On the Consistency among Prior, Posteriors, and Information Sets*

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March 17, 2023

Abstract

This paper studies implications of the consistency conditions among prior, posteriors, and information sets on introspective properties of qualitative belief induced from information sets. The main result reformulates the consistency conditions as: (i) the information sets, without any assumption, almost surely form a partition; and (ii) the posterior at a state is equal to the Bayes conditional probability given the corresponding information set. This paper also provides a tractable epistemic model which dispenses with the technical assumptions inherent in the standard epistemic model such as the countable number of information sets. Applications are agreement theorem, no-trade theorem, and the epistemic characterization of correlated equilibria. Implications are as follows. First, since qualitative belief reduces to fully introspective knowledge in the standard environment, a care must be taken when one studies non-veridical belief or non-introspective knowledge. Second, an information partition compatible with the consistency conditions is uniquely determined by the posteriors. Third, qualitative and probability-one beliefs satisfy truth axiom almost surely. The paper also sheds light on how the additivity of the posteriors yields negative introspective properties of beliefs.

Journal of Economic Literature Classification Numbers: C70, D83.
Keywords: Information Sets; Prior; Posteriors; Bayes Conditional Probability; Truth Axiom; Additivity

*I would like to thank Atsushi Kajii, Nobuo Koida, Massimo Marinacci, Sujoy Mukerji, Eric Pacuit, Burkhard Schipper, Takashi Ui, and the seminar participants at TARK 2019 and SWET/DTW 2019 for their insightful comments and discussions.

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1 Introduction

Agents in a strategic situation have two forms of beliefs. One is probabilistic beliefs, represented by a notion of types (Harsanyi, 1967-1968). The other is qualitative belief (or knowledge, if it is truthful), represented by information sets (Aumann, 1976). Consider, for instance, a dynamic game where each agent has knowledge about her past observations on the play, while she has probabilistic beliefs about her opponents' future plays. Qualitative belief plays a role when it comes to, say, studying consequences of common belief in rationality instead of common knowledge of rationality.[1] These two kinds of beliefs are well studied in a rather separate manner, and somewhat surprisingly, little has been known about how reasoning based on one form of beliefs influences the other.

This paper examines introspective properties of qualitative belief induced by information sets from its relation with prior and posterior beliefs. First, I link prior and posteriors in a way such that the prior probability of an event coincides with the expectation of the posterior probabilities of the event with respect to the prior. Second, I relate information sets and posteriors. An agent qualitatively believes (or knows) her own probabilistic beliefs. Also, if the agent qualitatively believes (or knows) something, then she believes it with probability one. I study how these linkages themselves yield introspective properties on qualitative beliefs.

Consider an agent, Alameda, who faces uncertainty about underlying states of the world. On the one hand, Alameda has a prior countably-additive probability measure. She also has a posterior probability measure at each realized state. These dictate her quantitative beliefs. While I analyze how the additivity of each posterior affects her reasoning, for now I assume each posterior to be countably additive. On the other hand, Alameda has a mapping, called a possibility correspondence. It associates, with each state of the world, the set of states that she considers possible (the information set) at that state. I, as an analyst, derive properties of information sets, instead of directly assuming them. The framework is fairly parsimonious.

The main result (Theorem 1) restates the consistency conditions as: (i) Alameda's information sets form a partition almost surely;[2] and (ii) her posterior at each state coincides with the Bayes conditional probability given her corresponding information set. While information sets are usually exogenously assumed to form a partition, the main result demonstrates that the consistency conditions on the agent's qualitative

[1]See, for instance, Dekel and Gul (1997) for the importance of capturing both knowledge and probabilistic beliefs. See, for example, Stalnaker (1994) for using qualitative and probabilistic beliefs for studying solution concepts of games. More recently, Bonanno (2008), Bonanno and Tsakas (2018), Fukuda (2023), Guarino and Ziegler (2022), Hillas and Samet (2020), and Samet (2013) study epistemic characterizations of solution concepts of games where agents possess qualitative beliefs, and show that common belief in rationality and common knowledge of rationality may lead to different predictions.

[2]For the precise definition of almost sureness in this context, see Section 2.3.4. Section 3 discusses the statement of the main theorem.

and quantitative beliefs alone determine Bayes updating.

The main theorem implies that the framework of this paper provides a tractable epistemic model which dispenses with the technical assumption inherent in the standard partitional model of knowledge and belief, namely, the one that the number of information sets (partition cells) is at most countable. Thus, the paper can provide generalizations of previously known results on a state space model of knowledge and belief. Especially, the paper extends Aumann (1976)'s agreement theorem, Milgrom and Stokey (1982)'s no-trade theorem, and Aumann (1987)'s epistemic characterization of correlated equilibria on a general measurable state space. As Green (2012) argues, the no-trade result of Milgrom and Stokey (1982), which is established under the assumption that the state space is finite, has been applied to a setting in which a state space is a continuum in the literature of market microstructure and rational-expectation equilibria.

While the main result and its applications have their own interest, I also derive a variety of theoretical implications. The first implication (Corollary 1) is that the consistency conditions uniquely determine the posterior at each state as the Bayes conditional probability given the associated information set.

The second implication of the main result (Corollary 2) is on the uniqueness of an information partition (i.e., partitional information sets) compatible with the consistency conditions in a standard model. This result justifies the use of the partition generated by the posteriors in the previous literature (e.g., Battigalli and Bonanno, 1999; Halpern, 1991; Tan and Werlang, 1988; Vassilakis and Zamir, 1993): if an agent is certain of her own posterior (Harsanyi, 1967-1968; Mertens and Zamir, 1985), then, at each state, she has to be able to infer the set of states that generate the realized posterior. The resulting sets form the unique information partition.

The third implication (Corollary 3) is on the introspective properties of qualitative belief. To see this, Alameda's introspective abilities in her qualitative belief are reflected in properties of her information sets. When her information sets form a partition, her qualitative belief becomes knowledge (true belief) with full introspection. Truth axiom obtains: she can only know what is true. Her knowledge satisfies positive introspection: if she knows something then she knows that she knows it. Her knowledge also satisfies negative introspection: if she does not know something then she knows that she does not know it.

Now, the third implication is that, in a standard countable model where the prior puts positive probability to every state, qualitative belief reduces to fully introspective knowledge. The consistency conditions alone determine this property, and thus qualitative belief reduces to fully introspective knowledge with no a priori assumption on qualitative belief. Also, the notions of qualitative and probability-one beliefs coincide, and each form of beliefs inherits the properties of the other form.

Corollary 3 itself yields the following three additional implications. Its first implication is on the evaluation of a solution concept of a game. If the analysts assume the consistency conditions among prior, posteriors, and information sets, qualitative belief reduces to knowledge even though the analysts would like to study, say, implications of common belief in rationality instead of common knowledge of rationality. That is, not only does qualitative belief reduce to knowledge for individual agents, but also common qualitative belief reduces to common knowledge (Corollary 5).

On a related point, second, if the analysts attempt to represent non-introspective knowledge violating negative introspection, then such non-introspective knowledge turns out to reduce to fully introspective knowledge. Such non-introspective knowledge is associated with unawareness: not only Alameda does not know an event E, but also she does not know that she does not know E (Modica and Rustichini, 1994, 1999). Previous negative results on describing a non-trivial form of unawareness in a standard possibility correspondence model are based on some direct link between qualitative belief (knowledge) and unawareness (e.g., Dekel, Lipman, and Rustichini, 1998; Modica and Rustichini, 1994). In contrast, Corollary 4 demonstrates that a link between qualitative and probabilistic beliefs alone makes qualitative belief fully introspective knowledge and thus makes a standard state-space model with knowledge and probabilistic beliefs incapable of describing a non-trivial form of unawareness.

Third, still another implication of Corollary 3 is the violation of consistency between ex-ante and ex-post analyses in a non-partitional model. Suppose that the analysts aim to study non-introspective knowledge of a "boundedly rational" agent who is ignorant of her own ignorance. Assume that knowledge implies probability-one belief. Also, assume that the agent is positively introspective on her own probabilistic beliefs with respect to her knowledge (i.e., if she believes an event with probability at least p (i.e., she p-believes the event), then she knows that she p-believes the event) in the same spirit as she has positive introspection on knowledge. Then, her prior and posteriors must violate the consistency condition, as the previous studies of non-partitional knowledge have demonstrated the inconsistency between ex-ante and ex-post evaluations of a decision problem.

The final and fourth implication (Corollary 6) is that, while qualitative and probability-one beliefs may differ, both satisfy truth axiom almost surely.[3] Likewise, common qualitative belief and common probability-one belief are also almost surely true.[4]

The paper also sheds light on the role of additivity in introspection (Propositions 5 and 6). Now, suppose that the agent's posteriors are non-additive. First, if she does not p-believe an event E, then she may not be certain that she does not p-believe E. An agent with additive posteriors, however, would be certain of her own probabilistic ignorance. Second, additivity also implies negative introspection on probabilistic

[3]For example, an agent believes with probability one (i.e., she is certain) that a random draw from $[0, 1]$ is an irrational number but she does not know it (Monderer and Samet, 1989). Recall, however, that Corollary 3 shows that, in a standard countable full-support environment, qualitative and probability-one beliefs coincide.

[4]For the precise definition of almost sureness in this context, see Section 2.3.4. Section 6.1 discusses this statement of the corollary.

beliefs with respect to qualitative belief. If the agent does not p-believe an event then she may not qualitatively believe that she does not p-believe it. Again, an agent with additive posteriors would qualitatively believe her own probabilistic ignorance.

While the main focus of this paper is how the posterior probabilities are derived from the prior probability measure and information sets, I also demonstrate in Proposition 8 that the similar consistency conditions indeed characterize conditional expectations. Namely, using the main result, I show that an agent's conditional expectation is derived as the expectation with respect to the Bayes conditional probability given her information set if prior, conditional expectations, and information sets satisfy the similar consistency conditions. Interestingly, since a posterior probability of an event is a conditional expectation of the indicator of the event, the consistency condition between a prior and posteriors turns out to be the law of iterated expectations. The prior expectation of (the indicator of) an event E is equal to the prior expectation of the conditional expectations of (the indicator of) E.

The paper is organized as follows. The rest of this section discusses related literature. Section 2 provides the framework. Section 3 demonstrates the main result. Section 4 studies special cases to show that the framework of this paper encompasses the standard model in the literature. Section 5 provides economic and game-theoretic applications. Section 6 studies theoretical implications of the main result. Section 7 provides further discussions and the concluding remarks. Proofs are in Appendix A. Appendix B provides additional supplementary examples.

Related Literature

This paper is related to the following four strands of literature: (i) derivation of Bayes updating from consistency between prior and posterior beliefs, (ii) interaction of knowledge and beliefs, (iii) non-partitional knowledge models, and (iv) the role of additivity in probabilistic reasoning.

In the first strand of literature on Bayes updating, the main result (Theorem 1) is closely related to Gaifman (1988), Mertens and Zamir (1985), and Samet (1999) in a purely probabilistic setting. In a single-agent perspective, these papers study how the consistency conditions between prior and posterior probabilities lead to Bayes updating. Section 3 will discuss how these papers relate to the main result (Theorem 1). As in Mertens and Zamir (1985), the studies of existence of a common prior (e.g., Bonanno and Nehring, 1999; Feinberg, 2000; Golub and Morris, 2017; Heifetz, 2006; Hellman, 2011; Morris, 1994; Nehring, 2001; Samet, 1998) also impose the consistency condition that the prior coincides with the expectation of the posteriors with respect to the prior.

The second is an extensive literature on the interaction between knowledge and beliefs in artificial intelligence, computer science, economics, game theory, logic, and philosophy. The consistency conditions between qualitative and probabilistic beliefs imposed in this paper are fairly common in economics and game theory in such contexts as epistemic characterizations of solution concepts for games, existence of a common prior, and canonical structures of agents' knowledge and beliefs (e.g., Aumann, 1999; Battigalli and Bonanno, 1999; Dekel and Gul, 1997; Meier, 2008). The validity of individual consistency conditions between knowledge and beliefs (i.e, "knowledge entails beliefs" and the knowledge of own beliefs) has been well studied since Hintikka (1962) and Lenzen (1978).

The third is on non-partitional knowledge that fails negative introspection. The studies of non-partitional structures include implications of common knowledge such as generalizations of agreement theorem of Aumann (1976), studies of solution concepts (Brandenburger, Dekel, and Geanakoplos, 1992; Geanakoplos, 2021), foundations for information processing by "boundedly rational" agents (Bacharach, 1985; Morris, 1996; Samet, 1990, 1992; Shin, 1993), and unawareness (Dekel, Lipman, and Rustichini, 1998; Modica and Rustichini, 1994, 1999).[5] See also Dekel and Gul (1997).

Fourth, I turn to the role of additivity in probabilistic introspection. In the decision theory literature, such papers as Ghirardato (2001) and Mukerji (1997) study foundations for non-additive beliefs in terms of an agent's imperfect information processing. Although the framework of this paper is quite different from these decision-theoretic papers on non-additive subjective expected utility models, these and this papers have the following similar intuition behind why the non-additivity is associated with the lack of introspection. An agent with non-additive beliefs cannot imagine what the possible states of the world are in her fullest extent.

2 An Epistemic Model

This section defines the framework. Section 2.1 defines an epistemic model, which consists of a prior, posteriors, and information sets defined on a state space. Section 2.2 introduces the probabilistic and non-probabilistic belief operators from the primitives of the model. Section 2.3 defines the consistency conditions among prior, posteriors, and information sets. Section 2.4 provides examples.

2.1 An Epistemic Model

This subsection defines an epistemic model to capture quantitative and qualitative beliefs of an agent. For ease of exposition, I start with presenting a single agent model.

An *epistemic model* (a *model*, for short) is a tuple $\overrightarrow{\Omega} := \langle \Omega, \Sigma, \mu, P, t \rangle$. First, Ω is a non-empty set of *states* of the world endowed with a σ-algebra Σ. Each element E of Σ is an *event*. I denote the complement of an event E by E^c or $\neg E$.

[5]Propositions 5 and 6 suggest that non-additivity may yield the violation of probabilistic negative introspection: if the agent is not certain of an event, she may not be certain that she is not certain of it. The implication of such violation of probabilistic negative introspection would also be interesting.

Second, $\mu : \Sigma \to [0,1]$ is a *prior* countably-additive probability measure. Thus, (Ω, Σ, μ) forms a probability space.

Third, $P : \Omega \to \Sigma$ is a *possibility correspondence* with the measurability condition that $\{\omega \in \Omega \mid P(\omega) \subseteq E\} \in \Sigma$ for each $E \in \Sigma$. It associates, with each state ω, the set of states considered possible at that state. For each $\omega \in \Omega$, I call $P(\omega)$ the *possibility set* (or the *information set*) at ω. Note that $P(\omega)$ is assumed to be an event about which the agent herself reasons. The measurability condition will be used, in Section 2.2, to introduce the qualitative belief operator. Thus, the possibility correspondence P dictates the agent's qualitative belief on $\langle \Omega, \Sigma \rangle$.

Fourth, $t : \Omega \times \Sigma \to [0,1]$ is a *type mapping* satisfying the following two measurability conditions. The type mapping t dictates the agent's quantitative beliefs on (Ω, Σ).

The first measurability condition is: for each $E \in \Sigma$, the mapping $t(\cdot, E) : \Omega \to [0,1]$ satisfies $(t(\cdot, E))^{-1}([p,1]) = \{\omega \in \Omega \mid t(\omega, E) \geq p\} \in \Sigma$ for all $p \in [0,1]$. That is, each $t(\cdot, E) : (\Omega, \Sigma) \to ([0,1], \mathcal{B}_{[0,1]})$ is measurable with respect to the Borel σ-algebra $\mathcal{B}_{[0,1]}$ on $[0,1]$. This assumption allows the agent to reason about whether her degree of belief in an event E is at least p. It will be used, in Section 2.2, to define the agent's p-belief operators.

The second measurability condition is: $[t(\omega)] := \{\omega' \in \Omega \mid t(\omega', \cdot) = t(\omega, \cdot)\} \in \Sigma$ for all $\omega \in \Omega$. Note that $t(\omega', \cdot) = t(\omega, \cdot)$ means $t(\omega', E) = t(\omega, E)$ for all $E \in \Sigma$. I use similar abbreviations throughout the paper. The set $[t(\omega)]$ consists of states $\tilde{\omega}$ indistinguishable from ω in that $t(\omega, \cdot) = t(\tilde{\omega}, \cdot)$: the agent's posteriors (i.e., quantitative beliefs) at ω and $\tilde{\omega}$ coincide. Intuitively, if the agent is perfectly certain of her quantitative beliefs (i.e., her type mapping t), then, at each state ω, she would be able to infer that the realization must be in $[t(\omega)]$. This second measurability assumption ensures each $[t(\omega)]$ to be an object of the agent's beliefs.

I remark that if Σ is generated by a countable algebra Σ_0 (i.e., an algebra which has at most countably many events) and if each $t(\omega, \cdot)$ is continuous with respect to both increasing and decreasing sequences of events and is monotone, then $[t(\omega)] \in \Sigma$ automatically holds.[6]

For each $\omega \in \Omega$, call $t(\omega, \cdot)$ the *type* at ω. If a state ω realizes, the type $t(\omega, \cdot)$ at ω assigns, with each event E, the agent's posterior (i.e., quantitative) belief in E. The idea behind the type mapping $t : \Omega \times \Sigma \to [0,1]$ is a Markov kernel when each $t(\omega, \cdot)$ is a countably-additive probability measure (Gaifman, 1988; Samet, 1998, 2000). Here, each type $t(\omega, \cdot)$ is assumed to be a general set function. I do not assume any property of a set function on each type $t(\omega, \cdot)$ at this point, as I first study how each type inherits the properties of the prior μ by imposing the link between the prior μ and the type mapping t.

[6]There is a countable algebra Σ_0 which generates Σ if there is a countable set of events which generates Σ.

2.2 Quantitative and Qualitative Belief Operators

I introduce quantitative and qualitative belief operators in an epistemic model. The agent's quantitative beliefs are represented by p-belief operators induced by the type mapping t, while her qualitative belief is represented by the qualitative belief operator induced by her possibility correspondence P. I also define introspective properties of quantitative and qualitative beliefs.

2.2.1 Quantitative Beliefs: p-Belief Operators

The agent's quantitative beliefs are captured by *p-belief operators* $(B^p)_{p\in[0,1]}$ (e.g., Monderer and Samet, 1989). For each $(E,p) \in \Sigma \times [0,1]$, define $B^p(E) := \{\omega \in \Omega \mid t(\omega, E) \geq p\} \in \Sigma$. The event $B^p(E)$ is the set of states at which the agent *p-believes* E, i.e., she assigns probability at least p to E.

I consider the following introspective property of quantitative beliefs. Namely, a model $\overrightarrow{\Omega}$ satisfies *Certainty of (p-)Beliefs* if $t(\cdot, [t(\cdot)]) = 1$ (Gaifman, 1988; Mertens and Zamir, 1985; Samet, 1999, 2000). Certainty of Beliefs states that the type at state ω puts probability one to the set of states indistinguishable from ω according to the type mapping t. To restate, if the agent has a perfect understanding of her own type mapping, she would be able to infer that, at each state ω, the true state is in $[t(\omega)]$ by unpacking the possible types.[7]

If $\overrightarrow{\Omega}$ satisfies Certainty of Beliefs and if each $t(\omega, \cdot)$ is monotone (i.e, $E \subseteq F$ implies $t(\omega, E) \leq t(\omega, F)$), then one can show: (i) $B^p(\cdot) \subseteq B^1 B^p(\cdot)$ and (ii) $(\neg B^p)(\cdot) \subseteq B^1(\neg B^p)(\cdot)$. Part (i) states that if the agent p-believes an event E then she 1-believes that she p-believes E. Part (ii), on the other hand, states that if the agent does not p-believe an event E then she 1-believes that she does not p-believe E. Thus, Certainty of Beliefs implies full introspection in the above sense.[8]

I remark that if Σ in a given model is generated by a countable algebra and if each $t(\omega, \cdot)$ is a countably-additive probability measure, then it follows from Samet (2000, Theorem 3) that the model satisfies Certainty of Beliefs if and only if (hereafter, abbreviated as iff) $B^p(\cdot) \subseteq B^1 B^p(\cdot)$.

2.2.2 Qualitative Belief: Qualitative Belief Operator

Next, I turn to the agent's qualitative belief. The possibility correspondence P induces the *qualitative belief operator* $K : \Sigma \to \Sigma$ defined by $K(E) := \{\omega \in \Omega \mid P(\omega) \subseteq E\} \in \Sigma$ for each $E \in \Sigma$. The event $K(E)$ is the set of states at which the agent qualitatively

[7] Technically, Gaifman (1988) and Samet (1999) require $t(\cdot, [t(\cdot)]) = 1$ μ-almost surely (i.e., $\mu(\{\omega \in \Omega \mid t(\omega, [t(\omega)]) = 1\}) = 1$). This paper defines Certainty of Beliefs for each state for the conceptual reason.

[8] Moreover, the idea of Certainty of Beliefs plays an important role in the construction of a universal Harsanyi type space (Mertens and Zamir, 1985) to formalize the idea that each agent is informed of her own type.

believes E. I often drop the mentioning of "qualitative" or "qualitatively" when it is clear from the context.

The qualitative belief operator K always satisfies the following well-known properties: (i) Monotonicity: $E \subseteq F$ implies $K(E) \subseteq K(F)$; (ii) (Countable) Conjunction: $\bigcap_{n \in \mathbb{N}} K(E_n) \subseteq K(\bigcap_{n \in \mathbb{N}} E_n)$; and (iii) Necessitation: $K(\Omega) = \Omega$. Monotonicity states that the agent believes any logical consequence of her belief. Countable Conjunction states that the agent believes any countable conjunction of her beliefs. Necessitation states that the agent believes any form of tautology Ω (e.g., $E \cup E^c$).

Other properties of the qualitative belief operator K correspond to properties of the possibility correspondence P. The following four properties are well known (e.g., Aumann, 1999; Dekel and Gul, 1997). First, K satisfies Truth Axiom ($K(E) \subseteq E$ for each $E \in \Sigma$) iff P is reflexive (i.e., $\omega \in P(\omega)$ for all $\omega \in \Omega$). Truth Axiom states that if the agent believes an event E at a state then E is true at that state. Second, K satisfies Consistency ($K(E) \subseteq (\neg K)(\neg E)$ for each $E \in \Sigma$) iff P is serial (i.e., $P(\cdot) \neq \emptyset$). Consistency states that if the agent believes an event E then she does not believe its negation. Note that Truth Axiom implies Consistency. Third, K satisfies Positive Introspection ($K(\cdot) \subseteq KK(\cdot)$) iff P is transitive (i.e., $\omega' \in P(\omega)$ implies $P(\omega') \subseteq P(\omega)$). Positive Introspection states that if the agent believes an event then she believes that she believes it. Fourth, K satisfies Negative Introspection ($(\neg K)(\cdot) \subseteq K(\neg K)(\cdot)$) iff P is Euclidean (i.e., $\omega' \in P(\omega)$ implies $P(\omega) \subseteq P(\omega')$). Negative Introspection states that if the agent does not believe an event then she believes that she does not believe it.

The collection of information sets $\{P(\omega)\}_{\omega \in \Omega}$ forms a partition of Ω (i.e., it is reflexive, transitive, and Euclidean) iff K satisfies Truth Axiom, Positive Introspection, and Negative Introspection. Note that Truth Axiom and Negative Introspection imply Positive Introspection.

In the epistemic model, no such introspective assumption on P is imposed a priori. This is in contrast to a standard partitional model of knowledge (e.g., Aumann, 1976). Thus, K need not be the "knowledge" operator that satisfies Truth Axiom, although I denote the qualitative belief operator by K to distinguish it from the p-belief operator B^p. Instead of assuming axioms on P, I derive properties of P from how qualitative and quantitative beliefs interact with each other.

2.3 Relations among Prior, Posteriors, and Information Sets

Throughout this subsection, fix a model $\overrightarrow{\Omega}$. This subsection relates the primitives of the model: (i) the prior μ and the type mapping t (i.e., prior and posterior beliefs) and (ii) the type mapping t and the possibility correspondence P (i.e., quantitative and qualitative beliefs).

2.3.1 Invariance: Consistency between Prior and Posterior Beliefs

I assume the invariance condition on the prior and the type mapping stating that the prior probability of an event E coincides with the expectation of the posteriors of E with respect to the prior. Formally, the model satisfies *Invariance* if

$$\mu(\cdot) = \int_{\Omega} t(\omega, \cdot)\mu(d\omega).$$

This consistency condition is especially used to characterize (existence of) a common prior as discussed in the Introduction. Note also that, in accordance with this literature, one can define the prior μ as a (countably-additive) probability measure that satisfies the Invariance condition. Under Invariance, $\mu(E) = 0$ iff $t(\cdot, E) = 0$ μ-almost surely.

2.3.2 Entailment and Self-Evidence of Beliefs: Consistency between Qualitative and Quantitative Beliefs

I introduce two introspective properties that relate qualitative and quantitative beliefs. These properties are commonly imposed in economics and game theory when knowledge and probabilistic beliefs are present.

The model $\overrightarrow{\Omega}$ satisfies *Entailment* if $t(\cdot, P(\cdot)) = 1$. Entailment states that, at each state, the agent assigns probability one to the set of states that she considers possible at that state. Since $\omega \in K(P(\omega))$, if each type $t(\omega, \cdot)$ is monotone then Entailment is expressed, in terms of operators, as the stronger condition $K(\cdot) \subseteq B^1(\cdot)$: if the agent qualitatively believes an event then she 1-believes it.[9] If each type $t(\omega, \cdot)$ is monotone and if $\{\omega\} \in \Sigma$ for all $\omega \in \Omega$, then Entailment implies that $\{\omega' \in \Omega \mid t(\omega, \{\omega'\}) > 0\} \subseteq P(\omega)$. In words, the agent considers ω' possible at ω whenever she assigns positive probability to $\{\omega'\}$ at that state. Also, Entailment implies that P is serial (i.e., $P(\cdot) \neq \emptyset$), which is equivalent to Consistency of K: $K(\cdot) \subseteq (\neg K \neg)(\cdot)$.

Next, $\overrightarrow{\Omega}$ satisfies *Self-Evidence of (p-)Beliefs* if $\omega' \in P(\omega)$ implies $t(\omega, \cdot) = t(\omega', \cdot)$, i.e., $P(\omega) \subseteq [t(\omega)]$. It provides a consistency requirement between the possibility correspondence P and the type mapping t: if the agent considers ω' possible at ω, then her quantitative beliefs at ω and ω' are identical. This condition is commonly imposed in economics and game theory when qualitative belief reduces to knowledge.

Self-Evidence refers to the property that is established in Proposition 6 of Section 7.1: the model satisfies Self-Evidence of Beliefs iff (i) whenever the agent p-beliefs an event, she qualitatively believes that she p-believes the event; and (ii) whenever the agent does not p-belief an event, she qualitatively believes that she does not p-believe the event.

Entailment and Self-Evidence of Beliefs imply Certainty of Beliefs, provided that each $t(\omega, \cdot)$ is monotone.

[9] If qualitative belief reduces to knowledge, then Entailment states that knowledge entails probability-one belief (see, e.g., Battigalli and Bonanno, 1999; Dekel and Gul, 1997; Hintikka, 1962).

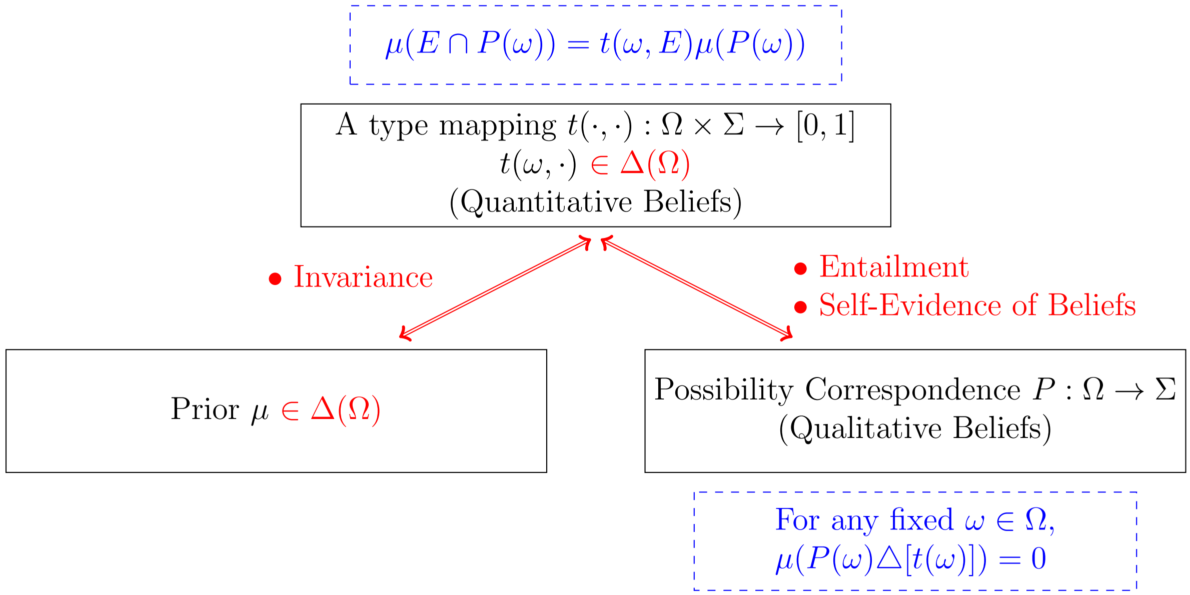

Figure 1: A Regular Epistemic Model $\langle \Omega, \Sigma, \mu, P, t \rangle$. The three primitives, the prior μ, the possibility correspondence P, and the type mapping t on the state space (Ω, Σ) are depicted by a solid rectangle. The consistency conditions (Invariance, Entailment, and Self-Evidence of Beliefs) are depicted by a two-way arrow between the corresponding primitives. The assumptions on the primitives (including the consistency conditions) are colored in red ($\Delta(\Omega)$ is the set of countably-additive probability measures on (Ω, Σ)). The main result (Theorem 1 in Section 3) is anticipated by blue dashed rectangles ($P(\omega) \triangle [t(\omega)]$ is the symmetric difference between $P(\omega)$ and $[t(\omega)]$).

2.3.3 A Regular Model

With these definitions in mind, the main result (Theorem 1) characterizes conditions on a given model under which the agent's type $t(\omega, \cdot)$ at each state coincides with the Bayes conditional probability $\mu(\cdot \mid P(\omega))$ given her information set at that state. To that end, call the model $\overrightarrow{\Omega}$ *regular* if each $t(\omega, \cdot)$ is a countably-additive probability measure and if the model satisfies Invariance (the consistency condition between the prior and the type mapping), Entailment and Self-Evidence of Beliefs ($t(\cdot, P(\cdot)) = 1$ and $P(\cdot) \subseteq [t(\cdot)]$, which are the consistency conditions between qualitative and quantitative beliefs). Figure 1 illustrates the regular model.

I also call a model *proper* if $\mu(P(\cdot)) > 0$.

2.3.4 Almost Sureness

I provide two technical definitions. First, an event E *is μ-almost surely included in* an event F, denoted by $E \subseteq_\mu F$, if $\mu(E \setminus F) = 0$. Informally, the set of states ω

which belong to E but which do not belong to F (i.e., the set of states ω that violate "$E \subseteq F$") has measure zero according to μ.

Second, an event E *is μ-almost surely equal to* an event F, denoted by $E =_\mu F$, if $\mu(E \triangle F) = 0$, where $\triangle$ denotes the symmetric-difference operation (i.e., $E \triangle F := (E \setminus F) \cup (F \setminus E)$). Thus, $E =_\mu F$ if(f) $E \subseteq_\mu F$ and $F \subseteq_\mu E$, as long as μ is additive. Informally, the set of states ω that violate "$E = F$" (either $\omega \in E \setminus F$ or $\omega \in F \setminus E$) has measure zero according to μ. Note that "$\mu(P(\omega) \triangle [t(\omega)]) = 1$" in Figure 1 can be rewritten as "$P(\omega) =_\mu [t(\omega)]$.[10]

I remark that, in a proper regular model, (i) it turns out (as a consequence of Theorem 1) that $\mu([t(\cdot)]) = \mu([P(\cdot)]) > 0$; and (ii) $\mu(E) = 0$ iff $t(\cdot, E) = 0$. Thus, $E \subseteq_\mu F$ iff $E \subseteq_{t(\omega,\cdot)} F$ (i.e., $t(\omega, E \triangle F) = 0$) for all $\omega \in \Omega$.

2.3.5 An Interactive Epistemic Model

To conclude this subsection, I remark that one can easily incorporate a set of agents into an epistemic model. Namely, let I be a non-empty countable set of agents. An *interactive epistemic model* is $\langle (\Omega, \Sigma, \mu), (P_i, t_i)_{i \in I} \rangle$ such that $\langle \Omega, \Sigma, \mu, P_i, t_i \rangle$ is a model for each agent $i \in I$.

2.4 Examples

Before presenting the main result, I provide examples for the following two purposes. First, Example 1 shows that Entailment, Self-Evidence of Beliefs, and Invariance are independent assumptions.

Example 1. I provide three models in which one of Entailment, Self-Evidence of Beliefs, and Invariance fails. Throughout the example, let $(\Omega, \Sigma) = (\{\omega_1, \omega_2, \omega_3\}, 2^\Omega)$, where 2^Ω is the power set of Ω.

First, let $\mu = (\frac{1}{2}, \frac{1}{2}, 0)$, and let $P(\omega) = \{\omega\}$ for each $\omega \in \Omega$, $t(\omega_1, \cdot) = (1, 0, 0)$, $t(\omega_2, \cdot) = (0, 1, 0)$, and $t(\omega_3, \cdot) = \mu$.[11] On the one hand, the agent considers ω to be the only possibility at each state ω. On the other hand, at state ω_3 which occurs with prior probability zero, her type at that state does not reflect her information $P(\omega_3) = \{\omega_3\}$. For the model to be regular, $t(\omega_3, \cdot) = (0, 0, 1)$.

Second, let $\mu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and let $P(\cdot) = \Omega$ and $t(\omega, \cdot) = \delta_\omega$ for every $\omega \in \Omega$, where δ_ω is the Dirac measure concentrated at ω (e.g., $t(\omega_1, \cdot) = (1, 0, 0)$). By construction, the model satisfies Invariance and Entailment but violates Self-Evidence of Beliefs.

[10]The statements "$E \subseteq_\mu F$" and "$E =_\mu F$" are also sometimes mentioned as "$E \subseteq F$ μ-almost surely" and "$E = F$ μ-almost surely," respectively (e.g., Hoffmann-Jørgensen, 1994; Monderer and Samet, 1989). As depicted in Figure 1, part of Theorem 1 asserts that, for each given $\omega \in \Omega$, $P(\omega) =_\mu [t(\omega)]$. Since the alternative notation "$P(\omega) = [t(\omega)]$ μ-almost surely" may be incorrectly interpreted as $\mu(\{\omega \in \Omega \mid P(\omega) = [t(\omega)]\}) = 1$, I use the specific notation "$P(\omega) =_\mu [t(\omega)]$." I am grateful to an anonymous referee and the handling editor for pointing out this issue.

[11]This example is from Brandenburger and Dekel (1987).

On the one hand, if the agent is informed of her own type mapping t, she would be able to invert t to distinguish each state ω. On the other hand, her possibility correspondence does not reflect this consideration. For the model to be regular, $P(\omega) = \{\omega\}$ for each $\omega \in \Omega$.

Third, let $\mu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and let $P(\omega_1) = P(\omega_3) = \{\omega_1, \omega_3\}$ and $P(\omega_2) = \{\omega_2\}$. Let $t(\omega_1, \cdot) = t(\omega_3, \cdot) = (\alpha, 0, 1-\alpha)$ with $\alpha \neq \frac{1}{2}$ and $t(\omega_2, \cdot) = (0, 1, 0)$. By construction, the model satisfies Entailment and Self-Evidence of Beliefs. According to the agent's possibility correspondence, she can distinguish $\{\omega_1, \omega_3\}$ and $\{\omega_2\}$. This is consistent with the facts that her type is distinguishable between $\{\omega_1, \omega_3\}$ and $\{\omega_2\}$ and that it is indistinguishable between ω_1 and ω_3. However, the model does not satisfy Invariance. The likelihood ratio between $\mu(\{\omega_1\})$ and $\mu(\{\omega_3\})$ is different from the one between $t(\omega_1, \cdot)$ and $t(\omega_3, \cdot)$ (even though the agent cannot distinguish between ω_1 and ω_3). For the model to be regular, $\alpha = \frac{1}{2}$. □

Second, while the standard partitional model of knowledge and belief (e.g., Aumann, 1976) makes the technical assumption that the partition $\{P(\omega)\}_{\omega \in \Omega}$ is countable, this paper does not have to impose such an assumption. The following examples illustrate that the framework of this paper can accommodate the "standard" Bayesian situations in which an agent's partition is uncountable.[12] While Example 2 is a general example, Example 3 is a particular one.

Example 2. Let $\Omega = \mathbb{R}^2$, and let Σ be the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^2}$ on Ω. Let μ be a prior, which is a countably-additive probability measure, on (Ω, Σ). Assume that μ is absolutely continuous with respect to the Lebesgue measure on (Ω, Σ). Thus, denote by f the probability density function.

Let $I = \{1, 2\}$. Let $P_1(\omega_1, \omega_2) = \{\omega_1\} \times \mathbb{R}$ and $P_2(\omega_1, \omega_2) = \mathbb{R} \times \{\omega_2\}$ for each $\omega = (\omega_1, \omega_2) \in \Omega$. That is, at state $\omega = (\omega_1, \omega_2)$, agent i is informed of the i-th coordinate ω_i.

For each $\omega = (\omega_1, \omega_2) \in \Omega$, let $t_i(\omega, \cdot)$ be the probability measure on (Ω, Σ) induced by the conditional distribution $f(\omega_{-i} \mid \omega_i)$ in the following sense: for each $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2) \in \Omega$,

$$t_i(\omega, (-\infty, \tilde{\omega}_1] \times (-\infty, \tilde{\omega}_2]) = \begin{cases} \displaystyle\int_{-\infty}^{\tilde{\omega}_{-i}} f(x \mid \omega_i) dx & \text{if } \tilde{\omega}_i \geq \omega_i \\ 0 & \text{if } \tilde{\omega}_i < \omega_i \end{cases}.$$

By construction, the model is regular. □

As a special case, I consider a bivariate normal distribution.

Example 3. Let (Ω, Σ) be as in Example 2. Let μ be the probability measure on (Ω, Σ) associated with the bivariate normal distribution $\mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$, where

[12] One can also generalize Example 1 to uncountable information sets.

ρ is the correlation coefficient. That is, for any $(\omega_1, \omega_2) \in \Omega$,

$$\mu((-\infty, \omega_1] \times (-\infty, \omega_2]) = \int_{-\infty}^{\omega_1} \int_{-\infty}^{\omega_2} \phi_\rho(x, y) dy dx,$$

where ϕ_ρ is the pdf:

$$\phi_\rho(x, y) = \frac{1}{\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right).$$

As in Example 2, let $I = \{1, 2\}$, and let $P_1(\omega_1, \omega_2) = \{\omega_1\} \times \mathbb{R}$ and $P_2(\omega_1, \omega_2) = \mathbb{R} \times \{\omega_2\}$ for each $\omega = (\omega_1, \omega_2) \in \Omega$.

For each $i \in I$ and $\omega = (\omega_1, \omega_2) \in \Omega$, since it is well-known that $f(\cdot \mid \omega_i)$ follows $\mathcal{N}(\rho\omega_i, 1 - \rho^2)$, agent i's type $t_i(\omega, \cdot)$ at ω is given by the unique measure such that, for each $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2) \in \Omega$,

$$t_i(\omega, (-\infty, \tilde{\omega}_1] \times (-\infty, \tilde{\omega}_2]) = \begin{cases} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \int_{-\infty}^{\tilde{\omega}_{-i}} \exp\left(-\frac{1}{2}\left(\frac{x - \rho\omega_i}{\sqrt{1-\rho^2}}\right)^2\right) dx & \text{if } \tilde{\omega_1} \geq \omega_i \\ 0 & \text{if } \tilde{\omega_1} < \omega_i \end{cases}.$$

□

Two remarks are in order. First, one can modify Example 2 so that Ω is a non-empty subset of $\mathbb{R}^2$. Second, one can also consider the case with a bivariate normal distribution in a different context. Suppose that a single agent is reasoning about the unknown mean x of a normal distribution $\mathcal{N}(x, \sigma_x^2)$. Assume that the variance σ_x^2 is known. The agent observes the outcome of a certain experiment $y = x + \varepsilon$, where ε is a noise following a normal distribution $\mathcal{N}(0, \sigma_\varepsilon^2)$. Let $\Omega = \mathbb{R}^2$ and $\Sigma = \mathcal{B}_{\mathbb{R}^2}$. The prior μ is given by the two normal distributions. At each state $\omega = (y, x) \in \Omega$, the agent's possibility set is $P(\omega) = \{y\} \times \mathbb{R}$. Letting $t(\omega, \cdot)$ be the product measure associated with the Dirac measure δ_y and a normal distribution $\mathcal{N}\left(\frac{\sigma_x^2 y + \sigma_\varepsilon^2 \mu_x}{\sigma_x^2 + \sigma_\varepsilon^2}, \frac{\sigma_x^2 \sigma_\varepsilon^2}{\sigma_x^2 + \sigma_\varepsilon^2}\right)$, the epistemic model $\langle \Omega, \Sigma, \mu, P, t \rangle$ is regular. Such "normal-normal updating" model is a workhorse tool, for instance, in the literature on market microstructure and rational-expectation equilibria. Section 5.2, in fact, extends the no-trade theorem in the framework of this paper.

3 Main Result

I present the main result that fully characterizes a regular model. A model is regular iff (i) each type $t(\omega, \cdot)$ is derived, through Bayes updating, from the prior μ conditional on the information set $P(\omega)$; and (ii) the information sets form a partition almost surely in the sense that $P(\omega) =_\mu [t(\omega)]$ (i.e., $\mu(P(\omega) \triangle [t(\omega)]) = 0$) for all $\omega \in \Omega$. Recall Figure 1.

Theorem 1. *If a model $\overrightarrow{\Omega}$ is regular, then (i) $\mu(E \cap P(\omega)) = \mu(P(\omega))t(\omega, E)$ for all $(\omega, E) \in \Omega \times \Sigma$, (ii) $P(\cdot) \subseteq [t(\cdot)]$, and (iii) for any fixed $\omega \in \Omega$, $P(\omega) \supseteq_\mu [t(\omega)]$. The converse holds when $\mu(P(\cdot)) > 0$ so that $t(\cdot, \cdot) = \mu(\cdot \mid P(\cdot))$ is well-defined.*

While Theorem 1 is interesting in itself, it generalizes the standard partitional model of knowledge and probabilistic beliefs in a way so that one does not have to impose the technical assumption that the information partitions are countable. Section 4 indeed shows that the framework of this paper is a generalization of the standard partitional model. Section 5 shows that well-known results such as the agreement theorem, the no-trade theorem, and the epistemic characterization of correlated equilibria all extend to the setting of this paper.

Four technical remarks on Theorem 1 are in order. First, one can require each $t(\omega, \cdot)$ only to be finitely additive in a regular model. The theorem does not hinge on the countable additivity of each $t(\omega, \cdot)$. Indeed, each finitely-additive type $t(\omega, \cdot)$ becomes countably additive as it inherits countable additivity from the prior μ by Part (i).

Second, suppose that μ and every $t(\omega, \cdot)$ are only finitely additive while keeping all the other assumptions.[13] The first part still holds. That is, the consistency conditions still imply that each type $t(\omega, \cdot)$ is the Bayes conditional probability $\mu(\cdot \mid P(\omega))$ whenever it is well-defined and that the collection of information sets $\{P(\omega)\}_{\omega \in \Omega}$ almost surely forms a partition (i.e., $P(\omega) =_\mu [t(\omega)]$ for all $\omega \in \Omega$). The second part holds when $\{[t(\cdot)]\}$ forms a finite partition. Section 7.1 and 7.2 study the role of additivity of the types $t(\omega, \cdot)$ and the prior μ, respectively.

Third, for each $\omega \in \Omega$, while the events $P(\omega)$ and $[t(\omega)]$ are equal with each other μ-almost surely, they are conceptually different. One dictates qualitative belief while the other quantitative belief.[14] In fact, since Part (iii) only requires $P(\omega) \supseteq_\mu [t(\omega)]$ for all $\omega \in \Omega$, it follows that $\{P(\omega)\}_{\omega \in \Omega}$ may fail to be a partition. Consequently, the qualitative belief operator K may violate Truth Axiom, Positive Introspection, and Negative Introspection.[15] The model, however, satisfies introspection of the form $B^p(\cdot) \subseteq KB^p(\cdot)$ and $(\neg B^p)(\cdot) \subseteq K(\neg B^p)(\cdot)$. Section 4.2 examines a special case where Ω is countable, $\Sigma = 2^\Omega$, and where $\mu(\{\cdot\}) > 0$. There, $P(\cdot) = [t(\cdot)]$ forms a partition, that is, the agent's qualitative belief satisfies Truth Axiom, Positive Introspection, and Negative Introspection. Moreover, qualitative belief and probability-one belief coincide as $K = B^1$. Section 6.1 compares the qualitative belief and probability-one belief operators in detail.

Fourth, I compare Parts (ii) and (iii). Both parts compare two sets $P(\omega)$ and $[t(\omega)]$ at each given state $\omega \in \Omega$. The possibility set $P(\omega)$ consists of states that the agent considers possible at ω, while the set $[t(\omega)]$ consists of states that the agent

[13] Note that the integral of a bounded measurable mapping with respect to μ is well-defined.

[14] Technically, in a regular model, one can show that $P(\omega) = \bigcap_{E \in \Sigma : \omega \in K(E)} E$ and $[t(\omega)] = \bigcap_{(E,p) \in \Sigma \times [0,1] : \omega \in B^p(E)} B^p(E)$.

[15] Example 6 in Appendix B is such an example.

cannot distinguish from ω according to her type mapping t. Part (ii) states that $P(\omega)$ is at least as "informative" as $[t(\omega)]$ in the sense that $P(\omega)$ is at least as fine as $[t(\omega)]$ by set inclusion. In contrast, Part (iii) establishes the weak sense in which the converse holds: $[t(\omega)]$ is at least as fine as $P(\omega)$ in the sense that $[t(\omega)]$ is μ-almost surely included in $P(\omega)$ (i.e., $[t(\omega)] \subseteq_\mu P(\omega)$, that is, $\mu([t(\omega)] \setminus P(\omega)) = 0$).

Here, it is important to compare two different sets $P(\omega)$ and $[t(\omega)]$ at each given state $\omega \in \Omega$ because they are pieces of information that the agent can process at each realized state ω. Indeed, in Part (iii), if one considers a different notion of μ-almost-sureness measuring the set of ω such that $P(\omega) \supseteq [t(\omega)]$ (i.e., $\{\omega \in \Omega \mid P(\omega) \supseteq [t(\omega)]\}$), then it may be the case that the μ-measure of such a set may not be equal to 1, provided that such a set is measurable.[16]

Theorem 1 relates to the following previous results on the consistency conditions between prior and posteriors where quantitative beliefs are sole primitives. First, Mertens and Zamir (1985) ask when an agent's posterior beliefs are derived from her (or common) prior conditional on her information. Mertens and Zamir (1985, Proposition 4.2) show that if a given quantitative belief model $\langle \Omega, \Sigma, \mu, t \rangle$ satisfies Invariance and Certainty of Beliefs then the agent's type $t(\omega, E)$ turns out to be the Bayes conditional probability $\mu(E \mid [t(\omega)])$ whenever it is well-defined.[17]

Second, Samet (1999) calls a quantitative belief model $\langle \Omega, \Sigma, \mu, t \rangle$ to be Bayesian if it satisfies Certainty of Beliefs and Invariance. Gaifman (1988) and Samet (1999) characterize a Bayesian model by the consistency requirement that the prior conditioned on some specification of the posterior beliefs must agree with the specification.[18]

These results and Theorem 1 of this paper *derive* Bayes updating from epistemic properties within a model. Theorem 1 states that, in an environment in which an agent's quantitative and qualitative beliefs are both present, the interaction between prior and posteriors (i.e., Invariance) and the ones between posteriors and possibility correspondence (i.e., Entailment and Self-Evidence of Beliefs) give rise to Bayes updating within the model, provided each type is countably (indeed, finitely) additive.

A possibility correspondence is almost surely unique: if $\langle \Omega, \Sigma, \mu, P, t \rangle$ and $\langle \Omega, \Sigma, \mu, P', t \rangle$ are regular models (where Ω, Σ, μ, and t are common), then, for each $\omega \in \Omega$, $P(\omega) =_\mu P'(\omega)$. Yet, the resulting qualitative belief operators may not necessarily satisfy $K(\cdot) =_\mu K'(\cdot)$.[19] Section 4.2 shows that if P in a regular model forms a partition then P is a unique one.

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[16] Example 6 in Appendix B is such an example.

[17] In the similar way to the proof of Theorem 1, the following can be established. For a given model $\overrightarrow{\Omega}$, let each $t(\omega, \cdot)$ be countably additive. Then, the model satisfies Invariance and Certainty of Beliefs iff $t(\cdot, \cdot) = \mu(\cdot \mid [t(\cdot)])$ whenever the right-hand side is well-defined.

[18] Formally, a model is Bayesian iff $\mu\left(E_i \mid \bigcap_{k=1}^{n} B^{p_k}(E_k)\right) \geq p_i$ for any $(p_i, E_i)_{i=1}^{n}$.

[19] Examples 5 and 6 in Appendix B can be seen as such examples.

4 Special Cases

This section shows that the framework of this paper encompasses the standard models in the literature. This section also demonstrates the uniqueness of an information partition in the standard cases and studies introspective properties of qualitative belief.

4.1 Standard Countable-Partition Models

The framework of this paper generalizes a standard countable-partition model: $\langle \Omega, \Sigma, \mu, P, t \rangle$ where P is an (at-most) countable partition with $\mu(P(\cdot)) > 0$ and $t(\omega, E) = \frac{\mu(E \cap P(\omega))}{\mu(P(\omega))}$. Such a countable model is, as in the e-mail game of Rubinstein (1989), important when one distinguishes common knowledge and finite levels of mutual knowledge.

I demonstrate two general results. First, if $\mu(P(\cdot)) > 0$ in a regular model then the type mapping t is uniquely determined by the other two ingredients (μ and P) through Bayes conditional probabilities. Second, if $\mu([t(\cdot)]) > 0$ then a partition P coincides with the one $[t(\cdot)]$ generated by the type mapping t.

Corollary 1. *Let $\langle \Omega, \Sigma, \mu, P, t \rangle$ and $\langle \Omega, \Sigma, \mu, P, t' \rangle$ be regular models (where Ω, Σ, μ, and P are common). Then, $t(\omega, \cdot) = t'(\omega, \cdot)$ for any $\omega \in \Omega$ with $\mu(P(\omega)) > 0$.*

Corollary 1 follows immediately from Theorem 1: $t(\omega, E) = \frac{\mu(E \cap P(\omega))}{\mu(P(\omega))} = t'(\omega, E)$ for all $E \in \Sigma$.

Next, observe that if a model $\langle \Omega, \Sigma, \mu, P, t \rangle$ is regular then so is $\langle \Omega, \Sigma, \mu, P', t \rangle$ with $P'(\cdot) = [t(\cdot)]$. While $K(\cdot) \subseteq K'(\cdot)$, the qualitative belief operator K' satisfies Truth Axiom, Positive Introspection, and Negative Introspection.

I show that if there is a partition $\{P(\omega)\}_{\omega \in \Omega}$ such that the model $\langle \Omega, \Sigma, \mu, P, t \rangle$ is regular and proper, then $P(\cdot) = [t(\cdot)]$. Roughly, if the analysts would like to introduce an agent's fully introspective knowledge (i.e., knowledge introduced by a partition) in a quantitative belief model $\langle \Omega, \Sigma, \mu, t \rangle$ with $\mu([t(\cdot)]) > 0$, then the unique possibility correspondence $P(\cdot)$ which makes the resulting model regular is $[t(\cdot)]$.

Corollary 2.

1. *Let $\overrightarrow{\Omega}$ be a model with $\mu(P(\cdot)) > 0$. The following are equivalent.*
   - *(a) $P(\cdot) = [t(\cdot)]$ and $t(\cdot, \cdot) = \mu(\cdot \mid P(\cdot))$.*
   - *(b) $\{P(\omega)\}_{\omega \in \Omega}$ is a partition; Invariance; Entailment; Self-Evidence of Beliefs; and each $t(\omega, \cdot)$ is a (countably) additive probability measure.*
2. *Let $\overrightarrow{\Omega}$ be a model satisfying Invariance, Entailment, Self-Evidence of Beliefs, and $\mu(P(\cdot)) > 0$. Suppose further that each $t(\omega, \cdot)$ is a (countably) additive probability measure. Then, $\{P(\omega)\}_{\omega \in \Omega}$ is a partition iff $P(\cdot) = [t(\cdot)]$.*
3. *Let $\overrightarrow{\Omega}$ be a model with (i) $P(\cdot) = [t(\cdot)]$, (ii) $\mu(P(\cdot)) > 0$, and (iii) each $t(\omega, \cdot)$ being a (countably) additive probability measure. The model satisfies $t(\cdot, \cdot) = \mu(\cdot \mid P(\cdot))$ iff it satisfies Entailment and Invariance.*

Parts (2) and (3) of Corollary 2 follow from Part (1). Part (2) establishes the uniqueness of the possibility correspondence P compatible with the consistency conditions. If the analysts introduce fully-introspective knowledge together with quantitative beliefs in a consistent way, the possibility correspondence $P(\cdot) = [t(\cdot)]$ is uniquely determined. Part (3) states that, under $P(\cdot) = [t(\cdot)]$, the "Bayes conditional property" (i.e., $t(\cdot, \cdot) = \mu(\cdot \mid P(\cdot))$) characterizes Entailment and Invariance. Section 6.2 studies the probabilistic definition of qualitative belief.

4.2 Discrete Regular Models: Partitional Properties of Qualitative Belief

Call a model $\overrightarrow{\Omega}$ *discrete* if Ω is countable, $\Sigma = 2^{\Omega}$, and if $\mu(\{\cdot\}) > 0$. Under this "standard" setting, the following corollary demonstrates that qualitative belief necessarily becomes knowledge and that probability-one belief and knowledge coincide.

Corollary 3. *1. A discrete model $\overrightarrow{\Omega}$ is regular iff (i) $P(\cdot) = [t(\cdot)]$ and (ii) $t(\cdot, \cdot) = \mu(\cdot \mid P(\cdot))$. In this case, $K = B^1$ (satisfies Truth Axiom, Positive Introspection, and Negative Introspection).*

2. Let a model $\overrightarrow{\Omega}$ satisfy the following: (i) $\{\omega\} \in \Sigma$ for all $\omega \in \Omega$; (ii) the model is regular; and (iii) $\mu(P(\cdot)) > 0$. Then, the model is discrete iff B^1 satisfies Truth Axiom. In this case, $K = B^1$ (satisfies Truth Axiom, Positive Introspection, and Negative Introspection).

Corollary 3 is related to Bonanno and Nehring (1998, 1999): they show that, in a finite state space, the existence of a common prior (assigning positive probability to each state) implies Truth Axiom of probability-one belief.

In a regular discrete model, the possibility correspondence $P(\cdot)$ exactly coincides with $[t(\cdot)]$, and knowledge and probability-one belief coincide with each other, irrespective of assumptions on information sets. Since $P(\cdot) = [t(\cdot)]$ forms a partition on the state space, $K = B^1$ satisfies Truth Axiom, Positive Introspection, Negative Introspection, (Countable) Conjunction, Monotonicity, and Necessitation. The operator K inherits Positive Introspection and Negative Introspection from B^1. In contrast, B^1 inherits the following form of strong conjunction property from K. For any collection of events $\mathcal{E} \in 2^{\Sigma}$ with $\bigcap \mathcal{E} \in \Sigma$, $\bigcap_{E \in \mathcal{E}} B^1(E) \subseteq B^1(\bigcap \mathcal{E})$.[20]

[20] To see that this conjunction property is strong, consider the following example in which the state space is uncountable (that is, the model is not discrete). Namely, take $(\Omega, \Sigma) = ([0,1], \mathcal{B}_{[0,1]})$, where $\mathcal{B}_{[0,1]}$ is the Borel σ-algebra on $[0,1]$. Take $\omega \in \Omega$. Let $E_{\omega'} = \Omega \setminus \{\omega'\}$ for each $\omega' \in \Omega \setminus \{\omega\}$. Suppose that, for each $\omega' \neq \omega$, the agent 1-believes $E_{\omega'}$ at ω. By this conjunction property, it follows that the agent 1-believes $\{\omega\} = \bigcap_{\omega' \in \Omega \setminus \{\omega\}} E_{\omega'}$ at ω. Thus, the agent would be omniscient in the sense that $B^1(E) = E$ for all $E \in \Sigma$.

Moreover, Corollary 3 implies that possibility coincides with assigning positive probability in the sense that $P(\omega) = \{\omega' \in \Omega \mid t(\omega, \{\omega'\}) > 0\}$ for all $\omega \in \Omega$.[21] Section 6.2 studies the probabilistic definition of qualitative belief.

Corollary 3 suggests that a care must be taken of the agent's qualitative belief if the analysts study an epistemic characterization of a solution concept for a game. Suppose that the analysts introduce qualitative belief instead of knowledge when they study, for example, implications of common belief in rationality instead of common knowledge of rationality.[22] Corollary 3 points to the importance of figuring out the relations among prior, posteriors, and information sets in a discrete model because qualitative belief reduces to (fully introspective) knowledge despite the analysts' purpose.

Next, a violation of fully introspective knowledge may also come from lack of introspection. Thus, suppose that the analysts would like to study knowledge of an agent who violates Negative Introspection: the agent does not know some event E at some state, and she does not know that she does not know it at that state.[23] In other words, consider unawareness of the agent in a state space model with knowledge and quantitative beliefs. In a discrete regular model, since K has to satisfy Negative Introspection, there is no state at which the agent is unaware of any event E (i.e., she does not know E and she does not know that she does not know E). I formulate it as an immediate corollary.

Corollary 4. *In a discrete regular model $\overrightarrow{\Omega}$, the agent is unaware of nothing:* $(\neg K)(\cdot) \cap (\neg K)^2(\cdot) = \emptyset$.

I discuss two points that the corollary makes on unawareness in standard state space models. First, the previous negative results on the possibility of describing a richer form of unawareness (e.g., Dekel, Lipman, and Rustichini, 1998; Modica and Rustichini, 1994) impose some direct links between knowledge and unawareness. Corollary 4 suggests that, once an agent possesses her quantitative belief, there is a new channel through which the agent can have fully introspective knowledge and thus she is fully aware of everything. Thus, the corollary sheds a new light on an additional limitation of a single-state-space model to represent a non-trivial form of unawareness.

Second, when it comes to representing non-introspective knowledge in a probabilistic environment, additivity of posteriors and the invariance condition would be strong so that knowledge eventually becomes fully introspective. To see this point,

[21] Halpern (1991) studies certainty (probability-one belief) by defining the "support relation" between two states by $t(\omega, \{\omega'\}) > 0$. Samet (1998) studies the (Markov transition) matrix generated by $(t(\omega, \{\omega'\}))_{\omega,\omega' \in \Omega}$. Morris (1996) derives qualitative belief from preferences, and under certain condition, the notion of possibility reduces to assigning positive probability.

[22] See, for instance, Bonanno (2008), Bonanno and Tsakas (2018), Fukuda (2023), Guarino and Ziegler (2022), Hillas and Samet (2020), Samet (2013), and Stalnaker (1994).

[23] See, for instance, Chen, Ely, and Luo (2012) and Fukuda (2021) for the possibility of representing unawareness on a standard state space.

suppose $P(\cdot) \neq [t(\cdot)]$ due to the failure of Negative Introspection (of K) in a discrete model as above. I consider non-partitional models in broader contexts, as the same conclusion as above can already be drawn with respect to solution concepts of games and the implications of common knowledge such as the agreement theorem (Aumann, 1976) in non-partitional models. Now, the model has to violate either Invariance, Entailment, Self-Evidence of Beliefs, or the assumption that each type $t(\omega, \cdot)$ is additive.

Since the information sets represent the agent's knowledge, assume Entailment: if she knows an event then she 1-believes the event. Also, assume Self-Evidence of Beliefs. Just as the agent's knowledge is positively introspective, if she p-believes an event E then she knows that she p-believes E. Thus, if each type $t(\omega, \cdot)$ is additive, then Corollary 3 implies that the model has to violate Invariance. Likewise, under Invariance, some type $t(\omega, \cdot)$ may be non-additive.

Consider a discrete model $\overrightarrow{\Omega} = \langle \Omega, \Sigma, \mu, P, t \rangle$ with $P(\cdot) \neq [t(\cdot)]$ and $t(\cdot, P(\cdot)) = 1$ (i.e., Entailment). The type at each state ω coincides with the Bayes conditional probability $t(\omega, E) = \frac{\mu(E \cap P(\omega))}{\mu(P(\omega))}$ for all (ω, E) iff the likelihood ratio between types coincides with the likelihood ratio between prior probabilities given an information set: $\frac{t(\omega,E)}{t(\omega,F)} = \frac{\mu(E \cap P(\omega))}{\mu(F \cap P(\omega))}$ for all (ω, E, F) with $\mu(F \cap P(\omega)) > 0$ and $t(\omega, F) > 0$. If this is the case, then each type $t(\omega, \cdot)$ is additive. If the model satisfies Self-Evidence of Beliefs, then it has to violate Invariance.

This observation sheds light on the comparison between ex-ante and ex-post analyses or the value of information in non-partitional knowledge models as discussed by, for example, Dekel and Gul (1997) and Geanakoplos (2021). This observation provides an intuition behind why "dynamic inconsistency" occurs in a non-partitional (i.e., reflexive and transitive) environment.[24] This observation also sheds light on quantitative belief updating in a non-partitional environment. In the literature studying non-partitional information sets, it has been assumed that an agent updates her probabilistic assessment according to the Bayes rule given a non-partitional information set as in the previous argument. As Dekel and Gul (1997, p.147) put it, however, "there are essentially no results that justify stapling traditional frameworks together with non-partitions."[25]

5 Applications

On the one hand, the standard partitional model of knowledge and probabilistic beliefs can be used to study epistemic characterizations of solution concepts such as Nash

[24]Gaifman (1988) also discusses the violation of a Bayesian belief model $\langle \Omega, \Sigma, \mu, t \rangle$ in terms of "dynamic inconsistency" between ex-ante and ex-post analyses.

[25]The probabilistic approach to unawareness by Heifetz, Meier, and Schipper (2013) considers an extended structure consisting of multiple sub-state-spaces. While an agent's beliefs satisfy Invariance within each sub-space, she exhibits unawareness in the entire enriched model.

and correlated equilibria, agreement and no-trade theorems, and various notions of rational-expectations equilibria. On the other hand, the model has a technical assumption that each agent's partition is at most countable. The technical contribution of this paper is to provide a tractable extension of the standard partitional model to an uncountable state space with uncountable information sets. In fact, the results obtained in the standard framework can be immediately generalized to the framework of this paper.

Among others, this section examines the following applications. Section 5.1 studies the notions of common belief and extends the agreement theorem of Aumann (1976), Monderer and Samet (1989), and Neeman (1996a). Section 5.2 generalizes the no-trade theorem of Milgrom and Stokey (1982) and Sonsino (1995). Section 5.3 generalizes Aumann (1974)'s epistemic characterization of correlated equilibria.[26]

Unless otherwise stated, let I be a non-empty countable set of agents. Recall that an interactive epistemic model is $\langle(\Omega,\Sigma,\mu),(P_i,t_i)_{i\in I}\rangle$ such that $\langle\Omega,\Sigma,\mu,P_i,t_i\rangle$ is a model for each agent $i\in I$. Denote by B_i^p and K_i agent i's p-belief operator and qualitative belief operator, respectively. Call the interactive epistemic model $\langle(\Omega,\Sigma,\mu),(P_i,t_i)_{i\in I}\rangle$ *regular* if $\langle\Omega,\Sigma,\mu,P_i,t_i\rangle$ is regular for each $i\in I$. Likewise, call the interactive epistemic model *discrete* if Ω is countable, $\Sigma=2^\Omega$, and if $\mu(\{\cdot\})>0$.

5.1 Common Qualitative and Quantitative Beliefs and Agreement Theorems

For a regular model, define the (iterative) common p-belief operator $\mathbb{C}^p:\Sigma\to\Sigma$ as $\mathbb{C}^p(\cdot):=\bigcap_{n\in\mathbb{N}}(B_I^p)^n(\cdot)$, where $B_I^p(\cdot):=\bigcap_{i\in I}B_i^p$ is the mutual p-belief operator. While $B_I^p(E)$ is the event that every agent p-believes E, the event $\mathbb{C}^p(E)$ is the set of states at which the agents commonly p-believe E, i.e., the agents believe E, they all believe that they all believe E, and so on *ad infinitum*.

Likewise, define the (iterative) common qualitative belief operator $\mathbb{C}:\Sigma\to\Sigma$ by $\mathbb{C}(\cdot):=\bigcap_{n\in\mathbb{N}}K_I^n(\cdot)$, where $K_I(\cdot):=\bigcap_{i\in I}K_i$ is the mutual qualitative belief operator. Since $K_I(\cdot)\subseteq B_I^1(\cdot)$, it can be seen that $\mathbb{C}(\cdot)\subseteq\mathbb{C}^1(\cdot)$. Also, it is well-known that $\mathbb{C}^p(\cdot)\subseteq B_I^p\mathbb{C}^p(\cdot)$ and $\mathbb{C}(\cdot)\subseteq K_I\mathbb{C}(\cdot)$.

An immediate consequence of Corollary 3 is that, in any discrete interactive epistemic model, common qualitative belief reduces to common knowledge without assuming, a priori, any assumption on agents' qualitative beliefs.

[26]For epistemic characterizations of solution concepts, this paper only studies Aumann (1987)'s characterization of correlated equilibria. For other solution concepts, for instance, epistemic characterizations of (mixed-strategy) Nash equilibria may call for a common prior (e.g., Aumann and Brandenburger, 1995; Barelli, 2009). In cooperative games or exchange economies with asymmetric information, for instance, Maus (2003) and Yannelis (1991) study various notions of a core in a partitional model. In the robustness literature, pioneering papers such as Kajii and Morris (1997b) and Morris, Rob, and Shin (1995) use a model with at-most countable information sets. It would be interesting to extend such results to an uncountable state space model.

Corollary 5. *Let $\langle(\Omega, \Sigma, \mu), (P_i, t_i)_{i\in I}\rangle$ be a discrete regular interactive epistemic model. Then, common qualitative belief and common 1-belief coincide: $\mathbb{C} = \mathbb{C}^1$. Moreover, $\mathbb{C} = \mathbb{C}^1$ satisfies Truth Axiom, Positive Introspection, and Negative Introspection. That is, common qualitative belief reduces to common knowledge.*

Next, in a regular model, the agents have correct mutual belief in what is commonly p-believed. The proposition below states that, whenever the agents mutually believe (either qualitatively or with probability 1) that they have common p-belief in an event, they have common p-belief in the event.

Proposition 1. *In a regular interactive epistemic model,*

$$K_I\mathbb{C}^p(\cdot) \subseteq B_I^1\mathbb{C}^p(\cdot) \subseteq \mathbb{C}^p(\cdot).$$

Next, I establish the following agreement theorem(s) (Aumann, 1976; Monderer and Samet, 1989; Neeman, 1996a) for a regular interactive epistemic model. The result holds as long as there exists some prior μ such that $\langle(\Omega, \Sigma, \mu), (P_i, t_i)_{i\in I}\rangle$ is regular.

Proposition 2. *Let $\langle(\Omega, \Sigma, \mu), (P_i, t_i)_{i\in I}\rangle$ be regular. Let $X_i := \{\omega \in \Omega \mid t_i(\omega, E) = r_i\}$ for each $i \in I$, and let $X := \bigcap_{i\in I} X_i$. If $\mu(\mathbb{C}^p(X)) > 0$, then $|r_i - r_j| \leq 1 - p$ for all $i, j \in I$. Especially, if $\mu(\mathbb{C}(X)) > 0$, then $r_i = r_j$ for all $i, j \in I$.*

If the regular model $\langle(\Omega, \Sigma, \mu), (P_i, t_i)_{i\in I}\rangle$ satisfies $\mu(P_i(\cdot)) > 0$ for some $i \in I$, then the following stronger agreement theorem holds: $\mathbb{C}^p(X) \neq \emptyset$ implies $|r_i - r_j| \leq 1 - p$ for all $i, j \in I$. Especially, if $\mathbb{C}(X) \neq \emptyset$, then $r_i = r_j$ for all $i, j \in I$.[27]

5.2 No-Trade Theorem

I extend Sonsino (1995)'s no-trade theorem to the framework of this paper, where a state space can be an arbitrary measurable space and each agent's information sets can be uncountable.[28] As claimed by Green (2012), the literature in market microstructure and rational-expectation equilibria sometimes invokes the no-trade theorem by Milgrom and Stokey (1982) (which is established on a finite state space) to a setting in which the underlying state space is a continuum.

In this subsection, let I be a non-empty finite set of agents. Let L be a non-empty finite set of commodities. Agent i's utility function $u_i : \Omega \times \mathbb{R}_+^{|L|} \to \mathbb{R}$ is measurable. Agent i's (state-contingent) initial endowment $e_i : \Omega \to \mathbb{R}_+^{|L|}$ is a measurable function

[27] Without $\mu(P_i(\cdot)) > 0$ for some $i \in I$, the statement requires $\mu(\mathbb{C}(X)) > 0$ even if Ω is finite. See Example 4 in Appendix B.

[28] The no-trade theorem of Sonsino (1995) extends the common knowledge assumption to common p-belief, where each agent's information sets are at most countable. See also Neeman (1996b) for another extension. Koutsougeras and Yannelis (2017) also study a no-trade theorem on a general measurable space, keeping the original common knowledge assumption of Milgrom and Stokey (1982).

which satisfies $P_i(\omega) \subseteq \{\omega' \in \Omega \mid e_i(\omega') = e_i(\omega)\}$ for each $\omega \in \Omega$. The assumption intends to capture that each agent qualitatively believes, at each state, her own endowment. Denote $e := (e_i)_{i \in I}$. A state-contingent consumption bundle of agent i is a measurable function $x_i : \Omega \to \mathbb{R}_+^{|L|}$. An allocation $x := (x_i)_{i \in I}$ is *feasible* (with respect to the initial endowment e) if $\sum_{i \in I} x_i(\cdot) \leq \sum_{i \in I} e_i(\cdot)$.

Agent i's ex-ante expected utility from x_i is

$$v_i(x_i) := \int_\Omega u_i(\omega, x_i(\omega)) d\mu,$$

provided that the right-hand side is well-defined. Agent i's ex-post expected utility at ω from x_i is

$$v_i(x_i \mid \omega) := \int_\Omega u_i(\tilde{\omega}, x_i(\tilde{\omega})) t_i(\omega, d\tilde{\omega}),$$

provided that the right-hand side is well-defined.

The initial allocation e is *ex-ante Pareto-optimal* if there is no other feasible allocation x satisfying (i) $v_i(x_i) \geq v_i(e_i)$ for all $i \in I$; and (ii) $v_j(x_j) > v_j(e_j)$ for some $j \in I$. Agent i *weakly ε-prefers* x_i to e_i at ω (written $x_i \succcurlyeq^\varepsilon_{i,\omega} e_i$) if $v_i(x_i \mid \omega) \geq v_i(e_i \mid \omega) + \varepsilon$. Likewise, agent i *strictly ε-prefers* x_i to e_i at ω (written $x_i \succ^\varepsilon_{i,\omega} e_i$) if $v_i(x_i \mid \omega) > v_i(e_i \mid \omega) + \varepsilon$. The agents *$\varepsilon$-prefer* x to e at ω (written $x \succ^\varepsilon_\omega e$) if (i) $x_i \succcurlyeq^\varepsilon_{i,\omega} e_i$ for all $i \in I$ and (ii) $x_j \succ^\varepsilon_{j,\omega} e_j$ for some $j \in I$.

With these in mind, the no-trade theorem below states that if the agents start with an ex-ante Pareto-optimal allocation e and if x is a feasible allocation, then the event that it is common p-belief that the agents ε-prefer x to e has μ-measure zero, provided that p is close enough to 1.

Proposition 3. *Let e and x be an ex-ante Pareto-optimal allocation and a feasible allocation, respectively, such that $\overline{M} := \max_{i \in I} \sup_{\omega \in \Omega}(u_i(\omega, x_i(\omega)) - u_i(\omega, e_i(\omega))) < \infty$. Let $(p, \varepsilon) \in (0, 1] \times (0, +\infty)$ be such that $\varepsilon > (1 - p)\overline{M}$. Then,*

$$\mu(\mathbb{C}^p(\{\omega \in \Omega \mid x \succ^\varepsilon_\omega e\})) = 0.$$

5.3 Correlated Equilibria

Let $\langle (A_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be a strategic game: A_i is agent i's action space endowed with a σ-algebra $\mathcal{A}_i$ with $\{a_i\} \in \mathcal{A}_i$ for every $a_i \in A_i$; and $u_i : A \to \mathbb{R}$ with $A := \bigtimes_{i \in I} A_i$ is agent i's measurable payoff function with respect to the product σ-algebra $\mathcal{A}$ on A. To ensure that agents' payoffs are well-defined, assume $\sup_{(a,i) \in A \times I} |u_i(a)| < \infty$.

A *correlated equilibrium* $\sigma^* = (\sigma^*_i)_{i \in I}$ over a regular (interactive epistemic) model $\langle (\Omega, \Sigma, \mu), (P_i, t_i)_{i \in I} \rangle$ is a profile of (measurable) strategies $\sigma^*_i : \Omega \to A_i$ satisfying, for each $i \in I$, (i) belief-in-strategy: $P(\omega) \subseteq [\sigma^*_i(\omega)] := \{\omega' \in \Omega \mid \sigma^*_i(\omega') = \sigma^*_i(\omega)\} \in \Sigma$ for every $\omega \in \Omega$; and (ii) (ex-ante) optimality:

$$\int_\Omega u_i(\sigma^*_i(\omega), \sigma^*_{-i}(\omega)) \mu(d\omega) \geq \int_\Omega u_i(\sigma_i(\omega), \sigma^*_{-i}(\omega)) \mu(d\omega)$$

for any (measurable) strategy $\sigma_i : \Omega \to A_i$ satisfying $P(\omega) \subseteq [\sigma_i(\omega)] \in \Sigma$ for each $\omega \in \Omega$. The belief-in-strategy condition means that agent i qualitatively believes that she takes an action $\sigma_i^*(\omega) \in A_i$ at ω.

Since the model is regular, the ex-ante optimality condition is equivalent to the ex-post one. That is, for each $i \in I$, the optimality condition reduces to: for each $a_i \in A_i$ and $\omega \in \Omega$,

$$\int_{P_i(\omega)} u_i(\sigma_i^*(\tilde{\omega}), \sigma_{-i}^*(\tilde{\omega}))t_i(\omega, d\tilde{\omega}) \geq \int_{P_i(\omega)} u_i(a_i, \sigma_{-i}^*(\tilde{\omega}))t_i(\omega, d\tilde{\omega}).$$

With this in mind, the agents commonly believe their Bayes rationality in a regular (interactive epistemic) model $\langle(\Omega, \Sigma, \mu), (P_i, t_i)_{i \in I}\rangle$ under a strategy profile $\sigma^* = (\sigma_i^*)_{i \in I}$ if σ^* is a profile of (measurable) strategies $\sigma_i^* : \Omega \to A_i$ satisfying (i) belief-in-strategy: $P(\omega) \subseteq [\sigma_i^*(\omega)] := \{\omega' \in \Omega \mid \sigma_i^*(\omega') = \sigma_i^*(\omega)\} \in \Sigma$ for every $\omega \in \Omega$; and (ii) Bayes rationality: for all $i \in I$, $a_i \in A_i$, and $\omega \in \Omega$,

$$\int_{\Omega} u_i(\sigma_i^*(\tilde{\omega}), \sigma_{-i}^*(\tilde{\omega}))t_i(\omega, d\tilde{\omega}) \geq \int_{\Omega} u_i(a_i, \sigma_{-i}^*(\tilde{\omega}))t_i(\omega, d\tilde{\omega}).$$

Then, σ^* is a correlated equilibrium over $\langle(\Omega, \Sigma, \mu), (P_i, t_i)_{i \in I}\rangle$. This generalizes Aumann (1987)'s epistemic characterization of correlated equilibria to any regular model.

6 Theoretical Implications

This section studies theoretical implications of Theorem 1. Especially, the section studies the differences between qualitative and probability-one beliefs. It is theoretically important to understand the differences between qualitative and probability-one beliefs. Section 6.1 studies almost-sure Truth Axiom of qualitative and probability-one beliefs. Section 6.2 studies the notion of qualitative belief from the standpoint of quantitative beliefs.

6.1 Almost-Sure Truth Axiom of Probability-One and Qualitative Beliefs

Truth Axiom is a strong axiom in that if one agent (say, Alice) knows that another agent (say, Bob) is rational then it must be the case that Bob is indeed rational. In the literature on epistemic characterizations of solution concepts of games where agents possess qualitative beliefs, papers such as Bonanno (2008), Bonanno and Tsakas (2018), Fukuda (2020, 2023), Guarino and Ziegler (2022), Hillas and Samet (2020), and Samet (2013) study the differences in predictions between common belief in rationality and common knowledge of rationality. Bonanno and Nehring (1998) study the role of Truth Axiom on the agreement theorem and the absence of unbounded gains from betting.

I show below that the probability-one belief and qualitative belief operators satisfy Truth Axiom μ-almost surely: $B^1(E) \subseteq_\mu E$ and $K(E) \subseteq_\mu E$ for all $E \in \Sigma$. In a regular interactive epistemic model, this implies that the common 1-belief and qualitative common belief operators also satisfy Truth Axiom μ-almost surely: $\mathbb{C}^1(E) \subseteq_\mu E$ and $\mathbb{C}(E) \subseteq_\mu E$ for all $E \in \Sigma$.

Corollary 6. *1. In any regular model, B^1 satisfies Truth Axiom μ-almost surely: $\mu(B^1(E) \setminus E) = 0$ for all $E \in \Sigma$. Consequently, K also satisfies Truth Axiom μ-almost surely.*

2. In any regular interactive epistemic model, the common qualitative belief operator $\mathbb{C}$ and the common 1-belief operator $\mathbb{C}^1$ satisfy Truth Axiom μ-almost surely.

To obtain almost-sure Truth Axiom of B^1, it is enough for a given model to satisfy Invariance, Certainty of Beliefs, and the property that each $t(\omega, \cdot)$ is monotone. Thus, this part of Corollary 6 also holds in a quantitative belief model $\langle \Omega, \Sigma, \mu, t \rangle$.

Brandenburger and Dekel (1987, Property P.4.) obtain a closely-related result. In their model, an agent's type mapping is introduced as a posterior conditional on a sub-σ-algebra that dictates her information. Then, the probability-one belief operator satisfies Truth Axiom $t(\omega, \cdot)$-almost surely for all $\omega \in \Omega$ (μ-almost surely as well). Also, Halpern (1991, Proposition 4.3) establishes that B^1 satisfies Truth Axiom μ-almost surely when the type mapping does not depend on states (i.e., $t(\omega, \cdot) = t(\omega', \cdot)$ for all $\omega, \omega' \in \Omega$).

Indeed, repeating almost-sure Truth Axiom for agents, for any sequence of agents $(i_j)_{j=1}^n$ in I, the operator $B_{i_n} \circ \cdots \circ B_{i_1}$ satisfies Truth Axiom μ-almost surely (for an event E, $(B_{i_n} \circ \cdots \circ B_{i_1})(E)$ is the event that agent i_1 believes ... the agent i_n believes E). Indeed, letting $E_j := (B_{i_j} \circ \cdots \circ B_{i_1})(E)$ for each $j \in \{1, \ldots, n\}$ and $E_0 := E$,

$$\mu((B_{i_n} \circ \cdots \circ B_{i_1})(E) \setminus E) = \mu(E_n \setminus E_0) \leq \sum_{j=1}^{n} \mu(E_j \setminus E_{j-1}) = 0.$$

6.2 A Probabilistic Definition of Qualitative Belief

I examine the differences between qualitative and quantitative beliefs. First, as is known in the literature (e.g., Monderer and Samet, 1989; Vassilakis and Zamir, 1993), it is not necessarily the case that $B^1(\cdot) =_\mu K(\cdot)$.[29]

Second, on a related point, while the probability-one belief operator satisfies Positive Introspection and Negative Introspection ($B^1(\cdot) \subseteq B^1 B^1(\cdot)$ and $(\neg B^1) \subseteq B^1(\neg B^1)(\cdot)$) by Certainty of Beliefs, it is not necessarily the case in a regular model

[29]Indeed, in a regular model, let $\omega \in \Omega$ satisfy the following properties: (i) $\omega' \in P(\omega)$ implies $P(\omega) \subseteq P(\omega')$; (ii) $\mu(P(\omega)) > 0$; and (iii) $\omega \in B^1(E) \setminus K(E)$. Then, $0 < \mu(P(\omega)) \leq \mu(B^1(E) \setminus K(E))$.

that the qualitative belief operator K satisfies both introspective properties μ-almost surely (i.e., $K(\cdot) \subseteq_\mu KK(\cdot)$ and $(\neg K) \subseteq_\mu K(\neg K)(\cdot)$ may fail).[30]

Below, I first show that qualitative and probability-one beliefs may differ in an uncountable model. Then, I move on to a sufficient condition under which the possibility correspondence that determines the agent's qualitative belief can be defined from the type mapping that determines her quantitative beliefs.

6.2.1 Do Qualitative and Probability-One Belief Coincide?

I show that qualitative and probability-one beliefs coincide if and only if possibility implies putting positive probability, i.e., if an agent considers a state ω' possible at ω, then she assigns positive probability to $\{\omega'\}$ at ω. This result may be incompatible with an uncountable state space because it restricts the support of the type at ω to be at most countable.

Proposition 4. *For all $\omega \in \Omega$, assume that $\{\omega\} \in \Sigma$ and that $t(\omega, \cdot)$ is monotone. Then, each of (1) and (2) implies (3). Assuming Entailment, all are equivalent.*

1. $P(\omega) \subseteq \{\omega' \in \Omega \mid t(\omega, \{\omega'\}) > 0\}$.
2. $(\neg K)(\neg E) \subseteq \bigcup_{r \in (0,1] \cap \mathbb{Q}} B^r(E)$.
3. $B^1(\cdot) \subseteq K(\cdot)$ *(consequently, $K = B^1$).*

In Proposition 4, Condition (1) means that if the agent considers ω' possible at ω, then her type at ω assigns positive probability to ω'. Condition (2) states that if the agent considers an event E possible (in the sense that she does not qualitatively believe its negation $\neg E$) then she believes E with positive probability. Under Entailment, each condition is equivalent to $K = B^1$.

Two remarks are in order. First, the converse of Condition (2), $(\neg K)(\neg E) \supseteq \bigcup_{r \in \mathbb{Q} \setminus \{0\}} B^r(E)$, states that if the agent believes an event E with positive probability, then she considers E possible (in the sense that she does not believe its negation $\neg E$). This property obtains in any regular model.[31]

Second, Proposition 4 states that, generally, $P(\omega)$ may not be the support of $t(\omega)$ (in the sense of the smallest closed set F satisfying $t(\omega, F) = 1$). For example, let each $t(\omega, \cdot)$ be the Lebesgue measure on $\Omega = [0, 1]$. The support is $\mathbb{R}$. If possibility reduces to putting positive probability, then Condition (1) implies that the agent assigns positive probability to any non-empty set, which is impossible. Similarly, probability-one and qualitative beliefs may not coincide if the underlying state space Ω is uncountable. Condition (1) implies that each $P(\omega)$ is at most countable. Hence, Proposition 4 implies the importance of introducing a possibility correspondence separately from a type mapping when an underlying state space is uncountable.

[30] Example 6 in Appendix B is such an example.

[31] This is because the model satisfies $t(\cdot, (P(\cdot))^c) = 0$ under Entailment and the additivity of each $t(\omega, \cdot)$.

6.2.2 Possibility Correspondence and Support of Types

Next, in a purely-probabilistic setting, I provide a technical measurability condition under which one can introduce a possibility correspondence (qualitative belief) from the support of each type, as previous studies have introduced qualitative belief in a probabilistic framework.[32]

To define the notion of the support of a type (which is a probability measure), let $(\Omega, \mathcal{O})$ be a Polish space (i.e., a complete and separable metric space with $\mathcal{O}$ being the collection of open sets), let $\Sigma := \mathcal{B}_\Omega(\mathcal{O})$ be the Borel σ-algebra, and let μ be a prior on (Ω, Σ). As is standard, introduce a topology $\mathcal{O}_\Delta$ on $\Delta(\Omega)$ by the sub-basis $\{\{\mu \in \Delta(\Omega) \mid \mu(E) > p\} \mid (E, p) \in \mathcal{O} \times [0, 1]\}$. Note that since Ω is Polish, this topology coincides with the weak-$*$ topology. Slightly abusing the notation, consider a continuous type mapping $t : (\Omega, \mathcal{O}) \to (\Delta(\Omega), \mathcal{O}_\Delta)$ (i.e., each $t(\omega, \cdot)$ is a countably-additive probability measure on Ω).

Now, each $t(\omega, \cdot)$ has its support $\operatorname{supp} t(\omega) = (\bigcup\{E \in \mathcal{O} \mid t(\omega, E) = 0\})^c \in \Sigma$. If $\operatorname{supp} t(\cdot)$ satisfies the measurability condition that $\{\omega \in \Omega \mid \operatorname{supp} t(\omega) \subseteq E\} \in \Sigma$ for each $E \in \Sigma$, then one can add the possibility correspondence $\operatorname{supp} t(\cdot)$ into a purely probabilistic model $\langle \Omega, \Sigma, \mu, t\rangle$.[33]

By definition, $t(\cdot, \operatorname{supp} t(\cdot)) = 1$. Since $\{\mu\}$ is closed, so is $[t(\cdot)] = t^{-1}(\{t(\cdot)\})$. Thus, $\operatorname{supp} t(\cdot) \subseteq [t(\cdot)]$ iff $t(\cdot, [t(\cdot)]) = 1$. The former condition states that the agent qualitatively believes her probabilistic beliefs (Certainty of Beliefs) when $t(\cdot, [t(\cdot)]) = 1$, provided that the qualitative belief (i.e., a possibility correspondence) is defined by the support of the type $t(\omega)$. Hence, if $\langle \Omega, \Sigma, \mu, t\rangle$ satisfies Invariance and $t(\cdot, [t(\cdot)]) = 1$, then the model $\langle (\Omega, \Sigma, \mu), t, \operatorname{supp} t(\cdot)\rangle$ is regular.[34]

In a (regular) model in which Self-Evidence of Beliefs holds, if $\omega' \in \operatorname{supp} t(\omega)$ then $t(\omega, \cdot) = t(\omega', \cdot)$ and thus $\operatorname{supp} t(\omega) = \operatorname{supp} t(\omega')$. Hence, the possibility correspondence $\operatorname{supp} t(\cdot)$ is transitive and Euclidean.[35]

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[32] See, for instance, Battigalli and Bonanno (1999), Bonanno and Nehring (1999), Halpern (1991), Tan and Werlang (1988), and Vassilakis and Zamir (1993).

[33] Section 6.2.1 has shown that this measurability condition may be a strong sufficient condition in some contexts.

[34] In particular, if Ω is countable and $\Sigma = 2^\Omega$, then one can introduce an agent's possibility correspondence by the support of t. However, note that it may not necessarily be the case that $\operatorname{supp}(\cdot) = [t(\cdot)]$. For example, let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mu = (\frac{1}{2}, \frac{1}{2}, 0)$, and $t(\omega, \cdot) = \mu$ for each $\omega \in \Omega$. Then, $\operatorname{supp}(\cdot) = \{\omega_1, \omega_2\} \subsetneq [t(\cdot)]$. Consequently, $K \neq B^1$. For instance, $K(\{\omega_1, \omega_2\}) = \emptyset \subsetneq \Omega = B^1(\{\omega_1, \omega_2\})$.

[35] Three remarks are in order. First, reflexivity may fail. As a simple example, let $(\Omega, \Sigma) = (\{\omega_1, \omega_2\}, 2^\Omega)$, and let $\mu = t(\omega, \cdot) = (1, 0)$ for each $\omega \in \Omega$. Second, as in Section 4.2, if Ω is countable, $\Sigma = 2^\Omega$, and $\mu(\{\cdot\}) > 0$, then $\operatorname{supp} t(\cdot)$ forms a partition. Third, without Self-Evidence of Beliefs, $\operatorname{supp} t(\cdot)$ may not necessarily be transitive or Euclidean. Let $(\Omega, \Sigma) = (\{\omega_1, \omega_2, \omega_3\}, 2^\Omega)$, and let $\mu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Let $t(\omega_1, \cdot) = (0, \frac{1}{2}, \frac{1}{2})$, $t(\omega_2, \cdot) = (\frac{1}{2}, 0, \frac{1}{2})$, and $t(\omega_3, \cdot) = (\frac{1}{2}, \frac{1}{2}, 0)$. Then, $\operatorname{supp} t(\cdot)$ is not reflexive, transitive, or Euclidean.

7 Discussions

This section provides further discussions. Section 7.1 studies the role of additivity of types. Section 7.2 studies the role of additivity of the prior. Section 7.3 shows that one can take agents' expectations instead of quantitative beliefs as a primitive. Section 7.4 studies conditional probability systems (CPS). Section 7.5 introduces regular conditional probabilities instead of types as a primitive. Section 7.6 provides concluding remarks. Throughout this section, unless otherwise stated, I focus on a single-agent epistemic model.

7.1 The Role of Additivity of Types

This subsection studies the role of additivity of types. I decompose the set $[t(\omega)]$ into two parts and impose the following stronger measurability conditions on a type mapping t: (i) $(\uparrow t(\omega)) := \{\omega' \in \Omega \mid t(\omega, \cdot) \leq t(\omega', \cdot)\} \in \Sigma$ and (ii) $(\downarrow t(\omega)) := \{\omega' \in \Omega \mid t(\omega', \cdot) \leq t(\omega, \cdot)\} \in \Sigma$ for all $\omega \in \Omega$. If $\tilde{\omega} \in (\uparrow t(\omega))$, then $t(\tilde{\omega}, E)$ is at least as high as $t(\omega, E)$ for any $E \in \Sigma$. By definition, $[t(\cdot)] = (\uparrow t(\cdot)) \cap (\downarrow t(\cdot))$.

Here I assume the measurability of $(\uparrow t(\cdot))$ and $(\downarrow t(\cdot))$ in order to see two different ways in which the agent can be certain of her probabilistic beliefs. By letting $[0,1]_{\mathbb{Q}} := [0,1] \cap \mathbb{Q}$, observe

$$(\uparrow t(\omega)) = \bigcap_{(p,E) \in [0,1]_{\mathbb{Q}} \times \Sigma : t(\omega,E) \geq p} \{\omega' \in \Omega \mid t(\omega', E) \geq p\} \text{ and} \tag{1}$$

$$(\downarrow t(\omega)) = \bigcap_{(p,E) \in [0,1]_{\mathbb{Q}} \times \Sigma : t(\omega,E) \leq p} \{\omega' \in \Omega \mid t(\omega', E) \leq p\}. \tag{2}$$

In the right-hand side of each expression, p can also range over $[0,1]$ instead of $[0,1]_{\mathbb{Q}}$. Also, Σ can be replaced with a countable algebra Σ_0 that generates Σ if each $t(\omega, \cdot)$ is continuous with respect to both increasing and decreasing sequences of events and is monotone. Then, $(\uparrow t(\cdot))$ and $(\downarrow t(\cdot))$ are measurable without assumption.

When the agent reasons about $(\uparrow t(\cdot))$, she only uses her positive belief of the form, "I believe an event with probability at least p." That is, when she does not believe an event E with probability at least p, she does not take this information into account in inferring the true state of the world. In contrast, when the agent reasons about $(\downarrow t(\cdot))$, she only uses her negative belief of the form, "I do not believe an event with probability more than p."

The distinction between $(\uparrow t(\cdot))$ and $[t(\cdot)]$ (and between $(\downarrow t(\cdot))$ and $[t(\cdot)]$) matters when the agent's belief is non-additive. That is, if each $t(\omega, \cdot)$ is additive, then $(\uparrow t(\cdot)) = (\downarrow t(\cdot)) = [t(\cdot)]$.[36] Note that, in this case, Part (ii) of the measurability condition is implied by Part (i).

[36]Suppose $t(\omega, \cdot) \leq t(\omega', \cdot)$. If $t(\omega, E) < t(\omega', E)$ for some $E \in \Sigma$, then $1 = t(\omega, E) + t(\omega, E^c) < t(\omega', E) + t(\omega', E^c) = 1$, a contradiction.

In fact, the distinction between $(\uparrow t(\cdot))$ and $[t(\cdot)]$ is related to Ghirardato (2001) and Mukerji (1997) in the decision theory literature, which characterize non-additivity from the agent's "perception" (see also Bonanno, 2002; Lipman, 1995). To see this point, interpret t as a mapping from Ω into the collection of set functions (with some given properties). On the one hand, $[t(\omega)]$ can be considered to be $t^{-1}(\{t(\omega)\}) := \{\tilde{\omega} \in \Omega \mid t(\omega)(\cdot) = t(\tilde{\omega})(\cdot)\}$. Thus, at state ω, the agent is assumed to be able to observe a singleton $\{t(\omega)\}$ so that she is able to infer that the true state is in $t^{-1}(\{t(\omega)\})$. On the other hand, $(\uparrow t(\omega))$ can be regarded as $t^{-1}(\{\mu \mid \mu(\cdot) \geq t(\omega)(\cdot)\}) := \{\tilde{\omega} \in \Omega \mid t(\tilde{\omega})(\cdot) \geq t(\omega)(\cdot)\}$. At state ω, the agent is assumed to be able to observe a (generally) non-singleton set $\{\mu \mid \mu(\cdot) \geq t(\omega)(\cdot)\}$. In Ghirardato (2001) and Mukerji (1997), the agent has a limited observation on a non-empty set of consequences or signals (instead of own beliefs here) so that her beliefs may be non-additive. If, in contrast, she has a perfect observation on a singleton set of a consequence or a signal, then her beliefs are finitely additive.

Full introspection associated with Certainty of Beliefs comes from the fact that, at each state ω, the agent always puts probability one to the set of states that are indistinguishable from ω. I now disentangle the negative introspective property $((\neg B^p)(\cdot) \subseteq B^1(\neg B^p)(\cdot))$ and Certainty of Beliefs.

A model $\overrightarrow{\Omega}$ satisfies *Certainty of Positive (p-)Beliefs* if $t(\cdot, (\uparrow t(\cdot))) = 1$. At each state ω, the agent puts probability one to the set of states $\tilde{\omega}$ with $t(\omega, \cdot) \leq t(\tilde{\omega}, \cdot)$. If the model satisfies Certainty of Positive Beliefs and if each $t(\omega, \cdot)$ is monotone, then $B^p(\cdot) \subseteq B^1 B^p(\cdot)$. Indeed, I show the sense in which Certainty of Positive Beliefs captures the positive introspective property $B^p(\cdot) \subseteq B^1 B^p(\cdot)$.

Proposition 5. *Let a model $\overrightarrow{\Omega}$ satisfy the following: (i) Σ is generated by a countable algebra Σ_0; (ii) each $t(\omega, \cdot)$ is monotone; and (iii) for each $\omega \in \Omega$, if $t(\omega, E_n) = 1$ for all $n \in \mathbb{N}$ then $t(\omega, \bigcap_{n \in \mathbb{N}} E_n) = 1$. Then:*

1. *The model satisfies Certainty of Positive Beliefs iff $B^p(\cdot) \subseteq B^1 B^p(\cdot)$.*
2. *Suppose further that $t(\cdot, \Omega) = 1$. Then, $t(\cdot, (\downarrow t(\cdot))) = 1$ iff $(\neg B^p)(\cdot) \subseteq B^1(\neg B^p)(\cdot)$.*

Proposition 5 provides the sense in which Certainty of Positive Beliefs characterizes the positive introspective property $B^p(\cdot) \subseteq B^1 B^p(\cdot)$, while (standard) Certainty of Beliefs implies both $B^p(\cdot) \subseteq B^1 B^p(\cdot)$ and $(\neg B^p)(\cdot) \subseteq B^1(\neg B^p)(\cdot)$.

Proposition 5 clarifies the role of additivity in introspection of quantitative beliefs. If each $t(\omega, \cdot)$ is additive then Certainty of Beliefs and Certainty of Positive Beliefs coincide as $(\uparrow t(\cdot)) = [t(\cdot)]$. Thus, under the setting of Proposition 5, if each $t(\omega, \cdot)$ is additive, then the positive introspective property $(B^p(\cdot) \subseteq B^1 B^p(\cdot))$ implies the negative introspective property $((\neg B^p)(\cdot) \subseteq B^1(\neg B^p)(\cdot))$.[37]

[37] This remark applies to the aforementioned setting where Σ is generated by a countable algebra and where each $t(\omega, \cdot)$ is a countably-additive probability measure.

If Σ is finite (note that an infinite σ-algebra Σ is necessarily uncountable), then Condition (iii) of Proposition 5 can be replaced as follows: $t(\omega, E) = t(\omega, F) = 1$ implies $t(\omega, E \cap F) = 1$. This condition is met, for instance, when a monotone type $t(\omega, \cdot)$ satisfies $t(\omega, E) + t(\omega, F) \leq t(\omega, E \cap F) + t(\omega, E \cup F)$, the property known as convexity (of a set function).

Next, $\overrightarrow{\Omega}$ satisfies *Self-Evidence of Positive (p-)Beliefs* if $\omega' \in P(\omega)$ implies $t(\omega, \cdot) \leq t(\omega', \cdot)$, i.e., $P(\omega) \subseteq (\uparrow t(\omega))$. It says: if the agent considers ω' possible at ω, then, as long as she assigns probability at least p to an event E at ω, she also assigns probability at least p to E at ω'. Again, Entailment and Self-Evidence of Positive Beliefs imply Certainty of Positive Beliefs, provided that each $t(\omega, \cdot)$ is monotone. If each $t(\omega, \cdot)$ is additive, then Self-Evidence of Positive Beliefs reduces to Self-Evidence of Beliefs: $P(\omega) \subseteq [t(\omega)]$ (i.e., if $\omega' \in P(\omega)$ then $t(\omega, \cdot) = t(\omega', \cdot)$). If the possibility correspondence P is symmetric (i.e., $\omega' \in P(\omega)$ implies $\omega \in P(\omega')$ or equivalently $E \subseteq K(\neg K)(\neg E)$ for all $E \in \Sigma$), then Self-Evidence of Positive Beliefs also reduces to Self-Evidence of Beliefs. I formulate Self-Evidence of Positive Beliefs so as to examine the effect of additivity of types. The following proposition shows that Self-Evidence of Positive Beliefs captures positive introspection.

Proposition 6. *1. The model satisfies Self-Evidence of Positive Beliefs iff* $B^p(\cdot) \subseteq K(B^p)(\cdot)$.

2. The model satisfies Self-Evidence of Negative Beliefs, i.e., $P(\cdot) \subseteq (\downarrow t(\cdot))$, *iff* $(\neg B^p)(\cdot) \subseteq K(\neg B^p)(\cdot)$.

Proposition 6 states that Self-Evidence of Beliefs means: whenever the agent p-believes an event E, she qualitatively believes that she p-believes E. Proposition 6 implies that $P(\cdot) \subseteq [t(\cdot)]$ iff $B^p(\cdot) \subseteq K(B^p)(\cdot)$ and $(\neg B^p)(\cdot) \subseteq K(\neg B^p)(\cdot)$. Proposition 6 again disentangles the role of additivity of types by considering $(\uparrow t(\cdot))$ and $(\downarrow t(\cdot))$.

7.2 Non-Additive Prior

This subsection briefly studies the case in which the prior μ is not necessarily additive. A set function $\nu : \Sigma \to [0, 1]$ is a *capacity* if $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$, and $\nu(E) \leq \mu(F)$ for any $E, F \in \Sigma$ with $E \subseteq F$. This section considers an epistemic model $\langle \Omega, \Sigma, \mu, P, t \rangle$ in which μ and each $t(\omega, \cdot)$ are a capacity. Below, I present an analogous result to Theorem 1 in this context.

Proposition 7. *Suppose the following: Invariance,* $P(\cdot) \subseteq [t(\cdot)]$, *and* $t(\omega, E) = t(\omega, E \cap P(\omega))$ *for all* $(\omega, E) \in \Omega \times \Sigma$. *Then,*

$$t(\omega, E)\mu(P(\omega)) = \mu(E \cap P(\omega)) \text{ for all } (\omega, E) \in \Omega \times \Sigma.$$

In the proposition, the assumption $t(\omega, E) = t(\omega, E \cap P(\omega))$ for all $(\omega, E) \in \Omega \times \Sigma$ is indeed equivalent to Entailment if each $t(\omega, \cdot)$ is additive. The proposition shows the robust sense in which Invariance, Entailment, and Self-Evidence of Beliefs characterize Bayesian updating. It would be interesting to characterize other updating rules when μ and each $t(\omega, \cdot)$ are a capacity, although it is beyond the scope of this paper.

7.3 Consistency among Prior, Expectations, and Information Sets

The paper has studied the consistency conditions among prior, posteriors, and information sets. In a regular model, each type $t(\omega, \cdot)$ turns out to be the (countably-additive) Bayes conditional probability $\mu(\cdot \mid P(\omega))$. Each type induces conditional expectations $\mathbb{E}_t[f \mid \omega] := \int_\Omega f(\tilde{\omega})t(\omega, d\tilde{\omega})$ of bounded measurable functions f. Thus, the analysts can study agents' higher-order expectations in the regular model.[38] Moreover, Invariance can be seen as the Law of Iterated Expectations (of indicator functions):

$$\mathbb{E}_\mu[\mathbb{I}_E] = \mu(E) = \int_\Omega t(\omega, E)\mu(d\omega) = \mathbb{E}_\mu \mathbb{E}_t[\mathbb{I}_E \mid \omega],$$

where $\mathbb{I}_E$ is the indicator of $E \in \Sigma$. Since μ and each $t(\omega, \cdot)$ are countably additive, Invariance is indeed seen as the the Law of Iterated Expectations: $\mathbb{E}_\mu[f] = \mathbb{E}_\mu \mathbb{E}_t[f \mid \omega]$ for any bounded measurable function f.

This subsection establishes how the consistency conditions among prior, conditional expectations, and information sets lead to the conditional expectation at a given state being equal to the expectation with respect to the Bayes conditional probability at that state. To that end, let $\mathcal{B}(\Omega, \Sigma)$ denote the set of bounded measurable functions $f : (\Omega, \Sigma) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$.

Call a mapping $\mathbb{E} : \mathcal{B}(\Omega, \Sigma) \to \mathcal{B}(\Omega, \Sigma)$ an *expectation mapping* if it satisfies the following two measurable conditions: (i) $\{\omega \in \Omega \mid \mathbb{E}[f \mid \omega] \geq r\} \in \Sigma$ for each $(f, r) \in \mathcal{B}(\Omega, \Sigma) \times \mathbb{R}$; and (ii) $[\mathbb{E}(\omega)] := \{\tilde{\omega} \in \Omega \mid \mathbb{E}(\cdot)(\tilde{\omega}) = \mathbb{E}(\cdot)(\omega)\} \in \Sigma$ for all $\omega \in \Omega$. Henceforth, denote $\mathbb{E}[f \mid \omega] = \mathbb{E}(f)(\omega)$ for any $(f, \omega) \in \mathcal{B}(\Omega, \Sigma) \times \Omega$.

An *(expectation) model* is a tuple $\overrightarrow{\Omega} = \langle \Omega, \Sigma, \mu, P, \mathbb{E} \rangle$ such that $\langle \Omega, \Sigma, \mu, P \rangle$ is a probability space endowed with a possibility correspondence (conditions on $\langle \Omega, \Sigma, \mu, P \rangle$ are the same as in epistemic models) and that $\mathbb{E}$ is an expectation mapping. The model satisfies *Entailment* if $\mathbb{E}[\mathbb{I}_{P(\cdot)} \mid \cdot] = 1$. The model satisfies *Self-Evidence of Expectations* if $P(\cdot) \subseteq [\mathbb{E}(\cdot)]$. The model satisfies Invariance (or the Law of Iterated Expectations) if

$$\mu(E) = \int_\Omega \mathbb{E}[\mathbb{I}_E \mid \omega]\mu(d\omega) \text{ for each } E \in \Sigma.$$

Call a model *regular* if it satisfies Entailment, Self-Evidence of Expectations, and the Law of Iterated Expectations, and if each $\mathbb{E}[\cdot \mid \cdot]$ satisfies the following four properties: (i) $f(\cdot) \geq 0$ implies $\mathbb{E}[f \mid \cdot] \geq 0$ (Non-negativity); (ii) $\mathbb{E}[c\mathbb{I}_\Omega \mid \cdot] = c$ for all

[38] See, for example, Golub and Morris (2017), Nehring (2001), Samet (1998, 2000), and Weinstein and Yildiz (2007) for interactive expectations. Fukuda (2023) constructs a canonical expectation space.

$c \in \mathbb{R}$ (Constancy); and (iii) $\mathbb{E}[f+g \mid \cdot] = \mathbb{E}[f \mid \cdot] + \mathbb{E}[g \mid \cdot]$ for all $f, g \in \mathcal{B}(\Omega, \Sigma)$ (Additivity); and (iv) $f_n \to \mathbb{I}_\emptyset$ implies $\mathbb{E}[f_n \mid \cdot] \to \mathbb{I}_\emptyset$ (Continuity). These four properties on $\mathbb{E}[\cdot \mid \cdot]$ are standard conditions that characterize the expectation operator with respect to a countably-additive probability measure (e.g., Samet, 2000).[39] The following proposition derives the Bayes conditional expectations from the consistency conditions.

Proposition 8. *If an expectation model $\overrightarrow{\Omega}$ is regular, then (i) $\mathbb{E}[f \mid \omega] = \int_\Omega f(\tilde{\omega}) t_{\mathbb{E}}(\omega, d\tilde{\omega})$ for each $(\omega, f) \in \Omega \times \mathcal{B}(\Omega, \Sigma)$, where $t_{\mathbb{E}}$ is a type mapping such that $t_{\mathbb{E}}(\omega, E)\mu(P(\omega)) = \mu(E \cap P(\omega))$, (ii) $P(\cdot) \subseteq [\mathbb{E}(\cdot)]$, and (iii) for any fixed $\omega \in \Omega$, $P(\omega) \supseteq_\mu [\mathbb{E}(\omega)]$. The converse holds if $\mu(P(\cdot)) > 0$.*

In a regular expectation model, since each $\mu(\cdot \mid P(\omega))$ is additive and since $\mu([\mathbb{E}(\cdot)] \mid P(\cdot)) = 1$, the law of iterated expectations holds: for any $f \in \mathcal{B}(\Omega, \Sigma)$,

$$\mathbb{E}[\mathbb{E}[f \mid \tilde{\omega}] \mid \omega] = \mathbb{E}[f \mid \omega] \text{ for any } \omega \in \Omega.$$

In a regular expectation model, the expectation mapping $\mathbb{E}[\cdot \mid \cdot]$ induces the type mapping $t_{\mathbb{E}}$ through $t_{\mathbb{E}}(\omega, E) := \mathbb{E}[\mathbb{I}_E \mid \omega]$, and the type mapping satisfies $t_{\mathbb{E}}(\omega, E) = \mu(E \mid P(\omega))$, provided $\mu(P(\omega)) > 0$. Each $t_{\mathbb{E}}(\omega, \cdot)$ is countably additive. Moreover, $[t_{\mathbb{E}}(\omega)] = [\mathbb{E}(\omega)] \in \Sigma$ because $[\mathbb{E}(\omega)] = \{\tilde{\omega} \in \Omega \mid \mathbb{E}[\mathbb{I}_E \mid \tilde{\omega}] = \mathbb{E}[\mathbb{I}_E \mid \omega] \text{ for all } E \in \Sigma\}$ holds. Self-evidence of Expectations turns out to be Self-Evidence of (p-)Beliefs in that $P(\cdot) \subseteq [t_{\mathbb{E}}(\cdot)]$. Entailment is expressed as $t_{\mathbb{E}}(\cdot, P(\cdot)) = 1$. Thus, if an expectation model $\langle \Omega, \Sigma, \mu, P, \mathbb{E} \rangle$ is regular then so is the epistemic model $\langle \Omega, \Sigma, \mu, P, t_{\mathbb{E}} \rangle$.

In a regular epistemic model, the type mapping t induces the expectation mapping $\mathbb{E}_t[f \mid \omega] := \int_\Omega f(\tilde{\omega}) t(\omega, d\tilde{\omega})$. Since $\mathbb{E}_t[\mathbb{I}_E \mid \omega] = t(\omega, E)$, it can be seen that $\mathbb{E}[f \mid \cdot] \in \mathcal{B}(\Omega, \Sigma)$ for any $f \in \mathcal{B}(\Omega, \Sigma)$. Indeed, $\langle \Omega, \Sigma, \mu, P, \mathbb{E}_t \rangle$ is a regular expectation model. It can be seen that $t = t_{\mathbb{E}_t}$ and $\mathbb{E} = \mathbb{E}_{t_{\mathbb{E}}}$.

With these in mind, I provide the proof sketch for Proposition 8. For any given regular expectation model, the corresponding regular epidemic model satisfies Theorem 1. Then, the corresponding expectation model (which reduces to the original expectation model) satisfies the statement of Proposition 8.

One can also establish the agreement theorem (Proposition 2) in a regular interactive expectation model $\langle (\Omega, \Sigma, \mu), (P_i, \mathbb{E}_i)_{i \in I} \rangle$. As the proof is similar to that of Proposition 2, it is omitted.

Corollary 7. *Let $\langle (\Omega, \Sigma, \mu), (P_i, \mathbb{E}_i)_{i \in I} \rangle$ be regular, and let $f : (\Omega, \Sigma) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be a bounded non-negative measurable function with $M := \sup_{\omega \in \Omega} f(\omega) < \infty$. Let $X_i := \{\omega \in \Omega \mid \mathbb{E}_i[f \mid \omega] = r_i\}$ for each $i \in I$, and let $X := \bigcap_{i \in I} X_i$. If $\mu(\mathbb{C}^p(X)) > 0$, then $|r_i - r_j| \leq (1-p)M$ for all $i, j \in I$. Especially, if $\mu(\mathbb{C}(X)) > 0$, then $r_i = r_j$ for all $i, j \in I$.*

[39] Note that Homogeneity ($\mathbb{E}[cf \mid \cdot] = c\mathbb{E}[f \mid \cdot]$ for all $(c, f) \in \mathbb{R} \times \mathcal{B}(\Omega, \Sigma)$) and consequently the linearity of the expectation mapping follow from Additivity and Continuity.

7.4 Prior-consistent Conditional Probability Systems

Suppose that an agent's beliefs at each state ω are dictated by a conditional probability system (CPS), a collection of types at ω for conditioning events. CPSs are used for epistemic analyses of dynamic games (e.g., Battigalli and Siniscalchi, 1999, 2002).[40] Suppose also that the agent has information sets at ω for conditioning events. The consistency conditions among the prior, the conditional type mapping, and the conditional possibility correspondence imply that the conditional type mapping forms a CPS derived from the prior and the conditional possibility correspondence.

Throughout the subsection, let (Ω, Σ) be a measurable space, and let $\mathcal{C}$ be a sub-σ-algebra of Σ. Let $\mathcal{C}^+ := \mathcal{C} \setminus \{\emptyset\}$. A collection of countably-additive probability measures $(\nu(\cdot \mid C))_{C \in \mathcal{C}^+}$ on (Ω, Σ) is a *conditional probability system (CPS)* if (i, *Normalization*) $\nu(C \mid C) = 1$ for all $C \in \mathcal{C}^+$ and if (ii, the *Chain Rule*) $\nu(E \mid D) = \nu(E \mid C)\nu(C \mid D)$ for all $(C, D, E) \in \mathcal{C}^+ \times \mathcal{C}^+ \times \Sigma$ with $E \subseteq C \subseteq D$.

Now, a model is a tuple $\overrightarrow{\Omega} := \langle \Omega, \Sigma, \mathcal{C}, \mu, (P(\cdot \mid C))_{C \in \mathcal{C}^+}, (t(\cdot, \cdot \mid C))_{C \in \mathcal{C}^+} \rangle$ with the following three properties. First, μ is a countably-additive prior probability measure on (Ω, Σ).

Second, a conditional possibility correspondence $P(\cdot \mid \cdot) : \Omega \times \mathcal{C}^+ \to \mathcal{C}^+$ satisfies the following three conditions: (i) $K(E \mid C) := \{\omega \in \Omega \mid P(\omega \mid C) \subseteq E\} \in \Sigma$ for each $(E, C) \in \Sigma \times \mathcal{C}^+$; (ii, Normalization) $P(\cdot \mid C) \subseteq C$ (i.e., $K(C \mid C) = \Omega$) for each $C \in \mathcal{C}^+$; and (iii, the (qualitative) Chain Rule) $P(\omega \mid C) = P(\omega \mid D) \cap C$ for any $(\omega, C, D) \in \Omega \times \mathcal{C}^+ \times \mathcal{C}^+$ with $C \subseteq D$ and $P(\omega \mid D) \cap C \neq \emptyset$. The (qualitative) Chain Rule implies $K(E \mid C) \cap K(C \mid D) \subseteq K(E \mid D)$ for any $(C, D, E) \in \mathcal{C}^+ \times \mathcal{C}^+ \times \Sigma$ with $E \subseteq C \subseteq D$.[41]

Third, a conditional type mapping $t(\cdot, \cdot \mid \cdot) : \Omega \times \Sigma \times \mathcal{C}^+ \to [0, 1]$ satisfies the following two measurability conditions: (i) $B^p(E \mid C) := \{\omega \in \Omega \mid t(\omega, E \mid C) \geq p\} \in \Sigma$ for each $(E, C, p) \in \Sigma \times \mathcal{C}^+ \times [0, 1]$; and (ii) $[t(\omega \mid C)] := \{\omega' \in \Omega \mid t(\omega, \cdot \mid C) = t(\omega', \cdot \mid C)\} \in \mathcal{C}^+$.

The model satisfies *Invariance* if, for any $C \in \mathcal{C}^+$,

$$\mu(E \cap C) = \int_C t(\omega, E \mid C)\mu(d\omega).$$

The model satisfies *Entailment* if $t(\cdot, P(\cdot \mid C) \mid C) = 1$ for each $C \in \mathcal{C}^+$. The model satisfies *Self-Evidence of (p-)Beliefs* if $P(\cdot \mid C) \subseteq [t(\cdot \mid C)]$ for every $C \in \mathcal{C}^+$. If each $t(\omega, \cdot \mid C)$ is monotone, then the model satisfies Entailment iff $K(\cdot \mid C) \subseteq B^1(\cdot \mid C)$. The model satisfies Self-Evidence of Beliefs iff $B^p(\cdot \mid C) \subseteq K(B^p(\cdot \mid C) \mid C)$ for all $(p, C) \in [0, 1] \times \mathcal{C}^+$.

The model is *regular* if each $t(\omega, \cdot \mid \cdot)$ is a CPS and if it satisfies Invariance, Entailment, and Self-Evidence of Beliefs. In the regular model, $t(\omega, [t(\omega \mid C)] \mid C) = 1$

[40] See also Fukuda (2023) and Guarino (2017) for canonical constructions of CPSs.

[41] The proof goes as follows. If $\omega \in K(E \mid C) \cap K(C \mid D)$ then $P(\omega \mid C) \subseteq E$ and $P(\omega \mid D) \subseteq C$. Then, $P(\omega \mid D) = P(\omega \mid D) \cap C \subseteq P(\omega \mid C) \subseteq E$. Thus, $\omega \in K(E \mid D)$.

for all $(\omega, C) \in \Omega \times \mathcal{C}^+$.[42]

With these in mind, one can extend Theorem 1 to the case with conditional probability systems.

Proposition 9. *If the model is regular, then (i)* $\mu(P(\omega \mid C))t(\omega, E \mid C) = \mu(E \cap P(\omega \mid C))$ *for all* $(\omega, E, C) \in \Omega \times \Sigma \times \mathcal{C}^+$*; (ii)* $P(\omega \mid C) \subseteq [t(\omega \mid C)] \cap C$ *for all* $(\omega, C) \in \Omega \times \mathcal{C}^+$*; and (iii) for all* $(\omega, C) \in \Omega \times \mathcal{C}^+$, $P(\omega \mid C) \supseteq_\mu [t(\omega \mid C)] \cap C$*. The converse holds when* $\mu(P(\cdot \mid \cdot)) > 0$*.*

Since the proof is similar to that of Theorem 1, the proof is omitted.

Three remarks are in order. First, Proposition 9 generalizes Theorem 1 in the sense that the theorem can be seen as a special case $\mathcal{C} = \{\emptyset, \Omega\}$.[43] Second, in the first part of the proposition, I have not used the fact that each $t(\omega, \cdot \mid \cdot)$ is a CPS. The other conditions imply that each $t(\omega, \cdot \mid \cdot)$ satisfies Normalization. If $\mu(P(\cdot \mid \cdot)) > 0$ then the other conditions also imply that each $t(\omega, \cdot \mid \cdot)$ satisfies the Chain Rule. Third, in a discrete model (i.e., Ω is countable, $\Sigma = 2^\Omega$, and $\mu(\{\cdot\}) > 0$), the conditional possibility correspondence is uniquely given by $P(\omega \mid C) = [t(\omega \mid C)] \cap C$.

One can also define an interactive epistemic model for CPSs and establish an agreement theorem for CPSs in a regular interactive model $\overrightarrow{\Omega} := \langle(\Omega, \Sigma, \mathcal{C}, \mu), ((P_i(\cdot \mid C), t_i(\cdot, \cdot \mid C))_{C \in \mathcal{C}^+})_{i \in I}\rangle$.[44]

Corollary 8. *Let* $\overrightarrow{\Omega}$ *be regular. If* $\mu(\mathbb{C}^p(\bigcap_{i \in I}\{\tilde{\omega} \in \Omega \mid t_i(\tilde{\omega}, E \mid D) = r_i\} \mid D)) > 0$*, then* $|r_i - r_j| \leq 1 - p$ *for all* $i, j \in I$*. Especially, if* $\mu(\mathbb{C}(\bigcap_{i \in I}\{\tilde{\omega} \in \Omega \mid t_i(\tilde{\omega}, E \mid D) = r_i\} \mid D)) > 0$*, then* $r_i = r_j$ *for all* $i, j \in I$*.*

In fact, the form of agreement theorem for CPSs given by Corollary 8 is a consequence of Proposition 2 in the sense that once a conditioning event D is fixed, $t_i(\cdot, \cdot \mid D)$ is a type mapping. Thus, the proof of the corollary is omitted.

7.5 Proper Regular Conditional Probabilities

Here I discuss the following previous approach to accommodate agents' prior and posterior beliefs on a general measurable space. Namely, Brandenburger and Dekel (1987) employ proper regular conditional probability. Roughly, under the assumption that an agent's qualitative belief reduces to fully introspective knowledge, the regular models are a subclass of models in which a type mapping is a proper regular conditional probability given a σ-algebra that represents the agent's knowledge.[45]

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[42] Di Tillio, Halpern, and Samet (2014) study the consequence of this introspection property.

[43] Technically, the assumption that $P(\cdot \mid \cdot) \in \mathcal{C}^+$ implies $P(\cdot \mid \cdot) \neq \emptyset$ (i.e., $K(E \mid \cdot) \cap K(\neg E \mid \cdot) = \emptyset$).

[44] See Tsakas (2018) for agreement theorems for CPSs.

[45] Although Kajii and Morris (1997a) provide still another approach, they take a different route from Brandenburger and Dekel (1987) and this paper. They define an equivalence relation on the collection of events Σ in a way so that events E and F are equivalent if $E =_\mu F$ (i.e., $\mu(E \triangle F) = 0$). Then, they define an agent's p-belief operator on the collection of equivalence classes instead of Σ.

A regular model in which P forms a partition induces a proper regular conditional probability. Conversely, a proper regular conditional probability on a σ-algebra that represents an agent's partitional knowledge yields a regular model.

Let $\langle \Omega, \Sigma, \mu, P, t \rangle$ be a regular model such that P forms a partition. Let $\mathcal{J} := \{E \in \Sigma \mid E = \bigcup_{\omega \in E} P(\omega)\}$ be a sub-σ-algebra of Σ that consists of self-evident events, i.e., $\mathcal{J} = \{E \in \Sigma \mid E \subseteq K(E)\}$.[46]

It can be seen that t is a proper regular conditional probability given $\mathcal{J}$, that is, t satisfies the following four conditions: (i) each $t(\omega, \cdot)$ is a countably-additive probability measure; (ii) each mapping $t(\cdot, E) : (\Omega, \mathcal{J}) \to ([0,1], \mathcal{B}_{[0,1]})$ is measurable; (iii) for any $(E, F) \in \Sigma \times \mathcal{J}$,

$$\mu(E \cap F) = \int_F t(\omega, E)\mu(d\omega);$$

and (iv) $t(\omega, E) = 1$ for any $(\omega, E) \in \Omega \times \mathcal{J}$ with $\omega \in E$.

Conversely, let $\mathcal{J}$ be a sub-σ-algebra of Σ satisfying the following three properties: (i) for each $\omega \in \Omega$, $P(\omega) := \bigcap\{E \in \mathcal{J} \mid \omega \in E\} \in \mathcal{J}$; (ii) $\{\omega \in \Omega \mid P(\omega) \subseteq E\} \in \Sigma$ for each $E \in \Sigma$; and (iii) $\mathcal{J} = \{E \in \Sigma \mid E = \bigcup_{\omega \in E} P(\omega)\}$.[47] Let t be a proper regular conditional probability on $\langle \Omega, \Sigma, \mu \rangle$ given $\mathcal{J}$ such that $[t(\omega)] \in \Sigma$ for each $\omega \in \Omega$. Now, one can show that $\langle \Omega, \Sigma, \mu, P, t \rangle$ is a regular model.

7.6 Concluding Remarks

This paper studied implications of the consistency conditions among prior, posteriors, and information sets on introspective properties of qualitative belief induced from information sets. The consistency conditions are: (i) the prior belief is equal to the expectation of the posterior beliefs (Invariance); (ii) qualitative belief entails probability-one belief (Entailment); and (iii) qualitative belief in one's own quantitative beliefs (Self-Evidence of Beliefs). The main benchmark result (Theorem 1) states that a model satisfies the consistency conditions iff the agent's information sets form a partition almost surely and her posteriors coincide with the Bayes conditional prob-

[46] The collection $\mathcal{J}$ may not necessarily be the σ-algebra generated by the partition P. Consider, for example, the case with the most informative partition $P(\omega) = \{\omega\}$ for each $\omega \in \Omega$. While $\mathcal{J} = \Sigma$, the σ-algebra generated by P is the countable-co-countable σ-algebra. See, for instance, Dubra and Echenique (2004), Fukuda (2019), Hérves-Beloso and Monteiro (2013), Lee (2018), and Tóbiás (2021) for representing the information content of a partition by a set-algebra.

[47] The idea behind these three assumptions is that not all σ-algebra represents the notion of knowledge that comes from an information partition. The first assumption guarantees that the smallest event $P(\omega)$ containing each ω exists. Since $\mathcal{J}$ is a σ-algebra, P forms a partition. The second assumption is the measurability condition of an (epistemic) model by which the knowledge operator associated with P is well-defined. The third assumption states that the given σ-algebra $\mathcal{J}$ is compatible with the notion of knowledge derived from P. Brandenburger and Dekel (1987) do not impose such properties on σ-algebras because they do not study knowledge coming from an information partition but probability-one belief.

abilities given the information sets. The theorem extends to the Bayes conditional expectations and conditional probability systems (Propositions 8 and 9).

The main theorem enables one to extend the standard partitional model of knowledge and probabilistic beliefs to a setting in which an agent's information sets may be uncountable in a tractable manner (e.g., the examples in Section 2.4). The paper thus extended the agreement theorem, the no-trade theorem, and the epistemic characterizations of correlated equilibria (Section 5).

The main theorem also has theoretical implications. First, the posterior at each state is uniquely determined whenever the prior puts positive probability to an information set (Corollary 1). Second, to introduce fully introspective knowledge in a quantitative belief model, the partition generated by the type mapping is a unique partition compatible with the consistency conditions (Corollary 2). This result justifies the definition of an information partition by the support of each type in the previous literature. Third, in a discrete model, the information sets necessarily form a partition (Corollary 3). I discussed its implications when the information sets do not form a partition, i.e., for qualitative belief violating Truth Axiom or non-introspective knowledge violating Negative Introspection (Corollaries 4 and 5). Forth, while qualitative and probability-one beliefs may differ, both satisfy Truth Axiom almost surely (Corollary 6).

Propositions 5 and 6 also studied how the additivity of types plays a role in negative introspection of beliefs. As avenues for future research, it is interesting to scrutinize a link between prior and posteriors (or an "updating rule") that is consistent with non-partitional information processing. In so doing, it is also interesting to explore the role of additivity. It would also be interesting to study the relation among prior, posteriors, and information sets in the context of a generalized state space model of unawareness.

A Appendix

A.1 Section 3

Proof of Theorem 1. Suppose that $\overrightarrow{\Omega}$ is regular. First, Self-Evidence of Beliefs implies Part (ii): $P(\cdot) \subseteq [t(\cdot)]$. Second, since the model is regular, it follows that each $t(\omega, \cdot)$ is monotone and the model satisfies Entailment. Then, the model satisfies Certainty of Beliefs. Third, since each $t(\omega, \cdot)$ is monotone and $t(\cdot, [t(\cdot)]^c) = 0$, I show $\mu(E \cap [t(\cdot)]) \leq t(\cdot, E)\mu([t(\cdot)])$. Observe that if $\tilde{\omega} \in [t(\omega)]^c$ then $[t(\omega)] \subseteq [t(\tilde{\omega})]^c$. Thus,

$$\int_{[t(\omega)]^c} t(\tilde{\omega}, [t(\omega)])\mu(d\tilde{\omega}) \leq \int_{[t(\omega)]^c} t(\tilde{\omega}, [t(\tilde{\omega})]^c)\mu(d\tilde{\omega}) = 0,$$

where the first inequality follows because each $t(\tilde{\omega}, \cdot)$ is monotone and the second because $t(\cdot, [t(\cdot)]^c) = 0$. Hence,

$$\begin{aligned}
\mu(E \cap [t(\omega)]) &= \int_{\Omega} t(\tilde{\omega}, E \cap [t(\omega)])\mu(d\tilde{\omega}) \\
&\leq \int_{[t(\omega)]} t(\tilde{\omega}, E)\mu(d\tilde{\omega}) + \int_{[t(\omega)]^c} t(\tilde{\omega}, [t(\omega)])\mu(d\tilde{\omega}) \\
&= \int_{[t(\omega)]} t(\tilde{\omega}, E)\mu(d\tilde{\omega}) = t(\omega, E)\mu([t(\omega)]).
\end{aligned}$$

The first equality follows from Invariance. The inequality follows because each $t(\tilde{\omega}, \cdot)$ is monotone and additive. The second equality follows from one of the previous arguments. The last equality follows because $t(\tilde{\omega}, E) = t(\omega, E)$ for all $\tilde{\omega} \in [t(\omega)]$.

Fourth, since each $t(\omega, \cdot)$ is additive and the model satisfies Certainty of Beliefs, it follows that $t(\cdot, E) = t(\cdot, E \cap [t(\cdot)])$ for all $E \in \Sigma$. Fifth, I show $\mu(E \cap [t(\cdot)]) \geq t(\cdot, E)\mu([t(\cdot)])$. Indeed,

$$\begin{aligned}
\mu(E \cap [t(\omega)]) &\geq \int_{[t(\omega)]} t(\tilde{\omega}, E \cap [t(\omega)])\mu(d\tilde{\omega}) \\
&= t(\omega, E \cap [t(\omega)])\mu([t(\omega)]) = t(\omega, E)\mu([t(\omega)]).
\end{aligned}$$

The first inequality follows from Invariance. The first equality follows from the definition of $[t(\omega)]$ and the second from one of the previous assertions.

Fifth, I obtain:

$$\mu(E \cap [t(\omega)]) = t(\omega, E)\mu([t(\omega)]). \tag{3}$$

Substituting $E = P(\omega)$ into Expression (3) yields $\mu(P(\omega)) = \mu([t(\omega)])$, given that the model satisfies Self-Evidence of Beliefs and Entailment. Thus, I obtain Part (iii): $[t(\cdot)] \supseteq_{\mu} P(\cdot)$. I also obtain Part (i): $\mu(E \cap P(\omega)) = t(\omega, E)\mu(P(\omega))$.

I show the converse assuming $\mu(P(\cdot)) > 0$. Each $t(\omega, \cdot) = \mu(\cdot \mid P(\omega))$ is a countably-additive probability measure. Entailment follows because $t(\omega, P(\omega)) =$

$\mu(P(\omega) \mid P(\omega)) = 1$. Self-Evidence of Beliefs holds by supposition. I show Invariance. Since $\mu([t(\omega)]) = \mu(P(\omega)) > 0$, let $([t(\omega_n)])_n$ be a countable partition of Ω. Since it follows from the assumptions that $t(\omega_n, E) = \mu(E \mid P(\omega_n)) = \mu(E \mid [t(\omega_n)])$, I obtain

$$\mu(E \cap [t(\omega_n)]) = \mu([t(\omega_n)])t(\omega_n, E) = \int_{[t(\omega_n)]} t(\omega', E)\mu(d\omega').$$

By summing over all n, I obtain Invariance, as desired. □

A.2 Section 4

Proof of Corollary 2. As Parts (2) and (3) follow from Part (1), I only prove Part (1). Also, it is sufficient to show that (1b) implies (1a). Self-Evidence of Beliefs implies $P(\cdot) \subseteq [t(\cdot)]$. Thus, the partition $\{P(\omega)\}_{\omega\in\Omega}$ is a refinement of $\{[t(\omega)]\}_{\omega\in\Omega}$. Suppose to the contrary that $\omega' \in [t(\omega)] \setminus P(\omega)$. Since $P(\omega) \cap P(\omega') = \emptyset$, it follows that $\mu(P(\omega)) < \mu(P(\omega)) + \mu(P(\omega')) = \mu(P(\omega') \sqcup P(\omega)) \leq \mu([t(\omega)])$, a contradiction. Thus, $[t(\cdot)] = P(\cdot)$. Since the model is regular and $\mu(P(\cdot)) > 0$, the other property, $t(\cdot, \cdot) = \mu(\cdot \mid P(\cdot))$, follows from Theorem 1. □

Proof of Corollary 3.

1. First, the equivalence follows from Theorem 1 and the supposition that $\mu(\{\cdot\}) > 0$. Second, it suffices to show $B^1(\cdot) \subseteq K(\cdot)$ as Entailment implies the converse set inclusion $K(\cdot) \subseteq B^1(\cdot)$. If $\omega \in B^1(E)$ then $t(\omega, E) = 1$. Then, $\mu(E \cap P(\omega)) = \mu(P(\omega))$, and thus $\mu(P(\omega) \cap E^c) = 0$. Since $\mu(\{\cdot\}) > 0$, I have $P(\omega) \cap E^c = \emptyset$, i.e., $P(\omega) \subseteq E$. Thus, $\omega \in K(E)$.

2. If the model is discrete, then the first part implies that $B^1 = K$ satisfies Truth Axiom. Conversely, let B^1 satisfy Truth Axiom. It suffices to show that $t(\omega, \{\omega\}) > 0$. Indeed, if this is the case, then $0 < \mu(P(\omega))t(\omega, \{\omega\}) = \mu(\{\omega\})$, Ω is countable, and $\Sigma = 2^\Omega$. Thus, suppose to the contrary that $t(\omega, \{\omega\}) = 0$ for some $\omega \in \Omega$. Then, $\omega \in B^1(P(\omega) \setminus \{\omega\}) \subseteq P(\omega) \setminus \{\omega\}$, a contradiction.

□

A.3 Section 5

Lemma 1. *In a regular model $\langle \Omega, \Sigma, \mu, P, t\rangle$, $B^p B^p(\cdot) \subseteq B^p(\cdot)$.*

Proof of Lemma 1. If $p = 0$ then the statement holds because $B^p(\cdot) = \Omega$. Let $p > 0$. If $\omega \in B^p(B^p(E))$ then $t(\omega, B^p(E)) \geq p > 0$. If $[t(\omega)] \cap B^p(E) = \emptyset$ then $B^p(E) \subseteq [t(\omega)]^c$. Then, $t(\omega, [t(\omega)]^c) \geq p > 0$, a contradiction to $t(\omega, [t(\omega)]) \geq t(\omega, P(\omega)) = 1$. Then, there is $\tilde{\omega} \in [t(\omega)] \cap B^p(E)$. This implies $t(\omega, E) = t(\tilde{\omega}, E) \geq p$, i.e., $\omega \in B^p(E)$. □

Note that, in a regular model, $B^p(\cdot) \subseteq KB^p(\cdot) \subseteq B^p B^p(\cdot) \subseteq B^p(\cdot)$.

Proof of Proposition 1. Since $K_I(\cdot) \subseteq B_I^1(\cdot)$, it is enough to show $B_I^1\mathbb{C}^p(\cdot) \subseteq \mathbb{C}^p(\cdot)$. For any $n \in \mathbb{N}$,

$$B_j^1(B_I^p)^n(\cdot) \subseteq B_j^1 B_j^p (B_I^p)^{n-1}(\cdot) \subseteq B_j^p B_j^p (B_I^p)^{n-1}(\cdot) \subseteq B_j^p (B_I^p)^{n-1}(\cdot),$$

where the last set inclusion follows from Lemma 1. It follows that $B_I^1(B_I^p)^n(\cdot) \subseteq (B_I^p)^n(\cdot)$. Taking the intersection with respect to $n \in \mathbb{N}$,

$$B_I^1\mathbb{C}^p(\cdot) = B_I^1\left(\bigcap_{n\in\mathbb{N}}(B_I^p)^n(\cdot)\right) \subseteq \bigcap_{n\in\mathbb{N}} B_I^1(B_I^p)^n(\cdot) \subseteq \bigcap_{n\in\mathbb{N}}(B_I^p)^n(\cdot) = \mathbb{C}^p(\cdot).$$

□

Proof of Proposition 2. Without loss, let $p > 0$. Let $F := \mathbb{C}^p(X)$. Observe $F \subseteq B_I^p(F)$, $F \subseteq B_I^p(X)$, and $\mu(F) > 0$.

If $[t_i(\omega')] \cap X_i \neq \emptyset$ then $[t_i(\omega')] \subseteq X_i$. If $\omega' \in B_i^p(X)$ then $t_i(\omega', X) \geq p > 0$, and thus $\emptyset \neq P_i(\omega') \cap X \subseteq [t_i(\omega')] \cap X$. This implies that if $\omega' \in B_i^p(X)$ then $[t_i(\omega')] \subseteq X_i$. Since $F \subseteq B_i^p(F) \subseteq B_i^p B_i^p(X) \subseteq B_i^p(X)$ (where the second set inclusion follows from Self-Evidence of Beliefs and Entailment, and the third from Lemma 1), $t_i(\omega', E) = r_i$ for any $\omega' \in B_i^p(F)$.

By Invariance and Certainty of Beliefs,

$$\begin{aligned}\mu(E \cap B_i^p(F)) &= \int_\Omega (\mathbb{I}_{B_i^p(F)}(\omega) t_i(\omega, E \cap B_i^p(F)) + \mathbb{I}_{(\neg B_i^p)(F)}(\omega) t_i(\omega, E \cap B_i^p(F)))\mu(d\omega) \\ &= \int_{B_i^p(F)} t_i(\omega, E)\mu(d\omega) = r_i \mu(B_i^p(F)).\end{aligned}$$

Thus, $\mu(E \mid B_i^p(F)) = r_i$.

Since $F \subseteq B_I^p(F)$, $B_i^p(F) \subseteq B_i^p B_I^p(F)$. Together with Invariance,

$$\mu(B_i^p(F) \cap B_j^p(F)) \geq \mu(B_I^p(F)) \geq \int_{B_i^p(F)} t_i(\omega, B_I^p(F))\mu(d\omega) \geq p\mu(B_i^p(F)).$$

Thus, $\mu(B_j^p(F) \mid B_i^p(F)) \geq p$. Then, for any event H,

$$\begin{aligned}\mu(H \mid B_i^p(F)) &= \frac{\mu(H \cap B_i^p(F) \cap B_j^p(F))}{\mu(B_i^p(F) \cap B_j^p(F))} \cdot \frac{\mu(B_i^p(F) \cap B_j^p(F))}{\mu(B_i^p(F))} \cdot \frac{\mu(H \cap B_i^p(F))}{\mu(H \cap B_i^p(F) \cap B_j^p(F))} \\ &\geq \mu(H \mid B_i^p(F) \cap B_j^p(F)) \cdot \mu(B_j^p(F) \mid B_i^p(F)) \\ &\geq p\mu(H \mid B_i^p(F) \cap B_j^p(F)).\end{aligned}$$

By letting $H = E$ and $H = E^c$,

$$(1-p) + p\mu(E \mid B_i^p(F) \cap B_j^p(F)) \geq r_i \geq p\mu(E \mid B_i^p(F) \cap B_j^p(F)).$$

By exchanging the role of i and j, I get $|r_i - r_j| \leq 1 - p$. □

Proof of Proposition 3. Define $A^\varepsilon := \{\omega \in \Omega \mid x \succ^\varepsilon_\omega e\}$ and $E := \mathbb{C}^p(A^\varepsilon)$. Then, $E \subseteq B^p_I(A^\varepsilon)$ and $E \subseteq B^p_I(E)$. If $\omega \in (\neg K_i)(\neg A^\varepsilon)$, then $\emptyset \neq P_i(\omega) \cap A^\varepsilon \subseteq [t_i(\omega)] \cap A^\varepsilon$. Thus, $(\neg K_i)(\neg A^\varepsilon) \subseteq \{\omega \in \Omega \mid x_i \succcurlyeq^\varepsilon_{i,\omega} e_i\}$.

Suppose to the contradiction that $\mu(E) > 0$. I show that the initial allocation e is not ex-ante Pareto optimal. If $\omega \in B^p_i(A^\varepsilon)$, then $t_i(\omega, A^\varepsilon) \geq p > 0$, and thus, $A^\varepsilon \cap P_i(\omega) \neq \emptyset$. Hence, $B^p_i(A^\varepsilon) \subseteq (\neg K_i)(\neg A^\varepsilon)$. Then, $E \subseteq B^p_I(A^\varepsilon) \subseteq \bigcap_{i \in I}(\neg K_i)(\neg A^\varepsilon) \subseteq \bigcap_{i \in I}\{\omega \in \Omega \mid x_i \succcurlyeq^\varepsilon_{i,\omega} e_i\}$. Hence, for every $\omega \in E$ and $i \in I$,

$$\int_{P_i(\omega)} (u_i(\tilde{\omega}, x_i(\tilde{\omega})) - u_i(\tilde{\omega}, e_i(\tilde{\omega}))t_i(\omega, d\tilde{\omega}) \geq \varepsilon > (1-p)\overline{M}.$$

On the other hand, if $\omega \in E$, then $\omega \in B^p_i(E)$ and thus $t_i(\omega, E^c \cap P_i(\omega)) = t_i(\omega, E^c) \leq 1 - p$. Thus, for all $\omega \in E$ and $i \in I$,

$$\int_{E^c \cap P_i(\omega)} (u_i(\tilde{\omega}, x_i(\tilde{\omega})) - u_i(\tilde{\omega}, e_i(\tilde{\omega}))t_i(\omega, d\tilde{\omega}) \leq (1-p)\overline{M}.$$

Hence, for all $\omega \in E$ and $i \in I$,

$$\int_{P_i(\omega) \cap E} (u_i(\tilde{\omega}, x_i(\tilde{\omega})) - u_i(\tilde{\omega}, e_i(\tilde{\omega}))t_i(\omega, d\tilde{\omega}) > 0.$$

Then, I have

$$\int_E (u_i(\omega, x_i(\omega)) - u_i(\omega, e_i(\omega))\mu(d\omega) = \int_E \int_{P_i(\omega) \cap E} (u_i(\tilde{\omega}, x_i(\tilde{\omega})) - u_i(\tilde{\omega}, e_i(\tilde{\omega}))t_i(\omega, d\tilde{\omega})\mu(d\omega) > 0.$$

This implies that e is not ex-ante Pareto-optimal. □

A.4 Section 6

Proof of Corollary 6.

1. For any $E \in \Sigma$,

$$\mu(B^p(E) \cap E^c) = \int_{B^p(E)} t(\omega, B^p(E) \cap E^c)\mu(d\omega) + \int_{(\neg B^p)(E)} t(\omega, B^p(E) \cap E^c)\mu(d\omega)$$
$$\leq \int_{B^p(E)} \underbrace{t(\omega, E^c)}_{\leq 1-p}\mu(d\omega) + \int_{(\neg B^p)(E)} \underbrace{t(\omega, B^p(E))}_{=0}\mu(d\omega) \leq (1-p)\mu(B^p(E)).$$

   By substituting $p = 1$, $\mu(K(E) \setminus E) \leq \mu(B^1(E) \setminus E) = 0$.

2. This part follows from $\mathbb{C}^1(\cdot) \subseteq B^1_i(\cdot)$ and $\mathbb{C}(\cdot) \subseteq K_i(\cdot)$.

□

Proof of Proposition 4. I show "(1) ⇒ (2)." If $\omega \in (\neg K)(\neg E)$, then $P(\omega) \cap E \neq \emptyset$. Thus, there is $\omega' \in P(\omega) \cap E$. Then, $t(\omega, E) \geq t(\omega, \{\omega'\}) > 0$ and thus $\omega \in \bigcup_{r \in \mathbb{Q} \setminus \{0\}} B^r(E)$. Conversely, I show "(2) ⇒ (1)." Let $\omega' \in P(\omega)$. Then, $\omega \in (\neg K)(\neg\{\omega'\}) \subseteq \bigcup_{r \in \mathbb{Q} \setminus \{0\}} B^r(\{\omega'\})$. Thus, $t(\omega, \{\omega'\}) > 0$.

I show "(1) ⇒ (3)." If $\omega \in B^1(E)$ then $t(\omega, E) = 1$. Then, $P(\omega) \subseteq \{\omega' \in \Omega \mid t(\omega, \{\omega'\}) > 0\} \subseteq E$. Thus, $\omega \in K(E)$. Conversely, I show "(3) ⇒ (1)" by assuming Entailment. Suppose $\omega' \in P(\omega)$ and $t(\omega, \{\omega'\}) = 0$. Then, together with Entailment, $\omega \in B^1(P(\omega) \setminus \{\omega'\})$. By Condition (3), $B^1(P(\omega) \setminus \{\omega'\}) \subseteq K(P(\omega) \setminus \{\omega'\})$. Then, $\omega \in K(P(\omega) \setminus \{\omega'\})$ implies $\omega' \in P(\omega) \subseteq P(\omega) \setminus \{\omega'\}$, a contradiction. □

A.5 Section 7

Proof of Proposition 5. 1. Assume Certainty of Positive Beliefs. If $\omega \in B^p(E)$ then $(\uparrow t(\omega)) \subseteq B^p(E)$. By Certainty of Positive Beliefs and (ii), $1 = t(\omega, (\uparrow t(\omega))) \leq t(\omega, B^p(E))$, i.e., $\omega \in B^1 B^p(E)$.

Conversely, $\omega \in B^{t(\omega,E)}(E) \subseteq B^1 B^{t(\omega,E)}(E)$ for any $(\omega, E) \in \Omega \times \Sigma$. By (i) and (iii), $t(\omega, (\uparrow t(\omega))) = 1$ follows because

$$\omega \in \bigcap_{E \in \Sigma_0} B^1 B^{t(\omega,E)}(E) \subseteq B^1 \left(\bigcap_{E \in \Sigma_0} \{\tilde{\omega} \in \Omega \mid t(\tilde{\omega}, E) \geq t(\omega, E)\} \right) = B^1(\uparrow t(\omega)).$$

2. If $\omega \in (\neg B^p)(E)$ then $(\downarrow t(\omega)) \subseteq (\neg B^p)(E)$. By (ii), $1 = t(\omega, (\downarrow t(\omega))) \leq t(\omega, (\neg B^p)(E))$, i.e., $\omega \in B^1(\neg B^p)(E)$.

Conversely, take $\omega \in \Omega$. I start with showing that $\{\tilde{\omega} \in \Omega \mid t(\tilde{\omega}, E) \leq p\} \subseteq B^1(\{\tilde{\omega} \in \Omega \mid t(\tilde{\omega}, E) \leq p\})$ for all $(E, p) \in \Sigma \times [0, 1]$. If $p = 1$, the statement follows from $t(\cdot, \Omega) = 1$. Thus, assume $p < 1$. For each $n \in \mathbb{N}$ with $p + \frac{1}{n} < 1$, let $E_n := (\neg B^{p+\frac{1}{n}})(E)$. Since $E_n \subseteq B^1(E_n)$, $\{\tilde{\omega} \in \Omega \mid t(\tilde{\omega}, E) \leq p\} = \bigcap_m E_m \subseteq B^1(\bigcap_m E_m) = B^1(\{\tilde{\omega} \in \Omega \mid t(\tilde{\omega}, E) \leq p\})$.

Now, I show $t(\omega, (\downarrow t(\omega))) = 1$. By the previous assertion, (i), and (iii), I have

$$\omega \in \bigcap_{E \in \Sigma_0} B^1(\{\tilde{\omega} \in \Omega \mid t(\tilde{\omega}, E) \leq t(\omega, E)\}) \subseteq B^1 \left(\bigcap_{E \in \Sigma_0} \{\tilde{\omega} \in \Omega \mid t(\tilde{\omega}, E) \leq t(\omega, E)\} \right)$$
$$= B^1(\downarrow t(\omega)),$$

which implies $t(\cdot, (\downarrow t(\omega))) = 1$.

□

Proof of Proposition 6. 1. Assume Self-Evidence of Positive Beliefs. If $\omega \in B^p(E)$, then $t(\omega', E) \geq t(\omega, E) \geq p$ for any $\omega' \in P(\omega)$. Thus, $P(\omega) \subseteq B^p(E)$, i.e., $\omega \in K(B^p)(E)$. Conversely, fix $(\omega, E) \in \Omega \times \Sigma$. Since $\omega \in B^{t(\omega,E)}(E) \subseteq K(B^{t(\omega,E)})(E)$, if $\omega' \in P(\omega)$ then $\omega' \in B^{t(\omega,E)}(E)$, i.e., $t(\omega', E) \geq t(\omega, E)$.

2. Assume $P(\cdot) \subseteq (\downarrow t(\cdot))$. If $\omega \in (\neg B^p)(E)$ and if $\omega' \in P(\omega)$, then $t(\omega', E) \leq t(\omega, E) < p$. Thus, $P(\omega) \subseteq (\neg B^p)(E)$, i.e., $\omega \in K(\neg B^p)(E)$. Conversely, let $\omega' \in P(\omega)$ and $E \in \Sigma$. If $t(\omega, E) = 1$ then $t(\omega', E) \leq t(\omega, E)$. If $t(\omega, E) < 1$, let $\varepsilon > 0$ be such that $q_\varepsilon := t(\omega, E) + \varepsilon < 1$. Since $\omega \in (\neg B^{q_\varepsilon})(E) \subseteq K(\neg B^{q_\varepsilon})(E)$, I have $\omega' \in P(\omega) \subseteq (\neg B^{q_\varepsilon})(E)$. That is, $t(\omega', E) < t(\omega, E) + \varepsilon$. Letting $\varepsilon \to 0$ yields $t(\omega', E) \leq t(\omega, E)$.

□

Proof of Proposition 7. Take any $(\omega, E) \in \Omega \times \Sigma$. On the one hand, it follows from Invariance that

$$\mu(E \cap P(\omega)) = \int_\Omega t(\tilde{\omega}, E \cap P(\omega))\mu(d\tilde{\omega}). \tag{4}$$

On the other hand, since $P(\cdot) \subseteq [t(\cdot)]$, it follows that

$$\begin{aligned} t(\tilde{\omega}, E \cap P(\omega)) &= \mathbb{I}_{[t(\omega)]}(\tilde{\omega})t(\tilde{\omega}, E \cap P(\omega)) + \mathbb{I}_{[t(\omega)]^c}(\tilde{\omega})t(\tilde{\omega}, E \cap P(\omega)) \\ &= \mathbb{I}_{[t(\omega)]}(\tilde{\omega})t(\omega, E \cap P(\omega)) + \mathbb{I}_{[t(\omega)]^c}(\tilde{\omega})\underbrace{t(\tilde{\omega}, E \cap P(\omega) \cap P(\tilde{\omega}))}_{t(\tilde{\omega}, \emptyset)=0} \\ &= \mathbb{I}_{[t(\omega)]}(\tilde{\omega})t(\omega, E \cap P(\omega)). \end{aligned}$$

Then, the Choquet integral given by Expression (4) reduces to

$$\begin{aligned} \mu(E \cap P(\omega)) &= \int_\Omega \mathbb{I}_{[t(\omega)]}(\tilde{\omega})t(\omega, E \cap P(\omega))\mu(d\tilde{\omega}) \\ &= t(\omega, E \cap P(\omega))\mu([t(\omega)]). \end{aligned} \tag{5}$$

Substituting $E = \Omega$ yields

$$\mu(P(\omega)) = t(\omega, \Omega \cap P(\omega))\mu([t(\omega)]) = t(\omega, \Omega)\mu([t(\omega)]) = \mu([t(\omega)]),$$

where the second equality follows from one of the assumptions and the third from the assumption that $t(\omega, \cdot)$ is a capacity (precisely, $t(\omega\Omega) = 1$). Hence, substituting $\mu(P(\omega)) = \mu([t(\omega)])$ into Expression (5) yields

$$\mu(E \cap P(\omega)) = t(\omega, E \cap P(\omega))\mu(P(\omega)),$$

as desired. □

B Additional Examples

Example 4. Let $(\Omega, \Sigma) = (\{\omega_1, \omega_2, \omega_3\}, 2^\Omega)$, and let $\mu = (1, 0, 0)$. Let $P_i(\omega_1) = \{\omega_1\}$ and $P_i(\omega_2) = P_i(\omega_3) = \{\omega_2, \omega_3\}$ for each $i \in I = \{1, 2\}$. Let $t_i(\omega_1, \cdot) = \mu$ and $t_i(\omega_2, \cdot) = t_i(\omega_3, \cdot) = (0, \alpha_i, 1 - \alpha_i)$, where $\alpha_1 \neq \alpha_2$. Then, $\mathbb{C}(\bigcap_{i \in I}\{\omega \in \Omega \mid t_i(\omega, \{\omega_2\}) = \alpha_i\}) = \{\omega_2, \omega_3\}$. $\square$

Example 5. As in Monderer and Samet (1989, p. 176), suppose that the agent is reasoning about the realization of a random draw from $[0, 1]$. Let $(\Omega, \Sigma, \mu) = ([0, 1], \mathcal{B}_{[0,1]}, \mu)$, where μ is the Lebesgue measure. Let $P(\cdot) = [0, 1]$, i.e., the agent considers every number possible at each realization. Her qualitative belief reduces to (degenerate) knowledge in that she only knows that the draw is from $[0, 1]$ at each state. Her type at each ω is $t(\omega, \cdot) = \mu(\cdot)$. By construction, the model is regular. At any realization, the agent does not know that the draw is an irrational number, as she does not observe the realization. She, however, believes with probability one that the draw is an irrational number. In fact, she 1-believes any event E at any state ω as long as $\mu(E) = 1$. Thus, for any $E \in \Sigma \setminus \{\Omega\}$ with $\mu(E) = 1$, it follows that $B^1(E) \setminus K(E) = \Omega$ and $\mu(B^1(E) \setminus K(E)) = 1$. For any such E and $p > 0$, $B^p(E) = \Omega \not\subseteq \emptyset = B^pK(E)$. While the agent's qualitative belief is fully introspective knowledge, probability-one belief and knowledge differ. While K satisfies Truth Axiom, B^1 violates Truth Axiom. Also, B^1 fails the strong conjunction property that K possesses: for any $\mathcal{E} \in 2^\Sigma$ with $\bigcap \mathcal{E} \in \Sigma$, $\bigcap_{E \in \mathcal{E}} K(E) \subseteq K(\bigcap \mathcal{E})$. $\square$

Example 6. As in Example 5, suppose that the agent is reasoning about the realization of a random draw from $\Omega = [0, 1]$. The agent's prior μ is the Lebesgue measure on $(\Omega, \Sigma) = ([0, 1], \mathcal{B}_{[0,1]})$. At each realization $\omega \in [\frac{1}{2}, 1] \cup ([0, 1] \cap \mathbb{Q})$, the agent considers $P(\omega) = [0, 1] \setminus \mathbb{Q}$ possible. At each $\omega \in [0, \frac{1}{2}) \setminus \mathbb{Q}$, the agent considers $P(\omega) = [0, 1]$ possible. The agent's type at each ω remains unchanged: $t(\omega, \cdot) = \mu(\cdot)$. The model is regular. The agent's qualitative belief violates all of Truth Axiom, Positive Introspection, and Negative Introspection. While Truth Axiom holds μ-almost surely (as a consequence of Corollary 6), the introspection properties do not even in this sense. For example, if $E = [0, 1] \setminus \mathbb{Q}$ then $\mu(K(E) \setminus KK(E)) = \frac{1}{2}$ and $\mu((\neg K)(E) \setminus K(\neg K)(E)) = \frac{1}{2}$. Also, $K(E) \not\subseteq B^1K(E)$. $\square$

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