = 1, 2$.[2] Figure 7 illustrates Proposition 8 for different values of (p_1, δ_2). For the symmetric case in which $p = p_i$, $\delta = \delta_i$, $x = x_i$, and $l = \ell_i$, the right panel of Figure 3 in Section 4.6 of the main text depicts the characterization for different values of (p, δ).

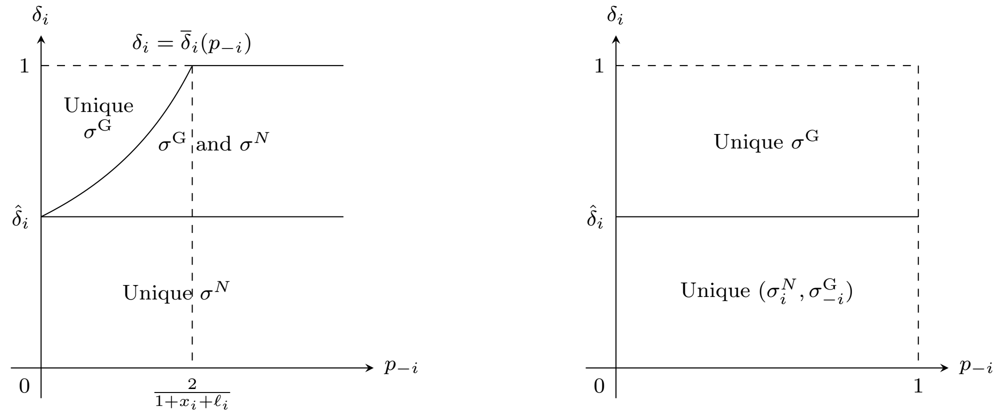
Proposition 8 implies that, when $x_i < 1$ for each $i = 1, 2$, we have

$$S = \{(\delta_1, \delta_2) \in (0, 1)^2 \mid \delta_i \notin [\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})] \text{ for some } i = 1, 2\}. \tag{A.4}$$

We remark that the set S is non-increasing in the parameters p_i, x_i, and ℓ_i in the following sense.

Remark 2. Suppose $x_j < 1$ for each $j = 1, 2$. Fix $i = 1, 2$.

[2] If $\overline{\delta}_{-i}(p_i) < \underline{\delta}_{-i}(p_i)$, then $\overline{\delta}_{-i}(p_i) = 0$ and $\underline{\delta}_{-i}(p_i) = 1$. Thus, σ^{G} is a unique PBE for any (δ_1, δ_2).



$\underline{\delta}_{-i}(p_i) < \delta_{-i} < \overline{\delta}_{-i}(p_i)$ $\delta_{-i} < \underline{\delta}_{-i}(p_i)$ or $\overline{\delta}_{-i}(p_i) < \delta_{-i}$

Figure 8: Illustration of Proposition 9 for different values of (p_{-i}, δ_i). *Left*: The case with $\underline{\delta}_{-i}(p_i) < \delta_{-i} < \overline{\delta}_{-i}(p_i)$. *Right*: The case with $\delta_{-i} < \underline{\delta}_{-i}(p_i)$ or $\overline{\delta}_{-i}(p_i) < \delta_{-i}$.

1. If $p_i \geq p_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i', p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i})$.

2. If $x_i \geq x_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i, p_{-i}, x_i', x_{-i}, \ell_i, \ell_{-i})$.

3. If $\ell_i \geq \ell_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i, p_{-i}, x_i, x_{-i}, \ell_i', \ell_{-i})$.

Finally, we consider the case in which $x_i > 1 > x_{-i}$ for some $i = 1, 2$.

Proposition 9. *Fix* $p_1, p_2 \in [0, 1]$. *Let* $\delta_1, \delta_2 \in (0, 1)$. *Suppose* $x_i > 1 > x_{-i}$ *for some* $i = 1, 2$.

1. *Suppose* $\delta_i > \hat{\delta}_i$.

   (a) *If* $\delta_{-i} \in (\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i))$ *and* $\delta_i < \overline{\delta}_i(p_{-i})$, *then both* σ^{G} *and* σ^N *are PBE.*

   (b) *If* $\delta_{-i} \notin [\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i)]$ *or* $\delta_i > \overline{\delta}_i(p_{-i})$, *then* σ^{G} *is a unique PBE.*

2. *Suppose* $\delta_i < \hat{\delta}_i$.

   (a) *If* $\delta_{-i} \in (\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i))$, *then* σ^N *is a unique PBE.*

   (b) *If* $\delta_{-i} \notin [\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i)]$, *then* $(\sigma_i^N, \sigma_{-i}^{\mathrm{G}})$ *is a unique PBE.*

The right panel of Figure 2 in Section 4.6 of the main text illustrates this proposition. The figure depicts the case in which $\underline{\delta}_{-i}(p_i) \leq \overline{\delta}_{-i}(p_i)$. Figure 8 illustrates PBE for different values of (p_{-i}, δ_i).

Proposition 9 implies that, fixing $i = 1, 2$ with $x_i > 1 > x_{-i}$, we have

$$S = \{(\delta_1, \delta_2) \in (0,1)^2 \mid \delta_i > \hat{\delta}_i \text{ and } (\delta_{-i} \notin [\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i)] \text{ or } \delta_i > \overline{\delta}_i(p_{-i}))\}. \quad \text{(A.5)}$$

We remark that the set S is non-increasing in the parameters p_i, x_i, and ℓ_i in the following sense.

Remark 3. Suppose there exists $j = 1, 2$ such that $x_j > 1 > x_{-j}$. For each $i = 1, 2$, the following hold.

1. If $p_i \geq p_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i', p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i})$.
2. If $x_i \geq x_i' > 1$ or $1 > x_i \geq x_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i, p_{-i}, x_i', x_{-i}, \ell_i, \ell_{-i})$.
3. If $\ell_i \geq \ell_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i, p_{-i}, x_i, x_{-i}, \ell_i', \ell_{-i})$.

B.3 Proofs for Supplementary Results

B.3.1 Proofs for Appendix B.1

Proof of Proposition 5. Suppose $\overline{q} = 1$. The proof of Proposition 2 goes through up to (4) under $\overline{q} = 1$. Now, (4) reduces to

$$x_i \leq \frac{1 - \delta_i^s}{1 - \delta_i} + \delta_i^s \min\{x_i, 0\},$$

which is equivalent to

$$\max\{(1 - \delta_i^s)x_i, x_i\} \leq \frac{1 - \delta_i^s}{1 - \delta_i}. \quad \text{(A.6)}$$

Now, suppose that $x_i \leq 1$ for each $i = 1, 2$. Then, the left-hand side of (A.6) is no greater than 1. Also, the right-hand side of (A.6) is no less than $\frac{1-\delta_i^1}{1-\delta_i} = 1$ because it is increasing in s. Hence, (A.6) holds. This implies that the continuation payoff from playing O is no less than the one from playing N. □

Proof of Proposition 6. We consider two cases.

Case 1. Consider any history $h_{i,t} \in \overline{H}_{i,t}$ such that $t = n \cdot T$ for some $n = 1, 2, \ldots$. The continuation payoff from N is $(1 - p_{-i})^T (1 - \delta_i) x_i$. The continuation payoff from O is

$$(1 - (1 - p_{-i})^T)(1 - \delta_i)(-\ell_i) + (1 - p_{-i})^T ((1 - \delta_i^T) \cdot 1 + \delta_i^T (1 - \delta_i)(1 - p_{-i})^T x_i).$$

The continuation payoff from N is no less than the one from O if and only if

$$(1-p_{-i})^T(1-\delta_i)x_i \geq (1-(1-p_{-i})^T)(1-\delta_i)(-\ell_i)+(1-p_{-i})^T((1-\delta_i^T)\cdot 1+\delta_i^T(1-\delta_i)(1-p_{-i})^T x_i),$$

or

$$(1-p_{-i})^T x_i \geq (1-(1-p_{-i})^T)(-\ell_i)+(1-p_{-i})^T\left(\frac{1-\delta_i^T}{1-\delta_i}+\delta_i^T(1-p_{-i})^T x_i\right).$$

There exists $\delta'' < 1$ such that this holds for all $\delta_i > \delta''$ if and only if:

$$(1-p_{-i})^T x_i \geq (1-(1-p_{-i})^T)(-\ell_i)+(1-p_{-i})^T\left(T+(1-p_{-i})^T x_i\right),$$

which we obtain by letting $\delta_i \to 1$. This is condition (A.1).

Case 2. Consider any history $h_{i,t} \in \overline{H}_{i,t}$ such that $t \neq n \cdot T$ for any $n = 1, 2, \ldots$. Let $s \in \{1, \ldots, T-1\}$ be such that the current period is $t = nT - s$ for some $n = 1, 2, \ldots$. If i plays N at $h_{i,t}$, then her continuation payoff is $(1-\delta_i)x_i$. If she plays O instead, then her continuation payoff is

$$(1-\delta_i^s)\cdot 1+\delta_i^s(1-p_{-i})^T(1-\delta_i)x_i. \tag{A.7}$$

Notice that this is a convex combination of 1 and $(1-p_{-i})^T(1-\delta_i)x_i$.

Case 2-1. If $1 \leq (1-p_{-i})^T(1-\delta_i)x_i$, then (A.7) is no greater than $(1-p_{-i})^T(1-\delta_i)x_i$. The continuation payoff from O is no less than the one from N if and only if

$$(1-p_{-i})^T(1-\delta_i)x_i \geq (1-\delta_i)x_i,$$

which is equivalent to $(1-p_{-i})^T \geq 1$, a contradiction. Hence, we need $1 > (1-p_{-i})^T(1-\delta_i)x_i$ for $\sigma^{(T,1)}$ to be a PBE.

Case 2-2. If $1 > (1-p_{-i})^T(1-\delta_i)x_i$, then (A.7) is minimized at $s = 1$, and the minimized value is

$$(1-\delta_i)\cdot 1+\delta_i(1-p_{-i})^T(1-\delta_i)x_i.$$

Thus, at any history $h_{i,t}$ that we consider in Case 2, the continuation payoff from O is no less than the one from N if and only if

$$(1-\delta_i)x_i \leq (1-\delta_i)\cdot 1 + \delta_i(1-p_{-i})^T(1-\delta_i)x_i,$$

which is equivalent to

$$x_i \leq 1 + \delta_i(1-p_{-i})^T x_i,$$

or

$$(1-p_{-i})^T \geq \frac{x_i - 1}{\delta_i x_i}.$$

Combining with $1 > (1-p_{-i})^T(1-\delta_i)x_i$, we need

$$\frac{x_i-1}{\delta_i x_i} \leq (1-p_{-i})^T < \frac{1}{(1-\delta_i)x_i}.$$

There exists $\delta'' < 1$ such that this holds for all $\delta_i > \delta''$ if and only if:

$$\frac{x_i-1}{x_i} < (1-p_{-i})^T,$$

which we obtain by letting $\delta_i \to 1$. This is condition (A.2).

□

B.3.2 Proofs for Appendix B.2

Proof of Proposition 7. For each $i = 1, 2$, when $x_i > 1$, it follows from the proof of Lemma 5 that $\hat{\delta}_i = \frac{x_i - 1}{x_i} > 0$ and $\hat{\delta}_i < \overline{\delta}_i(p_{-i})$. Also, for each $i = 1, 2$, since $\pi_i(0) = 1 < x_i$, we have $\underline{\delta}_i(p_{-i}) = 0$. Hence, $x_i < \pi_i(\delta_i)$ if $\delta_i > \overline{\delta}_i(p_{-i})$, and $x_i > \pi_i(\delta_i)$ if $\delta_i < \overline{\delta}_i(p_{-i})$.

Then, the proposition follows from Theorem 3. □

Proof of Proposition 8. Recall that, for each $i = 1, 2$, if $x_i < 1$ then $\hat{\delta}_i = 0$. Also, for each $i = 1, 2$, since $\pi_i(0) = 1 > x_i$, we have $\underline{\delta}_i(p_{-i}) > 0$.

Since $\hat{\delta}_i = 0$ for each $i = 1, 2$, it follows from Theorem 3 (1) and (3) that σ^{G} is a unique PBE if $x_i < \pi_i(\delta_i)$ for some $i = 1, 2$; and that both σ^{G} and σ^N are a PBE if $x_i > \pi_i(\delta_i)$ for all $i = 1, 2$.

Thus, it suffices to show the following two assertions for each $i = 1, 2$. First, if $\delta_i \in (\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i}))$, then $x_i > \pi_i(\delta_i)$. Second, if $\delta_i \notin [\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})]$, then $x_i < \pi_i(\delta_i)$.

Fix $i = 1, 2$. We consider the following three cases. As the first case, suppose $\underline{\delta}_i(p_{-i}) = 1$. Then, the first assertion vacuously follows because $(\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})) = \emptyset$. For the second assertion, it follows from the definition of $\underline{\delta}_i(p_{-i})$ that $x_i < \pi_i(\delta_i)$ for all $\delta_i \in (0, 1)$. Then, $x_i < \pi_i(\delta_i)$ for all $\delta_i \notin [\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})]$, as desired.

As the second case, suppose that $\underline{\delta}_i(p_{-i}) \neq 1$ and $\overline{\delta}_i(p_{-i}) = 1$. Then, $\underline{\delta}_i(p_{-i}) \in (0, 1)$. Also, $x_i \geq \pi_i(1)$ (otherwise, $\overline{\delta}_i(p_{-i}) < 1$). This means that the quadratic equation $x_i = \pi_i(\delta_i)$ has a unique interior solution in $(0, 1)$, which is $\underline{\delta}_i(p_{-i})$. Thus, $x_i < \pi_i(\delta_i)$ if $\delta_i \in (0, \underline{\delta}_i(p_{-i}))$, that is, $\delta_i \notin [\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})]$. Also, $x_i > \pi_i(\delta_i)$ if $\delta_i \in (\underline{\delta}_i(p_{-i}), 1)$, that is, $\delta_i \in (\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i}))$. Hence, the two assertions hold.

As the third case, suppose that $\underline{\delta}_i(p_{-i}) \neq 1$ and $\overline{\delta}_i(p_{-i}) \neq 1$. Then, $\underline{\delta}_i(p_{-i}) \in (0, 1)$. Also, $x_i < \pi_i(1)$ (otherwise, $\overline{\delta}_i(p_{-i}) = 1$). This means that the quadratic equation $x_i = \pi_i(\delta_i)$ has either (i) two interior solutions in $(0, 1)$, which are $\underline{\delta}_i(p_{-i})$ and $\overline{\delta}_i(p_{-i})$ with $\underline{\delta}_i(p_{-i}) < \overline{\delta}_i(p_{-i})$, or (ii) a unique solution in $(0, 1)$, which is $\underline{\delta}_i(p_{-i}) = \overline{\delta}_i(p_{-i})$. Thus, $x_i < \pi_i(\delta_i)$ if $\delta_i \in (0, \underline{\delta}_i(p_{-i})) \cup (\overline{\delta}_i(p_{-i}), 1)$, that is, $\delta_i \notin [\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})]$. Also, $x_i > \pi_i(\delta_i)$ if $\delta_i \in (\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i}))$. Hence, the two assertions hold.

In sum, for each of the three cases, the statement of the proposition holds. □

Proof of Proposition 9. Suppose that $x_i > 1 > x_{-i}$ for some $i = 1, 2$. Then, it follows from the proof of Lemma 5 that $\overline{\delta}_i(p_{-i}) > \hat{\delta}_i = \frac{x_i - 1}{x_i} > 0$. Also, $\hat{\delta}_{-i} = 0$. For player i, as in the proof of Proposition 7, $x_i < \pi_i(\delta_i)$ if $\delta_i > \overline{\delta}_i(p_{-i})$, and $x_i > \pi_i(\delta_i)$ if $\delta_i < \overline{\delta}_i(p_{-i})$. For player $-i$, similarly to the proof of Proposition 8, $x_{-i} < \pi_{-i}(\delta_{-i})$ if $\delta_{-i} \notin [\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i)]$, and $x_{-i} > \pi_{-i}(\delta_{-i})$ if $\delta_{-i} \in (\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i))$. Now, the statement of the proposition follows from Theorem 3. □

To prove Remarks 1 to 3, we provide the following two auxiliary results.

Lemma 10. *Fix $i = 1, 2$ with $x_i > 1$.*

1. *The threshold discount factor $\overline{\delta}_i$ is non-decreasing in p_{-i}, x_i, and ℓ_i.*

2. *We have $\overline{\delta}_i(p_{-i}) < 1$ if and only if $p_{-i} < \frac{2}{1 + x_i + \ell_i}$.*

3. *We have $\overline{\delta}_i(0) = \frac{x_i - 1}{x_i}$.*

Proof of Lemma 10. Take $i = 1, 2$ such that $x_i > 1$. Since the quadratic equation $x_i = \pi_i(\delta_i)$ has a positive coefficient on δ_i^2 and since $x_i > 1 = \pi_i(0)$, it follows that $\overline{\delta}_i(p_{-i})$ is the minimum of 1 and the positive solution of the quadratic equation $x_i = \pi_i(\delta_i)$:

$$\overline{\delta}_i(p_{-i}) = \min\left(1, \frac{-(1-(1+\ell_i)p_{-i}) + \sqrt{(1-(1+\ell_i)p_{-i})^2 + 4(1-p_{-i})(x_i-1)x_i}}{2x_i(1-p_{-i})}\right),$$

Also, we have $\underline{\delta}_i(p_{-i}) = 0$.

1. First, we show that $\overline{\delta}_i$ is non-decreasing in p_{-i}. If $p_{-i} \leq \frac{2}{1+x_i+\ell_i}$, then it can be seen that

$$\begin{aligned}\frac{\partial \overline{\delta}_i(p_{-i})}{\partial p_{-i}} &= \frac{\ell_i\sqrt{(1-(1+\ell_i)p_{-i})^2 + 4(1-p_{-i})(x_i-1)x_i} + 2(1-p_{-i})x_i(x_i-1) - \ell_i(1-(1+\ell_i)p_{-i})}{2(1-p_{-i})^2 x_i\sqrt{(1-(1+\ell_i)p_{-i})^2 + 4(1-p_{-i})(x_i-1)x_i}} \\ &\geq 0.\end{aligned}$$

   Also, if $p_{-i} < \frac{2}{1+x_i+\ell_i}$ then $\frac{\partial \overline{\delta}_i(p_{-i})}{\partial p_{-i}} > 0$. If $p_{-i} \geq \frac{2}{1+x_i+\ell_i}$, then $\overline{\delta}_i(p_{-i}) = 1$.

   Second, we show that $\overline{\delta}_i$ is non-decreasing in ℓ_i. If $\ell_i(> 0)$ satisfies $\ell_i \leq \frac{2}{p_{-i}} - (1+x_i)$, then it can be seen that

$$\frac{\partial \overline{\delta}_i}{\partial \ell_i} = p_{-i}\frac{1 - \frac{1-(1+\ell_i)p_{-i}}{\sqrt{(1-(1+\ell_i)p_{-i})^2+4(1-p_{-i})(x_i-1)x_i}}}{2x_i(1-p_{-i})} \geq 0.$$

   If $\ell_i \geq \frac{2}{p_{-i}} - (1+x_i)$, then $\overline{\delta}_i = 1$.

   Third, we show that $\overline{\delta}_i$ is non-decreasing in x_i. If $x_i(> 1)$ satisfies $x_i \leq \frac{2}{p_{-i}} - (1+\ell_i)$, then it can be seen that

$$\begin{aligned}\frac{\partial \overline{\delta}_i}{\partial x_i} &= \frac{1-(1+\ell_i)p_{-i} + \sqrt{(1-(1+\ell_i)p_{-i})^2 + 4(1-p_{-i})(x_i-1)x_i} + \frac{2(1-p_{-i})x_i(2x_i-1)}{\sqrt{(1-(1+\ell_i)p_{-i})^2+4(1-p_{-i})(x_i-1)x_i}}}{2x_i^2(1-p_{-i})} \\ &\geq 0.\end{aligned}$$

   If $x_i \geq \frac{2}{p_{-i}} - (1+\ell_i)$, then $\overline{\delta}_i = 1$.

2. Generally, $\overline{\delta}_i(p_{-i}) \leq 1$ if and only if $x_i \leq \pi_i(1)$, i.e., $p_{-i} \leq \frac{2}{1+x_i+\ell_i}$. It follows from the previous part that $\overline{\delta}_i(p_{-i})$ is strictly increasing when $p_{-i} < \frac{2}{1+x_i+\ell_i}$. Thus, $\overline{\delta}_i(p_{-i}) < 1$ if and only if $p_{-i} < \frac{2}{1+x_i+\ell_i}$.

3. When $p_{-i} = 0$,

$$\overline{\delta}_i(0) = \frac{-1 + \sqrt{1 + 4x_i(x_i - 1)}}{2x_i} = \frac{x_i - 1}{x_i}.$$

□

Lemma 11. *Fix $i = 1, 2$ with $x_i < 1$.*

1. *The threshold discount factor $\overline{\delta}_i$ is non-decreasing in p_{-i}, x_i, and ℓ_i.*
2. *The threshold discount factor $\underline{\delta}_i$ is non-increasing in p_{-i}, x_i, and ℓ_i.*
3. *We have $\overline{\delta}_i(0) = 0$ and $\underline{\delta}_i(0) = 1$.*

Proof of Lemma 11. Take $i = 1, 2$ such that $x_i < 1$. To prove the assertions, we find $\underline{\delta}_i$ and $\overline{\delta}_i$ for the following three cases.

Case 1. Suppose $x_i < 0$. Since the quadratic equation $x_i = \pi_i(\delta_i)$ has a negative coefficient on δ_i^2 and since $\pi_i(0) = 1 > x_i$, the quadratic equation has two solutions, one positive and one negative. The threshold discount factor $\underline{\delta}_i$ is the minimum of 1 and the larger solution:

$$\underline{\delta}_i(p_{-i}) = \min\left(1, \frac{-(1 - (1 + \ell_i)p_{-i}) - \sqrt{(1 - (1 + \ell_i)p_{-i})^2 + 4(1 - p_{-i})(x_i - 1)x_i}}{2x_i(1 - p_{-i})}\right). \tag{A.8}$$

We have $\underline{\delta}_i(p_{-i}) \leq 1$ if and only if $x_i \geq \pi_i(1)$, i.e., $p_{-i} \geq \frac{2}{1+x_i+\ell_i}$. We also have

$$\overline{\delta}_i(p_{-i}) = \begin{cases} 1 & \text{if } p_{-i} \geq \frac{2}{1+x_i+\ell_i} \\ 0 & \text{otherwise} \end{cases}. \tag{A.9}$$

Case 2. Suppose $x_i = 0$. Then, the equation $x_i = \pi_i(\delta_i)$ reduces to the linear equation $1 + (1 - (1 + \ell_i)p_{-i})\delta_i = 0$. Thus, we have

$$\underline{\delta}_i(p_{-i}) = \begin{cases} -\frac{1}{1-(1+\ell_i)p_{-i}} & \text{if } p_{-i} \geq \frac{2}{1+\ell_i} \\ 1 & \text{otherwise} \end{cases} \tag{A.10}$$

and

$$\overline{\delta}_i(p_{-i}) = \begin{cases} 1 & \text{if } p_{-i} \geq \frac{2}{1+\ell_i} \\ 0 & \text{otherwise} \end{cases}. \tag{A.11}$$

Note that $\underline{\delta}_i(p_{-i})$ given by equation (A.8) tends to the one given by equation (A.10) as x_i tends to 0 from below. Indeed,

$$\lim_{x_i \uparrow 0} \frac{-(1-(1+\ell_i)p_{-i}) - \sqrt{(1-(1+\ell_i)p_{-i})^2 + 4(1-p_{-i})(x_i-1)x_i}}{2x_i(1-p_{-i})}$$
$$= \lim_{x_i \uparrow 0} - \frac{\frac{4(1-p_{-i})(2x_i-1)}{2\sqrt{(1-(1+\ell_i)p_{-i})^2+4(1-p_{-i})(x_i-1)x_i}}}{2(1-p_{-i})} = \frac{1}{-(1-(1+\ell_i)p_{-i})},$$

where the first equality follows from l'Hôpital's theorem and the second follows because $p_{-i} \geq \frac{2}{1+\ell_i}$ implies $1-(1+\ell_i)p_{-i} < 0$. Note also that $\overline{\delta}_i(p_{-i})$ given by equation (A.9) tends to the one given by equation (A.11) as x_i tends to 0 from below.

Case 3. Suppose $x_i \in (0,1)$. We consider two subcases:

Case 3-1. Suppose $p_{-i}(1+\ell_i+x_i) \geq 2$ (i.e., $x_i \geq \pi_i(1)$). The quadratic equation $x_i = \pi_i(\delta_i)$ has one solution in $(0,1)$, which is $\underline{\delta}_i(p_{-i})$, and the other solution greater than 1. Thus,

$$\underline{\delta}_i(p_{-i}) = \frac{-(1-(1+\ell_i)p_{-i}) - \sqrt{(1-(1+\ell_i)p_{-i})^2 + 4(1-p_{-i})(x_i-1)x_i}}{2x_i(1-p_{-i})}. \tag{A.12}$$

Also,

$$\overline{\delta}_i(p_{-i}) = 1. \tag{A.13}$$

Note that, similarly to the previous argument, $\underline{\delta}_i(p_{-i})$ given by equation (A.12) tends to the one given by equation (A.10) as x_i tends to 0 from above. Note also that $\overline{\delta}_i(p_{-i})$ given by equation (A.13) tends to the one given by equation (A.11) as x_i tends to 0 from above.

Case 3-2. Suppose $p_{-i}(1+\ell_i+x_i) < 2$. We further consider two sub-cases:

Case 3-2-a. Suppose that the quadratic equation $x_i = \pi_i(\delta_i)$ has a solution. Then, the smaller solution of the quadratic equation $x_i = \pi_i(\delta_i)$ satisfies:

$$\frac{-(1-(1+\ell_i)p_{-i}) - \sqrt{(1-(1+\ell_i)p_{-i})^2 + 4(1-p_{-i})(x_i-1)x_i}}{2x_i(1-p_{-i})} > 1.$$

Thus, we have:

$$\underline{\delta}_i(p_{-i}) = 1 \tag{A.14}$$

and

$$\overline{\delta}_i(p_{-i}) = 0. \tag{A.15}$$

Case 3-2-b. Otherwise (i.e., if the quadratic equation $x_i = \pi_i(\delta_i)$ has no solution),

$$\underline{\delta}_i(p_{-i}) = 1 \tag{A.16}$$

and

$$\overline{\delta}_i(p_{-i}) = 0. \tag{A.17}$$

Note that as x_i tends to 0 from above, the equation $x_i = \pi_i(\delta_i)$ has no solution in $[0, 1]$. Thus, it can be seen that $\underline{\delta}_i(p_{-i})$ given by equation (A.16) tends to the one given by equation (A.10) as x_i tends to 0 from above. Likewise, $\overline{\delta}_i(p_{-i})$ given by equation (A.17) tends to the one given by equation (A.11) as x_i tends to 0 from above.

With these in mind, now we prove the assertions.

1. We show that $\overline{\delta}_i$ is non-decreasing in x_i. Observe that we have already shown that $\overline{\delta}_i$ is continuous in x_i. When $x_i < 0$, $\overline{\delta}_i$ given by equation (A.9) is non-decreasing in x_i. When $x_i \in (0, 1)$, $\overline{\delta}_i$ given by equation (A.13), (A.15), or (A.17) is non-decreasing in x_i (for equation (A.15), recall the proof of Lemma 10).

   Next, we show that $\overline{\delta}_i$ is non-decreasing in p_{-i} and ℓ_i. When $x_i < 0$, $\overline{\delta}_i$ given by equation (A.9) is non-decreasing in p_{-i} and ℓ_i. When $x_i \in (0, 1)$, $\overline{\delta}_i$ given by equation (A.13), (A.15), or (A.17) is non-decreasing in p_{-i} and ℓ_i (for equation (A.15), recall the proof of Lemma 10).

2. We show that $\underline{\delta}_i$ is non-increasing in x_i. Observe that we have already shown that $\underline{\delta}_i$ is continuous in x_i. When $x_i < 0$, we show that $\underline{\delta}_i$ given by equation (A.8)

is non-increasing in x_i. To that end, it is sufficient to consider $p_{-i} \geq \frac{2}{1+x_i+\ell_i}$. Then, $1-(1+\ell_i)p_{-i} < 0$ and we have

$$\frac{\partial \underline{\delta}_i}{\partial x_i} = \frac{1-(1+\ell_i)p_{-i} + \sqrt{(1-(1+\ell_i)p_{-i})^2 + 4(1-p_{-i})(x_i-1)x_i} - \frac{2(1-p_{-i})x_i(2x_i-1)}{\sqrt{(1-(1+\ell_i)p_{-i})^2+4(1-p_{-i})(x_i-1)x_i}}}{2x_i^2(1-p_{-i})}$$
$$\leq 0.$$

When $x_i \in (0,1)$, similarly, $\underline{\delta}_i$ given by equation (A.12), (A.14), or (A.16) is non-increasing in x_i.

Next, we show that $\underline{\delta}_i$ is non-increasing in p_{-i}. When $x_i < 0$, we show that $\underline{\delta}_i$ given by equation (A.8) is non-increasing in p_{-i}. To that end, it is sufficient to consider $p_{-i} \geq \frac{2}{1+x_i+\ell_i}$. Then, $1-(1+\ell_i)p_{-i} < 0$ and we have

$$\frac{\partial \underline{\delta}_i(p_{-i})}{\partial p_{-i}} = \frac{\ell_i\sqrt{(1-(1+\ell_i)p_{-i})^2 + 4(1-p_{-i})(x_i-1)x_i} - 2(1-p_{-i})x_i(x_i-1) + \ell_i(1-(1+\ell_i)p_{-i})}{2(1-p_{-i})^2 x_i \sqrt{(1-(1+\ell_i)p_{-i})^2 + 4(1-p_{-i})(x_i-1)x_i}}$$
$$\leq 0.$$

When $x_i \in (0,1)$, similarly, $\underline{\delta}_i$ given by equation (A.12), (A.14), or (A.16) is non-increasing in p_{-i}.

Next, we show that $\underline{\delta}_i$ is non-increasing in ℓ_i. When $x_i < 0$, we show that $\underline{\delta}_i$ given by equation (A.8) is non-increasing in ℓ_i. To that end, it is sufficient to consider $p_{-i} \geq \frac{2}{1+x_i+\ell_i}$. Then, $1-(1+\ell_i)p_{-i} < 0$ and we have

$$\frac{\partial \underline{\delta}_i}{\partial \ell_i} = p_{-i}\frac{1 + \frac{1-(1+\ell_i)p_{-i}}{\sqrt{(1-(1+\ell_i)p_{-i})^2+4(1-p_{-i})(x_i-1)x_i}}}{2x_i(1-p_{-i})} \leq 0.$$

When $x_i \in (0,1)$, similarly, $\underline{\delta}_i$ given by equation (A.12), (A.14), or (A.16) is non-increasing in ℓ_i.

3. Assume $p_{-i} = 0$. By inspection, for each possible case, we have $\underline{\delta}_i(0) = 1$ and $\overline{\delta}_i(0) = 0$.

□

Proof of Remark 1. Recall that the set S is given by equation (A.3). All the statements follow if, for each $j = 1, 2$, the threshold discount factors $\hat{\delta}_j$ and $\overline{\delta}_j$ are non-decreasing in x_j, p_{-j}, and ℓ_j. Since $x_j > 1$, $\hat{\delta}_j = \frac{x_j-1}{x_j}$ is indeed non-decreasing in x_j,

p_{-j}, and ℓ_j. For $\overline{\delta}_j$, it follows from Lemma 10 that $\overline{\delta}_j$ is non-decreasing in x_j, p_{-j}, and ℓ_j. □

Proof of Remark 2. Recall that the set S is given by equation (A.4). All the statements follow if, for each $i = 1, 2$, (i) the threshold discount factor $\overline{\delta}_i$ are non-decreasing in x_i, p_{-i}, and ℓ_i; and the threshold discount factor $\underline{\delta}_i$ are non-increasing in x_i, p_{-i}, and ℓ_i. For $\overline{\delta}_i$, it follows from Lemma 10 that $\overline{\delta}_i$ is non-decreasing in x_i, p_{-i}, and ℓ_i. For $\underline{\delta}_i$, it follows from Lemma 11 that $\underline{\delta}_i$ is non-increasing in x_i, p_{-i}, and ℓ_i. □

Proof of Remark 3. Take $i = 1, 2$ with $x_i > 1 > x_{-i}$. Recall that the set S is given by equation (A.5). All the statements follow if the following hold: (i) $\hat{\delta}_i$ and $\overline{\delta}_i$ are non-decreasing in x_i, p_{-i}, and ℓ_i; (ii) $\overline{\delta}_{-i}$ is non-decreasing in x_{-i}, p_i, and ℓ_{-i}; and (iii) $\underline{\delta}_{-i}$ is non-increasing in x_{-i}, p_i, and ℓ_{-i}. For (i), as shown in Remark 1, $\hat{\delta}_i = \frac{x_i - 1}{x_i}$ is non-decreasing in x_i, p_{-i}, and ℓ_i; and it follows from Lemma 10 that $\overline{\delta}_i$ is non-decreasing in x_i, p_{-i}, and ℓ_i. Statements (ii) and (iii) follow from Lemma 11. □