Who to Listen to? A Model of Endogenous Delegation Online Supplementary Appendix

William Fuchs[*] Satoshi Fukuda[†] Mahyar Sefidgaran[‡]

July 26, 2023

This Online Appendix provides the detailed proof of Theorem 1. Each subsection corresponds to each part of the theorem. Throughout this Online Appendix, for ease of exposition, denote by $F_i(\theta_i) := \frac{\theta_i - \underline{\theta}_i}{\overline{\theta}_i - \underline{\theta}_i}$ the cumulative distribution function of the uniform distribution on Θ_i. Likewise, denote by $f_i(\cdot) := \frac{1}{\overline{\theta}_i - \underline{\theta}_i}$ its probability density function.

E.1 Proof of Theorem 1

Throughout the proof of Theorem 1, we interchangeably use $\underline{\theta}_1 = 0$ and $\overline{\theta}_2 = 1$, respectively.

E.1.1 Part (1)

Assume $\overline{\theta}_1 \leq \gamma \mathbb{E}_{\theta_1}[\theta_1] + (1-\gamma)\mathbb{E}_{\theta_2}[\theta_2] \leq \underline{\theta}_2$. We consider the relaxed problem where the monotonicity constraint is ignored, and show that the constant allocation is optimal among such a class of allocations.

The proof consists of seven steps. The first step rewrites the relaxed problem by substituting the local IC constraints into the objective function. The second step formulates the Lagrangian. Denote by Λ_i the Lagrange multiplier associated with agent i's local IC constraint. The third step examines the first-order conditions. In the fourth to seven steps, we substitute $a^*(\cdot) = \gamma \mathbb{E}_{\theta_1}[\theta_1] + (1-\gamma)\mathbb{E}_{\theta_2}[\theta_2]$ and find the Lagrange multipliers (Λ_1, Λ_2) such that the first-order conditions are met.

Step 1. Consider the relaxed problem in which (Mon) is ignored. Thus, the problem is to maximize the sum of the agents' ex-ante utilities subject to their local IC constraints.

[*]McCombs School of Business, UT Austin, Universidad Carlos III Madrid, CEPR, and FTG.
[†]Department of Decision Sciences and IGIER, Bocconi University.
[‡]McCombs School of Business, UT Austin.

For agent 1, let the "reference" type of the local IC constraint be $\overline{\theta}_1$. For agent 2, let the "reference" type be $\underline{\theta}_2$. We show that the relaxed problem can be rewritten as follows:

$$\max_{a(\cdot)} \gamma \left(U_1(\overline{\theta}_1) - 2\mathbb{E}_\theta\left[(a(\theta)-\theta_1)(\theta_1-\underline{\theta}_1)\right]\right) + (1-\gamma)\left(U_2(\underline{\theta}_2) + 2\mathbb{E}_\theta\left[(a(\theta)-\theta_2)(\overline{\theta}_2-\theta_2)\right]\right)$$

subject to $U_1(\theta_1) = U_1(\overline{\theta}_1) - 2\int_{\theta_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\tau_1,\theta_2)-\tau_1\right] d\tau_1$ for each $\theta_1 \in [\underline{\theta}_1, \overline{\theta}_1]$ and

$$U_2(\theta_2) = U_2(\underline{\theta}_2) + 2\int_{\underline{\theta}_2}^{\theta_2} \mathbb{E}_{\theta_1}\left[a(\theta_1,\tau_2)-\tau_2\right] d\tau_2 \text{ for each } \theta_2 \in [\underline{\theta}_2, \overline{\theta}_2].$$

Since we ignore the monotonicity constraint, the local IC constraints are sufficient. Thus, it suffices to rewrite the objective function as above. Using the local IC constraint, we rewrite agent i's ex-ante expected utility as follows. For agent 1,

$$\begin{aligned}
\mathbb{E}_{\theta_1}\left[U_1(\theta_1)\right] &= \frac{1}{\overline{\theta}_1-\underline{\theta}_1}\int_{\underline{\theta}_1}^{\overline{\theta}_1} U_1(\theta_1)d\theta_1 = U_1(\overline{\theta}_1) - \frac{2}{\overline{\theta}_1-\underline{\theta}_1}\int_{\underline{\theta}_1}^{\overline{\theta}_1}\int_{\theta_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\tau_1,\theta_2)-\tau_1\right] d\tau_1 d\theta_1 \\
&= U_1(\overline{\theta}_1) - \frac{2}{\overline{\theta}_1-\underline{\theta}_1}\int_{\underline{\theta}_1}^{\overline{\theta}_1}\int_{\underline{\theta}_1}^{\tau_1} \mathbb{E}_{\theta_2}\left[a(\tau_1,\theta_2)-\tau_1\right] d\theta_1 d\tau_1 \\
&= U_1(\overline{\theta}_1) - \frac{2}{\overline{\theta}_1-\underline{\theta}_1}\int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\tau_1,\theta_2)-\tau_1\right](\tau_1-\underline{\theta}_1) d\tau_1 \\
&= U_1(\overline{\theta}_1) - 2\mathbb{E}_\theta\left[(a(\theta)-\theta_1)(\theta_1-\underline{\theta}_1)\right].
\end{aligned}$$

Similarly for agent 2, we have

$$\mathbb{E}_{\theta_2}\left[U_2(\theta_2)\right] = U_2(\underline{\theta}_2) + 2\mathbb{E}_\theta\left[(a(\theta)-\theta_2)(\overline{\theta}_2-\theta_2)\right].$$

Then, the reformulation follows because the objective function is written as $\gamma\mathbb{E}_{\theta_1}\left[U_1(\theta_1)\right] + (1-\gamma)\mathbb{E}_{\theta_2}\left[U_2(\theta_2)\right]$.

Step 2. To formulate the Lagrangian of the problem formulated in Step 1, we denote by Λ_i the Lagrange multiplier associated with agent i's (local) IC constraint. Theoretically, the Lagrange multiplier Λ_i is a function of bounded variation. Without loss of generality, we normalize Λ_i by setting $\Lambda_1(\underline{\theta}_1) = 0$ and $\Lambda_2(\overline{\theta}_2) = 0$.

In each of Steps 4 to 7, we conjecture and verify a specific functional form of Λ_i, from the first-order condition to be found in Step 3. In every step, the specific Λ_i is shown to have a density function λ_i on $[\underline{\theta}_i, \overline{\theta}_i]$. Then, each Λ_i can be written as follows:

$$\Lambda_1(\theta_1) = \int_{\underline{\theta}_1}^{\theta_1} \lambda_1(\tau_1) d\tau_1 \text{ for each } \theta_1 \in [\underline{\theta}_1, \overline{\theta}_1], \text{ and}$$

$$\Lambda_2(\theta_2) = -\int_{\theta_2}^{\overline{\theta}_2} \lambda_2(\tau_2) d\tau_2 \text{ for each } \theta_2 \in [\underline{\theta}_2, \overline{\theta}_2].$$

With this in mind, we define the Lagrangian as

$$\begin{aligned}
\mathcal{L} :=& \gamma \left(U_1(\overline{\theta}_1) - 2\mathbb{E}_\theta \left[(a(\theta) - \theta_1)(\theta_1 - \underline{\theta}_1)\right]\right) + (1-\gamma)\left(U_2(\underline{\theta}_2) + 2\mathbb{E}_\theta \left[(a(\theta) - \theta_2)(\overline{\theta}_2 - \theta_2)\right]\right) \\
&+ \int_{\underline{\theta}_1}^{\overline{\theta}_1} \left(U_1(\theta_1) - U_1(\overline{\theta}_1) + 2\int_{\theta_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\tau_1, \theta_2) - \tau_1\right] d\tau_1 \right) d\Lambda_1(\theta_1) \\
&+ \int_{\underline{\theta}_2}^{\overline{\theta}_2} \left(U_2(\theta_2) - U_2(\underline{\theta}_2) - 2\int_{\underline{\theta}_2}^{\theta_2} \mathbb{E}_{\theta_1}\left[a(\theta_1, \tau_2) - \tau_2\right] d\tau_2 \right) d\Lambda_2(\theta_2).
\end{aligned}$$

We show that the Lagrangian can be rewritten as:

$$\begin{aligned}
\mathcal{L} =& - \mathbb{E}_{\theta_2}\left[(a(\overline{\theta}_1, \theta_2) - \overline{\theta}_1)^2\right](\gamma - \Lambda_1(\overline{\theta}_1)) - 2\gamma\mathbb{E}_\theta\left[(a(\theta) - \theta_1)(\theta_1 - \underline{\theta}_1)\right] \\
&- \int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[(a(\theta) - \theta_1)^2\right]\lambda_1(\theta_1) d\theta_1 + 2\int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\theta) - \theta_1\right]\Lambda_1(\theta_1) d\theta_1 \\
&- \mathbb{E}_{\theta_1}\left[(a(\theta_1, \underline{\theta}_2) - \underline{\theta}_2)^2\right](1 - \gamma + \Lambda_2(\underline{\theta}_2)) + 2(1-\gamma)\mathbb{E}_\theta\left[(a(\theta) - \theta_2)(\overline{\theta}_2 - \theta_2)\right] \\
&- \int_{\underline{\theta}_2}^{\overline{\theta}_2} \mathbb{E}_{\theta_1}\left[(a(\theta) - \theta_2)^2\right]\lambda_2(\theta_2) d\theta_2 + 2\int_{\underline{\theta}_2}^{\overline{\theta}_2} \mathbb{E}_{\theta_1}\left[a(\theta) - \theta_2\right]\Lambda_2(\theta_2) d\theta_2. \qquad \text{(E.1)}
\end{aligned}$$

To see this, the part of the Lagrangian that corresponds to agent 1's local IC constraint is rewritten as:

$$\begin{aligned}
&\int_{\underline{\theta}_1}^{\overline{\theta}_1} \left(U_1(\theta_1) - U_1(\overline{\theta}_1) + 2\int_{\theta_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\tau_1, \theta_2) - \tau_1\right] d\tau_1 \right) \lambda_1(\theta_1) d\theta_1 \\
=& - U_1(\overline{\theta}_1)\Lambda_1(\overline{\theta}_1) + \int_{\underline{\theta}_1}^{\overline{\theta}_1} U_1(\theta_1)\lambda_1(\theta_1) d\theta_1 + 2\int_{\underline{\theta}_1}^{\overline{\theta}_1}\int_{\theta_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\tau_1, \theta_2) - \tau_1\right] d\tau_1 \lambda_1(\theta_1) d\theta_1 \\
=& - U_1(\overline{\theta}_1)\Lambda_1(\overline{\theta}_1) + \int_{\underline{\theta}_1}^{\overline{\theta}_1} U_1(\theta_1)\lambda_1(\theta_1) d\theta_1 + 2\int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\tau_1, \theta_2) - \tau_1\right] \int_{\underline{\theta}_1}^{\tau_1} \lambda_1(\theta_1) d\theta_1 d\tau_1 \\
=& - U_1(\overline{\theta}_1)\Lambda_1(\overline{\theta}_1) + \int_{\underline{\theta}_1}^{\overline{\theta}_1} U_1(\theta_1)\lambda_1(\theta_1) d\theta_1 + 2\int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\theta) - \theta_1\right]\Lambda_1(\theta_1) d\theta_1.
\end{aligned}$$

Similarly for agent 2, we have:

$$\int_{\underline{\theta}_2}^{\overline{\theta}_2}\left(U_2(\theta_2)-U_2(\underline{\theta}_2)-2\int_{\underline{\theta}_2}^{\theta_2}\mathbb{E}_{\theta_1}\left[a(\theta_1,\tau_2)-\tau_2\right]d\tau_2\right)\lambda_2(\theta_2)d\theta_2$$
$$=U_2(\underline{\theta}_2)\Lambda_2(\underline{\theta}_2)+\int_{\underline{\theta}_2}^{\overline{\theta}_2}U_2(\theta_2)\lambda_2(\theta_2)d\theta_2+2\int_{\underline{\theta}_2}^{\overline{\theta}_2}\mathbb{E}_{\theta_1}\left[a(\theta)-\theta_2\right]\Lambda_2(\theta_2)d\theta_2.$$

Substituting $U_i(\theta_i)=-\mathbb{E}_{\theta_{-i}}\left[(a(\theta_i,\theta_{-i})-\theta_i)^2\right]$, the Lagrangian reduces to Expression (E.1).

Step 3. We take the point-wise first-order condition for each $a(\theta)$. For any θ, the first-order condition is

$$\begin{aligned}&\frac{1}{\overline{\theta}_2-\underline{\theta}_2}\left\{\gamma\frac{\theta_1-\underline{\theta}_1}{\overline{\theta}_1-\underline{\theta}_1}+\lambda_1(\theta_1)(a(\theta)-\theta_1)-\Lambda_1(\theta_1)\right\}\\&+\frac{1}{\overline{\theta}_1-\underline{\theta}_1}\left\{-(1-\gamma)\frac{\overline{\theta}_2-\theta_2}{\overline{\theta}_2-\underline{\theta}_2}+\lambda_2(\theta_2)(a(\theta)-\theta_2)-\Lambda_2(\theta_2)\right\}\\&+\frac{(a(\overline{\theta}_1,\theta_2)-\overline{\theta}_1)(\gamma-\Lambda_1(\overline{\theta}_1))}{(\overline{\theta}_1-\underline{\theta}_1)(\overline{\theta}_2-\underline{\theta}_2)}\mathbb{I}(\theta_1=\overline{\theta}_1)+\frac{(a(\theta_1,\underline{\theta}_2)-\underline{\theta}_2)(1-\gamma+\Lambda_2(\underline{\theta}_2))}{(\overline{\theta}_1-\underline{\theta}_1)(\overline{\theta}_2-\underline{\theta}_2)}\mathbb{I}(\theta_2=\underline{\theta}_2)=0.\end{aligned}\tag{E.2}$$

From now on, we find (Λ_1,Λ_2) such that the first-order conditions are satisfied at

$$a^*=\gamma\mathbb{E}_{\theta_1}[\theta_1]+(1-\gamma)\mathbb{E}_{\theta_2}[\theta_2]=\frac{\gamma\overline{\theta}_1+(1-\gamma)(1+\underline{\theta}_2)}{2}.$$

For the rest of the proof, we consider the following four cases: (i) $\overline{\theta}_1<a^*<\underline{\theta}_2$; (ii) $\overline{\theta}_1=a^*<\underline{\theta}_2$; (iii) $\overline{\theta}_1<a^*=\underline{\theta}_2$; and (iv) $\overline{\theta}_1=a^*=\underline{\theta}_2(=1-\gamma)$.

Step 4. First, suppose $\overline{\theta}_1<a^*<\underline{\theta}_2$. We conjecture and verify $\Lambda_1(\overline{\theta}_1)=\gamma$ and $\Lambda_2(\underline{\theta}_2)=-(1-\gamma)$. Then, by the first-order conditions, there exist constants α_1 and α_2 such that

$$\alpha_1=\gamma\frac{\theta_1-\underline{\theta}_1}{\overline{\theta}_1-\underline{\theta}_1}+\lambda_1(\theta_1)(a^*-\theta_1)-\Lambda_1(\theta_1),\tag{E.3}$$
$$\alpha_2=-(1-\gamma)\frac{\overline{\theta}_2-\theta_2}{\overline{\theta}_2-\underline{\theta}_2}+\lambda_2(\theta_2)(a^*-\theta_2)-\Lambda_2(\theta_2),\text{ and}\tag{E.4}$$
$$0=\frac{\alpha_1}{\overline{\theta}_2-\underline{\theta}_2}+\frac{\alpha_2}{\overline{\theta}_1-\underline{\theta}_1}.$$

For agent 1, we show that

$$\Lambda_1(\theta_1) = \frac{\theta_1 - \underline{\theta}_1}{a^* - \theta_1}\left(-\frac{\gamma}{2}\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \alpha_1\right), \text{ where } \alpha_1 = \gamma\left(\frac{1}{2} + \frac{a^* - \overline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1}\right).$$

That is, we show:

$$\Lambda_1(\theta_1) = \gamma\frac{(\theta_1 - \underline{\theta}_1)(2a^* - \overline{\theta}_1 - \theta_1)}{2(a^* - \theta_1)(\overline{\theta}_1 - \underline{\theta}_1)}. \tag{E.5}$$

It can be seen that $\Lambda_1(\underline{\theta}_1) = 0$, $\Lambda_1(\overline{\theta}_1) = \gamma$, and that the function Λ_1 has its density:

$$\lambda_1(\theta_1) = \gamma\frac{(a^* - \theta_1)^2 + (a^* - \underline{\theta}_1)(a^* - \overline{\theta}_1)}{2(\overline{\theta}_1 - \underline{\theta}_1)(a^* - \theta_1)^2} \geq 0. \tag{E.6}$$

For agent 2, we show that

$$\Lambda_2(\theta_2) = -\frac{\overline{\theta}_2 - \theta_2}{a^* - \theta_2}\left(\frac{(1-\gamma)(\overline{\theta}_2 - \theta_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)} + \alpha_2\right), \text{ where } \alpha_2 = -(1-\gamma)\left(\frac{1}{2} - \frac{a^* - \underline{\theta}_2}{\overline{\theta}_2 - \underline{\theta}_2}\right).$$

That is, we show:

$$\Lambda_2(\theta_2) = (1-\gamma)\frac{(\overline{\theta}_2 - \theta_2)(\theta_2 + \underline{\theta}_2 - 2a^*)}{2(a^* - \theta_2)(\overline{\theta}_2 - \underline{\theta}_2)}. \tag{E.7}$$

It can be seen that $\Lambda_2(\underline{\theta}_2) = -(1-\gamma)$, $\Lambda_2(\overline{\theta}_2) = 0$, and that the function Λ_2 has its density:

$$\lambda_2(\theta_2) = (1-\gamma)\frac{(a^* - \theta_2)^2 + (a^* - \underline{\theta}_2)(a^* - \overline{\theta}_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)(a^* - \theta_2)^2} \geq 0. \tag{E.8}$$

Moreover, we have

$$\frac{\alpha_1}{\overline{\theta}_2 - \underline{\theta}_2} + \frac{\alpha_2}{\overline{\theta}_1 - \underline{\theta}_1} = \frac{2a^* - (\gamma\underline{\theta}_1 + (1-\gamma)(1 + \underline{\theta}_2))}{2(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)} = 0.$$

We start with agent 1. To prove Expression (E.5), observe that Expression (E.3) is a linear first-order differential equation. Since we have

$$\frac{d}{d\theta_1}\Lambda_1(\theta_1)(a^* - \theta_1) = \lambda_1(\theta_1)(a^* - \theta_1) - \Lambda_1(\theta_1) = \alpha_1 - \gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1},$$

integrating both sides from $\underline{\theta}_1$ to θ_1 yields

$$\Lambda_1(\theta_1)(a^* - \theta_1) = \alpha_1(\theta_1 - \underline{\theta}_1) - \frac{\gamma}{2}\frac{(\theta_1 - \underline{\theta}_1)^2}{\overline{\theta}_1 - \underline{\theta}_1}.$$

Hence, we have

$$\Lambda_1(\theta_1) = -\frac{\gamma(\theta_1 - \underline{\theta}_1)^2}{2(\overline{\theta}_1 - \underline{\theta}_1)(a^* - \theta_1)} + \alpha_1 \frac{\theta_1 - \underline{\theta}_1}{a^* - \theta_1} = \frac{\theta_1 - \underline{\theta}_1}{a^* - \theta_1} \left(-\frac{\gamma}{2} \frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \alpha_1 \right).$$

To get $\Lambda_1(\overline{\theta}_1) = \gamma$, we must have

$$\gamma = \frac{\overline{\theta}_1 - \underline{\theta}_1}{a^* - \overline{\theta}_1} \left(-\frac{\gamma}{2} + \alpha_1 \right), \text{ that is, } \alpha_1 = \gamma \left(\frac{1}{2} + \frac{a^* - \overline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} \right).$$

Then, we obtain:

$$\Lambda_1(\theta_1) = \gamma \frac{\theta_1 - \underline{\theta}_1}{a^* - \theta_1} \left(-\frac{1}{2} \frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \frac{1}{2} + \frac{a^* - \overline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} \right) = \gamma \frac{(\theta_1 - \underline{\theta}_1)(2a^* - \overline{\theta}_1 - \theta_1)}{2(a^* - \theta_1)(\overline{\theta}_1 - \underline{\theta}_1)},$$

as desired. We can obtain Expression (E.6) by taking the derivative of Λ_1.

Next, we move on to agent 2. To prove Expression (E.7), observe that Expression (E.4) is a linear first-order differential equation. Similarly to the case of agent 1, we have

$$\frac{d}{d\theta_2} \Lambda_2(\theta_2)(a^* - \theta_2) = \lambda_2(\theta_2)(a^* - \theta_2) - \Lambda_2(\theta_2) = \alpha_2 + (1 - \gamma) \frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2}.$$

Integrating both sides from θ_2 to $\overline{\theta}_2$ yields

$$-\Lambda_2(\theta_2)(a^* - \theta_2) = \alpha_2(\overline{\theta}_2 - \theta_2) + (1 - \gamma) \frac{(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)}.$$

Hence, we have

$$\Lambda_2(\theta_2) = -(1 - \gamma) \frac{(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)(a^* - \theta_2)} - \alpha_2 \frac{\overline{\theta}_2 - \theta_2}{a^* - \theta_2} = -\frac{\overline{\theta}_2 - \theta_2}{a^* - \theta_2} \left(\frac{(1 - \gamma)(\overline{\theta}_2 - \theta_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)} + \alpha_2 \right).$$

To get $\Lambda_2(\underline{\theta}_2) = -(1 - \gamma)$, we have to have

$$-(1 - \gamma) = -\frac{\overline{\theta}_2 - \underline{\theta}_2}{a^* - \underline{\theta}_2} \left(\frac{1 - \gamma}{2} + \alpha_2 \right), \text{ that is, } \alpha_2 = -(1 - \gamma) \left(\frac{1}{2} - \frac{a^* - \underline{\theta}_2}{\overline{\theta}_2 - \underline{\theta}_2} \right).$$

Then, we obtain:

$$\Lambda_2(\theta_2) = -(1 - \gamma) \frac{\overline{\theta}_2 - \theta_2}{a^* - \theta_2} \left(\frac{\overline{\theta}_2 - \theta_2}{2(\overline{\theta}_2 - \underline{\theta}_2)} - \frac{1}{2} + \frac{a^* - \underline{\theta}_2}{\overline{\theta}_2 - \underline{\theta}_2} \right) = (1 - \gamma) \frac{(\overline{\theta}_2 - \theta_2)(\theta_2 + \underline{\theta}_2 - 2a^*)}{2(a^* - \theta_2)(\overline{\theta}_2 - \underline{\theta}_2)},$$

as desired. We obtain Expression (E.8) by taking the derivative of Λ_2.

The Lagrangian reduces to:

$$\begin{aligned}\mathcal{L} =&\mathbb{E}_\theta\left[-\gamma(a(\theta)-\theta_1)^2\left(\frac{1}{2}+\frac{(a^*-\underline{\theta}_1)(a^*-\overline{\theta}_1)}{2(a^*-\theta_1)^2}\right)-\gamma(a(\theta)-\theta_1)\frac{(\theta_1-\underline{\theta}_1)(\overline{\theta}_1-\theta_1)}{(a^*-\theta_1)}\right]\\ &+\mathbb{E}_\theta\left[-(1-\gamma)(a(\theta)-\theta_2)^2\left(\frac{1}{2}+\frac{(a^*-\underline{\theta}_2)(a^*-\overline{\theta}_2)}{2(a^*-\theta_2)^2}\right)-(1-\gamma)(a(\theta)-\theta_2)\frac{(\theta_2-\underline{\theta}_2)(\overline{\theta}_2-\theta_2)}{(a^*-\theta_2)}\right].\end{aligned}$$

The Lagrangian is a concave function in a. By construction, the first-order conditions are satisfied at a^*.

Step 5. Second, suppose $\overline{\theta}_1 = a^* < \underline{\theta}_2$. By Expression (E.2), the first-order condition is:

$$\begin{aligned}&\frac{1}{\overline{\theta}_2-\underline{\theta}_2}\left\{\gamma\frac{\theta_1-\underline{\theta}_1}{\overline{\theta}_1-\underline{\theta}_1}+\lambda_1(\theta_1)(a^*-\theta_1)-\Lambda_1(\theta_1)\right\}\\ &+\frac{1}{\overline{\theta}_1-\underline{\theta}_1}\left\{-(1-\gamma)\frac{\overline{\theta}_2-\theta_2}{\overline{\theta}_2-\underline{\theta}_2}+\lambda_2(\theta_2)(a^*-\theta_2)-\Lambda_2(\theta_2)\right\}\\ &+\frac{(a^*-\underline{\theta}_2)}{(\overline{\theta}_1-\underline{\theta}_1)(\overline{\theta}_2-\underline{\theta}_2)}(1-\gamma+\Lambda_2(\underline{\theta}_2))\mathbb{I}(\theta_2=\underline{\theta}_2)=0.\end{aligned}$$

We conjecture and verify $\Lambda_2(\underline{\theta}_2) = -(1-\gamma)$. Then, there exist constants α_1 and α_2 with:

$$\begin{aligned}\alpha_1 &= \gamma\frac{\theta_1-\underline{\theta}_1}{\overline{\theta}_1-\underline{\theta}_1}+\lambda_1(\theta_1)(a^*-\theta_1)-\Lambda_1(\theta_1) \text{ for each } \theta_1\in[\underline{\theta}_1,\overline{\theta}_1],\\ \alpha_2 &= -(1-\gamma)\frac{\overline{\theta}_2-\theta_2}{\overline{\theta}_2-\underline{\theta}_2}+\lambda_2(\theta_2)(a^*-\theta_2)-\Lambda_2(\theta_2), \text{ and}\\ 0 &= \frac{\alpha_1}{\overline{\theta}_2-\underline{\theta}_2}+\frac{\alpha_2}{\overline{\theta}_1-\underline{\theta}_1}.\end{aligned}$$

For agent 2, we have the same differential equation as in Step 4 (except that $a^* = \frac{\gamma\underline{\theta}_1+(1-\gamma)(1+\underline{\theta}_2)}{2}$ coincides exactly with $\overline{\theta}_1$). Thus,

$$\Lambda_2(\theta_2) = -\frac{\overline{\theta}_2-\theta_2}{a^*-\theta_2}\left(\frac{(1-\gamma)(\overline{\theta}_2-\theta_2)}{2(\overline{\theta}_2-\underline{\theta}_2)}+\alpha_2\right), \text{ where } \alpha_2 = -(1-\gamma)\left(\frac{1}{2}-\frac{a^*-\underline{\theta}_2}{\overline{\theta}_2-\underline{\theta}_2}\right).$$

That is,

$$\Lambda_2(\theta_2) = \frac{(1-\gamma)(\overline{\theta}_2-\theta_2)(\theta_2+\underline{\theta}_2-2a^*)}{2(a^*-\theta_2)(\overline{\theta}_2-\underline{\theta}_2)}. \tag{E.9}$$

It can be seen that $\Lambda_2(\underline{\theta}_2) = -(1-\gamma)$, $\Lambda_2(\overline{\theta}_2) = 0$, and that the function Λ_2 has density:

$$\lambda_2(\theta_2) = (1-\gamma)\frac{(a^*-\theta_2)^2+(a^*-\underline{\theta}_2)(a^*-\overline{\theta}_2)}{2(\overline{\theta}_2-\underline{\theta}_2)(a^*-\theta_2)^2} \geq 0. \tag{E.10}$$

As in Step 4, we have:

$$\alpha_1 = \gamma\left(\frac{1}{2} + \frac{a^* - \overline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1}\right).$$

However, since $a^* = \overline{\theta}_1$, it reduces to

$$\alpha_1 = \frac{\gamma}{2}.$$

Then, the differential equation for Λ_1 is:

$$\frac{\gamma}{2} = \gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)(\overline{\theta}_1 - \theta_1) - \Lambda_1(\theta_1) \text{ for each } \theta_1 \in [\underline{\theta}_1, \overline{\theta}_1].$$

We show that there exists a differentiable function Λ_1 with $\Lambda_1(\underline{\theta}_1) = 0$ and $\Lambda_1(\overline{\theta}_1) = \frac{\gamma}{2}$. Indeed, since this first-order linear differential equation is rewritten as

$$\frac{d}{d\theta_1}\left[\Lambda_1(\theta_1)(\overline{\theta}_1 - \theta_1)\right] = \gamma\left(\frac{1}{2} - \frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1}\right),$$

integrating both sides from $\underline{\theta}_1$ to θ_1 yields

$$\Lambda_1(\theta_1) = \frac{\gamma(\theta_1 - \underline{\theta}_1)}{2(\overline{\theta}_1 - \underline{\theta}_1)}. \tag{E.11}$$

Thus, Λ_1 is a linear function. Then, we have:

$$\lambda_1(\theta_1) = \frac{\gamma}{2}\frac{1}{\overline{\theta}_1 - \underline{\theta}_1} \geq 0. \tag{E.12}$$

The Lagrangian is:

$$\begin{aligned}\mathcal{L} =& \mathbb{E}_\theta\left[-\frac{\gamma}{2}(a(\theta) - \theta_1)^2 - \gamma(a(\theta) - \theta_1)(\theta_1 - \underline{\theta}_1)\right] \\ &+ \mathbb{E}_\theta\left[-(1-\gamma)(a(\theta) - \theta_2)^2\left(\frac{1}{2} + \frac{(a^* - \underline{\theta}_2)(a^* - \overline{\theta}_2)}{2(a^* - \theta_2)^2}\right) - (1-\gamma)(a(\theta) - \theta_2)\frac{(\overline{\theta}_2 - \theta_2)(\theta_2 - \underline{\theta}_2)}{(a^* - \theta_2)}\right].\end{aligned} \tag{E.13}$$

The Lagrangian is a concave function in a. By construction, the first-order conditions are satisfied at a^*.

Step 6. Third, suppose $\overline{\theta}_1 < a^* = \underline{\theta}_2$. By Expression (E.2), the first-order condition is:

$$\begin{aligned}
&\frac{1}{\overline{\theta}_2 - \underline{\theta}_2}\left\{\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)(a^* - \theta_1) - \Lambda_1(\theta_1)\right\} \\
&+\frac{1}{\overline{\theta}_1 - \underline{\theta}_1}\left\{-(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} + \lambda_2(\theta_2)(a^* - \theta_2) - \Lambda_2(\theta_2)\right\} \\
&+\frac{(a^* - \overline{\theta}_1)}{(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)}(\gamma - \Lambda_1(\overline{\theta}_1))\mathbb{I}(\theta_1 = \overline{\theta}_1) = 0.
\end{aligned}$$

We conjecture and verify $\Lambda_1(\overline{\theta}_1) = \gamma$. Then, there exist constants α_1 and α_2 such that:

$$\begin{aligned}
\alpha_1 &= \gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)(a^* - \theta_1) - \Lambda_1(\theta_1), \\
\alpha_2 &= -(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} + \lambda_2(\theta_2)(a^* - \theta_2) - \Lambda_2(\theta_2), \text{ and} \\
0 &= \frac{\alpha_1}{\overline{\theta}_2 - \underline{\theta}_2} + \frac{\alpha_2}{\overline{\theta}_1 - \underline{\theta}_1}.
\end{aligned}$$

For agent 1, we have the same differential equation as in Step 4 (except that $a^* = \frac{\gamma\overline{\theta}_1 + (1-\gamma)(1+\underline{\theta}_2)}{2}$ coincides exactly with $\underline{\theta}_2$). Thus,

$$\Lambda_1(\theta_1) = \frac{\theta_1 - \underline{\theta}_1}{a^* - \theta_1}\left(-\frac{\gamma}{2}\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \alpha_1\right), \text{ where } \alpha_1 = \gamma\left(\frac{1}{2} + \frac{a^* - \overline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1}\right).$$

That is,

$$\Lambda_1(\theta_1) = \gamma\frac{(\theta_1 - \underline{\theta}_1)(2a^* - \overline{\theta}_1 - \theta_1)}{2(a^* - \theta_1)(\overline{\theta}_1 - \underline{\theta}_1)}. \tag{E.14}$$

It can be seen that $\Lambda_1(\underline{\theta}_1) = 0$, $\Lambda_1(\overline{\theta}_1) = \gamma$, and that the function Λ_1 has density:

$$\lambda_1(\theta_1) = \gamma\frac{(a^* - \theta_1)^2 + (a^* - \underline{\theta}_1)(a^* - \overline{\theta}_1)}{2(\overline{\theta}_1 - \underline{\theta}_1)(a^* - \theta_1)^2} \geq 0. \tag{E.15}$$

As in Step 4, we have:

$$\alpha_2 = -(1-\gamma)\left(\frac{1}{2} - \frac{a^* - \underline{\theta}_2}{\overline{\theta}_2 - \underline{\theta}_2}\right).$$

However, since $a^* = \underline{\theta}_2$, it reduces to

$$\alpha_2 = -\frac{1-\gamma}{2}.$$

Then, the differential equation for Λ_2 is:

$$-\frac{1-\gamma}{2} = -(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} + \lambda_2(\theta_2)(\underline{\theta}_2 - \theta_2) - \Lambda_2(\theta_2).$$

We show that there exists a differentiable function Λ_2 with $\Lambda_2(\underline{\theta}_2) = -\frac{1-\gamma}{2}$ and $\Lambda_2(\overline{\theta}_2) = 0$. Indeed, since this first-order linear differential equation is rewritten as

$$\frac{d}{d\theta_2}\left[\Lambda_2(\theta_2)(\underline{\theta}_2 - \theta_2)\right] = (1-\gamma)\left(\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} - \frac{1}{2}\right),$$

integrating both sides from θ_2 to $\overline{\theta}_2$ yields

$$\Lambda_2(\theta_2) = -(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{2(\overline{\theta}_2 - \underline{\theta}_2)}. \tag{E.16}$$

Thus, Λ_2 is a linear function. Then, we have:

$$\lambda_2(\theta_2) = \frac{1-\gamma}{2(\overline{\theta}_2 - \underline{\theta}_2)} \geq 0. \tag{E.17}$$

The Lagrangian is:

$$\begin{aligned}\mathcal{L} =& \mathbb{E}_\theta\left[-\gamma(a(\theta)-\theta_1)^2\left(\frac{1}{2} + \frac{(a^* - \underline{\theta}_1)(a^* - \overline{\theta}_1)}{2(a^* - \theta_1)^2}\right) - \gamma(a(\theta)-\theta_1)\frac{(\theta_1 - \underline{\theta}_1)(\overline{\theta}_1 - \theta_1)}{(a^* - \theta_1)}\right] \\ &+\mathbb{E}_\theta\left[-\frac{1-\gamma}{2}(a(\theta)-\theta_2)^2 + (1-\gamma)(a(\theta)-\theta_2)(\overline{\theta}_2 - \theta_2)\right].\end{aligned}$$

The Lagrangian is a concave function in a. By construction, the first-order conditions are satisfied at a^*.

Step 7. Fourth, suppose $\overline{\theta}_1 = a^* = \underline{\theta}_2 (= 1-\gamma)$. By Expression (E.2), the first-order condition reduces to:

$$\begin{aligned}&\frac{1}{\overline{\theta}_2 - \underline{\theta}_2}\left\{\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)(a^* - \theta_1) - \Lambda_1(\theta_1)\right\} \\ +&\frac{1}{\overline{\theta}_1 - \underline{\theta}_1}\left\{-(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} + \lambda_2(\theta_2)(a^* - \theta_2) - \Lambda_2(\theta_2)\right\} = 0.\end{aligned}$$

Thus, the first-order conditions imply that there exist constants α_1 and α_2 such that

$$\begin{aligned}
\alpha_1 =& \gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)(a^* - \theta_1) - \Lambda_1(\theta_1),\\
\alpha_2 =& -(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} + \lambda_2(\theta_2)(a^* - \theta_2) - \Lambda_2(\theta_2), \text{ and}\\
0 =& \frac{\alpha_1}{\overline{\theta}_2 - \underline{\theta}_2} + \frac{\alpha_2}{\overline{\theta}_1 - \underline{\theta}_1}.
\end{aligned}$$

Thus, the analyses in Steps 5 and 6 apply. We also see that the solutions Λ_1 and Λ_2 are obtained as the limit case of Step 4. For agent 1, we have:

$$\Lambda_1(\theta_1) = \frac{\theta_1 - \underline{\theta}_1}{a^* - \theta_1}\left(-\frac{\gamma}{2}\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \alpha_1\right), \text{ where } \alpha_1 = \gamma\left(\frac{1}{2} + \frac{a^* - \overline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1}\right) = \frac{\gamma}{2}.$$

That is,

$$\Lambda_1(\theta_1) = \gamma\frac{(\theta_1 - \underline{\theta}_1)(2a^* - \overline{\theta}_1 - \theta_1)}{2(a^* - \theta_1)(\overline{\theta}_1 - \underline{\theta}_1)} = \gamma\frac{\theta_1 - \underline{\theta}_1}{2(\overline{\theta}_1 - \underline{\theta}_1)}. \tag{E.18}$$

It can be seen that $\Lambda_1(\underline{\theta}_1) = 0$, $\Lambda_1(\overline{\theta}_1) = \frac{\gamma}{2}$, and that the function Λ_1 has its density:

$$\lambda_1(\theta_1) = \gamma\frac{(a^* - \theta_1)^2 + (a^* - \underline{\theta}_1)(a^* - \overline{\theta}_1)}{2(\overline{\theta}_1 - \underline{\theta}_1)(a^* - \theta_1)^2} = \frac{\gamma}{2(\overline{\theta}_1 - \underline{\theta}_1)} \geq 0. \tag{E.19}$$

For agent 2, we have:

$$\Lambda_2(\theta_2) = -\frac{\overline{\theta}_2 - \theta_2}{a^* - \theta_2}\left(\frac{(1-\gamma)(\overline{\theta}_2 - \theta_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)} + \alpha_2\right), \text{ where } \alpha_2 = -(1-\gamma)\left(\frac{1}{2} - \frac{a^* - \underline{\theta}_2}{\overline{\theta}_2 - \underline{\theta}_2}\right) = -\frac{1-\gamma}{2}.$$

That is,

$$\Lambda_2(\theta_2) = (1-\gamma)\frac{(\overline{\theta}_2 - \theta_2)(\theta_2 + \underline{\theta}_2 - 2a^*)}{2(a^* - \theta_2)(\overline{\theta}_2 - \underline{\theta}_2)} = -(1-\gamma)\frac{\overline{\theta}_2 - \theta_2}{2(\overline{\theta}_2 - \underline{\theta}_2)}. \tag{E.20}$$

It can be seen that $\Lambda_2(\underline{\theta}_2) = -\frac{1-\gamma}{2}$, $\Lambda_2(\overline{\theta}_2) = 0$, and the function Λ_2 has its density:

$$\lambda_2(\theta_2) = (1-\gamma)\frac{(a^* - \theta_2)^2 + (a^* - \underline{\theta}_2)(a^* - \overline{\theta}_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)(a^* - \theta_2)^2} = \frac{1-\gamma}{2(\overline{\theta}_2 - \underline{\theta}_2)} \geq 0. \tag{E.21}$$

Moreover, we have

$$\frac{\alpha_1}{\overline{\theta}_2 - \underline{\theta}_2} + \frac{\alpha_2}{\overline{\theta}_1 - \underline{\theta}_1} = \frac{\gamma(\overline{\theta}_1 - \underline{\theta}_1) - (1-\gamma)(\overline{\theta}_2 - \underline{\theta}_2)}{2(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)} = \frac{\gamma(1 - \gamma - 0) - (1-\gamma)(1 - (1-\gamma))}{2(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)} = 0.$$

The Lagrangian reduces to:

$$\mathcal{L} = -\mathbb{E}_\theta\left[\frac{\gamma}{2}(a(\theta)-\theta_1)^2 + \frac{1-\gamma}{2}(a(\theta)-\theta_2)^2 + \gamma(a(\theta)-\theta_1)(\theta_1-\underline{\theta}_1) - (1-\gamma)(a(\theta)-\theta_2)(\overline{\theta}_2-\theta_2)\right].$$

The Lagrangian is a concave function in a. By construction, the first-order conditions are satisfied at $a = a^*$. The proof is complete.

E.1.2 Part (2)

Suppose $\overline{\theta}_1 \leq \underline{\theta}_2$ and $\underline{\theta}_2 \leq \frac{2-\gamma}{1-\gamma}\overline{\theta}_1 - 1$. The latter condition is equivalent to $\overline{\theta}_1 \geq \frac{\gamma\overline{\theta}_1+(1-\gamma)(\underline{\theta}_2+1)}{2}$. Note that we interchangeably use $\underline{\theta}_1 = 0$ and $\overline{\theta}_2 = 1$, respectively. Denote by a^* the delegation solution for agent 1. Also, for ease of notation, we denote by $a^*(\theta_1) = a^*(\theta_1, \theta_2)$ as it does not depend on θ_2. Specifically, denoting by $k_1 = \frac{1-\gamma}{2-\gamma}(\underline{\theta}_2 + 1)$ agent 1's cap,

$$a^*(\theta_1) = \begin{cases} k_1 & \text{if } \theta_2 \in [0, k_1] \\ \theta_1 & \text{if } \theta_2 \in \left[k_1, \overline{\theta}_1\right] \end{cases}.$$

If $\overline{\theta}_1 = k_1$ then the delegation allocation $a^*(\cdot) = k_1$ reduces to the optimal (constant) allocation $a^*(\cdot) = \frac{\gamma\overline{\theta}_1+(1-\gamma)(\underline{\theta}_2+1)}{2}$ found in Theorem 1 (1). Thus, without loss of generality, we can assume $\overline{\theta}_1 > \frac{\gamma\overline{\theta}_1+(1-\gamma)(\underline{\theta}_2+1)}{2}$. Especially, $\overline{\theta}_1 > 1-\gamma$, as $\overline{\theta}_1 \leq \underline{\theta}_2$.

The proof consists of four steps. In the first step, we consider the following relaxed problem: for each agent i, the local IC constraint is imposed; and the allocation is required to be monotonic in agent 2's types given agent 1's types. We also set up the Lagrangian which incorporates agents' IC constraints. Denote by Λ_i the Lagrange multiplier associated with agent i's local IC constraint. Thus, the problem is to maximize the Lagrangian subject to the (relaxed) monotonicity constraint. In the second step, we explicitly incorporate the (relaxed) monotonicity constraint using the Lagrangian approach again. Denote by B the Lagrange multiplier associated with the monotonicity constraint on agent 2's types θ_2 for any given θ_1. The third step formulates the first-order conditions. The fourth step finds the multipliers $(\Lambda_1, \Lambda_2, B)$ under which the first-order conditions are met for the delegation allocation. Note that we have fewer cases than Part (1), which has dealt with the boundary cases.

Step 1. Consider the relaxed problem in which the monotonicity constraint is replaced by the one that an allocation a is monotonic in θ_2 for any given θ_1:

$$a(\theta_1, \theta_2) \text{ is non-decreasing in } \theta_2 \text{ for any given } \theta_1.$$

More precisely, we impose the following (relaxed) monotonicity constraint: for any given $\theta_1 \in [\underline{\theta}_1, \overline{\theta}_1]$,

$$\frac{\partial a(\theta_1, \theta_2)}{\partial \theta_2} \geq 0 \text{ for almost all } \theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]. \tag{E.22}$$

The problem is then to maximize the Lagrangian given by Expression (E.1) subject to the above (relaxed) monotonicity constraint (E.22). Denote by $\mathcal{L}_0$ Expression (E.1), the Lagrangian which incorporates agents' local IC constraints. Note that, as in the proof of Theorem 1 (1), we can assume, without loss of generality, $\Lambda_1(\underline{\theta}_1) = 0$ and $\Lambda_2(\overline{\theta}_2) = 0$.

Step 2. We now incorporate the monotonicity constraint (E.22) into the objective function $\mathcal{L}_0$. Thus, the Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = \mathcal{L}_0 + \int_{\underline{\theta}_1}^{\overline{\theta}_1} \int_{\underline{\theta}_2}^{\overline{\theta}_2} B(\theta_1, \theta_2) \frac{\partial a(\theta_1, \theta_2)}{\partial \theta_2} d\theta_2 d\theta_1,$$

where, for each $\theta_1 \in [0, \overline{\theta}_1]$, the function $B(\theta_1, \cdot)$ is the Lagrange multiplier associated with the monotonicity constraint $\frac{\partial a}{\partial \theta_2}(\theta_1, \cdot) \geq 0$. Hence, the Lagrangian is rewritten as:

$$\mathcal{L} =\mathcal{L}_0 + \int_{\underline{\theta}_1}^{\overline{\theta}_1} B(\theta_1, \overline{\theta}_2) a(\theta_1, \overline{\theta}_2) d\theta_1 - \int_{\underline{\theta}_1}^{\overline{\theta}_1} B(\theta_1, \underline{\theta}_2) a(\theta_1, \underline{\theta}_2) d\theta_1 - \int_{\underline{\theta}_1}^{\overline{\theta}_1} \int_{\underline{\theta}_2}^{\overline{\theta}_2} \frac{\partial B(\theta_1, \theta_2)}{\partial \theta_2} a(\theta_1, \theta_2) d\theta_2 d\theta_1. \tag{E.23}$$

Step 3. For each (θ_1, θ_2), the first-order condition at $a = a^*$ is:

$$\begin{aligned}
\frac{\partial B(\theta_1, \theta_2)}{\partial \theta_2} =& \frac{2}{\overline{\theta}_2 - \underline{\theta}_2} \left\{ -\gamma \frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} - \lambda_1(\theta_1)(a^*(\theta_1) - \theta_1) + \Lambda_1(\theta_1) \right\} \\
&+ \frac{2}{\overline{\theta}_1 - \underline{\theta}_1} \left\{ (1 - \gamma) \frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} - \lambda_2(\theta_2)(a^*(\theta_1) - \theta_2) + \Lambda_2(\theta_2) \right\} \\
&- \frac{1}{\overline{\theta}_2 - \underline{\theta}_2} \left\{ B(\theta_1, \underline{\theta}_2) + \frac{2}{\overline{\theta}_1 - \underline{\theta}_1} (a^*(\theta_1) - \underline{\theta}_2)(1 - \gamma + \Lambda_2(\underline{\theta}_2)) \right\} \mathbb{I}(\theta_2 = \underline{\theta}_2) \\
&+ \frac{B(\theta_1, \overline{\theta}_2)}{\overline{\theta}_2 - \underline{\theta}_2} \mathbb{I}(\theta_2 = \overline{\theta}_2).
\end{aligned} \tag{E.24}$$

Note that we have used $a^*(\overline{\theta}_1) = \overline{\theta}_1$.

Step 4. From now on, we will construct $(\Lambda_1, \Lambda_2, B)$ which satisfy the first-order conditions at a^*. We conjecture and verify the following:

$$B(\theta_1, \overline{\theta}_2) = 0 \text{ for all } \theta_1 \in [\underline{\theta}_1, \overline{\theta}_1], \text{ and} \tag{E.25}$$

$$\begin{aligned} B(\theta_1, \underline{\theta}_2) &= -\frac{2}{\overline{\theta}_1 - \underline{\theta}_1}(a^*(\theta_1) - \underline{\theta}_2)(1 - \gamma + \Lambda_2(\underline{\theta}_2)) \\ &= \begin{cases} -\frac{2}{\overline{\theta}_1 - \underline{\theta}_1}(k_1 - \underline{\theta}_2)(1 - \gamma + \Lambda_2(\underline{\theta}_2)) & \text{if } \theta_1 \in [0, k_1] \\ -\frac{2}{\overline{\theta}_1 - \underline{\theta}_1}(\theta_1 - \underline{\theta}_2)(1 - \gamma + \Lambda_2(\underline{\theta}_2)) & \text{if } \theta_1 \in \left[k_1, \overline{\theta}_1\right] \end{cases}. \end{aligned} \tag{E.26}$$

We substitute a^* into Expression (E.24) to get the following two forms of first-order conditions. First, for $\theta_1 \in [0, k_1]$,

$$\begin{aligned} \frac{\partial B(\theta_1, \theta_2)}{\partial \theta_2} &= \frac{2}{\overline{\theta}_2 - \underline{\theta}_2}\left\{-\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} - \lambda_1(\theta_1)\left(k_1 - \theta_1\right) + \Lambda_1(\theta_1)\right\} \\ &+ \frac{2}{\overline{\theta}_1 - \underline{\theta}_1}\left\{(1 - \gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} - \frac{d}{d\theta_2}\left[\Lambda_2(\theta_2)\left(k_1 - \theta_2\right)\right]\right\}. \end{aligned}$$

Second, for $\theta_1 \in \left[k_1, \overline{\theta}_1\right]$,

$$\begin{aligned} \frac{\partial B(\theta_1, \theta_2)}{\partial \theta_2} &= \frac{2}{\overline{\theta}_2 - \underline{\theta}_2}\left\{-\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \Lambda_1(\theta_1)\right\} - \frac{2\theta_1}{\overline{\theta}_1 - \underline{\theta}_1}\lambda_2(\theta_2) \\ &+ \frac{2}{\overline{\theta}_1 - \underline{\theta}_1}\left\{(1 - \gamma)\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} + \frac{d}{d\theta_2}\left[\Lambda_2(\theta_2)\theta_2\right]\right\}. \end{aligned}$$

We integrate each of the above equations with respect to θ_2 from θ_2 to $\overline{\theta}_2$. For $\theta_1 \in [0, k_1]$,

$$\begin{aligned} B(\theta_1, \theta_2) &= 2\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2}\left\{\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)\left(k_1 - \theta_1\right) - \Lambda_1(\theta_1)\right\} \\ &- \frac{2}{\overline{\theta}_1 - \underline{\theta}_1}\left\{(1 - \gamma)\frac{(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)} + \left(k_1 - \theta_2\right)\Lambda_2(\theta_2)\right\}. \end{aligned} \tag{E.27}$$

For $\theta_1 \in \left[k_1, \overline{\theta}_1\right]$,

$$\begin{aligned} B(\theta_1, \theta_2) &= 2\frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2}\left\{\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} - \Lambda_1(\theta_1)\right\} - \frac{2\theta_1}{\overline{\theta}_1 - \underline{\theta}_1}\Lambda_2(\theta_2) \\ &+ \frac{2}{\overline{\theta}_1 - \underline{\theta}_1}\left\{-(1 - \gamma)\frac{(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)} + \Lambda_2(\theta_2)\theta_2\right\}. \end{aligned} \tag{E.28}$$

We substitute $\theta_2 = \underline{\theta}_2$ into the above equations. For $\theta_1 \in [0, k_1]$,

$$B(\theta_1, \underline{\theta}_2) = 2\left\{\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + \lambda_1(\theta_1)\,(k_1 - \theta_1) - \Lambda_1(\theta_1)\right\} - \frac{2}{\overline{\theta}_1 - \underline{\theta}_1}\left\{(1-\gamma)\frac{\overline{\theta}_2 - \underline{\theta}_2}{2} + (k_1 - \underline{\theta}_2)\Lambda_2(\underline{\theta}_2)\right\}. \tag{E.29}$$

For $\theta_1 \in \left[k_1, \overline{\theta}_1\right]$,

$$B(\theta_1, \underline{\theta}_2) = 2\left\{\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} - \Lambda_1(\theta_1)\right\} - 2\frac{\theta_1 - \underline{\theta}_2}{\overline{\theta}_1 - \underline{\theta}_1}\Lambda_2(\underline{\theta}_2) - (1-\gamma)\frac{\overline{\theta}_2 - \underline{\theta}_2}{\overline{\theta}_1 - \underline{\theta}_1}. \tag{E.30}$$

Now, we solve for Λ_1. First, since $\Lambda_1(\underline{\theta}_1) = 0$, we start with the case when $\theta_1 \in [0, k_1]$. For $\theta_1 \in [0, k_1]$, it follows from Expressions (E.26) and (E.29) that

$$\lambda_1(\theta_1)\,(k_1 - \theta_1) - \Lambda_1(\theta_1) + \gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} + (1-\gamma)\frac{2k_1 - \underline{\theta}_2 - \overline{\theta}_2}{2(\overline{\theta}_1 - \underline{\theta}_1)} = 0.$$

This is a linear first-order differential equation. In fact, since we have

$$\frac{d}{d\theta_1}\left[\Lambda_1(\theta_1)\,(k_1 - \theta_1)\right] = -\gamma\frac{\theta_1 - \underline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1} - (1-\gamma)\frac{2k_1 - \underline{\theta}_2 - \overline{\theta}_2}{2(\overline{\theta}_1 - \underline{\theta}_1)},$$

integrating both sides from $\underline{\theta}_1$ to θ_1 yields

$$\begin{aligned}\Lambda_1(\theta_1)\,(k_1 - \theta_1) &= -\gamma\frac{(\theta_1 - \underline{\theta}_1)^2}{2(\overline{\theta}_1 - \underline{\theta}_1)} - (1-\gamma)\frac{(2k_1 - \underline{\theta}_2 - \overline{\theta}_2)(\theta_1 - \underline{\theta}_1)}{2(\overline{\theta}_1 - \underline{\theta}_1)} \\ &= -\frac{(\theta_1 - \underline{\theta}_1)(\gamma(\theta_1 - \underline{\theta}_1) + (1-\gamma)(2k_1 - \underline{\theta}_2 - \overline{\theta}_2))}{2(\overline{\theta}_1 - \underline{\theta}_1)}.\end{aligned}$$

Hence, for $\theta_1 \in [0, k_1]$, we have:

$$\Lambda_1(\theta_1) = -\frac{(\theta_1 - \underline{\theta}_1)(\gamma(\theta_1 - \underline{\theta}_1) + (1-\gamma)(2k_1 - \underline{\theta}_2 - \overline{\theta}_2))}{2(\overline{\theta}_1 - \underline{\theta}_1)(k_1 - \theta_1)}.$$

In order for $\Lambda_1(k_1)$ to be well-defined, we have to have

$$(1-\gamma)(2k_1 - \underline{\theta}_2 - \overline{\theta}_2) = -\gamma k_1, \text{ that is, } k_1 = \frac{1-\gamma}{2-\gamma}(1 + \underline{\theta}_2),$$

provided that $k_1 \neq 0$. Then, we obtain

$$\Lambda_1(\theta_1) = \frac{\gamma}{2}\frac{(\theta_1 - \underline{\theta}_1)}{(\overline{\theta}_1 - \underline{\theta}_1)}.$$

When $\theta_1 = k_1$, we have

$$\Lambda_1(k_1) = \frac{\gamma(k_1 - \underline{\theta}_1)}{2(\overline{\theta}_1 - \underline{\theta}_1)} = \frac{\gamma(1-\gamma)}{2(2-\gamma)} \frac{1+\underline{\theta}_2}{\overline{\theta}_1 - \underline{\theta}_1}.$$

Second, when $\theta_1 \in [k_1, \overline{\theta}_1]$, it follows from Expressions (E.26) and (E.30) that

$$\Lambda_1(\theta_1) = \frac{2\gamma(\theta_1 - \underline{\theta}_1) + (1-\gamma)(\overline{\theta}_2 - \underline{\theta}_2)}{2(\overline{\theta}_1 - \underline{\theta}_1)} = \frac{\theta_1}{\overline{\theta}_1 - \underline{\theta}_1} - \frac{1-\gamma}{2} \frac{1+\underline{\theta}_2}{\overline{\theta}_1 - \underline{\theta}_1} = \frac{2\theta_1 - (1-\gamma)(1+\underline{\theta}_2)}{2(\overline{\theta}_1 - \underline{\theta}_1)}.$$

It can be seen that

$$\Lambda_1(k_1) = \frac{\gamma(1-\gamma)(1+\underline{\theta}_2)}{2(2-\gamma)(\overline{\theta}_1 - \underline{\theta}_1)},$$

as desired. In sum, Λ_1 is a piece-wise-linear and continuous function given by

$$\Lambda_1(\theta_1) = \begin{cases} \dfrac{\gamma}{2} \dfrac{(\theta_1 - \underline{\theta}_1)}{(\overline{\theta}_1 - \underline{\theta}_1)} & \text{if } \theta_1 \in [0, k_1] \\ \dfrac{2\theta_1 - (1-\gamma)(1+\underline{\theta}_2)}{2(\overline{\theta}_1 - \underline{\theta}_1)} & \text{if } \theta_1 \in [k_1, \overline{\theta}_1] \end{cases}. \tag{E.31}$$

Also, we have

$$\lambda_1(\theta_1) = \begin{cases} \dfrac{\gamma}{2(\overline{\theta}_1 - \underline{\theta}_1)} \geq 0 & \text{if } \theta_1 \in [0, k_1) \\ \dfrac{1}{\overline{\theta}_1 - \underline{\theta}_1} \geq 0 & \text{if } \theta_1 \in (k_1, \overline{\theta}_1] \end{cases}. \tag{E.32}$$

Now, we substitute Λ_1 and λ_1 into Expressions (E.27) and (E.28) to rewrite B:

$$B(\theta_1,\theta_2) = \begin{cases} -\dfrac{2}{\overline{\theta}_1 - \underline{\theta}_1} \left\{ (1-\gamma)\dfrac{(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)} + (k_1 - \theta_2)\Lambda_2(\theta_2) \right\} + \dfrac{\gamma(k_1 - \underline{\theta}_1)(\overline{\theta}_2 - \theta_2)}{(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)} & \text{if } \theta_1 \in [0, k_1] \\ -\dfrac{2}{\overline{\theta}_1 - \underline{\theta}_1} \left\{ (1-\gamma)\dfrac{(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)} + (\theta_1 - \theta_2)\Lambda_2(\theta_2) \right\} + \dfrac{(1-\gamma)(1+\underline{\theta}_2 - 2\theta_1)(\overline{\theta}_2 - \theta_2)}{(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)} & \text{if } \theta_1 \in [k_1, \overline{\theta}_1] \end{cases}. \tag{E.33}$$

It can be seen that B satisfies Expressions (E.25) and (E.26).

Next, we conjecture and verify:

$$B(\theta_1, \theta_2) = 0 \text{ if } \theta_1 \in [0, k_1]. \tag{E.34}$$

In fact, it follows from Expression (E.33) that Expression (E.34) holds if and only if

$$\Lambda_2(\theta_2) = \frac{\gamma}{2} \frac{k_1 - \underline{\theta}_1}{k_1 - \theta_2} \frac{\overline{\theta}_2 - \theta_2}{\overline{\theta}_2 - \underline{\theta}_2} - \frac{(1-\gamma)(\overline{\theta}_2 - \theta_2)^2}{2(\overline{\theta}_2 - \underline{\theta}_2)(k_1 - \theta_2)} = -\frac{(1-\gamma)(\overline{\theta}_2 - \theta_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)} \left(1 + \frac{k_1 - \underline{\theta}_2}{k_1 - \theta_2}\right). \tag{E.35}$$

Note that we especially have

$$\Lambda_2(\underline{\theta}_2) = -(1-\gamma).$$

Also, we have

$$\lambda_2(\theta_2) = \frac{1-\gamma}{2(\overline{\theta}_2 - \underline{\theta}_2)} + \frac{(1-\gamma)(\underline{\theta}_2 - k_1)(\overline{\theta}_2 - k_1)}{2(\overline{\theta}_2 - \underline{\theta}_2)(k_1-\theta_2)^2} \geq 0. \tag{E.36}$$

Since we have found Λ_1 and Λ_2, we remark on the limit case in which $\overline{\theta}_1 = k_1$ holds, which is covered in the proof of Theorem 1 (1). In the limit case, $a^* = k_1$. For agent 2, Expressions (E.9) and (E.35) coincide, as it can be seen that

$$\frac{(1-\gamma)(\overline{\theta}_2 - \theta_2)(\theta_2 + \underline{\theta}_2 - 2k_1)}{2(k_1-\theta_2)(\overline{\theta}_2 - \underline{\theta}_2)} = \frac{(1-\theta_2)(\gamma k_1 - (1-\gamma)(1-\theta_2))}{2(k_1-\theta_2)(1-\underline{\theta}_2)}.$$

Generally, for agent 1, Expressions (E.11) and (E.31) coincide for $\theta_1 \in [0, k_1]$. Thus, in the particular limit case $\overline{\theta}_1 = k_1$, they coincide on Θ_1.

Substituting Expression (E.35) into Expression (E.33), we obtain B as:

$$B(\theta_1,\theta_2) = \begin{cases} 0 & \text{if } \theta_1 \in [0, k_1] \\ (1-\gamma)\dfrac{(\theta_2 - \underline{\theta}_2)(\theta_2 - 1)((1-\gamma)(\underline{\theta}_2+1) - (2-\gamma)\theta_1)}{(\overline{\theta}_1 - \underline{\theta}_1)(1-\underline{\theta}_2)((1-\gamma)(\underline{\theta}_2+1) - (2-\gamma)\theta_2)} & \text{if } \theta_1 \in \left[k_1, \overline{\theta}_1\right] \end{cases}. \tag{E.37}$$

where we have used $\overline{\theta}_2 = 1$. The Lagrange multiplier on the monotonicity constraint is zero when $\theta_1 \in [0, k_1]$.

Since the Lagrangian reduces to

$$\begin{aligned}\mathcal{L} =& -2\gamma \mathbb{E}_\theta\left[(a(\theta)-\theta_1)(\theta_1 - \underline{\theta}_1)\right] - \int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[(a(\theta)-\theta_1)^2\right]\lambda_1(\theta_1)d\theta_1 + 2\int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_{\theta_2}\left[a(\theta)-\theta_1\right]\Lambda_1(\theta_1)d\theta_1 \\ &+ 2(1-\gamma)\mathbb{E}_\theta\left[(a(\theta)-\theta_2)(\overline{\theta}_2 - \theta_2)\right] - \int_{\underline{\theta}_2}^{\overline{\theta}_2} \mathbb{E}_{\theta_1}\left[(a(\theta)-\theta_2)^2\right]\lambda_2(\theta_2)d\theta_2 + 2\int_{\underline{\theta}_2}^{\overline{\theta}_2} \mathbb{E}_{\theta_1}\left[a(\theta)-\theta_2\right]\Lambda_2(\theta_2)d\theta_2 \\ &- \int_{k_1}^{\overline{\theta}_1}\int_{\underline{\theta}_2}^{\overline{\theta}_2} \frac{\partial B(\theta_1,\theta_2)}{\partial \theta_2} a(\theta_1,\theta_2)d\theta_2 d\theta_1,\end{aligned}$$

the Lagrangian $\mathcal{L}$ is a concave function in a. Thus, the first-order conditions are sufficient.

E.1.3 Part (3)

The proof is similar to that of Part (2), exchanging the role of agents 1 and 2. Especially, we consider the relaxed monotonicity constraint which requires the allocation to be monotonic in agent 1's types given agent 2's types. Here, we only report the Lagrange multipliers. To that end, denote by k_2 agent 2's cap:

$$k_2 = \frac{\gamma\overline{\theta}_1 + 1 - \gamma}{1+\gamma}.$$

Now, for agent 1's local IC constraint, the multiplier Λ_1 is:

$$\Lambda_1(\theta_1) = \frac{\gamma}{2}\frac{\theta_1 - \underline{\theta}_1}{(\overline{\theta}_1 - \underline{\theta}_1)}\left\{1 + \frac{k_2 - \overline{\theta}_1}{k_2 - \theta_1}\right\}. \tag{E.38}$$

Especially, its density is:

$$\lambda_1(\theta_1) = \frac{\gamma}{2(\overline{\theta}_1 - \underline{\theta}_1)} + \frac{\gamma}{2}\frac{k_2 - \overline{\theta}_1}{\overline{\theta}_1 - \underline{\theta}_1}\frac{k_2 - \underline{\theta}_1}{(k_2 - \theta_1)^2} \geq 0. \tag{E.39}$$

For agent 2's local IC constraint, the multiplier Λ_2 is:

$$\Lambda_2(\theta_2) = \begin{cases} \dfrac{2\theta_2 - \gamma\overline{\theta}_1 - 2(1-\gamma)}{2(\overline{\theta}_2 - \underline{\theta}_2)} & \text{if } \theta_2 \in [\underline{\theta}_2, k_2] \\ -\dfrac{(1-\gamma)(\overline{\theta}_2 - \theta_2)}{2(\overline{\theta}_2 - \underline{\theta}_2)} & \text{if } \theta_2 \in [k_2, 1] \end{cases}. \tag{E.40}$$

Consequently, its density is:

$$\lambda_2(\theta_2) = \begin{cases} \dfrac{1}{\overline{\theta}_2 - \underline{\theta}_2} \geq 0 & \text{if } \theta_2 \in [\underline{\theta}_2, k_2) \\ \dfrac{1-\gamma}{2(\overline{\theta}_2 - \underline{\theta}_2)} \geq 0 & \text{if } \theta_2 \in (k_2, 1] \end{cases}. \tag{E.41}$$

For the Monotonicity constraint, denote by B the Lagrange multiplier associated with the monotonicity constraint on agent 1's types θ_1 (for any given θ_2). Then,

$$B(\theta_1, \theta_2) = \begin{cases} \gamma\dfrac{(\theta_1 - \underline{\theta}_1)(\theta_1 - \overline{\theta}_1)(1 - \gamma + \gamma\overline{\theta}_1 - (1+\gamma)\theta_2)}{(\overline{\theta}_1 - \underline{\theta}_1)(\overline{\theta}_2 - \underline{\theta}_2)(1 - \gamma + \gamma\overline{\theta}_1 - (1+\gamma)\theta_1)} & \text{if } \theta_2 \in [\underline{\theta}_2, k_2] \\ 0 & \text{if } \theta_2 \in [k_2, 1] \end{cases}. \tag{E.42}$$

Thus, the Lagrange multiplier on the monotonicity constraint is zero when $\theta_2 \in [k_2, 1]$.

Finally, we remark on the limit case in which $\underline{\theta}_2 = k_2$ holds, which is covered in the proof of Theorem 1 (1). For agent 1, Expressions (E.14) and (E.38) coincide. Generally, for agent 2, Expressions (E.16) and (E.40) coincide for $\theta_2 \in [k_2, 1]$. Thus, in the particular limit case $\underline{\theta}_2 = k_2$, they coincide on Θ_2.

E.1.4 Part (4)

The proof of Theorem 1 (4) is in two steps. First, if an optimal allocation depends on at most one agent's information then the optimal allocation satisfies ex-post IC constraints, including Monotonicity. In particular, the optimal allocation which depends on at most one agent's information is a constrained delegation solution. Note that the constrained delegation solution is continuous. Second, we find the unique optimal continuous ex-post IC allocation, which can depend on both agents' information. We show that this allocation depends on both agents' information and is better than the best allocation which depends on at most one agent's information.

Step 1. Consider an allocation a which depends on (at most) agent i's information. Denote by $a(\theta) = a(\theta_i)$. Denote by γ_i agent i's Pareto weight. Since the objective function is

$$-\int_{\underline{\theta}_i}^{\overline{\theta}_i}\int_{\underline{\theta}_{-i}}^{\overline{\theta}_{-i}} \left\{\gamma_i(a(\theta_i)-\theta_i)^2 + \gamma_{-i}(a(\theta_i)-\theta_{-i})^2\right\} dF_i(\theta_i)dF_{-i}(\theta_{-i}),$$

the problem reduces to:

$$\begin{aligned}\max_a -\int_{\underline{\theta}_i}^{\overline{\theta}_i} &(a(\theta_i) - (\gamma_i\theta_i + \gamma_{-i}\mathbb{E}_{\theta_{-i}}[\theta_{-i}]))^2 f_i(\theta_i)d\theta_i \\ \text{subject to } &-(a(\theta_i)-\theta_i)^2 \geq -(a(\hat{\theta}_i)-\theta_i)^2.\end{aligned}$$

Thus, without loss, we can restrict attention to ex-post IC constraints. In particular, since ex-post IC constraints imply Monotonicity, if an optimal allocation depends on at most one agent's types then it satisfies Monotonicity naturally without imposing it.

For this problem, similarly to Melumad and Shibano (1991), one can show that the optimal allocation a^i is a constrained delegation allocation with possibly two caps:

$$a^i(\theta_i) = \min\left(k_i^h, \max\left(\theta_i, k_i^\ell\right)\right) = \begin{cases} k_i^\ell & \text{if } \theta_i \in [\underline{\theta}_i, k_i^\ell] \\ \theta_i & \text{if } \theta_i \in [k_i^\ell, k_i^h] \\ k_i^h & \text{if } \theta_i \in [k_i^h, \overline{\theta}_i] \end{cases},$$

where $\underline{\theta}_i \leq k_i^\ell \leq k_i^h \leq \overline{\theta}_i$ (note that if $k_i^\ell = k_i^h$ then the allocation is constant). Also, note that a^i is continuous. Substituting the constrained delegation allocation a^i into the social welfare, we can find the optimal cutoffs (k_i^ℓ, k_i^h) by taking the first-order conditions. See Remark E.1 at the end of this subsection for the optimal allocations a^1 and a^2.

Step 2. The second step finds the unique optimal continuous ex-post IC allocation, which indeed depends on both agents' information and which yields strictly better social welfare than the best allocation in the first step. The proof of this step is in four sub-steps.

In the first sub-step, as Martimort and Semenov (2008) characterize continuous and ex-post IC allocations in the uniform-quadratic setting (see also Moulin, 1980), also in our setting of Θ, an allocation a is ex-post IC and continuous if and only if

$$a(\theta_1,\theta_2) = \min\left(x, \max(\theta_1, y_1), \max(\theta_2, y_2), \max(\theta_1, \theta_2, z)\right)$$

for some (x, y_1, y_2, z) with $z \le y_1, y_2 \le x$.

In the second sub-step, we show that, for an optimal continuous ex-post IC allocation, x and z can be dropped. Intuitively, if $x < \overline{\theta}_1$, then the allocation a does not use agent 1's information even when both agents' types lie in the common set $[x, \overline{\theta}_1]$, which is inefficient. Likewise, if $z > \underline{\theta}_2$, then the allocation a does not use agent 2's information even when both agents' types lie in the common set $[\underline{\theta}_2, z]$, which is inefficient.

Lemma E.1. *Suppose $\underline{\theta}_2 \le \overline{\theta}_1$. An optimal ex-post incentive-compatible and continuous allocation a satisfies*

$$a(\theta_1,\theta_2) = \min\left(\max(\theta_1, y_1), \max(\theta_2, y_2), \max(\theta_1, \theta_2)\right) \text{ for some } (y_1, y_2).$$

Proof of Lemma E.1. Suppose that x and z are essential in an optimal allocation, i.e., $\underline{\theta}_2 \le z \le y_1, y_2 \le x \le \overline{\theta}_1$. We show that it is optimal to set $x = \overline{\theta}_1$ and $z = \underline{\theta}_2$, in which case, it is without loss to drop x and z from the expression for a.

First, we determine the optimal constant x. The part that depends on x is

$$\begin{aligned} &-\frac{1}{\overline{\theta}_1(1-\underline{\theta}_2)} \int_x^{\overline{\theta}_1} \int_x^1 \left\{\gamma(x-\theta_1)^2 + (1-\gamma)(x-\theta_2)^2\right\} d\theta_2 d\theta_1 \\ &= -\frac{(\overline{\theta}_1 - x)(1-x)\left((x-(1-\gamma+\gamma\overline{\theta}_1))^2 + \gamma(1-\gamma)(1-\overline{\theta}_1)^2\right)}{3\overline{\theta}_1(1-\underline{\theta}_2)} \le 0, \end{aligned}$$

and the unique maximum (in $x \le \overline{\theta}_1$) is obtained at $x = \overline{\theta}_1$.

Second, to determine the optimal constant z, the part that depends on z is

$$\begin{aligned} &-\frac{1}{\overline{\theta}_1(1-\underline{\theta}_2)} \int_0^z \int_{\underline{\theta}_2}^z \left\{\gamma(z-\theta_1)^2 + (1-\gamma)(z-\theta_2)^2\right\} d\theta_2 d\theta_1 \\ &= -\frac{(z-(1-\gamma)\underline{\theta}_2)^2 + \gamma(1-\gamma)\underline{\theta}_2^2}{3\overline{\theta}_1(1-\underline{\theta}_2)}(z-\underline{\theta}_2)z \le 0. \end{aligned}$$

The unique maximum (in $z \geq \underline{\theta}_2$) is attained at $z = \underline{\theta}_2$. $\square$

In the third sub-step, we find the optimal continuous ex-post IC allocation.

Lemma E.2. *Suppose $\underline{\theta}_2 \leq \overline{\theta}_1$. The optimal ex-post incentive-compatible and continuous allocation a is*

$$a(\theta_1, \theta_2) = \min\left(\max(\theta_1, 1-\gamma), \max(\theta_2, \gamma\overline{\theta}_1 + (1-\gamma)\underline{\theta}_2), \max(\theta_1, \theta_2)\right).$$

Figures E.1 and E.2 illustrate the optimal continuous ex-post IC allocation (when $\underline{\theta}_2 < 1-\gamma < \overline{\theta}_1$). It can be seen that a indeed depends on both agents' information.

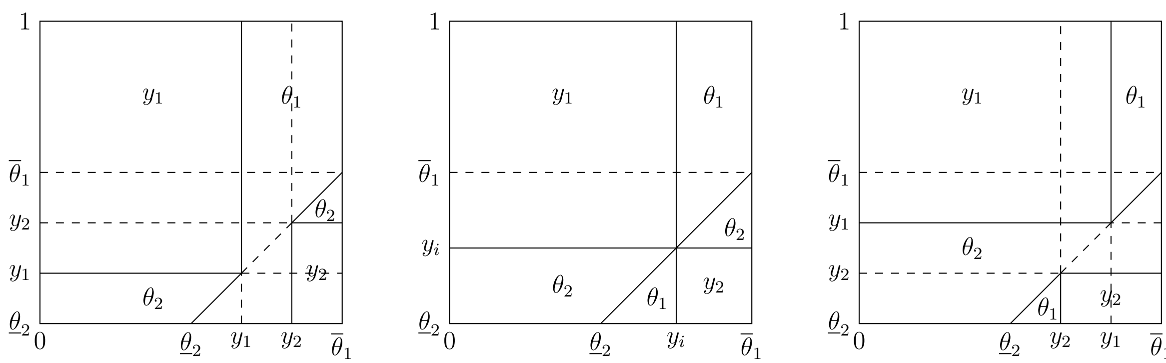

Figure E.1: Illustration of the Optimal Continuous Ex-Post IC Allocation (when $\underline{\theta}_2 < 1-\gamma < \overline{\theta}_1$). The optimal continuous ex-post IC allocation has constants $(y_1, y_2) = (1-\gamma, \gamma\overline{\theta}_1 + (1-\gamma)\underline{\theta}_2)$. The left panel depicts the optimal allocation when $\frac{1-\underline{\theta}_2}{1+\overline{\theta}_1-\underline{\theta}_2} < \gamma$, in which case $y_1 < y_2$. The central panel depicts the optimal allocation when $\frac{1-\underline{\theta}_2}{1+\overline{\theta}_1-\underline{\theta}_2} = \gamma$, in which case $y_1 = y_2$. The right panel depicts the optimal allocation when $\frac{1-\underline{\theta}_2}{1+\overline{\theta}_1-\underline{\theta}_2} > \gamma$, in which case $y_1 > y_2$.

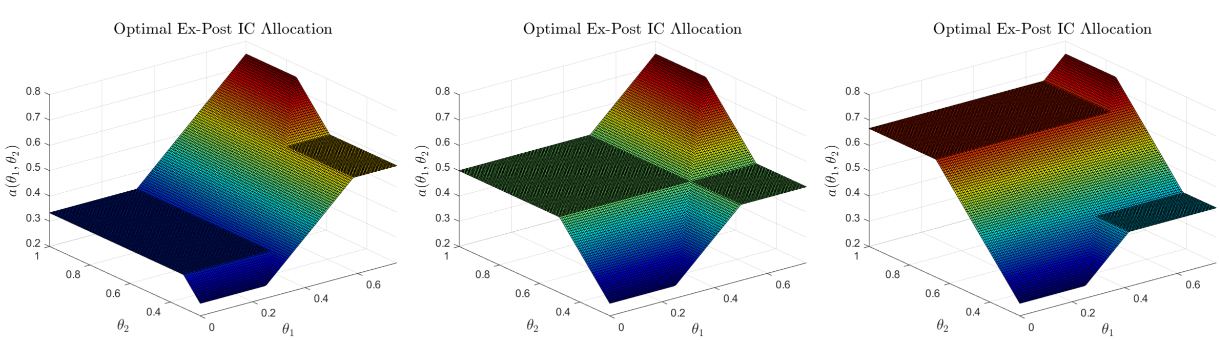

Figure E.2: Illustration of the Optimal Continuous Ex-Post IC Allocation: $\overline{\theta}_1 = \frac{3}{4}$ and $\underline{\theta}_2 = \frac{1}{4}$. The left panel depicts the optimal allocation when $\gamma = \frac{2}{3}$. The central panel depicts the optimal allocation when $\gamma = \frac{1}{2}$. The right panel depicts the optimal allocation when $\gamma = \frac{1}{3}$.

Proof of Lemma E.2. It follows from Lemma E.1 that the allocation reduces to

$$a(\theta_1,\theta_2)=\begin{cases}\min(\theta_2,\max(\theta_1,y_1))=\text{med}(\theta_1,\theta_2,y_1) & \text{if } \theta_1\le\theta_2\\ \min(\theta_1,\max(\theta_2,y_2))=\text{med}(\theta_1,\theta_2,y_2) & \text{if } \theta_2\le\theta_1\end{cases}.$$

First, the part of the social welfare that depends on y_1 is:

$$\begin{aligned}W_1:=&-\frac{\gamma}{\overline{\theta}_1(1-\underline{\theta}_2)}\int_{\underline{\theta}_2}^{y_1}\int_0^{\theta_2}(\theta_2-\theta_1)^2d\theta_1d\theta_2\\&-\frac{1}{\overline{\theta}_1(1-\underline{\theta}_2)}\int_0^{y_1}\int_{y_1}^{1}\left\{\gamma(y_1-\theta_1)^2+(1-\gamma)(y_1-\theta_2)^2\right\}d\theta_2d\theta_1\\&-\frac{1-\gamma}{\overline{\theta}_1(1-\underline{\theta}_2)}\int_{y_1}^{\overline{\theta}_1}\int_{\theta_1}^{1}(\theta_1-\theta_2)^2d\theta_2d\theta_1.\end{aligned}$$

It can be seen that W_1 is a quartic equation with a positive coefficient on y_1^4. Differentiating, its derivative is

$$\frac{\partial W_1}{\partial y_1}=\frac{y_1(y_1-(1-\gamma))(y_1-1)}{\overline{\theta}_1(1-\underline{\theta}_2)}.$$

Thus, the first-order condition yields three distinct roots which are ordered by $0<1-\gamma<1$. Hence, we obtain

$$y_1=1-\gamma.$$

Note that if we restrict attention to $y_1\in[\underline{\theta}_2,\overline{\theta}_1]$, then

$$y_1=\text{med}(\underline{\theta}_2,1-\gamma,\overline{\theta}_1)=\begin{cases}\overline{\theta}_1 & \text{if } \overline{\theta}_1\le 1-\gamma\\ 1-\gamma & \text{if } \underline{\theta}_2\le 1-\gamma\le\overline{\theta}_1\\ \underline{\theta}_2 & 1-\gamma\le\underline{\theta}_2\end{cases}.$$

However, since the value of $\text{med}(\theta_1,\theta_2,y_1)$ does not change, we can simply take $y_1=1-\gamma$.

Second, the part of the social welfare that depends on y_2 is:

$$\begin{aligned}W_2:=&-\frac{1-\gamma}{\overline{\theta}_1(1-\underline{\theta}_2)}\int_{\underline{\theta}_2}^{y_2}\int_{\underline{\theta}_2}^{\theta_1}(\theta_1-\theta_2)^2d\theta_2d\theta_1\\&-\frac{1}{\overline{\theta}_1(1-\underline{\theta}_2)}\int_{y_2}^{\overline{\theta}_1}\int_{\underline{\theta}_2}^{y_2}\{\gamma(y_2-\theta_1)^2+(1-\gamma)(y_2-\theta_2)^2\}d\theta_2d\theta_1\\&-\frac{\gamma}{\overline{\theta}_1(1-\underline{\theta}_2)}\int_{y_2}^{\overline{\theta}_1}\int_{y_2}^{\theta_1}(\theta_2-\theta_1)^2d\theta_2d\theta_1.\end{aligned}$$

It can be seen that W_2 is a quartic function in y_2 with a positive coefficient on y_2^4, and its derivative is

$$\frac{\partial W_2}{\partial y_2} = \frac{(y_2 - \underline{\theta}_2)(y_2 - \overline{\theta}_1)(y_2 - (\gamma\overline{\theta}_1 + (1-\gamma)\underline{\theta}_2))}{\overline{\theta}_1(1-\underline{\theta}_2)}.$$

Since the first-order condition yields the three distinct roots which are ordered as

$$\underline{\theta}_2 < \gamma\overline{\theta}_1 + (1-\gamma)\underline{\theta}_2 < \overline{\theta}_1,$$

we obtain the best constant y_2 as

$$y_2 = \gamma\overline{\theta}_1 + (1-\gamma)\underline{\theta}_2.$$

The proof is complete. Note that the optimal allocation is unique. □

Now, by construction, the optimal continuous ex-post IC allocation a yields strictly better social welfare than a^1 and a^2. The proof of Part (4) is complete.

To conclude this subsection, we make three remarks on the comparisons of social welfare. First, when $\Theta_1 = \Theta_2 = [0,1]$ and $\gamma = \frac{1}{2}$, Figure E.3 depicts the optimal (continuous) ex-post IC allocation, which is

$$a(\theta) = \text{med}\left(\theta_1, \theta_2, \frac{1}{2}\right).$$

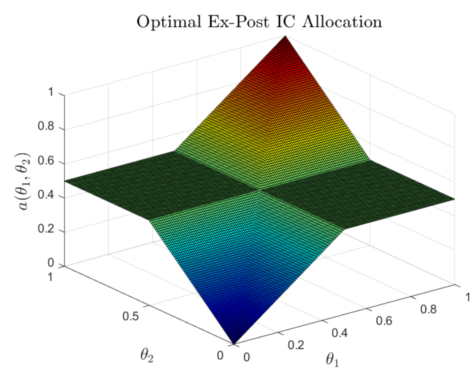

Figure E.3: The Optimal Continuous Ex-Post IC Allocation: $\overline{\theta}_1 = 1$, $\underline{\theta}_2 = 0$, and $\gamma = \frac{1}{2}$.

Second, notice that the best allocation which depends on at most one agent's information is written as

$$\begin{aligned} a^i(\theta_i) &= \min(k_i^h, \max(\theta_i, k_i^\ell)) \\ &= \min\left(k_i^h, \max(\theta_i, k_i^\ell), \max(\theta_{-i}, k_i^h), \max(\theta_1, \theta_2, k_i^\ell)\right). \end{aligned}$$

Note that the last term $\max(\theta_1, \theta_2, k_i^\ell)$ is redundant and a^i depends only on θ_i. Now, even

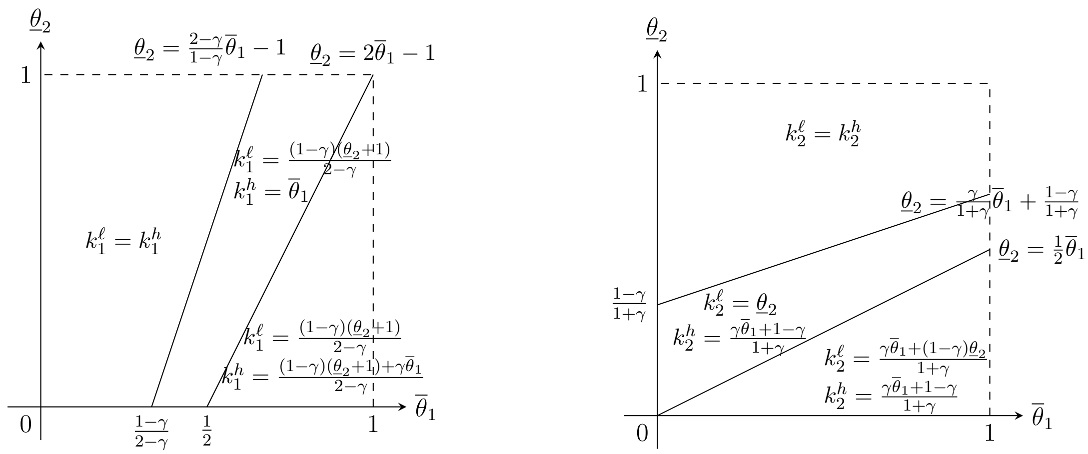

Figure E.4: Cutoffs for Delegation Allocations a^1 (Left) and $a^2(\cdot)$ (Right). In the left panel, if $k_1^\ell = k_1^h$ then the best allocation (which depends on at most agent 1's information) is the best constant allocation. Likewise, in the right panel, if $k_2^\ell = k_2^h$ then the best allocation (which depends on at most agent 2's information) is the best constant allocation.

the allocation

$$a(\theta_1, \theta_2) = \min\left(\max(\theta_i, k_i^\ell), \max(\theta_{-i}, k_i^h), \max(\theta_1, \theta_2)\right)$$

turns out to be a strict improvement, although it is cumbersome to consider many cases in which the cutoffs and the end-points of agents' type sets are ordered and also to prove the improvement for both agents (comparing the social welfare associated with a^1 and a^2 is quite cumbersome when γ may not be $\frac{1}{2}$). Intuitively, if both agents' types are above the higher cap k_1^h of agent 1 then it is better to make the allocation the minimum of the agents' types instead of the constant; and similarly, if both agents' types are below the lower cap k_2^ℓ of agent 2 then it is better to make the allocation the maximum of the types. The optimal continuous ex-post IC allocation is indeed of this form with different constants $1-\gamma$ and $\gamma\overline{\theta}_1 + (1-\gamma)(1+\underline{\theta}_2)$.

Third, to conclude the proof of Part (4) of Theorem 1, we provide the optimal cutoffs (k_i^ℓ, k_i^h) for the delegation allocations a^1 and a^2. Figure E.4 illustrates them.

Remark E.1. 1. Suppose that a depends on at most agent 1's types. Then, the optimal allocation a^1 is given as follows. If $\underline{\theta}_2 \geq \frac{2-\gamma}{1-\gamma}\overline{\theta}_1 - 1$, then the optimal allocation is the best constant allocation. If $\frac{2-\gamma}{1-\gamma}\overline{\theta}_1 - 1 \geq \underline{\theta}_2 \geq 2\overline{\theta}_1 - 1$, then the optimal allocation is the delegation allocation for agent 1 of the following form:

$$a^1(\theta_1) = \begin{cases} \frac{(1-\gamma)(\underline{\theta}_2+1)}{2-\gamma} & \text{if } \theta_1 \in \left[0, \frac{(1-\gamma)(\underline{\theta}_2+1)}{2-\gamma}\right] \\ \theta_1 & \text{if } \theta_1 \in \left[\frac{(1-\gamma)(\underline{\theta}_2+1)}{2-\gamma}, \overline{\theta}_1\right] \end{cases}.$$

If $2\overline{\theta}_1 - 1 \geq \underline{\theta}_2$, then the optimal allocation is the delegation allocation for agent 1 of the following form:

$$a^1(\theta_1) = \begin{cases} \frac{(1-\gamma)(\underline{\theta}_2+1)}{2-\gamma} & \text{if } \theta_1 \in \left[0, \frac{(1-\gamma)(\underline{\theta}_2+1)}{2-\gamma}\right] \\ \theta_1 & \text{if } \theta_1 \in \left[\frac{(1-\gamma)(\underline{\theta}_2+1)}{2-\gamma}, \frac{(1-\gamma)(\underline{\theta}_2+1)+\gamma\overline{\theta}_1}{2-\gamma}\right] \\ \frac{(1-\gamma)(\underline{\theta}_2+1)+\gamma\overline{\theta}_1}{2-\gamma} & \text{if } \theta_1 \in \left[\frac{(1-\gamma)(\underline{\theta}_2+1)+\gamma\overline{\theta}_1}{2-\gamma}, \overline{\theta}_1\right] \end{cases}.$$

2. Suppose that a depends on at most agent 2's types. Then, the optimal allocation a^2 is given as follows. If $\underline{\theta}_2 \geq \frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma}$, then the optimal allocation is the best constant allocation. If $\frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma} \geq \underline{\theta}_2 \geq \frac{\overline{\theta}_1}{2}$, then the optimal allocation is the delegation allocation for agent 2 of the following form:

$$a^2(\theta_2) = \begin{cases} \theta_2 & \text{if } \theta_2 \in \left[\underline{\theta}_2, \frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma}\right] \\ \frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma} & \text{if } \theta_2 \in \left[\frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma}, 1\right] \end{cases}.$$

If $\frac{\overline{\theta}_1}{2} \geq \underline{\theta}_2$, then the optimal allocation is the delegation allocation for agent 2 of the following form:

$$a^2(\theta_2) = \begin{cases} \frac{\gamma\overline{\theta}_1+(1-\gamma)\underline{\theta}_2}{1+\gamma} & \text{if } \theta_2 \in \left[\underline{\theta}_2, \frac{\gamma\overline{\theta}_1+(1-\gamma)\underline{\theta}_2}{1+\gamma}\right] \\ \theta_2 & \text{if } \theta_2 \in \left[\frac{\gamma\overline{\theta}_1+(1-\gamma)\underline{\theta}_2}{1+\gamma}, \frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma}\right] \\ \frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma} & \text{if } \theta_2 \in \left[\frac{\gamma\overline{\theta}_1+1-\gamma}{1+\gamma}, 1\right] \end{cases}.$$

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