An Information Correspondence Approach to Bridging Knowledge-Belief Representations in Economics and Mathematical Psychology*

Satoshi Fukuda†

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Abstract

This paper develops a model of interactive beliefs and knowledge which I call an information correspondence. The information correspondence assigns multiple information sets at each state. It reduces to a standard possibility correspondence when it assigns a unique information set at each state. This generalization allows one to analyze an agent who fails to believe the conjunction of her own beliefs or a tautology. While a possibility correspondence may not be able to represent probabilistic beliefs, this generalization enables one to study qualitative and probabilistic beliefs in a unified manner. The model also generalizes, in a mathematical sense, a knowledge representation in mathematical psychology known as a surmise function. The paper bridges seemingly different knowledge and belief representations in economics and mathematical psychology. The paper also connects the information correspondence model to knowledge and belief representations in computer science, logic, and philosophy.

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1 Introduction

Representations of beliefs and knowledge have been objects of study in such diverse disciplines as computer science and artificial intelligence, economics and game theory, logic and

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†Department of Decision Sciences and IGIER, Bocconi University, Milan 20136, Italy.

philosophy, and psychology, to name a few. This paper provides a model of interactive beliefs and knowledge, which I call an information correspondence model, with the following two objectives. As the first objective, the information correspondence model generalizes a possibility correspondence model in economics and game theory with the following two features: (i) the model can relax agents' logical reasoning ability while keeping tractability; and (ii) the model can represent agents' qualitative and probabilistic beliefs within the same framework. Second, this paper bridges, in a mathematical sense, seemingly different knowledge-belief representations between economics and mathematical psychology.

An information correspondence model, as in a possibility correspondence model (e.g., Aumann, 1976, 1999; Dekel and Gul, 1997; Geanakoplos, 2021; Morris, 1996), has the following three ingredients. The first is a set Ω of states of the world. Each state $\omega \in \Omega$ is supposed to be a complete description of the world in question. Second, each agent reasons about some aspects of the underlying states. Each property of the state space is represented as an event, which is a subset E of the state space Ω. Thus, the second ingredient is the collection of events, which determines the objects of agents' beliefs. Third, each agent has her information correspondence $\mathcal{I}$. It associates, with each state, possibly multiple information sets available at that state. The agent believes an event at a state if the event is implied by some information set at that state. If the information correspondence assigns a single information set at each state, then it reduces to the possibility correspondence.

An agent whose beliefs are represented by an information correspondence, unlike a possibility correspondence, may fail to believe the conjunction of events that she believes or a tautology. The information correspondence model is a tractable generalization of the possibility correspondence model that dispenses with such logical sophistication. To obtain tractability, the only requirement of the model is logical monotonicity: an agent believes logical consequences of her own beliefs. The information correspondence approach, by assuming logical monotonicity, enables the analysts to represent agents' beliefs without specifying the entire collection of events that an agent believes at each state.

The relaxation of agents' conjunction and necessitation properties, which are conceptually at odd with real human reasoning, technically allows the analysts to study both qualitative and probabilistic beliefs under the umbrella of the information correspondence approach. For example, in a dynamic game, while agents have knowledge about past observations, they form probabilistic beliefs about their opponents' future behaviors.[1] If an agent exhibits the arbitrary conjunction property (i.e., she believes an arbitrary conjunction of her beliefs) rendered by a possibility correspondence, such belief may not be an event (e.g., in a probabilistic framework where the collection of events forms a σ-algebra, such belief may not be measurable). As I will provide an economic example, such arbitrary conjunction property may indeed be at odd with probabilistic reasoning because, for example, the agent may not assign probability one to the arbitrary conjunction of events that she believes with probability one. Thus, a possibility correspondence model cannot necessarily represent probabilistic beliefs. In contrast, the information correspondence ap-

[1] See, for example, Dekel and Gul (1997) for the importance of capturing both knowledge and probabilistic beliefs in dynamic games.

proach can accommodate various forms of monotone probabilistic beliefs such as standard countably-additive beliefs, finitely-additive beliefs, and even non-additive beliefs.

The paper then characterizes logical and introspective properties of beliefs. I complete the discussion of the information correspondence approach by demonstrating the generality of the model. For given logical and introspective properties of agents' monotone beliefs, I demonstrate the equivalence between the information correspondence approach and the belief operator approach where an agent's belief operator maps each event E to the event that she believes E.

The second objective of this paper is to connect the information correspondence model to the knowledge representation in mathematical psychology known as a "surmise system" in the "knowledge space theory" developed by Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011).[2] Knowledge is a special form of belief that satisfies the truth axiom: if an agent "knows" an event at a state then the event holds at that state. This paper aims to link seemingly different (for their respective aims) knowledge representations in economics and mathematical psychology from a unified mathematical point of view.[3]

To make the connection, I introduce a "surmise system" while keeping the same notations. Since I give two different interpretations to each mathematical object, I attach the quotation mark in referring to the mathematical-psychology literature. The literature studies the knowledge of an agent (e.g., a high school student) regarding subsets of Ω, which consists of "questions" or "items." Thus, the ambient set Ω represents the entire body of knowledge in question (e.g., high-school algebra). While I make the formal connection in the sequel, a "surmise function" is a mapping $\mathcal{I}$ which associates, with each "question" ω, the collection of possibly multiple sets of "items" with the following interpretation: each set of such "items" serves as a possible set of prerequisites for the "question" ω. Thus, if the agent has mastered the "question" ω, then she must have mastered all the "questions" in at least one of the members of $\mathcal{I}(\omega)$. Multiplicity of members of $\mathcal{I}(\omega)$ means multiple ways to master the "question" ω. Thus, the "surmise function" $\mathcal{I}$ encodes all possible (thus not necessarily unique) ways to making inferences from each question.[4] This paper shows that an agent's belief described by a "surmise function" is knowledge that exhibits positive introspection (if she knows a set of "questions" then she knows that she knows it) and necessitation (the agent knows a tautology).

The motivation behind bridging these two different knowledge and belief represen-

[2]Doignon and Falmagne (1985) is the pioneering paper on the subject, and Doignon and Falmagne (2016) is a more recent survey article. Doignon and Falmagne (1999) is the first survey book, while Falmagne and Doignon (2011) is an enriched edition of it. See also the references therein.

[3]Fukuda (2019) represents an agent's truthful knowledge by a set algebra (a collection of events) such as a σ-algebra or a topology in terms of the agent's logical and introspective reasoning ability. The set algebra, in a particular setting, turns out to correspond to the "knowledge states" in Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011).

[4]In light of inferences, Shin (1993) identifies the notion of knowledge with logical provability. An agent knows an event when she can prove it from her "basic knowledge" through use of propositional logic. This paper allows for such an interpretation: an agent believes an event E at a state ω if she can "prove" (in terms of set inclusion) E from one of her information set $F \in \mathcal{I}(\omega)$ (which could be incorrect in that $\omega \notin F$).

tations comes from the observation that the two knowledge models in economics and mathematical psychology, which I demonstrate are firmly related, have evolved in quite different ways. Hence, it would be interesting to connect these two seemingly different knowledge and belief representations under the same mathematical framework.

On the one hand, one feature that is not seen in interactive epistemology in economics and game theory is the development of empirical assessments of an agent's beliefs and knowledge based on the formal model. The mathematical psychology literature (see, Doignon and Falmagne (2016) and Falmagne and Doignon (2011) for surveys) has been attempting at constructing and testing a formal knowledge representation in practical contexts. A particular case of "knowledge space theory" referred to as "learning space theory" has developed the assessments of students' knowledge about their academic subjects. For example, the web-based system called **ALEKS** ("**A**ssessment of **LE**arning in **K**nowledge **S**paces") has been used by "millions of students in schools and colleges, and by home schooled students" (Doignon and Falmagne, 2016).

On the other hand, the economics literature has provided features that have not been developed in mathematical psychology. One is the consideration of interactive higher-order beliefs, i.e., an agent's belief about other agents' beliefs. Another is unawareness. In one formulation, an agent is unaware of an event if she does not know it and she does not know that she does not know it. In another formulation, the agent is unaware of an event if she lacks the conception that determines the event.[5]

This paper is also closely related to knowledge and belief representations referred to as monotone neighborhood systems in computer science, logic, and philosophy (e.g., Chellas, 1980; Fagin et al., 2003; Pacuit, 2017) and related models of limited reasoning in computer science. A neighborhood system (also called a Montague-Scott structure) is a mapping that associates, with each state of the world, the entire collection of events that an agent believes. A monotone neighborhood system is a neighborhood system such that the agent's belief is logically monotone (i.e., if the agent believes an event then she believes any of its logical consequences). An information correspondence is a "generator" of a monotone neighborhood system, and thus it can describe an agent's beliefs without specifying the entire collection of events that she believes at each state. A monotone neighborhood system is considered to be an information correspondence.

In economics and game theory, such papers as Heifetz (1996, 1999) and Lismont and Mongin (1994a,b) use monotone neighborhood systems to represent notions of common belief and common knowledge (e.g., Aumann, 1976; Friedell, 1969). This paper instead formalizes logical and introspective properties of individual agents' beliefs and knowledge. I also briefly discuss how to introduce notions of common belief and common knowledge into the framework of this paper. Salonen (2009a) studies a canonical syntactical interactive belief representation by capturing each agent's beliefs as a collection of propositions that she believes.

Monotone neighborhood systems and models of limited reasoning have been studied in computer science, logic, and philosophy. The closest paper in this literature is the logic of

[5]Pioneering papers on unawareness include Fagin and Halpern (1987) and Modica and Rustichini (1994, 1999). Some recent models include Heifetz, Meier, and Schipper (2006, 2013). See Schipper (2015) for a survey.

local reasoning (or the "society-of-minds") approach by Fagin and Halpern (1987).[6] They study a "boundedly rational" agent who fails to believe (or know) the conjunctions of her own belief (or knowledge). The agent is endowed with a collection of multiple information sets at each state, and she focuses on one information set possible at each time. While an information correspondence is regarded as a purely semantic counterpart of their model, I demonstrate that it can capture probabilistic beliefs by defining it on an appropriate set algebraic structure. This paper takes one step further to characterize various logical and introspective properties. This paper also connects it to the mathematical psychology literature.

The paper is organized as follows. Section 2 formally defines information correspondences. It also demonstrates that information correspondences generalize possibility correspondences. It also provides examples of information correspondences that cannot be captured by possibility correspondences. Section 3 analyzes various properties of information correspondences. Section 4 studies the equivalence among an information correspondence and alternative knowledge and belief representations in economics and mathematical psychology. Section 5 studies an economic application in which an agent's beliefs are induced from her preferences. It provides an economic example of information correspondences that cannot be captured by possibility correspondences. Section 6 provides concluding remarks. Proofs are relegated to Appendix A. Appendix B provides some additional results.

2 An Information Correspondence

I represent agents' beliefs (knowledge if it is truthful) on a state space. A *state space* is a pair $(\Omega, \mathcal{D})$, where Ω is a set of *states* of the world and where $\mathcal{D}$ is a collection of *events*, i.e., subsets of states. Throughout the paper, to accommodate qualitative and probabilistic beliefs in the same framework and possibly at the same time, I assume that $(\Omega, \mathcal{D})$ is a measurable space, that is, $\mathcal{D}$ is a σ-algebra on Ω.[7] Thus, $\mathcal{D}$ contains the empty set $\emptyset$ and the entire state space Ω and is closed under complementation and under countable union (and consequently countable intersection).[8] Denote by $\mathcal{P}(\Omega)$ the power set of Ω. The power set $\mathcal{P}(\Omega)$ is also a σ-algebra. Denote the complement of $E \in \mathcal{P}(\Omega)$ by E^c or $\neg E$.

Below, Section 2.1 defines an information correspondence. Section 2.2 provides examples.

[6]Thijsse (1993) (see also Thijsse, 1992, Chapter 6.6) calls their model a cluster model. Also, Fagin et al. (2003, Chapter 9.6) and Meyer and Hoek (1995, Chapter 2.9) study this approach.

[7]For example, Meier (2008) studies knowledge and σ-additive probabilistic beliefs on a σ-algebra.

[8]I make a technical remark that the analysis of this paper carries over to the following more general class. Letting κ be an infinite cardinal, a collection $\mathcal{D}$ of subsets of Ω is a *κ-algebra* (on Ω) if $\mathcal{D}$ contains $\emptyset$ and Ω and if $\mathcal{D}$ is closed under complementation and under arbitrary union (and intersection) of any non-empty sub-collection with cardinality less than κ. For instance, an $\aleph_0$-algebra is an algebra of sets where $\aleph_0$ is the least infinite cardinal. An $\aleph_1$-algebra is a σ-algebra, where $\aleph_1$ is the least uncountable cardinal. Meier (2006) studies a canonical representation of agents' finitely-additive beliefs on a κ-algebra. Fukuda (2023) studies a canonical representations of agents' qualitative beliefs on a κ-algebra. The analysis also carries over to the class of complete algebras: $\mathcal{D}$ is a *complete algebra* (on Ω) if $\mathcal{D}$ contains $\emptyset$ and Ω and if $\mathcal{D}$ is closed under complementation and is closed under arbitrary union (and intersection).

2.1 Definition of an Information Correspondence

I represent each agent's belief (or knowledge) on a state space by an information correspondence. For ease of exposition, unless otherwise stated, I restrict attention to a single agent. The information correspondence retains the spirit of a possibility correspondence in the sense that the agent's belief is logically entailed from her information. The agent is a logical reasoner in that she believes any logical consequence of her own belief. Thus, the only requirement is logical monotonicity, and I dispense with the conjunction and necessitation properties endemic in possibility correspondences. The agent may believe events E and F without believing the conjunction $E \cap F$. She may fail to believe a tautology Ω.

The information correspondence $\mathcal{I}$ associates, with each state ω, a collection of events $\mathcal{I}(\omega) \in \mathcal{P}(\mathcal{D})$ that can be a source of beliefs at the state ω in the following sense: the agent believes an event E at the state ω if there is an event $F \in \mathcal{I}(\omega)$ which is included in E. Each element of $\mathcal{I}(\omega)$ can be understood as a piece of information available to the agent at state ω. Call E an *information set* at ω if $E \in \mathcal{I}(\omega)$.

The information correspondence is a mapping $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ satisfying a certain regularity condition. To define the regularity condition, for any $\Gamma \in \mathcal{P}(\mathcal{D})$, define

$$\uparrow \Gamma := \{E \in \mathcal{D} \mid \text{there is } F \in \Gamma \text{ with } F \subseteq E\}.$$

If Γ is the agent's collection of information at a particular state (i.e., $\Gamma = \mathcal{I}(\omega)$ for some $\omega \in \Omega$) then $E \in \uparrow \Gamma$ means that E is entailed from some piece of information $F \in \Gamma$. I call such Γ an *information collection* in the sense that it is a collection of information sets. Note that $\uparrow \Gamma$ is closed under monotonicity (precisely, set inclusion) in that $\uparrow\uparrow \Gamma \subseteq \uparrow \Gamma$.

With these in mind, I formally define an information correspondence. An *information correspondence* on a state space $(\Omega, \mathcal{D})$ is a mapping $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ which satisfies the regularity condition that, for each $E \in \mathcal{D}$,

$$\{\omega \in \Omega \mid E \in \uparrow \mathcal{I}(\omega)\} \in \mathcal{D}. \tag{1}$$

Condition (1) states that the set of states at which the agent has information to support an event E itself is an event. I define the left-hand side of Expression (1) as:

$$B_{\mathcal{I}}(E) := \{\omega \in \Omega \mid E \in \uparrow \mathcal{I}(\omega)\}. \tag{2}$$

Thus, $B_{\mathcal{I}}(E)$ is the event that (i.e., the set of states at which) the agent *believes* E. Define the *belief operator* $B_{\mathcal{I}} : \mathcal{D} \to \mathcal{D}$ derived from $\mathcal{I}$ through Equation (2). Condition (1) grantees that $B_{\mathcal{I}}$ is a well-defined operator. Higher-order beliefs are generated through iterating the belief operator.

I make five remarks on the belief operator $B_{\mathcal{I}}$. First, observe that $\uparrow \mathcal{I}(\omega)$ is exactly the collection of events that the agent believes at ω. Put differently,

$$\uparrow \mathcal{I}(\omega) = \{E \in \mathcal{D} \mid \omega \in B_{\mathcal{I}}(E)\},$$

that is, $E \in \uparrow \mathcal{I}(\omega)$ if and only if (hereafter, often abbreviated as iff) $\omega \in B_{\mathcal{I}}(E)$. Also, one can regard $\uparrow \mathcal{I}$ itself as an information correspondence, i.e., a mapping from Ω into $\mathcal{P}(\mathcal{D})$ such that $(\uparrow \mathcal{I})(\omega) = \uparrow \mathcal{I}(\omega)$ for each $\omega \in \Omega$.

Second, on a related point, the mapping $\uparrow \mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ coincides with a monotone neighborhood system. A neighborhood system assigns, with each state, the collection of events that the agent believes. A neighborhood system is monotone when the agent's belief operator satisfies Monotonicity (i.e., if the agent believes E and if E implies F (i.e., $E \subseteq F$) then she believes F). Thus, the information correspondence $\mathcal{I}$ is identified as a monotone neighborhood system iff $\mathcal{I} = \uparrow \mathcal{I}$.

Third, the belief operator $B_{\mathcal{I}}$ satisfies Monotonicity: $E \subseteq F$ implies $B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}(F)$. That is, if the agent believes an event E at a state and if E implies F (in the sense of $E \subseteq F$), then she believes F at that state. Section 4.1 establishes the equivalence between an information correspondence and a monotone belief operator.

Fourth, the agent considers an event E *possible* at a state ω if she does not believe E^c at ω. Denote by $L_{\mathcal{I}}$ the agent's possibility operator: $L_{\mathcal{I}}(E) := (\neg B_{\mathcal{I}})(E^c) \in \mathcal{D}$ for all $E \in \mathcal{D}$.[9] The event that the agent considers E possible is

$$L_{\mathcal{I}}(E) = \{\omega \in \Omega \mid F \cap E \neq \emptyset \text{ for all } F \in \mathcal{I}(\omega)\}. \tag{3}$$

The agent considers E possible at ω when her information set at ω is always not inconsistent with E. Section 4.2 connects the possibility operator to a closure operator of Doignon and Falmagne (1985, 1999) and Falmagne and Doignon (2011) in mathematical psychology.

Fifth, if $\mathcal{I}$ is singleton-valued (i.e., if $\mathcal{I}(\cdot) = \{P(\cdot)\}$) then it reduces to the possibility correspondence $P : \Omega \to \mathcal{D}$ such (i) that each information/possibility set $P(\omega)$ is an event and (ii) that $\mathcal{I}$ satisfies the regularity condition that $\{\omega \in \Omega \mid P(\omega) \subseteq E\} \in \mathcal{D}$ for each $E \in \mathcal{D}$.[10]

More generally, I introduce the condition under which $\mathcal{I}$ is identified with a possibility correspondence. Namely, $\mathcal{I}$ satisfies the *Kripke property* if each $\mathcal{I}(\omega)$ contains a minimum element, i.e., there is $P(\omega) \in \mathcal{I}(\omega)$ such that $P(\omega) \subseteq E$ for all $E \in \mathcal{I}(\omega)$. If $\mathcal{I}$ satisfies the Kripke property, then the agent's beliefs are represented by $\mathcal{I}(\cdot) = \{P(\cdot)\}$, because the collection $\uparrow \mathcal{I}(\omega)$ of events that the agent believes at a state ω satisfies

$$\uparrow \mathcal{I}(\omega) = \{E \in \mathcal{D} \mid P(\omega) \subseteq E\} = \uparrow \{P(\omega)\}.$$

The following proposition formalizes this argument in terms of the belief operator, and demonstrates that an information correspondence is identified as a possibility correspondence under the Kripke property.

Proposition 1. *An information correspondence $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ satisfies the Kripke property iff for each $\omega \in \Omega$, there is $P(\omega) \in \mathcal{I}(\omega)$ such that $\omega \in B_{\mathcal{I}}(E)$ iff $P(\omega) \subseteq E$ for all $E \in \mathcal{D}$.*

[9] Possibility (or compatibility) is often considered to be the dual of knowledge or belief (e.g., Fagin et al., 2003; Hintikka, 1962).

[10] The regularity condition requires the belief operator induced by the possibility correspondence P to be well-defined. Thus, the possibility correspondence P has to satisfy the regularity condition. See Samet (2010) for examples of a state space $(\Omega, \mathcal{D})$ in which $\mathcal{D}$ is an algebra of sets and in which a partitional possibility correspondence $P : \Omega \to \mathcal{P}(\Omega)$ (i.e., a partition cell $P(\omega)$ may not be an event) does not induce a well-defined knowledge operator from $\mathcal{D}$ into itself.

To conclude this subsection, I make a connection to a "surmise function" in Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011). They study the knowledge of an agent regarding subsets of Ω, which consists of "questions" or "items." Again, note that I keep the same notations in order to make it easier to see the connections and that I append the quotation mark to the terminologies in the mathematical-psychology literature. Table A.1 in Appendix A.1 lists the correspondence of terminologies.

Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011) introduce a "surmise system" $(\Omega, \mathcal{I})$ as a way to model an agent's knowledge. A "surmise function" is a mapping $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{P}(\Omega))$ (i.e., $\mathcal{D} = \mathcal{P}(\Omega)$) which satisfies certain logical and introspective properties of knowledge to be discussed in Section 3. Each $\mathcal{I}(\omega)$ is interpreted as encoding all possible (not necessarily unique) ways of inferring a correct response to the "question" ω. Put differently, if the agent is capable of solving the "question" ω, then there exists $E \in \mathcal{I}(\omega)$ such that she is capable of solving all the "questions" in E. Such E (a member of $\mathcal{I}(\omega)$) is referred to as a "clause" (a "background" or a "foundation") for the "question" ω.

2.2 Examples of Information Correspondences

So far, the previous subsection has defined an information correspondence, and Proposition 1 has shown that the information correspondence generalizes a possibility correspondence. This subsection provides two examples of an information correspondence that cannot be reduced to a possibility correspondence.

2.2.1 Dispensing with Conjunction or Necessitation

The first example demonstrates that one can dispense with the agent's conjunctive ability (i.e., the agent believes the conjunction of what she believes) by having multiple information sets. The example also shows that one can dispense with the necessitation property (i.e., the agent believes a tautology) by allowing the information correspondence to be empty-valued.

Example 1. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Let $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ be such that $\mathcal{I}(\omega_1) = \{\{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}\}$ and $\mathcal{I}(\omega_2) = \mathcal{I}(\omega_3) = \emptyset$. The agent believes $\{\omega_1, \omega_2\}$ and $\{\omega_1, \omega_3\}$ at state ω_1 but shes does not believe $\{\omega_1, \omega_2\} \cap \{\omega_1, \omega_3\}$ at that state. The failure of the conjunction property comes from the specification that an information collection $\mathcal{I}(\omega)$ may not be closed under intersection. At state ω_2 or ω_3, she does not believe anything at all, and thus $B_{\mathcal{I}}(\Omega) = \{\omega_1\}$. Necessitation (i.e., $B_{\mathcal{I}}(\Omega) = \Omega$) fails because the information correspondence is empty-valued at some states. By inspection, one can show:

$$B_{\mathcal{I}}(E) = \begin{cases} \{\omega_1\} & \text{if } E \in \{\{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \Omega\} \\ \emptyset & \text{otherwise} \end{cases}. \tag{4}$$

The belief operator $B_{\mathcal{I}}$ cannot be induced by a possibility correspondence because any belief operator that is induced by a possibility correspondence has to satisfy the conjunction and necessitation properties. □

This example also relates to the literature on unawareness. In the literature on unawareness (on a standard state space), Dekel, Lipman, and Rustichini (1998) show that a standard state space model cannot represent a non-trivial form of unawareness if an underlying knowledge operator satisfies the monotonicity and necessitation properties and if underlying knowledge and unawareness operators satisfy certain desirable properties. Thus, for any standard possibility correspondence model, in which the derived knowledge operator satisfies the monotonicity and necessitation properties, knowledge and unawareness cannot satisfy certain desirable properties. In contrast, the information correspondence approach can drop Necessitation.[11] In fact, Modica and Rustichini (1994, Section 4) provide an example of a two-states model of unawareness in which the agent's knowledge fails Necessitation. Their knowledge operator can be induced from a simplified version of this example in which $\omega_2 = \omega_3$.

2.2.2 Modeling Qualitative and Quantitative Beliefs in a Unified Manner

The second example demonstrates that an information correspondence, unlike a possibility correspondence, can capture probabilistic beliefs. That is, the second demonstrates that an information correspondence can capture both qualitative and quantitative beliefs in a unified manner.

Example 2. Consider a measurable space $(\Omega, \mathcal{D}) = ([0,1], \mathcal{B}_{[0,1]})$, where $\mathcal{B}_{[0,1]}$ is the Borel σ-algebra on $[0,1]$. For simplicity, suppose that the agent's beliefs are dictated by the Lebesgue measure μ on $([0,1], \mathcal{B}_{[0,1]})$ at every state.

I first show that her probability-one belief is not induced by a possibility correspondence model. Suppose to the contrary that her probability-one belief is induced by a possibility correspondence $P : \Omega \to \mathcal{P}(\Omega)$: she believes an event $E \in \mathcal{D}$ with probability one at a state ω iff $P(\omega) \subseteq E$.[12] For each event $E_r := \Omega \setminus \{r\} \in \mathcal{D}$ with $r \in \Omega$, the agent assigns probability-one belief to E_r at each ω. Thus, $P(\omega) \subseteq E_r$ for all $r \in \Omega$. Taking the intersection over all $r \in \Omega$, one has $P(\omega) = \emptyset$. Thus, her probability-one belief operator $B : \mathcal{D} \to \mathcal{D}$ satisfies $B(E) = \Omega$ for all $E \in \mathcal{D}$, that is, she assigns probability one to any event. This is impossible. Since the agent assigns probability one to each event E_r, one can generally only assert that she assigns probability one to any *countable* intersection. Thus, this contradiction comes from the arbitrary conjunction rendered by the possibility correspondence.

Next, I construct an information correspondence that can capture the agent's probability-one belief. Let $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ be such that $\mathcal{I}(\omega) := \{E \in \mathcal{D} \mid \mu(E) = 1\}$ for each $\omega \in \Omega$.

[11] Fukuda (2021) studies unawareness without the conjunction or necessitation properties, in the framework in which agents' knowledge and unawareness operators are primitives on standard and generalized state spaces. On a standard state space, as Section 4 shows, the assumptions on one's beliefs (knowledge) in this paper are more general than those assumed in Fukuda (2021). It would be an interesting avenue for future research to generalize the framework of information correspondences to a generalized state space model of unawareness such as Heifetz, Meier, and Schipper (2006, 2008).

[12] Here, I establish the negative result even if I do not impose the assumption that each $P(\omega)$ is measurable (i.e., an event).

This is an information correspondence because, for any $E \in \mathcal{D}$,

$$B_{\mathcal{I}}(E) = \begin{cases} \emptyset & \text{if } \mu(E) \in [0,1) \\ \Omega & \text{if } \mu(E) = 1 \end{cases},$$

and thus $B_{\mathcal{I}}(E) \in \mathcal{D}$ for all $E \in \mathcal{D}$. If the agent believes an event E with probability one at a state ω, then, since $\mu(E) = 1$, I have $E \in \mathcal{I}(\omega)$ and thus $\omega \in B_{\mathcal{I}}(E)$. Conversely, if $\omega \in B_{\mathcal{I}}(E)$ then there is $F \in \mathcal{I}(\omega)$ (i.e., $\mu(F) = 1$) such that $F \subseteq E$. Thus, the agent believes E with probability one at state ω. □

More generally, let $(\Omega, \mathcal{D})$ be an arbitrary measurable space, and let $t : \Omega \times \mathcal{D} \to [0,1]$ be a function with the following two properties: (i) for each $\omega \in \Omega$, the mapping $t(\omega, \cdot) : \mathcal{D} \to [0,1]$ is a monotone set function that dictates the agent's probabilistic beliefs at ω (e.g., a non-additive, finitely-additive, or countably-additive probability measure);[13] and (ii) $\{\omega \in \Omega \mid t(\omega, E) \geq p\} \in \mathcal{D}$ for all $(p, E) \in [0,1] \times \mathcal{D}$. The first condition states that each $t(\omega, \cdot)$ represents the agent's beliefs at ω. The second is the regularity condition that the set of states at which the agent believes an event E with probability at least p is also an event.

For each state ω, the set function $t(\omega, \cdot)$ is referred to as the agent's *type* at ω, and the mapping $t : \Omega \times \mathcal{D} \to [0,1]$ is referred to as the agent's *type mapping*. For each $p \in [0,1]$, the agent's *p-belief operator* (Friedell, 1969; Monderer and Samet, 1989) $B^p : \mathcal{D} \to \mathcal{D}$ is defined as $B^p(E) := \{\omega \in \Omega \mid t(\omega, E) \geq p\}$ for each $E \in \mathcal{D}$.

Now, for each $p \in [0,1]$, define the mapping $\mathcal{I}_p : \Omega \to \mathcal{P}(\mathcal{D})$ by

$$\mathcal{I}_p(\omega) := \{E \in \mathcal{D} \mid t(\omega, E) \geq p\} \text{ for each } \omega \in \Omega.$$

The mapping $\mathcal{I}_p$ is an information correspondence because

$$B_{\mathcal{I}_p}(E) := \{\omega \in \Omega \mid E \in \uparrow \mathcal{I}_p(\omega)\} = B^p(E) \in \mathcal{D} \text{ for each } E \in \mathcal{D}.$$

Since the p-belief operator approach is equivalent to the type space approach (Samet, 2000) and since the information correspondence approach can represent the agent's p-belief operators, the information correspondence approach can accommodate both probabilistic and non-probabilistic beliefs in a unified manner. The information correspondence approach thus enables one to introduce both qualitative and probabilistic beliefs in, for example, a dynamic game where agents retain knowledge of past observations and beliefs in future actions. The information correspondence approach also enables one to study similarities and differences between qualitative and probabilistic beliefs.[14]

[13] A pioneering work on non-additive beliefs is Schmeidler (1989).

[14] Here, I show that probabilistic beliefs that cannot be represented by a possibility correspondence model can be represented by an information correspondence model. Thus, both probabilistic and non-probabilistic beliefs (including knowledge) can be introduced under the same framework. I remark that there is a whole strand of literature studying how properties of knowledge are technically and conceptually different from those of probability-one beliefs. See, for instance, Brandenburger and Dekel (1987), Dubra and Echenique (2004), Fukuda (2019), Hérves-Beloso and Monteiro (2013), Lee (2018), and Tóbiás (2021).

3 Properties of an Information Correspondence

This section represents three kinds of properties of beliefs for an information correspondence. The first is logical properties such as consistency, necessitation, and conjunction properties. The second is introspective properties such as the truth axiom. These two kinds of properties generalize possibility correspondence models that dispense with the conjunction and necessitation properties. The third kind is "Belief-in" properties. When an agent's belief is not represented by a possibility correspondence but by an information correspondence, while her belief violates certain logical properties, she can still falsely believe that her belief satisfies the logical properties.

3.1 Logical Properties

I introduce four well-known logical properties of beliefs to an information correspondence: No-Contradiction, Consistency, Necessitation, and Non-empty Countable Conjunction. In terms of the belief operator, No-Contradiction states that the agent never believes a contradiction in the form of the empty set. Consistency states that the agent does not believe an event E and its negation E^c at the same time. Necessitation means that the agent always believes a tautology of the form of the entire set. Non-empty Countable Conjunction states that whenever the agent believes each event from a non-empty countable collection of events, she believes its conjunction.

While No-Contradiction and Consistency are equivalent under the possibility correspondence model, they are not equivalent under the possibility correspondence model (Consistency is stronger than No-Contradiction in the information correspondence model). In the possibility correspondence model, qualitative belief is usually assumed to satisfy Consistency (equivalently, No-Contradiction).[15] The previous section has examined Necessitation and Non-empty Countable Conjunction.

Hereafter in this section, fix a state space $(\Omega, \mathcal{D})$. I define these logical properties in a way so that an information correspondence $\mathcal{I}$ satisfies a logical property if each information collection $\mathcal{I}(\omega)$ satisfies it.[16] With this in mind, I introduce the following four logical properties to an information collection Γ.

1. An information collection $\Gamma \in \mathcal{P}(\mathcal{D})$ satisfies *No-Contradiction* if $\emptyset \notin \Gamma$. In words, Γ does not contain a contradiction in the form of the empty set.

2. The information collection Γ satisfies *Consistency* (or it is *serial*) if $E \cap F \neq \emptyset$ for any $E, F \in \Gamma$. In words, any pair of information $(E, F) \in \Gamma^2$ is not contradictory with each other. Consistency implies No-Contradiction.

[15] For instance, Consistency plays an important role in formalizing weak-dominance rationality in Bonanno and Tsakas (2018).

[16] When it comes to probabilistic beliefs represented by a type mapping, recall that a type mapping t associates, with each state ω, a probability measure $t(\omega, \cdot)$ which is assumed to satisfy, for instance, Necessitation: $t(\omega, \Omega) = 1$. That is, logical properties of beliefs (i.e., the type mapping) are encoded as logical properties of realized beliefs $t(\omega, \cdot)$.

3. The information collection Γ satisfies *Necessitation* if $\Gamma \neq \emptyset$. That is, Γ contains some information, and thus a tautology is inferred from it.

4. The information collection Γ satisfies *Non-empty Countable Conjunction* if, for any $\{F_n\}_{n\in\mathbb{N}} \subseteq \Gamma$, there is $F \in \Gamma$ with $F \subseteq \bigcap_{n\in\mathbb{N}} F_n$.[17] Intuitively, for any given family of information, the information collection Γ is rich enough to have another information implying the conjunction of the given family.

Again, an information correspondence $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ satisfies a given logical property if every $\mathcal{I}(\omega)$ satisfies it. For instance, the information correspondence $\mathcal{I}$ satisfies No-Contradiction if $\emptyset \notin \mathcal{I}(\omega)$ for all $\omega \in \Omega$.

Recalling that $\uparrow \Gamma$ corresponds to the collection of events that the agent believes by making inferences from Γ, I formalize the logical properties in terms of $\uparrow \Gamma$ instead of the primitive Γ. As a consequence, the proposition below demonstrates that each logical property of an information correspondence $\mathcal{I}$ embodies the intended definition of the logical property of the belief operator $B_{\mathcal{I}}$.

Proposition 2. *Let Γ be an information collection, and let $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ be an information correspondence.*

1. *(a) Γ satisfies No-Contradiction iff $\uparrow \Gamma$ satisfies it.*

   (b) $\mathcal{I}$ satisfies No-Contradiction iff $\uparrow \mathcal{I}$ satisfies it iff $B_{\mathcal{I}}(\emptyset) = \emptyset$.

2. *(a) Γ satisfies Consistency iff $\uparrow \Gamma$ satisfies it iff $E^c \notin \uparrow \Gamma$ for any $E \in \uparrow \Gamma$.*

   (b) $\mathcal{I}$ satisfies Consistency iff $\uparrow \mathcal{I}$ satisfies it iff $B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(E^c)$ for all $E \in \mathcal{D}$.

3. *(a) Γ satisfies Necessitation iff $\uparrow \Gamma$ satisfies it iff $\Omega \in \uparrow \Gamma$.*

   (b) $\mathcal{I}$ satisfies Necessitation iff $\uparrow \mathcal{I}$ satisfies it iff $B_{\mathcal{I}}(\Omega) = \Omega$.

4. *(a) Γ satisfies Non-empty Countable Conjunction iff $\uparrow \Gamma$ satisfies it iff $\uparrow \Gamma$ is closed under non-empty countable intersection: $\bigcap_{n\in\mathbb{N}} F_n \in \uparrow \Gamma$ for any $\{F_n\}_{n\in\mathbb{N}} \in \mathcal{P}(\uparrow \Gamma)$.*

   (b) $\mathcal{I}$ satisfies Non-empty Countable Conjunction iff $\uparrow \mathcal{I}$ satisfies it iff $\bigcap_{n\in\mathbb{N}} B_{\mathcal{I}}(F_n) \subseteq B_{\mathcal{I}}\left(\bigcap_{n\in\mathbb{N}} F_n\right)$ for any $\{F_n\}_{n\in\mathbb{N}} \in \mathcal{P}(\mathcal{D})$.

Proposition 2 establishes the following two points. First, "part (a)" of Proposition 2 shows that the information collection Γ satisfies a given logical property if and only if the information collection $\uparrow \Gamma$ satisfies it. Thus, the information correspondence $\mathcal{I}$ satisfies a given logical property if and only if the information correspondence $\uparrow \mathcal{I}$ satisfies it. In other words, the logical properties are preserved under the operation of taking "$\uparrow$." This implies that the four logical properties of an information correspondence $\mathcal{I}$ are defined in a way such that the properties only depend on $\uparrow \mathcal{I}$. That is, if two information correspondences

[17] One can define variants of this conjunction property. For instance, Γ satisfies *Non-empty Finite Conjunction* if, for any $F_1, F_2 \subseteq \Gamma$, there is $F \in \Gamma$ with $F \subseteq F_1 \cap F_2$. For ease of exposition, I mainly focus on Non-empty Countable Conjunction.

$\mathcal{I}$ and $\mathcal{I}'$ induce the same beliefs in that $\uparrow \mathcal{I} = \uparrow \mathcal{I}'$, then the information correspondences $\mathcal{I}$ and $\mathcal{I}'$ share the same logical properties.

Second, "part (b)" of Proposition 2 demonstrates that each logical property of the information correspondence $\mathcal{I}$ captures the intended logical property of the belief operator $B_{\mathcal{I}}$. Henceforth, $B_{\mathcal{I}}$ is said to satisfy a given logical property (e.g., No-Contradiction) if $\mathcal{I}$ satisfies it. As an example, the belief operator defined by Equation (4) satisfies No-Contradiction and Consistency but fails Necessitation and Non-empty Countable Conjunction. For (1b), No-Contradiction means that there is no state at which the agent believes a contradiction in the form of $\emptyset$. For (2b), Consistency means that if the agent believes E, then she does not believe its negation E^c. Consistency implies No-Contradiction because $B_{\mathcal{I}}(\emptyset) \subseteq B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)$ by monotonicity of $B_{\mathcal{I}}$. For (3b), Necessitation means that the agent always believes a tautology in the form of Ω. It can be seen that $B_{\mathcal{I}}$ satisfies Necessitation iff $B_{\mathcal{I}}(B_{\mathcal{I}}(\Omega)) = \Omega$. That is, the agent always believes a tautology if and only if she always believes that she believes a tautology.[18] For (4b), Non-empty Countable Conjunction means that if the agent believes each of a non-empty countable collection of events, then she believes its conjunction. Under Non-empty Countable (in fact, Finite) Conjunction, Consistency and No-Contradiction are equivalent. This way, Proposition 2 generalized the logical properties of the possibility correspondence model.

3.2 Introspective Properties

Next, I introduce the following four well-known introspective properties of beliefs to an information correspondence. They generalize the reflexivity, transitivity, Euclideanness, and symmetry axioms of a possibility correspondence to an information correspondence. The axioms of reflexivity, transitivity, and Euclideanness are well-studied in the possibility correspondence model in economics and game theory. The symmetry axiom is well-studied in computer science, logic, and philosophy.

In terms of the belief operator, reflexivity implies Truth Axiom: if the agent "knows" an event at a state then the event has to hold at that state. Truth Axiom distinguishes belief and knowledge in that belief can be false while knowledge has to be true. Transitivity implies Positive Introspection: if the agent believes an event then she believes that she believes it. Euclideanness implies Negative Introspection: if the agent does not believe an event then she believes that she does not believe it. Symmetry characterizes the following form of introspection: if the agent considers it possible that she believes an event E at a state, then the event has to hold true at that state. As I will discuss below, one of the technical contributions of this paper is to provide the formulations of reflexivity, transitivity, Euclideanness, and symmetry.

I start with providing the technical formulations.

1. An information correspondence $\mathcal{I}$ is *reflexive* (or satisfies *Truth Axiom*) if, for any $(\omega, E) \in \Omega \times \mathcal{D}$, $E \in \mathcal{I}(\omega)$ implies $\omega \in E$. That is, $\mathcal{I}(\omega) \subseteq \{E \in \mathcal{D} \mid \omega \in E\}$ for any

[18] The proof goes as follows. If $B_{\mathcal{I}}$ satisfies Necessitation, then it follows from $\Omega = B_{\mathcal{I}}(\Omega)$ that $\Omega = B_{\mathcal{I}}(\Omega) = B_{\mathcal{I}}(B_{\mathcal{I}}(\Omega))$. Conversely, if $\Omega = B_{\mathcal{I}}(B_{\mathcal{I}}(\Omega))$, then it follows from $B_{\mathcal{I}}(\Omega) \subseteq \Omega$ that $\Omega = B_{\mathcal{I}}(B_{\mathcal{I}}(\Omega)) \subseteq B_{\mathcal{I}}(\Omega) \subseteq \Omega$.

$\omega \in \Omega$. In words, the agent's information is always correct at each state.

2. The information correspondence $\mathcal{I}$ is *transitive* (or satisfies *Positive Introspection*) if, for any $(\omega, E) \in \Omega \times \mathcal{D}$ with $E \in \mathcal{I}(\omega)$, there is $F \in \mathcal{I}(\omega)$ such that if $\omega' \in F$ then there is $E' \in \mathcal{I}(\omega')$ with $E' \subseteq E$. Put differently, for any information E at a state, there is another information F at the same state (which can possibly be E itself) such that E is always supported as long as F is true.

3. The information correspondence $\mathcal{I}$ is *Euclidean* (or satisfies *Negative Introspection*) if the following holds. If $(\omega, E) \in \Omega \times \mathcal{D}$ satisfies $E^c \cap F \neq \emptyset$ for all $F \in \mathcal{I}(\omega)$, then there is $F' \in \mathcal{I}(\omega)$ such that if $\omega' \in F'$ then $E^c \cap F \neq \emptyset$ for any $F \in \mathcal{I}(\omega')$. In words, at a state ω, if there is no information that supports E, then there exists a piece of information that supports that there is no information that supports E.

   Taking contrapositive, the Euclidean property of $\mathcal{I}$ is also characterized as follows. Let $(\omega, E) \in \Omega \times \mathcal{D}$. If, for any $F \in \mathcal{I}(\omega)$, there are $\omega' \in F$ and $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then there is $F \in \mathcal{I}(\omega)$ with $F \subseteq E$. This means that if the agent does not believe that she does not believe an event E, then she believes E.

4. I introduce a property that turns out to be equivalent to the Euclidean property for reflexive and transitive information correspondences. Namely, $\mathcal{I}$ is *symmetric* if the following obtains. Let $(\omega, E) \in \Omega \times \mathcal{D}$. If, for any $F \in \mathcal{I}(\omega)$, there are $\omega' \in F$ and $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $\omega \in E$. This condition states that if the agent does not believe that she does not believe E then E is true. Equivalently, $\mathcal{I}$ is symmetric if and only if, for any $(\omega, E) \in \Omega \times \mathcal{D}$ with $\omega \in E$, there is $F \in \mathcal{I}(\omega)$ such that $\omega' \in F$ implies $F' \cap E \neq \emptyset$ for all $F' \in \mathcal{I}(\omega')$.

Now, I restate the introspective properties.

Proposition 3. *Let $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ be an information correspondence. For each introspective property, the following are all equivalent.*

1. (a) *$\mathcal{I}$ is reflexive.*
   (b) *$\uparrow \mathcal{I}$ is reflexive.*
   (c) *$B_{\mathcal{I}}(E) \subseteq E$ for all $E \in \mathcal{D}$.*

2. (a) *$\mathcal{I}$ is transitive.*
   (b) *$\uparrow \mathcal{I}$ is transitive.*
   (c) *For any $\omega \in \Omega$, if $E \in \uparrow \mathcal{I}(\omega)$ then $\{\omega' \in \Omega \mid E \in \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega)$.*
   (d) *$B_{\mathcal{I}}(\cdot) \subseteq B_{\mathcal{I}} B_{\mathcal{I}}(\cdot)$.*

3. (a) *$\mathcal{I}$ is Euclidean.*
   (b) *$\uparrow \mathcal{I}$ is Euclidean.*
   (c) *If $E \not\in \uparrow \mathcal{I}(\omega)$ for some $(\omega, E) \in \Omega \times \mathcal{D}$, then $\{\omega' \in \Omega \mid E \not\in \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega)$.*
   (d) *$(\neg B_{\mathcal{I}})(\cdot) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(\cdot)$.*

4. (a) $\mathcal{I}$ *is symmetric.*

   (b) $\uparrow \mathcal{I}$ *is symmetric. That is, let* $(\omega, E) \in \Omega \times \mathcal{D}$*, and suppose that, for any* $F \in \uparrow \mathcal{I}(\omega)$*, there is* $\omega' \in F$ *with* $E \in \uparrow \mathcal{I}(\omega')$*. Then* $\omega \in E$*.*

   (c) $(\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E) \subseteq E$ *for all* $E \in \mathcal{D}$*.*

Similarly to Proposition 2, Proposition 3 establishes the following two points. First, the information correspondence $\mathcal{I}$ satisfies a given introspective property if and only if the information correspondence $\uparrow \mathcal{I}$ satisfies the given property. Put differently, the introspective properties are also preserved under the operation of taking "$\uparrow$." Thus, if information correspondences $\mathcal{I}$ and $\mathcal{I}'$ induce the same beliefs in that $\uparrow \mathcal{I} = \uparrow \mathcal{I}'$, then $\mathcal{I}$ and $\mathcal{I}'$ share the introspective properties.

Second, as briefly discussed above, each introspective property of the information correspondence $\mathcal{I}$ captures the intended introspective property of the belief operator $B_{\mathcal{I}}$. Henceforth, $B_{\mathcal{I}}$ is said to satisfy a given introspective property (e.g., Truth Axiom) if $\mathcal{I}$ satisfies it.

For Positive Introspection and Negative Introspection, Thijsse (1993) proves the equivalence of (2c) and (2d) and that of (3c) and (3d). Proposition 3 (2) and (3) provide the conditions on the primitive $\mathcal{I}$ under which the resulting belief satisfies Positive Introspection and Negative Introspection.

Next, I discuss an implication of the symmetry axiom. If $\mathcal{I}$ is symmetric, then the resulting property on $B_{\mathcal{I}}$ is often referred to as the axiom B in logic (e.g., Chellas, 1980). It can be seen that if $\mathcal{I}$ is reflexive and transitive, then $\mathcal{I}$ is symmetric if and only if it is Euclidean.[19] This argument generalizes the equivalence of the Euclidean and symmetry properties for a reflexive and transitive possibility correspondence to a reflexive and transitive information correspondence.

An interesting implication of symmetry is that if $\mathcal{I}$ is serial, symmetric, and transitive, then it satisfies all the other properties mentioned so far.[20]

Corollary 1. *Let* $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ *be a serial, symmetric, and transitive information correspondence. Then,* $\mathcal{I}$ *satisfies all the properties mentioned in Propositions 2 and 3.*

In Corollary 1, the assumption that the information correspondence $\mathcal{I}$ is serial and transitive, i.e., $B_{\mathcal{I}}$ satisfies Consistency and Positive Introspection, is common in economics and game theory, e.g., when one studies qualitative beliefs of agents. The corollary states that if, in addition, $\mathcal{I}$ satisfies symmetry, then $B_{\mathcal{I}}$ satisfies Truth Axiom. It is also interesting that, then, $B_{\mathcal{I}}$ satisfies Non-empty Countable Conjunction.

I discuss another corollary of Proposition 3. In the information correspondence model, Negative Introspection implies Necessitation and the axiom B (property (4c) in Proposition 3) implies Necessitation. For the first assertion, if $\mathcal{I}$ is Euclidean then it satisfies

[19] One can also recast this statement in terms of the belief operator $B_{\mathcal{I}}$ as follows: if $B_{\mathcal{I}}$ satisfies Monotonicity, Truth Axiom, and Positive Introspection, then (4c) and (3d) in Proposition 3 are equivalent. The proof goes as follows. (4c) implies $(\neg B_{\mathcal{I}})(E) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})B_{\mathcal{I}}(E)$. Since $B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}B_{\mathcal{I}}(E)$ by Positive Introspection and since $B_{\mathcal{I}}$ is monotone, it follows that $B_{\mathcal{I}}(\neg B_{\mathcal{I}})B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E)$. Thus, $(\neg B_{\mathcal{I}})(E) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E)$, as desired. Conversely, Truth Axiom and Negative Introspection yield $E^c \subseteq (\neg B_{\mathcal{I}})(E^{cc}) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E^{cc})$.

[20] In fact, the information correspondence $\mathcal{I}$ satisfies all the properties in Section 3.3 as well.

Necessitation as $\mathcal{I}(\omega) = \emptyset$ leads to a contradiction.[21] Similarly for the second assertion, if $\mathcal{I}$ is symmetric then it satisfies Necessitation as $\mathcal{I}(\omega) = \emptyset$ leads to a contradiction.[22]

The rest of this subsection discusses the assumptions on a surmise function in Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011). To that end, I introduce two additional properties of an information correspondence $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$. First, $\mathcal{I}$ is *strongly transitive* if, for any $(\omega, E) \in \Omega \times \mathcal{D}$ with $E \in \mathcal{I}(\omega)$, if $\omega' \in E$ then there is $E' \in \mathcal{I}(\omega')$ with $E' \subseteq E$. This is the transitivity condition in Doignon and Falmagne (1985, Definition 3.5).

Second, $\mathcal{I}$ satisfies the minimality condition if $E = F$ for any $E, F \in \mathcal{I}(\omega)$ with $E \subseteq F$. The idea behind the minimality condition is that, if $E \in \mathcal{I}(\omega)$ is not minimal in that there is $F \in \mathcal{I}(\omega)$ with $F \subsetneq E$, then E is redundant in $\mathcal{I}(\omega)$ in that $\uparrow \mathcal{I}(\omega) = \uparrow (\mathcal{I}(\omega) \setminus \{E\})$.

Now, a "surmise function" is a mapping $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{P}(\Omega))$ (i.e., $\mathcal{D} = \mathcal{P}(\Omega)$) satisfying (i) reflexivity, (ii) strong transitivity, (iii) Necessitation, and (iv) minimality. The agent whose knowledge is represented by a "surmise function" satisfies Truth Axiom, Positive Introspection, and Necessitation.

Three remarks on strong transitivity are in order. First, for a singleton-valued information correspondence, transitivity and strong transitivity are equivalent. Second, while strong transitivity implies transitivity, the converse may not be true. Example A.1 in Appendix A.4 provides an example of $\mathcal{I}$ which is transitive but not strongly transitive. Third, Example A.2 in Appendix A.4 shows that $\uparrow \mathcal{I}$ may not be strongly transitive even if $\mathcal{I}$ is.[23]

Finally, I remark on Negative Introspection. While a standard partitional (i.e., reflexive, transitive, and Euclidean) possibility correspondence in economics and game theory, by construction, presupposes Negative Introspection, a "surmise function" in the "knowledge space theory" (Doignon and Falmagne, 1985, 1999, 2016; Falmagne and Doignon, 2011) does not presuppose Negative Introspection.[24] As Fagin et al. (2003, Chapter 3) argue

[21]In terms of the belief operator, the proof goes as follows. Suppose to the contrary that there is $\omega \in (\neg B_{\mathcal{I}})(\Omega)$. Then, it follows from Negative Introspection and Monotonicity that $\omega \in B_{\mathcal{I}}(\neg B_{\mathcal{I}})(\Omega) \subseteq B_{\mathcal{I}}(\Omega)$, a contradiction.

[22]In terms of the belief operator, the proof goes as follows. Suppose to the contrary that there is $\omega \in (\neg B_{\mathcal{I}})(\Omega)$. Then, it follows from Monotonicity and the axiom B (property (4c) in Proposition 3) that $\omega \in (\neg B_{\mathcal{I}})(\Omega) \subseteq (\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(\emptyset) \subseteq \emptyset$, a contradiction.

[23]Fagin and Halpern (1987) consider the following form of transitivity that implies strong transitivity: if $\omega' \in E \in \mathcal{I}(\omega)$ then $E \in \mathcal{I}(\omega')$. Thijsse (1993, Example 4) provides an example where this stronger form of transitivity is not necessary for characterizing Positive Introspection. While it can be seen that the monotone information correspondence $\mathcal{I} = \uparrow \mathcal{I}$ in his example satisfies transitivity but violates strong transitivity, for future use, I provide examples of reflexive information correspondences in Examples A.1 and A.2 in Appendix A.4. On a related point, Fagin and Halpern (1987) also consider the following stronger Euclidean property: $\omega' \in E \in \mathcal{I}(\omega)$ implies $\mathcal{I}(\omega') \subseteq \mathcal{I}(\omega)$. Thijsse (1993, Example 5) demonstrates that this stronger Euclidean property does not necessarily characterize Negative Introspection. One of the technical contributions of this paper is to provide the technical formulations of transitivity and Euclideanness as demonstrated in Proposition 3.

[24]First, while partitional knowledge models are prevalent in economics and game theory, non-partitional (reflexive and transitive) possibility correspondence models have also been studied. See, for example, Dekel and Gul (1997), Geanakoplos (2021), Morris (1996), and Shin (1993) for foundations for such non-partitional information processing and characterizations of solution concepts in games. Also, an agent whose knowledge satisfies Negative Introspection cannot be unaware of any event in the sense that if

that "there is no one "true" notion of knowledge" and that "the appropriate notion depends on the application," I believe that the aforementioned difference between economics and mathematical psychology comes from the different contexts in which knowledge is analyzed in these distinct fields.

3.3 "Belief-in" Properties

Finally, this subsection introduces and characterizes "Belief-in" properties: Belief in Correct Belief, Belief in Consistency, Belief in Perfect Reasoning, and Belief in Non-empty Countable Conjunction. Belief in Correct Belief states that the agent believes that her belief satisfies Truth Axiom even if her belief may violate it. Belief in Consistency states that the agent believes that her belief satisfies Consistency even if her belief may violate it. Belief in Perfect Reasoning states that the agent believes that her belief satisfies "modus ponens" (i.e., if she believes E and $(\neg E) \cup F$, then she believes F) even if her belief may violate it. Belief in Non-empty Countable Conjunction states that the agent believes that her belief satisfies Non-empty Countable Conjunction even if her belief may violate it.

Such properties are interesting in its own when it comes to "boundedly rational" agents, as Fagin and Halpern (1987) extensively study some of such properties. Also, such properties play some roles in epistemic characterizations of solution concepts of games and other applications such as no-trade results. This subsection studies the following four such "Belief-in" properties.

1. An information correspondence $\mathcal{I}$ is *secondary reflexive* (or satisfies *Belief in Correct Belief*) if, for any $(\omega, E) \in \Omega \times \mathcal{D}$, there is $F \in \mathcal{I}(\omega)$ such that if $\omega' \in F$ and there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $\omega' \in E$. Roughly, there is always information indicating that if the agent believes E then E is true.

2. The information correspondence $\mathcal{I}$ is *secondary serial* (or satisfies *Belief in Consistency*) if, for any $(\omega, E) \in \Omega \times \mathcal{D}$, there is $F \in \mathcal{I}(\omega)$ such that if $\omega' \in F$ and there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $H \cap E \neq \emptyset$ for all $H \in \mathcal{I}(\omega')$. Roughly, there is always information implying that if the agent believes E then she does not believe the negation E^c.

3. The information correspondence $\mathcal{I}$ satisfies *Belief in Perfect Reasoning* if, for any $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$, there is $G \in \mathcal{I}(\omega)$ with the following property: if $\omega' \in G$, $E' \subseteq E$ for some $E' \in \mathcal{I}(\omega')$, and if $F' \subseteq (\neg E) \cup F$ for some $F' \in \mathcal{I}(\omega')$, then there is $G' \in \mathcal{I}(\omega')$ such that $G' \subseteq F$. Roughly, there is always information implying that if the agent believes E and $(\neg E) \cup F$ then she believes F.

4. The information correspondence $\mathcal{I}$ satisfies *Belief in Non-empty Countable Conjunction* if, for any $\omega \in \Omega$ and for any non-empty countable collection $\{F_n\}_{n \in \mathbb{N}}$ of events, there is $G \in \mathcal{I}(\omega)$ such that if $\omega' \in G$ and $\{F_n\}_{n \in \mathbb{N}} \subseteq \mathcal{I}(\omega')$, then there is $G' \in \mathcal{I}(\omega')$

she does not know an event then she knows that she does not know it. See footnote 5 for unawareness. Second, Negative Introspection has also been investigated in other fields such as logic and philosophy. For example, Hintikka (1962, Chapter 3.8) rejects it.

with $G' \subseteq \bigcap_{n\in\mathbb{N}} F_n$. Roughly, there is always information that implies that her belief satisfies Non-empty Countable Conjunction.

The following proposition characterizes the Belief-in properties.

Proposition 4. *Let $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ be an information correspondence. For each introspective property, the following are all equivalent.*

1. (a) *$\mathcal{I}$ is secondary reflexive.*
   (b) *$\uparrow \mathcal{I}$ is secondary reflexive. That is, for any $(\omega, E) \in \Omega \times \mathcal{D}$, there is $F \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F$, then $E \in \uparrow \mathcal{I}(\omega')$ implies $\omega' \in E$.*
   (c) *$\Omega = B_{\mathcal{I}}((\neg B_{\mathcal{I}})(E) \cup E)$ for any $E \in \mathcal{D}$.*

2. (a) *$\mathcal{I}$ is secondary serial.*
   (b) *$\uparrow \mathcal{I}$ is secondary serial. That is, for any $(\omega, E) \in \Omega \times \mathcal{D}$, there is $F \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F$ and $E \in \uparrow \mathcal{I}(\omega')$, then $E^c \notin \uparrow \mathcal{I}(\omega')$.*
   (c) *$\Omega = B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)))$ for any $E \in \mathcal{D}$.*

3. (a) *$\mathcal{I}$ satisfies Belief in Perfect Reasoning.*
   (b) *$\uparrow \mathcal{I}$ satisfies Belief in Perfect Reasoning. That is, for any $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$, there is $G \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$, $E \in \uparrow \mathcal{I}(\omega')$, and if $(\neg E) \cup F \in \uparrow \mathcal{I}(\omega')$, then $F \in \uparrow \mathcal{I}(\omega')$.*
   (c) *$B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F)) = \Omega$ for any $E, F \in \mathcal{D}$.*

4. *Belief in Perfect Reasoning is equivalent to Belief in Non-empty Finite Conjunction. Generally, the following are equivalent.*
   (a) *$\mathcal{I}$ satisfies Belief in Non-empty Countable Conjunction.*
   (b) *$\uparrow \mathcal{I}$ satisfies Belief in Non-empty Countable Conjunction. That is, for any $\omega \in \Omega$ and $\{F_n\}_{n\in\mathbb{N}} \in \mathcal{P}(\mathcal{D})$, there is $G \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$ and $\{F_n\}_{n\in\mathbb{N}} \subseteq \uparrow \mathcal{I}(\omega')$, then $\bigcap_{n\in\mathbb{N}} F_n \in \uparrow \mathcal{I}(\omega')$.*
   (c) *$B_{\mathcal{I}}(\neg(\bigcap_{n\in\mathbb{N}} B_{\mathcal{I}}(F_n)) \cup B_{\mathcal{I}}(\bigcap_{n\in\mathbb{N}} F_n)) = \Omega$ for any $\{F_n\}_{n\in\mathbb{N}} \in \mathcal{P}(\mathcal{D})$.*

Similarly to Propositions 2 and 3, Proposition 4 establishes the following two points. First, the information correspondence $\mathcal{I}$ satisfies a given "Belief-in" property if and only if the information correspondence $\uparrow \mathcal{I}$ satisfies the given property. Put differently, the "Belief-in" properties are also preserved under the operation of taking "$\uparrow$." Thus, if information correspondences $\mathcal{I}$ and $\mathcal{I}'$ induce the same beliefs in that $\uparrow \mathcal{I} = \uparrow \mathcal{I}'$, then $\mathcal{I}$ and $\mathcal{I}'$ share the introspective properties.

Second, each "Belief-in" property of the information correspondence $\mathcal{I}$ captures the intended property of the belief operator $B_{\mathcal{I}}$. Henceforth, $B_{\mathcal{I}}$ is said to satisfy a given introspective property (e.g., Belief in Correct Belief) if $\mathcal{I}$ satisfies it.

Belief in Correct Belief means that the agent always believes that either she does not believe an event E or otherwise E entails. Likewise, Belief in Consistency means that the agent always believes that her belief is consistent. Moreover, Belief in Perfect Reasoning states that the agent always believes that if she believes E and E implies F then she believes F.[25] Belief in Non-empty Countable Conjunction states that the agent always believes that if she believes each F_n then she believes its conjunction $\bigcap_{n \in \mathbb{N}} F_n$.

Under Necessitation, Belief in Correct Belief is a weakening of Truth Axiom. Belief in Correct Belief, especially its inter-subjective version $\Omega = B_{\mathcal{I}_j}((\neg B_{\mathcal{I}_i})(E) \cup E)$ (where $\mathcal{I}_i$ and $\mathcal{I}_j$ are information correspondences of agents i and j)[26] plays an important role in the existence of common prior (and consequently Aumann (1976)'s Agreement theorem) and an epistemic characterization of backward induction as an implication of common knowledge of rationality a la Aumann (1995) in a possibility correspondence model: see, for example, Bonanno and Nehring (1998a,b) and Samet (2013).[27]

I remark that Negative Introspection and Monotonicity imply Belief in Correct Belief (or, if $\mathcal{I}$ is Euclidean then it is secondary reflexive). For any $(\omega, E) \in \Omega \times \mathcal{D}$, Negative Introspection implies either $E \in \uparrow \mathcal{I}(\omega)$ or $(\neg B_{\mathcal{I}})(E) \in \uparrow \mathcal{I}(\omega)$. In either case, $(\neg B_{\mathcal{I}})(E) \cup E \in \uparrow \mathcal{I}(\omega)$.

It can be seen that if $\mathcal{I}$ satisfies any of Belief in Correct Belief, Belief in Consistency, Belief in Perfect Reasoning, or Belief in Non-empty Countable Conjunction, then it satisfies Necessitation. In other words, the failure of Necessitation implies that of each of these properties.[28] In fact, the information correspondence of Example 1 in Section 2.2 fails Necessitation. While it satisfies Truth Axiom, it violates all the other introspective and Belief-in properties.

Next, I show that Belief in Non-empty Countable Conjunction is at least as strong as Belief in Perfect Reasoning by showing that Belief in Perfect Reasoning is equivalent to Belief in Non-empty Finite Conjunction that is formulated below.

Proposition 5. *Belief in Non-empty Finite Conjunction: for any $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$, there is $G \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$ and $E, F \in \uparrow \mathcal{I}(\omega')$ then $E \cap F \in \uparrow \mathcal{I}(\omega')$.*

Next, observe that under Necessitation, Consistency implies Belief in Consistency. Likewise, under Necessitation, Truth Axiom and Non-empty Countable Conjunction imply Belief in Correct Belief and Belief in Non-empty Countable Conjunction, respectively. I provide an example in Example A.3 in Appendix A.5 that, while $B_{\mathcal{I}}$ violates Consistency, Truth Axiom, and Non-empty Countable Conjunction, the agent can still believe in these

[25] Fagin and Halpern (1987) study the notion of a narrow-minded agent. Call the agent narrow-minded if, for any $\omega \in \Omega$, there is $E \in \mathcal{I}(\omega)$ such that $\omega' \in E$ implies $\mathcal{I}(\omega') = \{E\}$. As in Fagin and Halpern (1987), this axiom implies Belief in Consistency and Belief in Perfect Reasoning. In contrast, parts (2) and (3) of Proposition 2 fully characterize these two properties.

[26] One can analogously characterize this property: For any (ω, E), there is $F \in \uparrow \mathcal{I}_j(\omega)$ such that if $\omega' \in F$ then $E \in \uparrow \mathcal{I}_i(\omega')$ implies $\omega' \in E$.

[27] Specifically, Samet (2013) shows that, in the framework of Aumann (1995), common belief in rationality (i.e., Truth Axiom (i.e., reflexivity) is dropped from common knowledge of rationality) still implies backward induction outcomes when agents' beliefs satisfy an interpersonal version of secondary reflexivity $\Omega = B_{\mathcal{I}_j}((\neg B_{\mathcal{I}_i})(E) \cup E)$. Also, Bonanno and Nehring (1998a) study a similar property and characterize it as the absence of unbounded gains from betting. Bonanno and Nehring (1998b) study another similar condition to study correlated equilibria under incomplete information.

[28] Recall also that if $\mathcal{I}$ is Euclidean or symmetric then it satisfies Necessitation.

properties. Thus, not only can the information correspondence capture the agent whose belief fails Non-empty Countable Conjunction, but also it can capture the very same agent who believes that her own belief satisfies it.

Finally, I remark on the introspective properties of $\mathcal{I}$ when $\mathcal{I}$ is singleton-valued. Since the agent's belief satisfies Monotonicity, Necessitation, and Non-empty Countable Conjunction, $\mathcal{I}(\cdot) = \{P(\cdot)\}$ satisfies Belief in Perfect Reasoning and Belief in Non-empty Countable Conjunction. Now, the introspective properties reduce to the standard definition on the possibility correspondence P. First, $\mathcal{I}$ is reflexive iff $\omega \in P(\omega)$ (for all $\omega \in \Omega$). Second, $\mathcal{I}$ is secondary reflexive iff $\omega' \in P(\omega)$ implies $\omega' \in P(\omega')$. Third, $\mathcal{I}$ is secondary serial iff $\omega' \in P(\omega)$ implies $P(\omega') \neq \emptyset$. Fourth, $\mathcal{I}$ is transitive iff $P(\omega') \subseteq P(\omega)$ for any $\omega' \in P(\omega)$. Fifth, $\mathcal{I}$ is Euclidean iff $\omega' \in P(\omega)$ implies $E^c \cap P(\omega') \neq \emptyset$ for any $E \in \mathcal{D}$ with $E^c \cap P(\omega) \neq \emptyset$. It can be seen that $\mathcal{I}$ is Euclidean iff $\omega' \in P(\omega)$ implies $P(\omega) \subseteq P(\omega')$. Sixth, $\mathcal{I}$ is symmetric if $\omega \in P(\omega')$ implies $\omega' \in P(\omega)$. On a related point, as to logical properties, $\mathcal{I}$ is serial iff $P(\cdot) \neq \emptyset$. Also, $\mathcal{I}$ satisfies No-Contradiction iff $P(\cdot) \neq \emptyset$. No-Contradiction and Consistency are equivalent with each other because $\mathcal{I}(\cdot) = \{P(\cdot)\}$ satisfies Non-empty Countable Conjunction.

4 Equivalence among Knowledge-Belief Representations

The previous section has shown that $B_{\mathcal{I}}$ inherits the logical, introspective, and "Belief-in" properties of beliefs imposed on a given $\mathcal{I}$. Section 4.1 completes the equivalence between an information correspondence and a monotone belief operator. This implies that the previous results involving (monotone) belief operators can be replicated under the framework of information correspondences. Section 4.2 formally studies a "surmise function" as a reflexive and transitive information correspondence.

I begin with introducing a particular type of events known as self-evident events in the literature. Letting B be a given monotone belief operator, an event $E \in \mathcal{D}$ is *self-evident* if $E \subseteq B(E)$, i.e., the agent believes E whenever E is true. Denote the collection of self-evident events by $\mathcal{J}_B := \{E \in \mathcal{D} \mid E \subseteq B(E)\}$. If an information correspondence $\mathcal{I}$ is given, then an event E is self-evident (i.e., $E \subseteq B_{\mathcal{I}}(E)$) if and only if for any $\omega \in E$, there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq E$.[29] In Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011), a self-evident event turns out to be what they call a "knowledge state." The "knowledge state" is interpreted as a set of "questions" that an agent is capable of solving. The collection of knowledge states is referred to as the

[29] Using self-evidence, one can introduce common belief among a set I of agents, where I is at most countable. For each agent i, let $\mathcal{I}_i$ be her information correspondence. Following the characterization of Monderer and Samet (1989), an event E is *common belief* among agents I at a state ω if there is an event F self-evident to every $i \in I$ such that $\omega \in F \subseteq \bigcap_{i \in I} B_{\mathcal{I}_i}(E)$. If E is common belief, then everybody believes E, everybody believes that everybody believes E, and so on *ad infinitum*. As discussed in the Introduction, Heifetz (1996, 1999) and Lismont and Mongin (1994a,b) study common belief using monotone neighborhood systems. Fukuda (2020) studies various properties of common belief in a framework in which no properties of individual beliefs are assumed.

"knowledge structure."[30]

4.1 Information Correspondences and Belief Operators

I define an information correspondence $\mathcal{I}$ from a given monotone belief operator B in such a way that the induced belief operator $B_{\mathcal{I}}$ coincides with the original operator B. Formally, for a given monotone belief operator $B : \mathcal{D} \to \mathcal{D}$, an information correspondence $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ is a *generator* of B (or $\mathcal{I}$ *induces* B) if $B = B_{\mathcal{I}}$.

Generally, a given monotone belief operator B has multiple generators. Information correspondences $\mathcal{I}$ and $\mathcal{I}'$ satisfying $\uparrow \mathcal{I} = \uparrow \mathcal{I}'$ induce the same belief operator $B_{\mathcal{I}} = B_{\mathcal{I}'}$. As the next proposition shows, the simplest way to find a generator of B is to consider the information correspondence $\mathcal{I}_B : \Omega \to \mathcal{P}(\mathcal{D})$ defined by

$$\mathcal{I}_B(\omega) := \{E \in \mathcal{D} \mid \omega \in B(E)\} \text{ for each } \omega \in \Omega. \tag{5}$$

Henceforth, I define the information correspondence $\mathcal{I}_B$ induced from a (monotone) belief operator B through Equation (5). Since B satisfies Monotonicity, $\mathcal{I}_B = \uparrow \mathcal{I}_B$ and consequently $B = B_{\mathcal{I}_B}$. Moreover, if B has multiple generators, then any generator $\mathcal{I}$ is included in $\mathcal{I}_B$ in the sense that $\mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}(\cdot) = \uparrow \mathcal{I}_B(\cdot) = \mathcal{I}_B(\cdot)$. If the given monotone belief operator B satisfies Truth Axiom and Positive Introspection, then the proposition demonstrates that one can restrict attention to the self-evident events.

Proposition 6. *Let $(\Omega, \mathcal{D})$ be a measurable space.*

1. *If $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ is an information correspondence, then $B_{\mathcal{I}}$ inherits the properties of beliefs imposed on $\mathcal{I}$ and $\uparrow \mathcal{I}(\cdot) = \uparrow \mathcal{I}_{B_{\mathcal{I}}}(\cdot)$. Conversely, if B is a monotone belief operator, then $\mathcal{I}_B$ is a generator of B, i.e., $B = B_{\mathcal{I}_B}$. Any generator $\mathcal{I}$ of B satisfies the properties of beliefs imposed on B and $\mathcal{I}(\cdot) \subseteq \mathcal{I}_B(\cdot)$.*

2. *Let $B : \mathcal{D} \to \mathcal{D}$ satisfy Monotonicity, Truth Axiom, and Positive Introspection. Define $\mathcal{I}_{\mathcal{J}_B} : \Omega \to \mathcal{P}(\mathcal{D})$ by $\mathcal{I}_{\mathcal{J}_B}(\omega) := \{E \in \mathcal{J}_B \mid \omega \in B(E)\}$ for each $\omega \in \Omega$. Then, $\mathcal{I}_{\mathcal{J}_B}$ is a reflexive and (strongly) transitive information correspondence that generates B.*

Multiplicity of generators can be used to compare agents' beliefs. For agents i and j, let $\mathcal{I}_i$ and $\mathcal{I}_j$ be generators of B_i and B_j, respectively. Then, $B_i(\cdot) \subseteq B_j(\cdot)$ iff $\uparrow \mathcal{I}_i(\cdot) \subseteq \uparrow \mathcal{I}_j(\cdot)$. This is also equivalent to: for any $\omega \in \Omega$ and $E \in \mathcal{I}_i(\omega)$, there is $F \in \mathcal{I}_j(\omega)$ such that $F \subseteq E$. In mathematical psychology, Doignon and Falmagne (1999, Definition 3.16) call it an "attribution order" (see also Doignon and Falmagne, 1985, Definition 3.3).

Moreover, $B_i(\cdot) \subseteq B_j(\cdot)$ implies $\uparrow \mathcal{I}_{B_i}(\cdot) \subseteq \uparrow \mathcal{I}_{B_j}(\cdot)$. Also, $\uparrow \mathcal{I}_i(\cdot) \subseteq \uparrow \mathcal{I}_j(\cdot)$ implies $B_{\mathcal{I}_i}(\cdot) \subseteq B_{\mathcal{I}_j}(\cdot)$ through Equation (2). Hence, $\uparrow \mathcal{I}_i(\cdot) \subseteq \uparrow \mathcal{I}_{B_j}(\cdot)$ iff $B_{\mathcal{I}_i}(\cdot) \subseteq B_j(\cdot)$. Following Doignon and Falmagne (1985, 1999) and Falmagne and Doignon (2011), which compare different knowledge-belief representations by a Galois connection in order theory, Remark B.1 in Appendix B formalizes this argument as a Galois connection.

[30]Fukuda (2019) shows the one-to-one correspondence between a belief (knowledge) operator satisfying Truth Axiom, Positive Introspection, and Monotonicity and its self-evident collection.

Proposition 6 (1) could also hold for probabilistic beliefs. Samet (2000) provides the conditions on p-belief operators to induce a type mapping. Meier (2006) and Zhou (2010) extend the conditions to finitely-additive beliefs.

4.2 Reflexive and Transitive Information Correspondences

I study a "surmise function" as a reflexive and transitive information correspondence because the "surmise function" $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{P}(\Omega))$ is regarded as an information correspondence on $(\Omega, \mathcal{P}(\Omega))$ satisfying (i) reflexivity, (ii) strong transitivity, (iii) Necessitation, and (iv) minimality. Doignon and Falmagne (1999, Theorems 3.10 and 6.25) show a one-to-one correspondence between a "surmise function" and a "granular" "knowledge structure."[31] Here, I show a general one-to-one correspondence between a reflexive and (strongly) transitive information correspondence and the collection of self-evident events on a state space by dropping the minimality condition. Observe also that, by Proposition 3, there is a one-to-one correspondence between belief (knowledge) operators satisfying Monotonicity, Truth Axiom, and Positive Introspection and reflexive and transitive information correspondences.

Proposition 7. *Let $(\Omega, \mathcal{D})$ be a measurable space.*

1. (a) *Let $\mathcal{J} \in \mathcal{P}(\mathcal{D})$ satisfy*

$$\{\omega \in \Omega \mid \text{there is } F \in \mathcal{J} \text{ with } \omega \in F \subseteq E\} \in \mathcal{J} \text{ for each } E \in \mathcal{D}. \tag{6}$$

The following mapping $\mathcal{I}_{\mathcal{J}} : \Omega \to \mathcal{P}(\mathcal{D})$ is a reflexive and (strongly) transitive information correspondence:

$$\mathcal{I}_{\mathcal{J}}(\omega) := \{E \in \mathcal{J} \mid \omega \in E\} \text{ for each } \omega \in \Omega. \tag{7}$$

(b) *If $\Omega \in \mathcal{J}$, then $\mathcal{I}_{\mathcal{J}}(\cdot) \neq \emptyset$.*

2. (a) *Conversely, if $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ is a reflexive and transitive information correspondence, then $\mathcal{J}_{\mathcal{I}} \in \mathcal{P}(\mathcal{D})$ defined below satisfies Condition (6):*

$$\mathcal{J}_{\mathcal{I}} := \{E \in \mathcal{D} \mid \text{if } \omega \in E \text{ then there is } F \in \mathcal{I}(\omega) \text{ with } F \subseteq E\}. \tag{8}$$

(b) *If $\mathcal{I}(\cdot) \neq \emptyset$, then $\Omega \in \mathcal{J}_{\mathcal{I}}$.*

3. (a) *Moreover, starting from $\mathcal{J}$, $\mathcal{J} = \mathcal{J}_{\mathcal{I}_{\mathcal{J}}}$.*

(b) *Starting from $\mathcal{I}$, $\uparrow \mathcal{I} = \uparrow \mathcal{I}_{\mathcal{J}_{\mathcal{I}}}$.*

[31] A "knowledge structure" $\mathcal{J}$ is *granular* (Doignon and Falmagne, 1999, Definition 1.35) if, for any $(\omega, E) \in \Omega \times \mathcal{J}$ with $\omega \in E$, there is a minimal $F \in \mathcal{J}$ with $\omega \in F \subseteq E$. Doignon and Falmagne (1985, Theorem 3.7) establish the equivalence between a "surmise function" and a "knowledge structure" when Ω is finite.

In mathematical psychology, Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011) characterize the collection $\mathcal{J}$ of "knowledge states" as a collection of events which are closed under arbitrary union. In contrast, Proposition 7 requires $\mathcal{J}$ to satisfy Condition (6). Now, the closure under arbitrary union turns out to be equivalent to Condition (6) if $\mathcal{D}$ is closed under arbitrary union for the following two observations: (i) if $\mathcal{D}$ is closed under arbitrary union, the set of states in Condition (6) reduces to $\bigcup\{F \in \mathcal{J} \mid F \subseteq E\}$; and (ii) $E = \bigcup\{F \in \mathcal{J} \mid F \subseteq E\}$ for each $E \in \mathcal{J}$. Generally, Condition (6) is equivalent to the existence of a maximal event in $\mathcal{J}$ that is included in a given event $E \in \mathcal{D}$. In economics, Fukuda (2019), Salonen (2009b), and Samet (2010) study this maximality property to obtain set-algebraic representations of knowledge.

In Proposition 7, since $\mathcal{I}_{\mathcal{J}}$ defined by Equation (7) is strongly transitive and since strong transitivity implies transitivity, the proposition also establishes the equivalence between a collection of self-evident events and a reflexive and strongly transitive information correspondence. Example A.4 in Appendix A.7, however, shows that $\uparrow \mathcal{I}_{\mathcal{J}}$ may not necessarily be strongly transitive even if $\mathcal{I}_{\mathcal{J}}$ is. Thus, strong transitivity is not necessarily preserved under the operation of taking "$\uparrow$." This also means that the reflexive and transitive information correspondence $\mathcal{I} :=\uparrow \mathcal{I}_{\mathcal{J}}$ (i.e., $\mathcal{I}(\omega) := \{E \in \mathcal{D} \mid \text{there is } F \in \mathcal{J} \text{ with } \omega \in F \subseteq E\}$) can also establish the part of Proposition 7 in place of Equation (7).

Doignon and Falmagne (1999, Theorem 6.25) (and Doignon and Falmagne, 1985, Theorem 3.7) establish the correspondence between a "surmise function" and a "knowledge structure" (i.e., a reflexive and transitive information correspondence and a collection of self-evident events in my context) in terms of a Galois connection. This amounts to proving: $\uparrow \mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}_{\mathcal{J}}(\cdot)$ iff $\mathcal{J}_{\mathcal{I}} \subseteq \mathcal{J}$ for each $\mathcal{I}$ and $\mathcal{J}$. I demonstrate in Remark B.2 in Appendix B that the pair of mappings defined through Equations (7) and (8) forms a Galois connection.

Next, consider a connection between Propositions 6 (2) and 7. If a given monotone belief operator B satisfies Truth Axiom and Positive Introspection, then the information correspondence $\mathcal{I}_{\mathcal{J}_B} = \{E \in \mathcal{J}_B \mid \omega \in B(E)\}$ in Proposition 6 (2) is indeed equal to $\mathcal{I}_{\mathcal{J}_B} = \{E \in \mathcal{J}_B \mid \omega \in E\}$ in Proposition 7. Note that it can be seen that $\mathcal{J}_B$ satisfies Condition (6).

I remark on the further connections with the "knowledge space theory" of Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011). First, let an information correspondence $\mathcal{I}$ be $\mathcal{I}(\cdot) = \{P(\cdot)\}$ (singleton-valued). If $\omega' \in P(\omega)$, then the agent considers ω' *possible* at state ω. Thus, P induces a binary relation also known as an accessibility (or possibility) relation in computer science, logic, and philosophy (e.g., Chellas, 1980; Fagin et al., 2003). Suppose further that P is reflexive and transitive. In mathematical psychology, if P is reflexive and transitive, then the reflexive and transitive binary relation induced by P turns out to be a "surmise (or precedence) relation" (Doignon and Falmagne, 1985, 1999, 2016; Falmagne and Doignon, 2011). In their context, if $\omega' \in P(\omega)$, then it can be surmised from a correct response to "question" ω that a correct response to "question" ω' is given.

Second, suppose that $\mathcal{I}$ is reflexive and transitive. It turns out that the possibility operator $L_{\mathcal{I}} : \mathcal{D} \to \mathcal{D}$ defined in Equation (3) satisfies the following three properties: (i) $E \subseteq F$ implies $L_{\mathcal{I}}(E) \subseteq L_{\mathcal{I}}(F)$; (ii) $E \subseteq L_{\mathcal{I}}(E)$; and (iii) $L_{\mathcal{I}}L_{\mathcal{I}}(\cdot) \subseteq L_{\mathcal{I}}(\cdot)$. The operator

$L_{\mathcal{I}}$ is related to the notion of a closure operator, and a tuple $(\Omega, \{E \in \mathcal{D} \mid L_{\mathcal{I}}(E) \subseteq E\}) = (\Omega, \{E \in \mathcal{D} \mid E^c \in \mathcal{J}_{B_{\mathcal{I}}}\})$ is related to the notion of a closure space (Doignon and Falmagne, 1985, 1999; Falmagne and Doignon, 2011).

5 An Economic Application: Deriving Qualitative Belief from Preferences

This section provides an economic application of information correspondences. As in Morris (1996), this section derives an agent's qualitative belief (or knowledge) from her underlying preferences. If an underlying state space is rich (i.e., uncountable), her qualitative belief may not satisfy the conjunction property for an arbitrary number of events. While Morris (1996)'s analysis is restricted to finite state spaces, this section shows that the agent's qualitative belief can be derived from her preferences on an arbitrary state space using an information correspondence.

Throughout the section, fix a compact space Ω, and consider a state space $(\Omega, \mathcal{D})$, where $\mathcal{D}$ is the Borel σ-algebra. For simplicity, assume that the set of consequences is $\mathbb{R}$. An *act* is a continuous mapping $x : \Omega \to \mathbb{R}$. Denote by $\mathcal{A}$ the set of acts. For any $x \in \mathcal{A}$, $\omega \in \Omega$, and $E \in \mathcal{D}$, denote $x_\omega := x(\omega)$ and $x_E := (x(\omega))_{\omega \in E}$. Denote by $z = (x_E, y_{E^c})$ an act that satisfies $z_\omega = x_\omega$ for all $\omega \in E$ and $z_{\omega'} = y_{\omega'}$ for all $\omega' \in E^c$.

Consider an agent who has preference (i.e., complete and transitive) relations $(\succcurlyeq_\omega)_{\omega \in \Omega}$ on acts $\mathcal{A}$: $x \succcurlyeq_\omega y$ means that, at state ω, the act x is at least as good as y. The preference relations $(\succcurlyeq_\omega)_{\omega \in \Omega}$ *induce* a (qualitative) belief operator $B_{\succcurlyeq} : \mathcal{D} \to \mathcal{D}$ if

$$B_{\succcurlyeq}(E) := \{\omega \in \Omega \mid (x_E, y_{E^c}) \succcurlyeq_\omega (x_E, z_{E^c}) \text{ for all } x, y, z \in \mathcal{A}\} \in \mathcal{D}.$$

As in Morris (1996, Theorem 2), $B_{\succcurlyeq}$ can be shown to satisfy: (i) Monotonicity, (ii) Necessitation, and (iii) Non-empty Finite Conjunction (see footnote 17 for the definition). By Monotonicity, $B_{\succcurlyeq}$ can be represented by an information correspondence.

Now, I show that the belief operator $B_{\succcurlyeq}$ may not necessarily satisfy the Kripke property, i.e., it may not be induced by a possibility correspondence. To see this point, the preference relations $(\succcurlyeq_\omega)_{\omega \in \Omega}$ have a (state-independent) *expected utility representation* if there exists a strictly increasing and continuous utility function $u : \mathbb{R} \to \mathbb{R}$ and a "type" mapping $t : \Omega \to \Delta(\Omega)$ (for ease of notation, denote $t(\omega, E) = t(\omega)(E)$ as in Section 2.2) satisfying $\{\omega \in \Omega \mid t(\omega, E) \geq p\} \in \mathcal{D}$ such that[32]

$$x \succcurlyeq_\omega y \text{ iff } \int_\Omega (u \circ x)(\omega') t(\omega, d\omega') \geq \int_\Omega (u \circ y)(\omega') t(\omega, d\omega').$$

If this is the case, an event E is believed if and only if E is assigned probability one:

$$B_{\succcurlyeq}(E) = \{\omega \in \Omega \mid t(\omega, E) = 1\}.$$

[32]Conceptually, the requirement on t guarantees that the agent's p-beliefs be well-defined. Technically, the requirement is innocuous. For example, let Ω be a compact Polish space. Endowing the weak-$*$ topology with $\Delta(\Omega)$, any continuous mapping $t : \Omega \to \Delta(\Omega)$ satisfies the above-mentioned property.

As a simple example, let $\Omega = [0, 1]$, and suppose that preference relations $(\succcurlyeq_\omega)_{\omega \in \Omega}$ are represented by $u(\omega) = \omega$ and $t(\omega, \cdot) = \mu$ where μ is the Lebesgue measure on Ω. Then, as seen in Section 2.2, $B_{\succcurlyeq}$ is not induced by a possibility correspondence (but an information correspondence) because the qualitative belief operator $B_{\succcurlyeq}$ violates the Kripke property.

6 Conclusion

This paper developed an information correspondence that represents an agent's beliefs about underlying states of the world. It associates, with each state, a set of possibly multiple information sets at that state. Conceptually, it can capture beliefs that may fail the conjunction or necessitation properties. It can also capture both qualitative and quantitative beliefs (e.g., knowledge and probability-one belief) in a unified manner. If it has a unique information set at each state, then it reduces to a possibility correspondence. The paper characterized the logical and introspective properties of beliefs.

This paper connected seemingly different knowledge-belief representations by demonstrating that a "surmise function" in mathematical psychology (Doignon and Falmagne, 1985, 1999, 2016; Falmagne and Doignon, 2011) can be seen as a particular information correspondence. This paper thus provided various logical and introspective properties of a "surmise function." I hope that this paper spurs further ideas in both economics and mathematical psychology as discussed in the introduction.

One interesting direction of future study is to explore interactions of knowledge and probabilistic beliefs, especially belief update on available information. In a standard possibility correspondence model, an agent's type at a state is usually the posterior probability measure conditional on the information set at that state. On a related point, one can study various properties of non-additive beliefs (e.g., in the sense of Dempster, 1967; Shafer, 1976) in the framework of this paper. Still another direction is to develop an information correspondence on a generalized state space of the unawareness structure developed by Heifetz, Meier, and Schipper (2006, 2013). In their generalized state space model, a state space consists of multiple subspaces, and a possibility correspondence on such generalized state space can represent an agent's unawareness satisfying certain properties. The information correspondence approach can strip away certain properties of possibility correspondences.

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A Appendix

A.1 List of notations and terminologies

Table A.1 provides a list of notations and terminologies: for each key notation of the paper, it provides the corresponding terminologies in economics and mathematical psychology side-by-side.

Notation	Terminology in the Paper	Corresponding Terminology in the Knowledge Space Theory
Ω	State space (Set of states of the world)	Domain (of the body of knowledge)
$\omega \in \Omega$	State	Item or question
$E \in \mathcal{D}$	Event	Set of items (or questions)
$\mathcal{J}$	Collection of self-evident events	Knowledge structure
$E \in \mathcal{J}$	Self-evident event	Knowledge state
$\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$	Information correspondence	Surmise function $(\Omega, \mathcal{I})$: Surmise system
$E \in \mathcal{I}(\omega)$	Information set at ω	Clause (background or foundation) for ω
$\omega' \in P(\omega)$ $(\mathcal{I}(\cdot) = \{P(\cdot)\})$	Possibility (or accessibility) relation	Surmise relation

Table A.1: A list of notations and terminologies. The first column lists key notations of this paper. The second does their terminologies. The third does the corresponding terminologies in Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011).

A.2 Section 2.1

Proof of Proposition 1. Let $\mathcal{I}$ be an information correspondence satisfying the Kripke property. Fix $\omega \in \Omega$, and let $P(\omega)$ be the minimum element of $\mathcal{I}(\omega)$. For each $E \in \mathcal{D}$, $\omega \in B_{\mathcal{I}}(E)$ iff $P(\omega) \subseteq E$. Conversely, suppose that, for each $\omega \in \Omega$, there is $P(\omega) \in \mathcal{I}(\omega)$ such that $\omega \in B_{\mathcal{I}}(E)$ iff $P(\omega) \subseteq E$. Fix $\omega \in \Omega$. If $E \in \mathcal{I}(\omega)$, then $\omega \in B_{\mathcal{I}}(E)$ and thus $P(\omega) \subseteq E$. Hence, $P(\omega)$ is the minimum element of $\mathcal{I}(\omega)$. □

A.3 Section 3.1

Proof of Proposition 2. By (a) of each logical property, it follows that, in (b), $\mathcal{I}$ satisfies a given logical property if and only if $\uparrow \mathcal{I}$ satisfies the given property. Thus, for (b), it suffices to show that $\uparrow \mathcal{I}$ satisfies a given property if and only if $B_{\mathcal{I}}$ satisfies it.

1. (a) I show that if Γ satisfies No-Contradiction then so does $\uparrow \Gamma$ by contraposition. If $\emptyset \in \uparrow \Gamma$, then there is $E \in \Gamma$ with $E \subseteq \emptyset$, i.e., $\emptyset \in \Gamma$. Conversely, if $\emptyset \notin \uparrow \Gamma$, then $\emptyset \notin \Gamma$ because $\Gamma \subseteq \uparrow \Gamma$.

(b) For each $\omega \in \Omega$, $\emptyset \in\uparrow \mathcal{I}(\omega)$ iff $\omega \in B_{\mathcal{I}}(\emptyset)$. Thus, $\emptyset \notin\uparrow \mathcal{I}(\omega)$ for all $\omega \in \Omega$ iff $B_{\mathcal{I}}(\emptyset) = \emptyset$.

2. (a) Let Γ satisfy Consistency. Suppose to the contrary that there are $E, F \in\uparrow \Gamma$ such that $E \cap F = \emptyset$. Then, there are $E', F' \in \Gamma$ such that $E' \subseteq E$ and $F' \subseteq F$. Thus, $E' \cap F' \subseteq E \cap F = \emptyset$, a contradiction. If $\uparrow \Gamma$ satisfies Consistency then $E^c \notin\uparrow \Gamma$ for any $E \in\uparrow \Gamma$. Finally, suppose that $E^c \notin\uparrow \Gamma$ for any $E \in\uparrow \Gamma$. Suppose to the contrary that there are $E, F \in \Gamma$ with $E \cap F = \emptyset$. Since $F \subseteq E^c$, it follows that $E, E^c \in\uparrow \Gamma$, a contradiction.

   (b) I show that if $\mathcal{I}$ satisfies Consistency then $B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(E^c)$. If $\omega \in B_{\mathcal{I}}(E)$ then $E \in\uparrow \mathcal{I}(\omega)$. Since $E^c \notin\uparrow \mathcal{I}(\omega)$, I have $\omega \in (\neg B_{\mathcal{I}})(E^c)$. Conversely, assume $B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(E^c)$. Take any $\omega \in \Omega$ and $E \in\uparrow \mathcal{I}(\omega)$. Since $\omega \in B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(E^c)$, I have $E^c \notin\uparrow \mathcal{I}(\omega)$.

3. (a) If $\Gamma \neq \emptyset$ then $\uparrow \Gamma \neq \emptyset$. If $\uparrow \Gamma \neq \emptyset$, then there is $E \in\uparrow \Gamma$. Then there is some $F \in \Gamma$ (with $F \subseteq E$), and thus $\Gamma \neq \emptyset$.

   (b) The statement follows because $\Omega \in\uparrow \mathcal{I}(\omega)$ for all $\omega \in \Omega$ iff $B_{\mathcal{I}}(\Omega) = \Omega$.

4. (a) Since $\uparrow \Gamma$ is closed under set inclusion, $\uparrow \Gamma$ satisfies Non-empty Countable Conjunction if and only if $\uparrow \Gamma$ is closed under non-empty countable intersection.

   (b) Take any non-empty countable collection $\{F_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}$. Observe that $\bigcap_{n\in\mathbb{N}} B_{\mathcal{I}}(F_n) \subseteq B_{\mathcal{I}}(\bigcap_{n\in\mathbb{N}} F_n)$ iff, for any $\omega \in \Omega$, if $F_n \in\uparrow \mathcal{I}(\omega)$ for all $n \in \mathbb{N}$ then $\bigcap_{n\in\mathbb{N}} F_n \in\uparrow \mathcal{I}(\omega)$.

□

A.4 Section 3.2

Proof of Proposition 3. 1. First, I show that (1a) implies (1b). If $E \in\uparrow \mathcal{I}(\omega)$ then there is $F \in \mathcal{I}(\omega)$ with $F \subseteq E$. Then, $\omega \in F \subseteq E$.

Second, I show that (1b) implies (1c). If $\omega \in B_{\mathcal{I}}(E)$, then $E \in\uparrow \mathcal{I}(\omega)$, and thus $\omega \in E$.

Finally, I show that (1c) implies (1a). If $E \in \mathcal{I}(\omega) \subseteq\uparrow \mathcal{I}(\omega)$, then $\omega \in B_{\mathcal{I}}(E) \subseteq E$.

2. First, I show that (2a) implies (2b). If $E \in\uparrow \mathcal{I}(\omega)$, then there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq E$. Then, there is $G \in \mathcal{I}(\omega)$ such that if $\omega' \in G$ then there is $F' \in \mathcal{I}(\omega') \subseteq\uparrow \mathcal{I}(\omega')$ such that $F' \subseteq F \subseteq E$.

   Second, I show that (2b) implies (2c). If $E \in\uparrow \mathcal{I}(\omega)$, then there is $F \in\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F$ then there is $E' \in\uparrow \mathcal{I}(\omega')$ such that $E' \subseteq E$. Then, since $F \subseteq \{\omega' \in \Omega \mid E \in\uparrow \mathcal{I}(\omega')\}$, it follows $\{\omega' \in \Omega \mid E \in\uparrow \mathcal{I}(\omega')\} \in\uparrow \mathcal{I}(\omega)$.

   Third, I show that (2c) implies (2d). If $\omega \in B_{\mathcal{I}}(E)$, then $E \in\uparrow \mathcal{I}(\omega)$. Then, $\{\omega' \in \Omega \mid E \in\uparrow \mathcal{I}(\omega')\} \in\uparrow \mathcal{I}(\omega)$, i.e., $\omega \in B_{\mathcal{I}}B_{\mathcal{I}}(E)$.

Finally, I show that (2d) implies (2a). Fix $\omega \in \Omega$ and $E \in \mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$. Then, $\omega \in B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}B_{\mathcal{I}}(E)$, and thus $\{\omega' \in \Omega \mid E \in \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega)$. Now, there is $F \in \mathcal{I}(\omega)$ such that if $\omega' \in F$ then $E \in \uparrow \mathcal{I}(\omega')$, i.e., there is $E' \in \mathcal{I}(\omega')$ with $E' \subseteq E$.

3. First, I show that (3a) implies (3b). Let $(\omega, E) \in \Omega \times \mathcal{D}$ be such that $E^c \cap F \neq \emptyset$ for all $F \in \uparrow \mathcal{I}(\omega)$. Since $\mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$ and since $\mathcal{I}(\omega)$ is Euclidean, there is $F' \in \mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F'$ then $\emptyset \neq E^c \cap F$ for all $F \in \uparrow \mathcal{I}(\omega')$.

   Second, I show that (3b) implies (3c). If $E \not\in \uparrow \mathcal{I}(\omega)$, then $E^c \cap F \neq \emptyset$ for all $F \in \uparrow \mathcal{I}(\omega)$. Then, there is $F' \in \uparrow \mathcal{I}(\omega)$ such that $F' \subseteq \{\omega' \in \Omega \mid E \not\in \uparrow \mathcal{I}(\omega')\}$, i.e., $\{\omega' \in \Omega \mid E \not\in \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega)$.

   Third, I show that (3c) implies (3d). If $\omega \in (\neg B_{\mathcal{I}})(E)$, then $E \not\in \uparrow \mathcal{I}(\omega)$. Then, $\{\omega' \in \Omega \mid E \not\in \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega)$, i.e., $\omega \in B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E)$.

   Finally, I show that (3d) implies (3a). Let $(\omega, E) \in \Omega \times \mathcal{D}$ be such that $E^c \cap F \neq \emptyset$ for all $F \in \mathcal{I}(\omega)$. Since $E \not\in \uparrow \mathcal{I}(\omega)$, I get $\omega \in (\neg B_{\mathcal{I}})(E) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E)$, i.e., $\{\omega' \in \Omega \mid E \not\in \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega)$. Thus, there is $F' \in \mathcal{I}(\omega)$ such that if $\omega' \in F$ then $E \not\in \uparrow \mathcal{I}(\omega')$, i.e., $E^c \cap F \neq \emptyset$ for all $F \in \mathcal{I}(\omega')$.

4. First, I show that (4a) implies (4b). Fix $(\omega, E) \in \Omega \times \mathcal{D}$. Suppose that, for any $F \in \uparrow \mathcal{I}(\omega)$, there is $\omega' \in F$ with $E \in \uparrow \mathcal{I}(\omega')$. Then, for any $F \in \mathcal{I}(\omega)$, there are $\omega' \in F$ and $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$. Thus, $\omega \in E$.

   Second, I show that (4b) implies (4c). Fix $E \in \mathcal{D}$. If $\omega \in (\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E)$, then for any $F \in \uparrow \mathcal{I}(\omega)$, $F \not\subseteq (\neg B_{\mathcal{I}})(E)$, i.e., there are $\omega' \in F$ and $E \in \uparrow \mathcal{I}(\omega')$. Then, $\omega \in E$.

   Finally, I show that (4c) implies (4a). Fix $(\omega, E) \in \Omega \times \mathcal{D}$. Suppose that, for any $F \in \mathcal{I}(\omega)$, there are $\omega' \in F$ and $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$. Thus, for any $F \in \mathcal{I}(\omega)$, $F \not\subseteq (\neg B_{\mathcal{I}})(E)$. Then, $\omega \in (\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E) \subseteq E$.

□

Proof of Corollary 1. First, I show below that $\mathcal{I}$ is reflexive. Second, then, as mentioned in the main text, since $\mathcal{I}$ is symmetric and transitive, it is Euclidean. Third, Negative Introspection (i.e., Euclideanness), Truth Axiom (i.e., reflexivity), and Monotonicity of $B_{\mathcal{I}}$ imply Necessitation and Non-empty Countable Conjunction (Fukuda, 2019, Corollary 1). Then, all the other properties follow.

Hence, I show that $\mathcal{I}$ is reflexive. Let $E \in \mathcal{I}(\omega)$. To show $\omega \in E$, by symmetry, it is enough to prove that if $F \in \mathcal{I}(\omega)$ then there are $\omega' \in F$ and $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$. Let $F \in \mathcal{I}(\omega)$. By transitivity, there is $G \in \mathcal{I}(\omega)$ such that if $\omega' \in G$ then there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$. Since $\mathcal{I}$ is symmetric, take $\omega' \in G \cap F \neq \emptyset$.

Alternatively, $B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E) \subseteq E$ follows from Positive Introspection (the first set inclusion), Consistency (the second), and Symmetry (the third). □

Example A.1. I provide an example where $\mathcal{I}$ is not strongly transitive but transitive. I define a reflexive and transitive information correspondence. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Define $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ as $\mathcal{I}(\omega_1) = \mathcal{I}(\omega_3) = \{\{\omega_1, \omega_3\}\}$ and $\mathcal{I}(\omega_2) = \{\{\omega_2\}, \{\omega_1, \omega_2\}\}$. By construction, $\mathcal{I}$ is reflexive.

To see that $\mathcal{I}$ is not strongly transitive, take $E = \{\omega_1, \omega_2\} \in \mathcal{I}(\omega_2)$ and $\omega_1 \in E$. Then, $\{\omega_1, \omega_3\} \in \mathcal{I}(\omega_1)$ and $\{\omega_1, \omega_3\} \not\subseteq E = \{\omega_1, \omega_2\}$. It can be seen, however, that $\mathcal{I}$ is transitive. This can also be verified by the fact that $B_{\mathcal{I}}$ satisfies Positive Introspection:

$$B_{\mathcal{I}}(E) = \begin{cases} \emptyset & \text{if } E \in \{\emptyset, \{\omega_1\}, \{\omega_3\}\} \\ \{\omega_2\} & \text{if } E \in \{\{\omega_2\}, \{\omega_1, \omega_2\}, \{\omega_2, \omega_3\}\} \\ \{\omega_1, \omega_3\} & \text{if } E = \{\omega_1, \omega_3\} \\ \Omega & \text{if } E = \Omega \end{cases}. \tag{A.1}$$

Two additional remarks are in order. First, observe that $\mathcal{I}$ satisfies the Kripke property. Thus, consider $\mathcal{I}'(\cdot) = \{P(\cdot)\}$, where $P(\omega_1) = \{\omega_1, \omega_3\}$, $P(\omega_2) = \{\omega_2\}$, and $P(\omega_3) = \{\omega_1, \omega_3\}$. Now, $\mathcal{I}'$ is strongly transitive.

Second, $B_{\mathcal{I}}$ satisfies all the four logical properties defined in Section 3.1. Also, $B_{\mathcal{I}}$ satisfies all the four introspective properties defined in Section 3.2 (it satisfies all the "Belief-in" properties in Section 3.3 as well). □

Example A.2. I provide an example where $\mathcal{I}$ is strongly transitive but $\uparrow \mathcal{I}$ is not as in Example A.1. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Define $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ as follows:

$$\mathcal{I}(\omega) = \begin{cases} \{\{\omega_1, \omega_3\}, \Omega\} & \text{if } \omega = \omega_1 \text{ or } \omega = \omega_3 \\ \{\{\omega_2\}, \Omega\} & \text{if } \omega = \omega_2 \end{cases}. \tag{A.2}$$

First, I show that $\mathcal{I}$ is strongly transitive. Let $\omega \in \Omega$, $E \in \mathcal{I}(\omega)$, and $\omega' \in E$. There is $F = E \in \mathcal{I}(\omega')$ such that $F \subseteq E$. Second, $\uparrow \mathcal{I}$ is written as follows:

$$\uparrow \mathcal{I}(\omega) = \begin{cases} \{\{\omega_1, \omega_3\}, \Omega\} & \text{if } \omega = \omega_1 \text{ or } \omega = \omega_3 \\ \{\{\omega_2\}, \{\omega_2, \omega_3\}, \Omega\} & \text{if } \omega = \omega_2 \end{cases}. \tag{A.3}$$

Third, I show that $\uparrow \mathcal{I}$ is not strongly transitive. Take $E = \{\omega_2, \omega_3\} \in \mathcal{I}(\omega_2)$ and $\omega_3 \in E$. Then, $\{\omega_1, \omega_3\} \not\subseteq \{\omega_2, \omega_3\} = E$ and $\Omega \not\subseteq \{\omega_2, \omega_3\} = E$. I remark that, since $\mathcal{I}$ is transitive, it follows from Proposition 3 that $\uparrow \mathcal{I}$ is transitive. I also remark that the belief operator $B_{\mathcal{I}}$ in this example coincides with that defined by Equation (A.1) in Example A.1. □

A.5 Section 3.3

Proof of Proposition 4. 1. First, I show that (1a) implies (1b). Take $(\omega, E) \in \Omega \times \mathcal{D}$. There is $F \in \mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$ with the following property: if $\omega' \in F$ and $E \in \uparrow \mathcal{I}(\omega')$ then $F' \in \mathcal{I}(\omega')$ for some $F' \subseteq E$. Then, $\omega' \in E$.

Second, I show that (1b) implies (1c). Take $(\omega, E) \in \Omega \times \mathcal{D}$. There is $F \in \mathcal{I}(\omega)$ such that $F \subseteq (\neg B_{\mathcal{I}})(E) \cup E$. Thus, $\omega \in B_{\mathcal{I}}((\neg B_{\mathcal{I}})(E) \cup E)$, and hence $\Omega = B_{\mathcal{I}}((\neg B_{\mathcal{I}})(E) \cup E)$.

Finally, I show that (1c) implies (1a). Take any $(\omega, E) \in \Omega \times \mathcal{D}$. Since $\omega \in \Omega = B_{\mathcal{I}}((\neg B_{\mathcal{I}})(E) \cup E)$, there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq (\neg B_{\mathcal{I}})(E) \cup E$. If $\omega' \in F$ and if there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then, since $\omega' \in B_{\mathcal{I}}(E)$, I have $\omega' \in E$.

2. First, I show that (2a) implies (2b). Fix $(\omega, E) \in \Omega \times \mathcal{D}$. There is $F \in \mathcal{I}(\omega) \subseteq\uparrow \mathcal{I}(\omega)$ with the following property: if $\omega' \in F$ and if $E \in\uparrow \mathcal{I}(\omega')$ and thus $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $H \cap E \neq \emptyset$ for all $H \in\uparrow \mathcal{I}(\omega')$. Thus, $E^c \notin\uparrow \mathcal{I}(\omega')$.

   Second, I show that (2b) implies (2c). Fix $E \in \mathcal{D}$, and take $\omega \in \Omega$. I show that $\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)) = (\neg B_{\mathcal{I}})(E) \cup (\neg B_{\mathcal{I}})(E^c) \in\uparrow \mathcal{I}(\omega)$. By supposition, there is $F \in\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F$ and if $\omega' \in B_{\mathcal{I}}(E)$ then $\omega' \in (\neg B_{\mathcal{I}})(E^c)$. Thus, $F \subseteq \neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c))$, and hence $\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)) \in\uparrow \mathcal{I}(\omega)$.

   Finally, I show that (2c) implies (2a). Fix $(\omega, E) \in \Omega \times \mathcal{D}$. Since $\omega \in \Omega = B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)))$, there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq \neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c))$. Thus, if $\omega' \in F$ and if there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $\omega' \in (\neg B_{\mathcal{I}})(E^c)$. Thus, $H \cap E \neq \emptyset$ for all $H \in \mathcal{I}(\omega')$.

3. First, I show that (3a) implies (3b). For any $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$, there is $G \in \mathcal{I}(\omega) \subseteq\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$, $E \in\uparrow \mathcal{I}(\omega')$, and if $(\neg E) \cup F \in\uparrow \mathcal{I}(\omega')$, then $F \in\uparrow \mathcal{I}(\omega')$.

   Second, I show that (3b) implies (3c). Fix $E, F \in \mathcal{D}$ and $\omega \in \Omega$. There is $G \in\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$, $\omega' \in B_{\mathcal{I}}(E)$, and if $\omega' \in B_{\mathcal{I}}((\neg E) \cup F)$, then $\omega' \in B_{\mathcal{I}}(F)$. Thus, $G \subseteq \neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F)$. Hence, $\omega \in B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F))$.

   Finally, I show that (3c) implies (3a). Take $(E, F) \in \mathcal{D}^2$ and $\omega \in \Omega = B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F))$. There is $G \in \mathcal{I}(\omega)$ such that if $\omega' \in G$ then $\omega' \in \neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F)$. If $\omega' \in G$, $E' \in \mathcal{I}(\omega')$ for some $E' \subseteq E$, and if $F' \in \mathcal{I}(\omega')$ for some $F' \subseteq (\neg E) \cup F$, then $\omega' \in B_{\mathcal{I}}(F)$, i.e., there is $G' \in \mathcal{I}(\omega')$ with $G' \subseteq F$.

4. Throughout the proof, take $\{F_n\}_{n \in \mathbb{N}} \in \mathcal{P}(\mathcal{D})$. I show that (4a) implies (4b). Fix $\omega \in \Omega$. There is $G \in \mathcal{I}(\omega) \subseteq\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$ and if $\{F_n\}_{n \in \mathbb{N}} \subseteq\uparrow \mathcal{I}(\omega')$, then there is $E_F \in \mathcal{I}(\omega')$ with $E_F \subseteq F_n$ for each $n \in \mathbb{N}$. Thus, there is $G' \in \mathcal{I}(\omega') \subseteq\uparrow \mathcal{I}(\omega')$ such that $G' \subseteq \bigcap_{n \in \mathbb{N}} F_n$.

   Next, I show that (4b) implies (4c). For each $\omega \in \Omega$, there is $G \in\uparrow \mathcal{I}(\omega)$ such that $G \subseteq \neg(\bigcap_{n \in \mathbb{N}} B_{\mathcal{I}}(F_n)) \cup B_{\mathcal{I}}(\bigcap_{n \in \mathbb{N}} F_n)$. Hence, $\omega \in B_{\mathcal{I}}(\neg(\bigcap_{n \in \mathbb{N}} B_{\mathcal{I}}(F_n)) \cup B_{\mathcal{I}}(\bigcap_{n \in \mathbb{N}} F_n))$.

   Next, I show that (4c) implies (4a). Take $\omega \in \Omega = B_{\mathcal{I}}(\neg(\bigcap_{n \in \mathbb{N}} B_{\mathcal{I}}(F_n)) \cup B_{\mathcal{I}}(\bigcap_{n \in \mathbb{N}} F_n))$. There is $G \in \mathcal{I}(\omega)$ such that if $\omega' \in G$ and if, for each $n \in \mathbb{N}$, there is $E_F \in \mathcal{I}(\omega')$ with $E_F \subseteq F_n$, then $\omega' \in B_{\mathcal{I}}(\bigcap_{n \in \mathbb{N}} F_n)$. Thus, there is $G' \in \mathcal{I}(\omega')$ with $G' \subseteq \bigcap_{n \in \mathbb{N}} F_n$.

□

Proof of Proposition 5. Suppose that $\mathcal{I}$ satisfies Belief in Perfect Reasoning, and take $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$. There is $G \in\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$, $E \in\uparrow \mathcal{I}(\omega)$, and $F \in\uparrow \mathcal{I}(\omega')$, then, since $F \subseteq (\neg E) \cup (E \cap F)$, I have $(\neg E) \cup (E \cap F) \in\uparrow \mathcal{I}(\omega')$ and thus $E \cap F \in\uparrow \mathcal{I}(\omega')$.

Conversely, suppose that $\mathcal{I}$ satisfies Belief in Non-empty Finite Conjunction. Take $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$. There is $G \in\uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$, $E \in\uparrow \mathcal{I}(\omega)$, and

$(\neg E) \cup F \in\uparrow \mathcal{I}(\omega')$, then $E \cap F = E \cap ((\neg E) \cup F) \in\uparrow \mathcal{I}(\omega')$. Since $E \cap F \subseteq F$, I get $F \in\uparrow \mathcal{I}(\omega')$. □

Example A.3. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Define $\mathcal{I} : \Omega \to \mathcal{P}(\mathcal{D})$ as follows:

$$\mathcal{I}(\omega) = \begin{cases} \{\{\omega_1, \omega_2\}, \{\omega_2, \omega_3\}\} & \text{if } \omega = \omega_1 \\ \{\{\omega_2\}\} & \text{if } \omega = \omega_2 \\ \{\emptyset\} & \text{if } \omega = \omega_3 \end{cases}.$$

The belief operator $B_{\mathcal{I}}$ is given as follows:

$$B_{\mathcal{I}}(E) = \begin{cases} \{\omega_3\} & \text{if } E \in \{\emptyset, \{\omega_1\}, \{\omega_3\}, \{\omega_1, \omega_3\}\} \\ \{\omega_2, \omega_3\} & \text{if } E = \{\omega_2\} \\ \Omega & \text{if } E \in \{\{\omega_1, \omega_2\}, \{\omega_2, \omega_3\}, \Omega\} \end{cases}.$$

The operator $B_{\mathcal{I}}$ violates No-Contradiction because $\emptyset \in \mathcal{I}(\omega_3)$. In fact, $B_{\mathcal{I}}(\emptyset) = \{\omega_3\}$. Consequently, it violates Truth Axiom and Consistency (indeed, $B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c) = \{\omega_3\}$ for any $E \in \mathcal{D}$). The operator violates Non-empty Countable Conjunction: $B_{\mathcal{I}}(\{\omega_1, \omega_2\}) \cap B_{\mathcal{I}}(\{\omega_2, \omega_3\}) = \Omega \not\subseteq \{\omega_2, \omega_3\} = B_{\mathcal{I}}(\{\omega_2\})$. In contrast, it can be seen that $\mathcal{I}$ is secondary reflexive and secondary serial. It also satisfies Belief in Non-empty Countable Conjunction and consequently Belief in Perfect Reasoning. □

A.6 Section 4.1

Proof of Proposition 6.

1. By Propositions 2, 3, and 4, $B_{\mathcal{I}}$ satisfies the logical and introspective properties of beliefs imposed on $\mathcal{I}$. Next, $\mathcal{I}_{B_{\mathcal{I}}}(\omega) = \{E \in \mathcal{D} \mid \omega \in B_{\mathcal{I}}(E)\} = \{E \in \mathcal{D} \mid E \in\uparrow \mathcal{I}(\omega)\} =\uparrow \mathcal{I}(\omega)$ for all $\omega \in \Omega$. Since $B_{\mathcal{I}}$ is monotone, $\uparrow \mathcal{I}_{B_{\mathcal{I}}}(\cdot) = \mathcal{I}_{B_{\mathcal{I}}}(\cdot) =\uparrow \mathcal{I}(\cdot)$. Conversely, $B_{\mathcal{I}_B}(E) = \{\omega \in \Omega \mid E \in\uparrow \mathcal{I}_B(\omega)\} = B(E)$ for each $E \in \mathcal{D}$. By Propositions 2, 3, and 4, any generator $\mathcal{I}$ of B satisfies the logical and introspective properties of beliefs imposed on B. As argued in the main text, $\mathcal{I}(\cdot) \subseteq\uparrow \mathcal{I}(\cdot) =\uparrow \mathcal{I}_B(\cdot) = \mathcal{I}_B(\cdot)$.

2. First, I show that $\mathcal{I}_{\mathcal{J}_B}$ is an information correspondence that generates B, i.e., $B = B_{\mathcal{I}_{\mathcal{J}_B}}$. Take $E \in \mathcal{D}$ and $\omega \in B(E)$. Since $\omega \in B(E) \subseteq BB(E)$, it follows that $B(E) \in \mathcal{I}_{\mathcal{J}_B}(\omega)$ and $E \in\uparrow \mathcal{I}_{\mathcal{J}_B}(\omega)$. Thus, $\omega \in B_{\mathcal{I}_{\mathcal{J}_B}}(E)$. Conversely, if $\omega \in B_{\mathcal{I}_{\mathcal{J}_B}}(E)$ then there is $F \in \mathcal{I}_{\mathcal{J}_B}(\omega)$ with $\omega \in F \subseteq E$. Then, $\omega \in F \subseteq B(F) \subseteq B(E)$. Second, $\mathcal{I}_{\mathcal{J}_B}$ is by construction reflexive. Third, I show that $\mathcal{I}_{\mathcal{J}_B}$ is strongly transitive. Let $E \in \mathcal{I}_{\mathcal{J}_B}(\omega)$, i.e., $\omega \in E = B(E)$. If $\omega' \in E$ then there is $E' = E \in \mathcal{I}_{\mathcal{J}_B}(\omega')$ such that $E' \subseteq E$.

□

A.7 Section 4.2

Proof of Proposition 7.

1. (a) Let $\mathcal{J}$ satisfy Condition (6). First, I show that $\mathcal{I}_{\mathcal{J}}$ is an information correspondence. For each $E \in \mathcal{D}$,

$$\begin{aligned}&\{\omega \in \Omega \mid \text{there is } F \in \mathcal{I}_{\mathcal{J}}(\omega) \text{ such that } F \subseteq E\}\\ &= \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J} \text{ such that } \omega \in F \subseteq E\} \in \mathcal{J} \subseteq \mathcal{D}.\end{aligned}$$

   Second, $\mathcal{I}_{\mathcal{J}}$ is reflexive by construction. Third, $\mathcal{I}_{\mathcal{J}}$ is strongly transitive. For any $E \in \mathcal{I}_{\mathcal{J}}(\omega)$ and $\omega' \in E$, $E' = E \in \mathcal{I}_{\mathcal{J}}(\omega')$ satisfies $\omega' \in E' \subseteq E$.

   (b) Fourth, if $\Omega \in \mathcal{J}$, then $\Omega \in \mathcal{I}_{\mathcal{J}}(\omega)$ for any $\omega \in \Omega$.

2. (a) Conversely, I show that $\mathcal{J}_{\mathcal{I}}$ satisfies Condition (6). For each $E \in \mathcal{D}$, let $B_{\mathcal{J}_{\mathcal{I}}}(E) := \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_{\mathcal{I}} \text{ such that } \omega \in F \subseteq E\}$. I show that $B_{\mathcal{J}_{\mathcal{I}}}(E) \in \mathcal{J}_{\mathcal{I}}$, i.e., if $\omega \in B_{\mathcal{J}_{\mathcal{I}}}(E)$, then there is $F' \in \mathcal{I}(\omega)$ such that $F' \subseteq B_{\mathcal{J}_{\mathcal{I}}}(E)$. Let $\omega \in B_{\mathcal{J}_{\mathcal{I}}}(E)$. There is $F \in \mathcal{J}_{\mathcal{I}}$ such that $\omega \in F \subseteq E$. Since $\omega \in F$, there is $E' \in \mathcal{I}(\omega)$ such that $E' \subseteq F$. Since $\mathcal{I}$ is transitive, there is $F' \in \mathcal{I}(\omega)$ such that if $\omega' \in F'$ then there is $G \in \mathcal{I}(\omega')$ such that $\omega' \in G \subseteq E' \subseteq F \subseteq E$. Thus, $\omega' \in B_{\mathcal{J}_{\mathcal{I}}}(E)$. Hence, I get $F' \subseteq B_{\mathcal{J}_{\mathcal{I}}}(E)$, as desired.

   (b) If $\mathcal{I}(\cdot) \neq \emptyset$, then for any $\omega \in \Omega$, there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq \Omega$. Thus, $\Omega \in \mathcal{J}_{\mathcal{I}}$.

3. (a) Next, let $E \in \mathcal{J}$. For any $\omega \in E$, I have $E \in \mathcal{I}_{\mathcal{J}}(\omega)$ and $E \subseteq E$. Thus, $E \in \mathcal{J}_{\mathcal{I}_{\mathcal{J}}}$. Conversely, if $E \in \mathcal{J}_{\mathcal{I}_{\mathcal{J}}}$, then

$$\begin{aligned}E &= \{\omega \in \Omega \mid \text{there is } F \in \mathcal{I}_{\mathcal{J}}(\omega) \text{ such that } F \subseteq E\}\\ &= \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J} \text{ such that } \omega \in F \subseteq E\} \in \mathcal{J}.\end{aligned}$$

   (b) Finally, let $\mathcal{I}$ be given. First, I show $\uparrow \mathcal{I}_{\mathcal{J}_{\mathcal{I}}}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$. If $E \in \mathcal{I}_{\mathcal{J}_{\mathcal{I}}}(\omega)$, then $\omega \in E \in \mathcal{J}_{\mathcal{I}}$. Thus, $E \in \uparrow \mathcal{I}(\omega)$. This implies $\mathcal{I}_{\mathcal{J}_{\mathcal{I}}}(\omega) \subseteq \uparrow \mathcal{I}_{\mathcal{J}_{\mathcal{I}}}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$. Conversely, suppose that $E \in \uparrow \mathcal{I}(\omega)$. Since $\mathcal{I}$ is transitive, $E' := \{\omega' \in \Omega \mid E \in \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega)$. Thus, there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq E'$. Since $\mathcal{I}$ is reflexive, $\omega \in F \subseteq E'$. If $\omega' \in E'$ then $\omega' \in \{\omega'' \in \Omega \mid E' \in \uparrow \mathcal{I}(\omega'')\}$. That is, if $\omega' \in E'$ then there is $G \in \mathcal{I}(\omega')$ such that $G \subseteq E'$. Thus, $E' \in \mathcal{J}_{\mathcal{I}}$. Since $\omega \in E'$, I have $E' \in \mathcal{I}_{\mathcal{J}_{\mathcal{I}}}(\omega)$. By reflexivity, $E' \subseteq E$. Thus, $E \in \uparrow \mathcal{I}_{\mathcal{J}_{\mathcal{I}}}(\omega)$.

□

Example A.4. Consider the belief operator B defined as in Equation (A.1) in Example A.1. Let $\mathcal{J} := \mathcal{J}_B = \{\emptyset, \{\omega_2\}, \{\omega_1, \omega_3\}, \Omega\}$. I get the information correspondence $\mathcal{I}_{\mathcal{J}}$ as in Equation (A.2) in Example A.2. Then, $\uparrow \mathcal{I}_{\mathcal{J}}$ is given as in Equation (A.3) in Example A.2. The arguments in Example A.2 show that, while $\mathcal{I}_{\mathcal{J}}$ is strongly transitive, $\uparrow \mathcal{I}_{\mathcal{J}}$ is not. □

B Additional Results

Remark B.1. I formulate the equivalence between information correspondences and belief operators in Proposition 6 as a Galois connection. To that end, let $(\mathbb{I}, \leq_{\mathbb{I}})$ be the collection of information correspondences on a state space $(\Omega, \mathcal{D})$ with the following pre-order (i.e., reflexive and transitive order): $\mathcal{I} \leq_{\mathbb{I}} \mathcal{I}'$ if and only if, for each $\omega \in \Omega$ and $E \in \mathcal{I}(\omega)$, there is $F \in \mathcal{I}'(\omega)$ such that $F \subseteq E$. In other words, $\uparrow \mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}'(\cdot)$. Let $(\mathbb{B}, \leq_{\mathbb{B}})$ be the collection of monotone belief operators from $\mathcal{D}$ into itself. Let $\alpha : (\mathbb{I}, \leq_{\mathbb{I}}) \to (\mathbb{B}, \leq_{\mathbb{B}})$ be such that $\alpha(\mathcal{I}) = B_{\mathcal{I}}$ defined as in Equation (2). I show that α is order-preserving. If $\mathcal{I} \leq_{\mathbb{I}} \mathcal{I}'$ then $\uparrow \mathcal{I} \leq_{\mathbb{I}} \uparrow \mathcal{I}'$ and thus $B_{\mathcal{I}} \leq_{\mathbb{B}} B_{\mathcal{I}'}$ by Equation (2). Next, let $\beta : (\mathbb{B}, \leq_{\mathbb{B}}) \to (\mathbb{I}, \leq_{\mathbb{I}})$ be $\beta(B) = \mathcal{I}_B$ as in Equation (5). By construction, β is order-preserving, i.e., if $B \leq_{\mathbb{B}} B'$ then $\mathcal{I}_B \leq_{\mathbb{I}} \mathcal{I}_{B'}$.

Now, I show that (α, β) is a Galois connection, that is, the order-preserving maps α and β on pre-ordered spaces satisfy

$$\mathcal{I} \leq_{\mathbb{I}} \beta(B) \text{ if and only if } \alpha(\mathcal{I}) \leq_{\mathbb{B}} B \text{ for any } (\mathcal{I}, B) \in \mathbb{I} \times \mathbb{B}.$$

Indeed, if $\alpha(\mathcal{I}) \leq_{\mathbb{B}} B$ then $\mathcal{I} = \mathcal{I}_{B_{\mathcal{I}}} \leq_{\mathbb{I}} \mathcal{I}_B = \beta(B)$. Conversely, if $\mathcal{I} \leq_{\mathbb{I}} \beta(B)$ then $\alpha(\mathcal{I}) = B_{\mathcal{I}} \leq_{\mathbb{B}} B_{\mathcal{I}_B} = B$. □

Remark B.2. Let $(\mathbb{I}^{rt}, \leq^{rt})$ be the collection of reflexive and transitive information correspondences with the following pre-order as in Remark B.1: $\mathcal{I} \leq^{rt} \mathcal{I}'$ if and only if, for each $\omega \in \Omega$ and $E \in \mathcal{I}(\omega)$, there is $F \in \mathcal{I}'(\omega)$ such that $F \subseteq E$. In other words, $\uparrow \mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}'(\cdot)$. Let $(\mathbb{J}, \subseteq)$ be the space consisting of collections of events $\mathcal{J} (\in \mathcal{P}(\mathcal{D}))$ satisfying Condition (6). Define $\alpha : (\mathbb{I}^{rt}, \leq^{rt}) \to (\mathbb{J}, \subseteq)$ by $\alpha(\mathcal{I}) = \mathcal{J}_{\mathcal{I}}$ as in Equation (8). I show that the mapping α is order-preserving. Let $\mathcal{I} \leq^{rt} \mathcal{I}'$, and take $E \in \mathcal{J}_{\mathcal{I}}$. If $\omega \in E$ then there is $F' \in \mathcal{I}'(\omega)$ such that $F' \subseteq E$. Thus, $E \in \mathcal{J}_{\mathcal{I}'}$. Next, define $\beta : (\mathbb{J}, \subseteq) \to (\mathbb{I}, \leq^{rt})$ by $\beta(\mathcal{J}) = \mathcal{I}_{\mathcal{J}}$ as in Equation (7). I show that β is order-preserving. If $\mathcal{J} \subseteq \mathcal{J}'$, then $\mathcal{I}_{\mathcal{J}}(\cdot) \subseteq \mathcal{I}_{\mathcal{J}'}(\cdot)$. This implies $\mathcal{I}_{\mathcal{J}} \leq^{rt} \mathcal{I}_{\mathcal{J}'}$.

Now, I establish (α, β) is a Galois connection, that is,

$$\mathcal{I} \leq^{rt} \beta(\mathcal{J}) \text{ if and only if } \alpha(\mathcal{I}) \subseteq \mathcal{J} \text{ for any } (\mathcal{I}, \mathcal{J}) \in \mathbb{I}^{rt} \times \mathbb{J}.$$

If $\mathcal{I} \leq^{rt} \beta(\mathcal{J}) = \mathcal{I}_{\mathcal{J}}$ then $\alpha(\mathcal{I}) \subseteq \alpha \circ \beta(\mathcal{J}) = \mathcal{J}_{\mathcal{I}_{\mathcal{J}}} = \mathcal{J}$. Conversely, if $\alpha(\mathcal{I}) \subseteq \mathcal{J}$ then $\mathcal{I} \leq^{rt} \beta(\mathcal{J})$ because $\uparrow \mathcal{I} = \uparrow \mathcal{I}_{\mathcal{J}_{\mathcal{I}}} \leq^{rt} \uparrow \mathcal{I}_{\mathcal{J}}$. □