Strategic Games with a Possibility Correspondence Model of Belief and Unawareness*

Satoshi Fukuda†

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Abstract

The primary objective of this paper is, within a possibility correspondence model of belief and unawareness, to provide an epistemic characterization of iterated elimination of strictly dominated actions (IESDA) in a game with unawareness as an implication of rationality and common belief in rationality. To that end, the second objective of the paper is to axiomatize a possibility correspondence model of belief and unawareness on a generalized state space by underlying properties of belief and unawareness. The paper fully characterizes properties of a possibility correspondence that yields the corresponding properties of the induced belief and unawareness operators. Conversely, the paper analyzes conditions on given belief and unawareness operators which generate a well-defined possibility correspondence, which, in turn, induces the original belief and unawareness operators. This way, the paper connects the epistemic analysis on the one hand and the generalized-state-space model of unawareness on the other hand.

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†Department of Decision Sciences and IGIER, Bocconi University.

1 Introduction

Iterated elimination of strictly dominated actions (IESDA) is one of the most fundamental solution concepts in game theory. For one reason, IESDA is characterized as an implication of rationality and common belief in rationality (e.g., Brandenburger and Dekel, 1987; Tan and Werlang, 1988). If players are rational, then they would never take actions that are strictly dominated. If, in addition, they believe that they are rational, then they would never take actions that are strictly dominated in a subgame in which strictly dominated actions (in the original game) have already been eliminated. Higher-order mutual beliefs in rationality correspond to higher-order elimination of strictly-dominated actions.

However, in standard game theory, predictions under IESDA hinge on the assumption that every player is aware of the game that they play. Consider the following two-player strategic game represented by the left panel of Table 1. Rikky, the row player, has three actions r_1, r_2, and r_3. Charlie, the column player, has three actions c_1, c_2, and c_3. Without any consideration of their unawareness, their payoffs are represented by the left panel of the table. In contrast, the right panel depicts the strategic game in which both players are unaware of the first action.

	c_1	c_2	c_3
r_1	$2, 2$	$2, 1$	$2, 0$
r_2	$1, 2$	$-1, -1$	$3, 0$
r_3	$0, 2$	$0, 3$	$0, 0$

	c_2	c_3
r_2	$-1, -1$	$3, 0$
r_3	$0, 3$	$0, 0$

Table 1: A Game with Unawareness. The left panel depicts the underlying game. The right panel depicts the game in which both players are unaware of the first action.

Without any consideration of unawareness, the unique prediction under IESDA is the action profile (r_1, c_1). If both players are rational, no player takes their third action, which is strictly dominated by the first action. As each player believes that the opponent is rational, in the subgame in which both players face the first two actions, the second action is strictly dominated by the first action.

What would happen if each player takes into account the possibility that the opponent is unaware of the first action? To consider this possibility, take the (sub-)game in which the players face the second and the third actions (i.e., the right panel of Table 1). In this (sub-)game, any action profile in $\{r_2, r_3\} \times \{c_2, c_3\}$ would survive IESDA because no action is strictly dominated. Thus, going back to the underlying game, Rikki (resp. Charlie), who would never choose the (strictly-dominated) third action, would not eliminate the second action as she (resp. he) would take into account the possibility that the opponent is unaware of the first action and thus chooses the third action. In this case, the elimination procedure for the underlying game would stop after eliminating the players' third action, yielding $\{r_1, r_2\} \times \{c_1, c_2\}$ as the set of action profiles in the underlying game that would survive IESDA.

In contrast, as depicted in the right panel of Table 2, consider the case in which both players are unaware of the third action. The left panel of Table 2 is kept intact.

	c_1	c_2	c_3
r_1	$2,2$	$2,1$	$2,0$
r_2	$1,2$	$-1,-1$	$3,0$
r_3	$0,2$	$0,3$	$0,0$

	c_1	c_2
r_1	$2,2$	$2,1$
r_2	$1,2$	$-1,-1$

Table 2: A Game with Unawareness. The left panel depicts the underlying game. The right panel depicts the game in which both players are unaware of the third action.

In this case, irrespective of whether the opponent is unaware of the third action, the opponent would not choose the third action. Thus, in the underlying game, after the third action is eliminated for both players, in the subgame in which the players face the first two actions, the unique action profile that would survive IESDA would be (r_1, c_1). In this case, the unique prediction in the underlying game is (r_1, c_1).

This paper has two objectives. The primary objective of this paper is to provide an epistemic characterization of iterated elimination of strictly dominated actions (IESDA) in a game with unawareness as an implication of rationality and common belief in rationality. This paper formally defines the notion of rationality and IESDA for strategic games with unawareness.

To provide epistemic analyses of strategic games with unawareness, the second objective of the paper is to axiomatize a possibility correspondence model of unawareness on a generalized state space by underlying properties of beliefs. In fact, this paper develops the first framework with which to analyze players' qualitative beliefs in the context of the generalized-state-space possibility correspondence model of unawareness a la Heifetz, Meier, and Schipper (2006).

Put differently, this paper connects the epistemic analysis on the one hand and the generalized-state-space model of unawareness on the other hand. The generalized state-space model of unawareness has two ingredients. The first is a generalized state space. While a "standard" state space is nothing but a non-empty set, the generalized state space consists of mutually disjoint subspaces which are ordered by an amount of concepts available within each subspace. The second is players' possibility correspondences on the generalized state space. Each player's possibility correspondence associates, with each state of the world, the set of states within some subspace that the player considers possible at that state. Contrary to the one in a standard state space, a player's possibility correspondence in the generalized state space can take into account of the concepts available to the player at a given state. For instance, in the context of Table 1, one can consider a generalized state space that consists of two subspaces, one in which the players are aware of the underlying game (the left panel), and the other in which the players are unaware of the first action (the right panel).

Accordingly, the players' possibility correspondences capture their beliefs (knowledge), possibility, and unawareness while keeping track of concepts available at each state. Especially, it can analyze two sources of unawareness, the one due to the lack of beliefs and the other due to the lack of concepts.

The first part of the paper (Sections 2 and 3) fully characterizes the generalized-state-space possibility correspondence model in terms of underlying properties of belief and unawareness. First, let players' possibility correspondences on a generalized state space be given. I fully characterize necessary-and-sufficient conditions under which the possibility correspondences induce well-defined belief and unawareness operators. Then, I go on to characterize properties of induced belief and unawareness operators from properties of possibility correspondences one by one. That is, for a given property of belief or unawareness, I characterize the corresponding property of a player's possibility correspondence. To complete the equivalence between possibility correspondences and belief and unawareness operators, I provide necessary-and-sufficient conditions on a player's belief and unawareness operators that generate a well-defined possibility correspondence, which, in turn, induces the original operators.

This approach provides a wide variety of unawareness structures that respect different combinations of assumptions on players' belief and unawareness. For a theoretical point of view, this sheds light on how each property of a possibility correspondence/belief operator generates the corresponding property on (un)awareness in the generalized-state-space possibility correspondence model a la Heifetz, Meier, and Schipper (2006). For a practical point of view, if the analysts have a particular generalized state space and players' belief and unawareness in mind, then this papers' approach provides a way to construct possibility correspondences that induce the given belief and unawareness operators.

One can readily introduce common belief into the framework of this paper. Once the generalized-state-space possibility correspondence model is developed, the second part of the paper (Section 4) studies epistemic characterizations of iterated elimination of strictly dominated actions (IESDA) as an implication of rationality and common belief in rationality for strategic games with unawareness. I also study the role of introspective properties of belief and unawareness on the epistemic analysis, proving the usefulness of the axiomatization in the first part of the paper.

The paper is structured as follows. The rest of the Introduction discusses the related literature. Section 2 provides a possibility correspondence model of qualitative belief and unawareness on a generalized state space. Section 3 introduces the notion of common belief. Section 4 studies implications of rationality and common belief in rationality. Section 5 provide concluding remarks. The proofs are relegated to Appendix A. Appendix B provides supplementary results.

Related Literature

This paper is related to the burgeoning strands of literature on unawareness, especially the one on modeling a non-trivial form of unawareness and the one on applications of unawareness to game theory.

Modeling Unawareness. I first discuss the strands of literature on modeling players' interactive unawareness, ever since such pioneering work as Fagin and Halpern (1987) and Modica and Rustichini (1994, 1999). Researchers have been trying to provide general and tractable models of unawareness in order to understand the concepts themselves and to apply the models to economic (e.g., contractual) problems. Since Dekel, Lipman, and Rustichini (1998) has shown that a standard state-space model may be inadequate for representing a non-trivial form of unawareness, there is a strand of literature that studies interactive unawareness in a rich structure.

In this regard, this paper extends the generalized-state-space possibility correspondence model of Heifetz, Meier, and Schipper (2006), one of the most general models of unawareness to date, in a way such that one can freely choose properties of belief and unawareness.[1] Proposition 1 identifies a necessary-and-sufficient condition on a possibility correspondence which induces a well-defined belief operator in standard and generalized state space models. In the context of standard state spaces, this paper extends such papers as Morris (1996) to the context of generalized state spaces. In the context of generalized state spaces, this paper also generalizes Board, Chung, and Schipper (2011) and Grant et al. (2015) in the sense that this paper presupposes only the minimal conditions on players' beliefs such as Monotonicity. Indeed, this paper provides the first formal framework with which to analyze players' qualitative beliefs.

This paper belongs to a literature that characterizes properties of beliefs (and unawareness) by corresponding properties of a possibility correspondence in standard and generalized state-space models. In the context of standard state spaces, a broad literature in such various fields as computer science, economics, game theory, logic, and philosophy is devoted to characterizing properties of beliefs. I extend these results to the context of generalized state spaces (e.g., Propositions 2, 3, and 4). I also identify (minimum) sets of axioms on knowledge/belief that fully characterize the original unawareness structure of Heifetz, Meier, and Schipper (2006), which has been missing in the literature (Remark 10). I also identify an axiom (which I call Generalized Negative Introspection) that distinguishes standard-state-space non-partitional models and the generalized-state-space model of Heifetz, Meier, and Schipper (2006) in that Generalized Negative Introspection in a standard-state-space model reduces to Negative Introspection and hence full awareness. Also, Generalized Negative Intro-

[1]There are other papers that represent unawareness in a richer structure. Among others, see: Board and Chung (2021, 2022), Board, Chung, and Schipper (2011), Galanis (2011, 2013), Halpern (2001), Halpern and Rêgo (2008, 2009), Heifetz, Meier, and Schipper (2008, 2013b), Heinsalu (2014), and Li (2009). See Schipper (2015) for an overview.

spection also connects two notions of awareness in a generalized-state-space model. In one notion, a player is aware of E when either she believes E or she believes that she does not believe E (Heifetz, Meier, and Schipper, 2006; Modica and Rustichini, 1994, 1999). In the other, the player is aware of an event E when she believes the tautology of a form E or the negation of E (e.g., Board, Chung, and Schipper, 2011; Heifetz, Meier, and Schipper, 2008). I show that Generalized Negative Introspection connects these two notions. I examine a variety of properties of unawareness which turn out to be equivalent to Generalized Negative Introspection. More importantly, I also study implications of these properties on game-theoretic solution concepts (e.g., Propositions 7 and 8). These introspective properties of beliefs are essential for a player to be aware of or to believe her own rationality.

This paper formally defines the notion of possibility, with the presence of unawareness, in a generalized-state-space model. In a standard possibility correspondence model, a player considers an event E possible when she does not believe the negation of E. However, as Modica and Rustichini (1999) suggest, how does a player consider an event E possible when she is indeed unaware of (the negation of) E (and accordingly she does not believe the negation of E)? Modica and Rustichini (1999) define possibility in a way such that a player considers an event E possible when she does not believe the negation of E and she is aware of E (in the sense of believing E or believing not believing E). I define a possibility operator from a possibility correspondence and show in Proposition 9 (in Appendix B) that a player considers an event E possible when she does not believe the negation of E and when she believes the tautology of a form E or the negation of E (a tautology phrased by the concepts involving E). If the player's beliefs satisfy Generalized Negative Introspection, as discussed, the latter condition reduces to the fact that the player is aware of E in the sense of Modica and Rustichini (1999). The notion of possibility is used to characterize players' rationality: player i is rational at a state if, for any action of which she is aware at the state, she considers it possible that taking the action prescribed by the given strategy is at least as good as the given action provided that the other players, who may be unaware of certain aspects of the game of which player i is aware, follow the prescribed strategies.

Game Theory with Unawareness. This paper also belongs to a fast-growing strand of literature on games with unawareness, especially the one that studies solution concepts of games with unawareness and their epistemic characterizations. Pioneering papers include, but are not limited to: Feinberg (2021), Grant and Quiggin (2013), Halpern and Rêgo (2014), Heifetz, Meier, and Schipper (2013a), Meier and Schipper (2014, 2023), Rêgo and Halpern (2012), and Sadzik (2021).[2] In relation to this paper, which studies rationalizability in complete-information static games with unawareness, Čopič and Galeotti (2006) and Feinberg (2021) study equilibrium notions

[2]There is also a fast-growing literature that applies unawareness to contractual problems. Some pioneering papers are: Auster (2013), Filiz-Ozbay (2012), Modica, Rustichini, and Tallon (1998), Von Thadden and Zhao (2012), and Zhao (2008, 2011).

for static games with complete (and incomplete) information. Meier and Schipper (2014) and Sadzik (2021) study Bayesian games and Bayes Nash equilibria with unawareness. Heifetz, Meier, and Schipper (2013a, 2021) and Meier and Schipper (2023) study rationalizable solution concepts in dynamic games with unawareness. To the best of my knowledge, this paper is the first paper that studies implication of common belief in rationality within the framework of possibility correspondence models of belief and unawareness.

In the context of static games with unawareness, Perea (2022) studies implications of common belief in rationality in a strategic game with unawareness in a type structure. The procedure of iterated elimination of strictly dominated actions in a game with unawareness in this paper is similar to the corresponding procedure in Perea (2022).[3] While the innovation in Perea (2022) is to introduce players' reasoning about types and "views," his paper relies on a type structure on a standard state space. Consequently, his paper would not allow for modeling unawareness as a lack of conception and interactive unawareness. In fact, this paper is the first paper in the literature that formalizes common belief in rationality and its epistemic characterization in a generalized-state-space model of unawareness. This paper is also the first paper that connects the literature on possibility correspondence models of belief and unawareness on the one hand and games with unawareness on the other.[4] Charness and Sontuoso (2023) provide a new rationalizable solution concept for static games with unawareness and experimentally investigate the solution concept. Sasaki (2016) study Nash equilibria for static games with unawareness.

While this paper focuses on static games, the literature on dynamic games with unawareness has also been growing. A dynamic game with unawareness may induce a new problem that players' representations of the game may change during the play of the game. Fukuda and Kamada (2023) study a repeated game in which players may be initially unaware of a certain action. Guarino (2020) introduces the notion of conditional probability systems into dynamic games with unawareness and provides an epistemic characterization of extensive-form rationalizability. Schipper (2021) and Tada (2022) study a "discovery process" in games with unawareness. Again, while this paper focuses on static games, the possibility correspondence model of belief and unawareness developed in the first part of the paper would be useful to study epistemic characterizations of solution concepts for dynamic games.

While this paper focuses on qualitative instead of probabilistic beliefs, this paper contributes to developing a possibility correspondence model of beliefs on a generalized state space and in fact it would be possible to incorporate probabilistic beliefs

[3]Moreover, the solution procedure in this paper is also similar to Bach and Perea (2021) and Meier and Schipper (2023), although the above-mentioned papers study different settings (a game with incomplete information without any consideration of unawareness and a dynamic game with unawareness, respectively) and consequently the procedures have some differences.

[4]Meier and Schipper (2014) connects the generalized-state-space type space model of probabilistic beliefs and unawareness and analyses of Bayesian games with unawareness.

into the framework of this paper. In this regard, papers such as Galanis (2015, 2016, 2018) and Heifetz, Meier, and Schipper (2013b) study the value of information and speculative trades in a generalized state-space model of unawareness. These papers and this paper share the difficulty of properly modeling players' strategies on a generalized state space in which they may be unaware of certain aspects of the underlying environment.

In epistemic game theory, there is a large strand of literature that studies solution concepts of games where players possess qualitative instead of probabilistic beliefs. The literature also studies the role of Truth Axiom, i.e., compares the implications of common belief in rationality versus common knowledge of rationality. See, for instance, Bonanno (2008, 2015), Bonanno and Tsakas (2018), Fukuda (2023b), Guarino and Ziegler (2022), Hillas and Samet (2020), Samet (2013), and Stalnaker (1994). None of these papers incorporates players' interactive unawareness. In fact, this paper extends one of the best-known solution concepts of games, iterated elimination of strictly dominated actions, as an implication of rationality and common belief in rationality to the generalized-state-space possibility-correspondence model. Moreover, this paper enables one to study the role of Truth Axiom on a generalized state space. Further discussions are provided in Section 4.5.

2 A Generalized Possibility Correspondence Model

This section establishes the one-to-one correspondence between a possibility correspondence and a belief operator. Section 2.1 defines a generalized state space that consists of multiple subspaces. Section 2.2 defines a possibility correspondence on the generalized state space. As in the previous literature, the possibility correspondence induces a belief operator. Conversely, Section 2.3 constructs a possibility correspondence from a given belief operator. Section 2.4 characterizes properties of beliefs by a possibility correspondence. Throughout this section, fix a non-empty set of players I and a generic player $i \in I$.

2.1 A Generalized State Space

A *generalized state space* $\langle((S_\lambda, \mathcal{D}_\lambda))_{\lambda\in\Lambda}, \succeq, r\rangle$ has the following three ingredients. First, for each $\lambda \in \Lambda$, $(S_\lambda, \mathcal{D}_\lambda)$ is a complete algebra of sets: $\mathcal{D}_\lambda$ is a collection of subsets of S_λ which contains S_λ and which is is closed under arbitrary union, arbitrary intersection, and complementation. The power set $\mathcal{D}_\lambda = \mathcal{P}(S_\lambda)$ is a special case. Without loss, assume that $\mathcal{S} := \{S_\lambda\}_{\lambda\in\Lambda}$ are disjoint. Call each $(S_\lambda, \mathcal{D}_\lambda)$ a *subspace*. I often denote by $(S, \mathcal{D})$ a generic subspace. When the underlying $\mathcal{D}_\lambda$ is clear, I interchangeably use $(S_\lambda, \mathcal{D}_\lambda)$ and S_λ to refer to a subspace. Henceforth, I omit the outermost parentheses to denote by $(S_\lambda, \mathcal{D}_\lambda)_{\lambda\in\Lambda}$ the collection of subspaces. The set of *states of the world* is the entire union $\Omega := \bigcup_{\lambda\in\Lambda} S_\lambda$. For any state $\omega \in \Omega$, denote by $S(\omega) \in \mathcal{S}$ the unique subspace with $\omega \in S(\omega)$.

Second, $\succeq$ is a partial order on subspaces $\mathcal{S}$ and is read as: S_λ is *at least as expressive as* S_μ if $S_\lambda \succeq S_\mu$. Assume that $\langle \mathcal{S}, \succeq \rangle$ is a complete lattice. I denote $S_\lambda \succ S_\mu$ if $S_\lambda \succeq S_\mu$ and $S_\lambda \neq S_\mu$.

Third, $r := (r_S^{S'})_{S' \succeq S}$ is a collection of a surjective measurable projection $r_S^{S'} : (S', \mathcal{D}') \to (S, \mathcal{D})$ defined for each pair $(S, S') \in \mathcal{S}^2$ with $S' \succeq S$. Assume that: (i) $(r_S^{S'})^{-1}(D) \in \mathcal{D}'$ for all $D \in \mathcal{D}$ (i.e., each projection is measurable, meaning that a set of states $D \in \mathcal{D}$ described in a less expressive space S can also be captured in a more expressive space S' through the projection); (ii) each r_S^S is the identity mapping; and (iii) the projections commute: if $S'' \succeq S' \succeq S$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$.

I introduce the following notations. Fix $S, S' \in \mathcal{S}$ with $S' \succeq S$. First, denote $w_S := r_S^{S'}(\omega)$ for any $\omega \in S'$. Second, for any $D \in \mathcal{D}'$, let $D_S := r_S^{S'}(D)$, which is not necessarily an element of $\mathcal{D}$.

Objects of players' beliefs and unawareness are described as events. An *event* is a pair $(D^\uparrow, S) \in \mathcal{P}(\Omega) \times \mathcal{S}$ such that

$$D^\uparrow := \bigcup \{ (r_S^{S'})^{-1}(D) \in \mathcal{P}(\Omega) \mid S' \succeq S \text{ for some } S' \in \mathcal{S} \} \text{ for some } D \in \mathcal{D}.$$

Denote the collection of events by $\mathcal{E}$.

Fix an event $(D^\uparrow, S_\lambda)$. Call S_λ the *base space* of $(D^\uparrow, S_\lambda)$ (or simply $D^\uparrow$), and denote $S(D^\uparrow, S_\lambda) = S_\lambda$ (or simply $S(D^\uparrow) = S_\lambda$). Next, call D the *basis* of $D^\uparrow$. For an event $(E, S(E))$, denote by $D(E)$ the basis of E, i.e., $E = D^\uparrow(E) := (D(E))^\uparrow$.[5]

For ease of notation, I often write an event $\overline{D}^\uparrow := (D^\uparrow, S(D^\uparrow))$ by adding the over line to suppress the base space to which it belongs. Also, I denote by $\overline{\emptyset}^S$ (or simply by $\emptyset^S$) the event $(\emptyset^S, S) := (\emptyset^\uparrow, S) = (\emptyset, S)$.

I introduce the following four operations on $\mathcal{E}$. The first is an informational partial order $\leqslant$ on events:

$$\overline{E} \leqslant \overline{F} \text{ if and only if (hereafter, iff) } E \subseteq F \text{ and } S(E) \succeq S(F).$$

If $\overline{E} \leqslant \overline{F}$, then $\overline{F}$ is *at least as coarse as* $\overline{E}$ (or $\overline{E}$ is *at least as refined as* $\overline{F}$). In a standard state space, this order reduces to the set inclusion. The greatest element is $(\Omega, \inf \mathcal{S}) = ((\inf \mathcal{S})^\uparrow, \inf \mathcal{S})$, while the least element is $(\emptyset, \sup \mathcal{S})$.

Second, for any collection of events $(D_x^\uparrow, S_x)_{x \in X}$, define the *conjunction* as:

$$\bigwedge_{x \in X} (D_x^\uparrow, S_x) := \left(\bigwedge_{x \in X} D_x^\uparrow, \sup_{x \in X} S_x \right) := \left(\left(\bigcap_{x \in X} (r_{S_x}^{\sup_{x \in X} S_x})^{-1}(D_x) \right)^\uparrow, \sup_{x \in X} S_x \right) \in \mathcal{E}.$$

It turns out that $\bigwedge_{x \in X} D_x^\uparrow = \bigcap_{x \in X} D_x^\uparrow$. Thus, $\bigwedge_{x \in X} (D_x^\uparrow, S_x)$ is the infimum of $(D_x^\uparrow, S_x)_{x \in X}$ in $\langle \mathcal{E}, \leqslant \rangle$, i.e., the finest event as coarse as every $(D_x^\uparrow, S_x)$. Hence, $\langle \mathcal{E}, \leqslant \rangle$ is a complete lattice.

[5] I will apply this convention, $D^\uparrow(\cdot) := (D(\cdot))^\uparrow$, to other similar notations to omit parentheses.

Third, define the *negation* of an event $(D^\uparrow, S)$ by

$$\neg(D^\uparrow, S) := (\neg D^\uparrow, S) \in \mathcal{E}, \text{ where } \neg D^\uparrow := (S \setminus D)^\uparrow.$$

By definition, $\neg\neg(D^\uparrow, S) = (D^\uparrow, S)$. By letting $\neg\emptyset^S := S^\uparrow$ and $\neg S^\uparrow := \emptyset^S$, one can unambiguously write $\neg\neg D^\uparrow = D^\uparrow$ for any $(D^\uparrow, S) \in \mathcal{E}$.

Fourth, I define the *disjunction* of a collection of events $(D_x^\uparrow, S_x)_{x\in X}$ as:

$$\bigvee_{x\in X}(D_x^\uparrow, S_x) := \left(\bigvee_{x\in X} D_x^\uparrow, \sup_{x\in X} S_x\right) := \left(\left(\bigcup_{x\in X}(r_{S_x}^{\sup_{x\in X} S_x})^{-1}(D_x)\right)^\uparrow, \sup_{x\in X} S_x\right).$$

The disjunction and conjunction of events satisfy the "De Morgan Laws." In particular,

$$\bigvee_{x\in X}(D_x^\uparrow, S_x) = \left(\neg\left(\bigwedge_{x\in X}(\neg D_x^\uparrow)\right), \sup_{x\in X} S_x\right) \in \mathcal{E}.$$

Since the base space of the disjunction $\bigvee_{x\in X}(D_x^\uparrow, S_x)$ is defined as $\sup_{x\in X} S_x$, the disjunction is typically different from the supremum of $(D_x^\uparrow, S_x)_{x\in X}$ in $\langle\mathcal{E}, \leqslant\rangle$ unless otherwise $\inf_{x\in X} S_x = \sup_{x\in X} S_x$. In fact, the disjunction $\bigvee_{x\in X}(D_x^\uparrow, S_x)$ is the finest event in $\sup_{x\in X} S_x$ at least as coarse as every $(D_x^\uparrow, S_x)$.

2.2 A Possibility Correspondence Induces Belief and Awareness Operators

This subsection defines player i's belief and awareness operators through her possibility correspondence. Section 2.2.1 defines the possibility correspondence. Section 2.2.2 defines the belief operator through the possibility correspondence. Sections 2.2.3 and 2.2.4 define the two awareness operators, one through the lack of conception and the other through the lack of beliefs.

2.2.1 Possibility Correspondence

Player i's possibility correspondence associates, with each state of the world, the event that player i considers possible at ω. Formally, player i's *possibility correspondence* is a mapping

$$\overline{\Pi}_i^\uparrow : \Omega \ni \omega \mapsto \overline{\Pi}_i^\uparrow(\omega) = (\Pi_i^\uparrow(\omega), \sigma_i(\omega)) \in \mathcal{E},$$

where $\sigma_i : \Omega \to \mathcal{S}$ is player i's *awareness function*. The possibility correspondence presupposes the mapping $\overline{\Pi}_i : \Omega \ni \omega \mapsto \overline{\Pi}_i(\omega) = (\Pi_i(\omega), \sigma_i(\omega)) \in \mathcal{P}(\Omega) \times \mathcal{S}$ with $\Pi_i(\cdot) \in \mathcal{D}_{\sigma_i(\cdot)}$ (i.e., $\Pi_i(\omega) \in \mathcal{D}_{\sigma_i(\omega)}$ for all $\omega \in \Omega$). Player i considers state ω' *possible* at a given state $\omega \in \Omega$ if $\omega' \in \Pi_i(\omega)$. This definition is slightly different from Heifetz, Meier, and Schipper (2006) in that I explicitly specify at the outset the base space of $\Pi_i^{\uparrow}(\cdot)$ by introducing the awareness function.[6]

I define the following four properties of the possibility correspondence.

Definition 1. The possibility correspondence $\overline{\Pi}_i^{\uparrow}$ is *well-defined* if it satisfies the following four properties.

1. Regularity: $\{\omega \in S \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant (E,S)\} \in \mathcal{D}$ for each $(E,S) \in \mathcal{E}$.

2. Projections Preserve Ignorance (PPI): If $S \preceq S(\omega)$, then $\overline{\Pi}_i^{\uparrow}(\omega) \leqslant \overline{\Pi}_i^{\uparrow}(\omega_S)$.

3. Projections Preserve Beliefs (PPB): Let $S \preceq S(\omega)$. If $\overline{\Pi}_i^{\uparrow}(\omega) \leqslant (E,S)$ for some $E \in \mathcal{P}(\Omega)$ with $(E,S) \in \mathcal{E}$, then $\overline{\Pi}_i^{\uparrow}(\omega_S) \leqslant (E,S)$.

4. Confinedness: $\sigma_i(\cdot) \preceq S(\cdot)$.

First, Regularity guarantees that each subspace $(S,\mathcal{D})$ is rich enough to incorporate players' beliefs (i.e., $\{\omega \in S \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant (E,S)\}$) within itself. Regularity is equivalent to: $(\{\omega \in S \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant (E,S)\}^{\uparrow}, S) \in \mathcal{E}$ for each $(E,S) \in \mathcal{E}$.

In a standard-state-space model (i.e., $\mathcal{S} = \{S\}$) in which the state space Ω is endowed with a complete algebra $\mathcal{D}$ of sets, Regularity means that $\{\omega \in S \mid \Pi_i(\omega) \subseteq E\} \in \mathcal{D}$ for each $E \in \mathcal{D}$. Thus, letting $B_{\Pi_i}(E) := \{\omega \in S \mid \Pi_i(\omega) \subseteq E\}$ for each $E \in \mathcal{D}$, Regularity is the sole condition that guarantees that the player's belief operator $B_{\Pi_i} : \mathcal{D} \to \mathcal{D}$ is well-defined.

Second, while Regularity is a condition on individual subspaces, PPI and PPB state how projections preserve each player's beliefs among upper and lower subspaces. PPI states that each player's beliefs are preserved in upper spaces. Take any $S(\omega) \succeq S$. If player i "believes" an event at ω_S, then she "believes" it at ω.

Third, PPB says that projections preserve the player's beliefs in lower spaces. Let $S(\omega) \succeq S$. Then, PPB states that if player i "believes" an event (E,S) at ω then she believes it at ω_S. Under Confinedness to be discussed, one can show that PPB is rewritten as follows:

- PPB under Confinedness: Let $S \preceq \sigma_i(\omega) \preceq S(\omega)$. Then, (i) $S = \sigma_i(\omega_S)$; and (ii) $\Pi_i(\omega_S) \subseteq D$ for any $D \in \mathcal{D}$ such that $\Pi_i(\omega) \subseteq (r_S^{\sigma_i(\omega)})^{-1}(D)$.

In contrast, PPI implies: $\Pi_i(\omega) \subseteq (r^{\sigma_i(\omega)}_{\sigma_i(\omega_S)})^{-1}(\Pi_i(\omega_S))$. Thus, PPB generalizes the original definition by Heifetz, Meier, and Schipper (2006, "Projections Preserve Knowledge"): $\overline{\Pi}_i(\omega_S) = ((\Pi_i(\omega))_S, S)$ when $S \preceq \sigma_i(\omega) \preceq S(\omega)$. This condition holds when $\mathcal{D} = \mathcal{P}(S)$ under Confinedness and PPI.

[6] The idea of capturing the possibility correspondence as a pair of mappings is conceptually related to the idea of having a pair of information and awareness functions that jointly maps a state into an event in the "object-based unawareness" structure of Board and Chung (2021, 2022) and Board, Chung, and Schipper (2011).

Fourth, Confinedness is a condition on the awareness function. As discussed, a part of the definition of Confinedness in Heifetz, Meier, and Schipper (2006) is built into the definition of the possibility correspondence $\overline{\Pi}_i^{\uparrow}$ itself. Accordingly, the definition of Confinedness takes into account only of the other part of their Confinedness: the base space of $\Pi_i^{\uparrow}(\omega)$ (i.e., the subspace $\sigma_i(\omega)$) is at least as less expressive as $S(\omega)$.

2.2.2 Belief Operator

I define the (qualitative) belief operator from a given possibility correspondence. To that end, for a given possibility correspondence $\overline{\Pi}_i^{\uparrow}$, I first define the set of states $B_{\overline{\Pi}_i}(E)$ at which an event $\overline{\Pi}_i^{\uparrow}(\omega)$ is at least as refined as an event $\overline{E}$:

$$B_{\overline{\Pi}_i}(E) := \{\omega \in \Omega \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant \overline{E}\}.$$

Especially, $\omega \in B_{\overline{\Pi}_i}(\Pi_i^{\uparrow}(\omega))$.

The possibility correspondence $\overline{\Pi}_i^{\uparrow}$ *induces* the belief operator $\overline{B}_{\overline{\Pi}_i} : \mathcal{E} \to \mathcal{E}$ if

$$\overline{B}_{\overline{\Pi}_i}(E, S) := (B_{\overline{\Pi}_i}(E), S) \in \mathcal{E} \text{ for all } (E, S) \in \mathcal{E}.$$

I require the base space of $\overline{B}_{\overline{\Pi}_i}(E, S)$ to be equal to that of (E, S). Conceptually, this means that if (E, S) is an event, the belief of (E, S) is also captured within the same space S. When $\overline{\Pi}_i^{\uparrow}$ induces the belief operator $\overline{B}_{\overline{\Pi}_i}$, player i *believes* an event $\overline{E}$ at a state $\omega \in \Omega$ if $\omega \in B_{\overline{\Pi}_i}(E)$.

When does the possibility correspondence $\overline{\Pi}_i^{\uparrow}$ induce the belief operator $\overline{B}_{\overline{\Pi}_i}$? I present a benchmark result stating that the four basic conditions of Regularity, PPI, PPB, and Confinedness are necessary and sufficient for the possibility correspondence to induce the belief operator (Board, Chung, and Schipper, 2011; Fukuda, 2021; Grant et al., 2015).

Remark 1. The following are equivalent.

1. $\overline{\Pi}_i^{\uparrow}$ is well-defined (i.e., it satisfies Regularity, PPI, PPB, and Confinedness).
2. $B_{\overline{\Pi}_i}(D^{\uparrow}) = (\{\omega \in S \mid \Pi_i(\omega) \subseteq D \text{ and } \sigma_i(\omega) = S\})^{\uparrow}$ for any $(D^{\uparrow}, S) \in \mathcal{E}$.
3. $\overline{\Pi}_i^{\uparrow}$ induces $\overline{B}_{\Pi_i}$: $\overline{B}_{\Pi_i}(D^{\uparrow}, S) = (B_{\overline{\Pi}_i}(D^{\uparrow}), S) \in \mathcal{E}$ for any $(D^{\uparrow}, S) \in \mathcal{E}$.

The intuition behind Remark 1 is that the possibility correspondence induces the belief operator exactly when each subspace can describe, by using the possibility correspondence restricted on the subspace, the player's beliefs within its subspace. Specifically, $\overline{\Pi}_i^{\uparrow}$ induces $\overline{B}_{\overline{\Pi}_i}$ when $\overline{B}_{\overline{\Pi}_i}(E, S)$ is the event generated from the set of states ω in S such that the possibility set $\Pi_i(\omega)$ resides in S and implies $D(E)$ in the sense of set inclusion. Equivalently, $B_{\overline{\Pi}_i}(E) = \{\omega \in S \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant (E, S)\}^{\uparrow}$.

Suppose the possibility correspondence $\overline{\Pi}_i^{\uparrow}$ induces the belief operator $\overline{B}_{\overline{\Pi}_i}$. Then, one can show the following properties of $\overline{B}_{\overline{\Pi}_i}$.

Remark 2. The belief operator $\overline{B}_{\overline{\Pi}_i}$ satisfies the following.

1. Decomposition: $S(\overline{B}_{\overline{\Pi}_i}(\overline{E})) = S(\overline{E})$ for all $\overline{E} \in \mathcal{E}$.

2. Monotonicity: $\overline{B}_{\overline{\Pi}_i}(\overline{E}) \leqslant \overline{B}_{\overline{\Pi}_i}(\overline{F})$ for any $\overline{E}, \overline{F} \in \mathcal{E}$ with $\overline{E} \leqslant \overline{F}$.

3. Conjunction: $\bigwedge_{x \in X} \overline{B}_{\overline{\Pi}_i}(\overline{E}_x) \leqslant \overline{B}_{\overline{\Pi}_i}\left(\bigwedge_{x \in X} \overline{E}_x\right)$.

4. Necessitation: $\overline{B}_{\overline{\Pi}_i}(\Omega, \inf \mathcal{S}) = (\Omega, \inf \mathcal{S})$.

When I stress that the belief operator $\overline{B}_{\overline{\Pi}_i}$ satisfies some of the above properties, I append "$\overline{B}_{\overline{\Pi}_i}$-" or "B-" to each property (e.g., $\overline{B}_{\overline{\Pi}_i}$-Decomposition or B-Decomposition). The proof of Remark 2 is omitted, as each statement follows from the definition of the possibility correspondence.

Two further technical remarks are in order. First, Necessitation can be seen as a special case of Conjunction with $X = \emptyset$. Second, Monotonicity and Conjunction are jointly equivalent to Conjunction with equality: $\bigwedge_{x \in X} \overline{B}_{\overline{\Pi}_i}(\overline{E}_x) = \overline{B}_{\overline{\Pi}_i}\left(\bigwedge_{x \in X} \overline{E}_x\right)$.

2.2.3 Awareness Operator induced by an Awareness Function

I define two kinds of awareness operators. Here I study the first kind of the awareness operator, the one that is induced by the awareness function. The awareness function σ_i *induces* an awareness operator $\overline{A}_{\sigma_i} : \mathcal{E} \to \mathcal{E}$ if $\overline{A}_{\sigma_i}(\overline{E}) := (A_{\sigma_i}(E), S(E)) \in \mathcal{E}$ for any $\overline{E} \in \mathcal{E}$, where

$$A_{\sigma_i}(E) := \{\omega \in \Omega \mid \sigma_i(\omega) \succeq S(E)\}$$

(Board, Chung, and Schipper, 2011; Heifetz, Meier, and Schipper, 2008). Player i is ($\overline{A}_{\sigma_i}$-)*aware* of an event $\overline{E}$ at a state $\omega \in \Omega$ if $\omega \in A_{\sigma_i}(E)$. This notion of awareness emphasizes the awareness of the concept in the following sense: player i is ($\overline{A}_{\sigma_i}$-)aware of $\overline{E}$ at ω if the subspace she is aware of at ω is at least as expressive as the subspace $S(E)$; hence, she is aware of the "vocabulary" to describe what $\overline{E}$ is. By definition, $\omega \in A_{\sigma_i}(\sigma_i^{\uparrow}(\omega))$. If $\overline{\Pi}_i^{\uparrow}$ induces $\overline{B}_{\overline{\Pi}_i}$, then σ_i does $\overline{A}_{\sigma_i}$, and $A_{\sigma_i}(E) = B_{\overline{\Pi}_i}(S^{\uparrow}(E))$.

Similarly to Definition 1, I define the following four properties of the awareness function σ_i.

Definition 2. The awareness function σ_i is *well-defined* if it satisfies the following four properties.[7]

1. A-Regularity: $\{\omega \in S \mid \sigma_i(\omega) \succeq S\} \in \mathcal{D}$ for each $S \in \mathcal{S}$.

2. Non-Increasing Awareness (NIA): If $S \preceq S(\omega)$, then $\sigma_i(\omega_S) \preceq \sigma_i(\omega)$.

3. Projections Preserve Awareness (PPA): If $S \preceq S(\omega)$ and $S \preceq \sigma_i(\omega)$, then $S \preceq \sigma_i(\omega_S)$.

[7] Technically, the awareness function σ_i is well-defined in the sense of Definition 2 iff the possibility correspondence $(\emptyset, \sigma_i(\cdot))$ is well-defined in the sense of Definition 1.

4. Confinedness: $\sigma_i(\cdot) \preceq S(\cdot)$.

A-Regularity is a special case of Regularity pertaining only to σ_i, by which each subspace S is rich enough to incorporate players' awareness within itself. It is equivalent to: $(\{\omega \in S \mid \sigma_i(\omega) \succeq S\}^{\uparrow}, S) \in \mathcal{E}$ for any $S \in \mathcal{S}$.

Non-Increasing Awareness (NIA) (Grant et al., 2015; Heifetz, Meier, and Schipper, 2006) is a part of PPI pertaining only to σ_i. Thus, for any $S \preceq S(\omega)$, if player i is "$\overline{A}_{\sigma_i}$-aware" of an event at ω_S then she is "$\overline{A}_{\sigma_i}$-aware" of it at ω. NIA is also expressed as: $\omega \in A_{\sigma_i}(\sigma_i^{\uparrow}(w_S))$ for any $S \preceq S(\omega)$.

Projections Preserve Awareness (PPA) is a part of PPB pertaining only to σ_i. Thus, for any $S \preceq S(\omega)$, if player i is "$\overline{A}_{\sigma_i}$-aware" of an event (E, S) at ω then she is "$\overline{A}_{\sigma_i}$-aware" of it at ω_S.

To see the role of Confinedness, for a well-defined awareness function σ_i, Confinedness could be expressed as $\overline{A}_{\sigma_i}(\overline{E}) \leqslant \overline{S}^{\uparrow}(E)$.

Now, I provide a benchmark result stating that the awareness function σ_i induces the awareness operator $\overline{A}_{\sigma_i}$ iff it satisfies the basic properties.

Remark 3. The following are equivalent.

1. σ_i is well-defined (i.e., it satisfies A-Regularity, NIA, PPA, and Confinedness).

2. $A_{\sigma_i}(E) = (\{\omega \in S \mid \sigma_i(\omega) = S\})^{\uparrow}$ for any $(E, S) \in \mathcal{E}$.

3. σ_i induces $\overline{A}_{\sigma_i}$: $\overline{A}_{\sigma_i}(E, S) = (A_{\sigma_i}(E), S) \in \mathcal{E}$ for any $(E, S) \in \mathcal{E}$.

The intuition behind Remark 3 is similar to that of Remark 1. Technically, it is also a corollary of Remark 1 when $\overline{\Pi}_i^{\uparrow}(\cdot) = (\emptyset, \sigma_i(\cdot))$. Thus, the proof is omitted.

The awareness function induces the awareness operator exactly when each subspace can describe, by using the awareness function restricted on the subspace, the player's awareness within its subspace. Specifically, σ_i induces $\overline{A}_{\sigma_i}$ when $\overline{A}_{\sigma_i}(E, S)$ is the event generated from the set of states ω in S such that the awareness function assigns $\sigma_i(\omega) = S$. It turns out to be equivalent to: $A_{\sigma_i}(E) = \{\omega \in S \mid \sigma_i(\omega) \succeq S\}^{\uparrow}$.

Suppose the awareness function σ_i induces the awareness operator $\overline{A}_{\sigma_i} : \mathcal{E} \to \mathcal{E}$. Then:

Remark 4. The awareness operator $\overline{A}_{\sigma_i} : \mathcal{E} \to \mathcal{E}$ satisfies the following.

1. Decomposition: $S(\overline{A}_{\sigma_i}(\overline{E})) = S(\overline{E})$ for all $\overline{E} \in \mathcal{E}$.

2. Independence: $\overline{A}_{\sigma_i}(E, S) = \overline{A}_{\sigma_i}(E', S)$ for any $(E, S), (E', S) \in \mathcal{E}$.

3. Subspace-Monotonicity: $\overline{A}_{\sigma_i}(\overline{S}^{\uparrow}) \leqslant \overline{A}_{\sigma_i}(\overline{S'}^{\uparrow})$ for any $S, S' \in \mathcal{S}$ with $S \preceq S'$.

4. Subspace-Conjunction: $\bigwedge_{x \in X} \overline{A}_{\sigma_i}(\overline{S}_x^{\uparrow}) \leqslant \overline{A}_{\sigma_i}(\overline{S}^{\uparrow})$, where $S = \sup_{x \in X} S_x$.

5. Awareness-at-the-Bottom: $\overline{A}_{\sigma_i}(E, \inf \mathcal{S}) = (\Omega, \inf \mathcal{S})$ for any $(E, \inf \mathcal{S}) \in \mathcal{E}$.

When I stress that $\overline{A}_{\sigma_i}$ satisfies the above properties, I append "$\overline{A}_{\sigma_i}$-" or "A-" to them (e.g., $\overline{A}_{\sigma_i}$-Decomposition or A-Decomposition). These properties follow from Remark 2 through $\overline{A}_{\sigma_i}(E,S) = \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow})$. Thus, the proof is omitted.

Independence states that the awareness of an event depends only on its base space. Awareness-at-the-Bottom implies full awareness in the subspace $\inf \mathcal{S}$. For the case with the awareness operator $\overline{A}_{\sigma_i}$, Conjunction and Monotonicity reduce to Subspace-Conjunction and Subspace-Monotonicity, respectively.

Two further technical remarks on Remark 4 are in order. First, given Independence, Subspace-Conjunction and Subspace-Monotonicity are jointly equivalent to: $\bigwedge_{x\in X} \overline{A}_{\sigma_i}(\overline{E}_x) = \overline{A}_{\sigma_i}\left(\bigwedge_{x\in X} \overline{E}_x\right)$. Second, given Independence, Subspace-Monotonicity can be replaced with Monotonicity: $\overline{A}_{\sigma_i}(\overline{E}) \leqslant \overline{A}_{\sigma_i}(\overline{F})$ for any $\overline{E}, \overline{F} \in \mathcal{E}$ with $\overline{E} \leqslant \overline{F}$.

Next, I summarize the joint properties of the belief and awareness operators $\overline{B}_{\overline{\Pi}_i}$ and $\overline{A}_{\sigma_i}$. To that end, for any given awareness operator $\overline{A}_i : \mathcal{E} \to \mathcal{E}$, define the *unawareness operator* $\overline{U}_i : \mathcal{E} \to \mathcal{E}$ as

$$\overline{U}_i(\cdot) := (\neg \overline{A}_i)(\cdot).$$

Denote $\overline{U}_i(E,S) = (U_i(E), S)$, i.e., $U_i(E) = (\neg A_i)(E)$. Player i is *unaware* of $\overline{E}$ at ω if $\omega \in U_i(E)$. Thus, if $\overline{\Pi}_i^{\uparrow}$ induces $\overline{B}_{\overline{\Pi}_i}$, then, for all $(E,S) \in \mathcal{E}$,

$$\overline{U}_{\sigma_i}(E) := (U_{\sigma_i}(E), S) := ((\neg A_{\sigma_i})(E), S) = ((\neg B_{\overline{\Pi}_i})(S^{\uparrow}), S) \in \mathcal{E}.$$

The unawareness operator $\overline{U}_{\sigma_i}$ satisfies *Plausibility*: if player i is unaware of an event, then she does not believe it and she does not believe that she does not believe it. That is,

$$\overline{U}_{\sigma_i}(\cdot) \leqslant (\neg \overline{B}_{\overline{\Pi}_i})(\cdot) \wedge (\neg \overline{B}_{\overline{\Pi}_i})^2(\cdot), \text{ where } (\neg \overline{B}_{\overline{\Pi}_i})^2(\cdot) = (\neg \overline{B}_{\overline{\Pi}_i})(\neg \overline{B}_{\overline{\Pi}_i})(\cdot).$$

Also, Awareness-at-the-Bottom implies that

$$\overline{U}_{\sigma_i}(E, \inf \mathcal{S}) = (\emptyset, \inf \mathcal{S}) \text{ for any } (E, \inf \mathcal{S}) \in \mathcal{E}.$$

That is, for any event in the least expressive subspace, there is no state at which player i is unaware of the event.

I provide joint properties of $\overline{B}_{\overline{\Pi}_i}$ and $\overline{A}_{\sigma_i}$.

Remark 5. The operators $\overline{B}_{\overline{\Pi}_i}$ and $\overline{A}_{\sigma_i}$ satisfy the following.

1. Symmetry: $\overline{A}_{\sigma_i}(\overline{E}) = \overline{A}_{\sigma_i}(\neg \overline{E})$.

2. AU-Introspection: $\overline{U}_{\sigma_i}(\cdot) \leqslant \overline{U}_{\sigma_i}\overline{U}_{\sigma_i}(\cdot)$, with equality.

3. Conjunction: $\bigwedge_{x\in X} \overline{A}_{\sigma_i}(\overline{E}_x) \leqslant \overline{A}_{\sigma_i}\left(\bigwedge_{x\in X} \overline{E}_x\right)$, with equality.

4. AB-Self Reflection: $\overline{A}_{\sigma_i}\overline{B}_{\overline{\Pi}_i}(\cdot) = \overline{A}_{\sigma_i}(\cdot)$.

5. AA-Self Reflection: $\overline{A}_{\sigma_i}\overline{A}_{\sigma_i}(\cdot) = \overline{A}_{\sigma_i}(\cdot)$.

6. Correct-Belief-in-Awareness: $\overline{B}_{\overline{\Pi}_i}\overline{A}_{\sigma_i}(\cdot) \leqslant \overline{A}_{\sigma_i}(\cdot)$.

7. Strong Plausibility: $\overline{U}_{\sigma_i}(\cdot) \leqslant \bigwedge_{n\in\mathbb{N}}(\neg\overline{B}_{\overline{\Pi}_i})^n(\cdot)$.

When I stress the operators (e.g., $\overline{B}_{\overline{\Pi}_i}$ and $\overline{A}_{\sigma_i}$) with which the above properties satisfy, I replace B, A, and U with the corresponding operators (e.g., $\overline{B}_{\overline{\Pi}_i}$, $\overline{A}_{\sigma_i}$, and $\overline{U}_{\sigma_i}$). The proof of Remark 5 is in Appendix A.1.

Symmetry states that player i is aware of an event $\overline{E}$ iff she is aware of its negation $\neg\overline{E}$. Roughly, Modica and Rustichini (1994) show that $\overline{A}_{\overline{\Pi}_i}$-Symmetry leads to a trivial form of unawareness in a standard-state-space model. AU-Introspection states that if player i is unaware of an event then she is unaware of being unaware of it. Dekel, Lipman, and Rustichini (1998) show that $\overline{A}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection, together with $\overline{U}_{\overline{\Pi}_i}$-Plausibility and an axiom called $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection, lead to a trivial form of unawareness in a standard-state-space model. In contrast, $\overline{A}_{\sigma_i}\overline{U}_{\sigma_i}$-Introspection holds with equality. AB-Self Reflection states that player i is aware of an event iff she is aware of believing it. Likewise, AA-Self Reflection states that player i is aware of an event iff she is aware of being aware of it. Correct-Belief-in-Awareness states that if she believes that she is aware of an event then she is aware of it. Strong Plausibility means that if player i is unaware of an event then she does not believe it, she does not believe that she does not believe it, and so on *ad infinitum*.

2.2.4 Awareness Operator defined by the Belief Operator

I define an awareness operator from the lack of beliefs. If $\overline{\Pi}_i^{\uparrow}$ induces $\overline{B}_{\overline{\Pi}_i}$, then I define the *awareness operator* $\overline{A}_{\overline{\Pi}_i} : \mathcal{E} \to \mathcal{E}$ as

$$\overline{A}_{\overline{\Pi}_i}(\overline{E}) := \overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \text{ for each } \overline{E} \in \mathcal{E}.$$

Define $A_{\overline{\Pi}_i}(E) := B_{\overline{\Pi}_i}(E) \cup B_{\overline{\Pi}_i}(\neg B_{\overline{\Pi}_i})(E)$ so that $\overline{A}_{\overline{\Pi}_i}(E, S) = (A_{\overline{\Pi}_i}(E), S)$.

Player i is ($\overline{A}_{\overline{\Pi}_i}$-)aware of an event $\overline{E}$ at a state $\omega \in \Omega$ if $\omega \in A_{\overline{\Pi}_i}(E)$. Two remarks are in order. First, the awareness operator $\overline{A}_{\overline{\Pi}_i}$ coincides with the original definition by Modica and Rustichini (1994, 1999): $\overline{A}_{\overline{\Pi}_i}(\overline{E}) = \overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee ((\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}))$. Second, $B_{\overline{\Pi}_i}(\cdot) \subseteq A_{\overline{\Pi}_i}(\cdot) \subseteq A_{\sigma_i}(\cdot)$.

Define the unawareness operator $\overline{U}_{\overline{\Pi}_i}$ from $\overline{A}_{\overline{\Pi}_i}$:

$$\overline{U}_{\overline{\Pi}_i}(E, S) := (U_{\overline{\Pi}_i}(E), S) := ((\neg A_{\overline{\Pi}_i})(E), S) \in \mathcal{E} \text{ for each } (E, S) \in \mathcal{E}.$$

By definition, the unawareness operator $\overline{U}_{\overline{\Pi}_i}$ satisfies Plausibility (with equality):

$$\overline{U}_{\overline{\Pi}_i}(\cdot) = (\neg\overline{B}_{\overline{\Pi}_i})(\cdot) \wedge (\neg\overline{B}_{\overline{\Pi}_i})^2(\cdot).$$

2.3 A Belief Operator Induces a Possibility Correspondence

The previous subsection has defined the belief operator induced by the possibility correspondence. Conversely, this subsection shows that, when a belief operator is given, there exist a unique possibility correspondence that induces the given belief operator. It also shows that, when an awareness operator is given, there exist a unique awareness function that induces the given awareness operator.

Definition 3. Let $\overline{B}_i : \mathcal{E} \ni (E,S) \mapsto \overline{B}_i(E,S) := (B_i(E),S) \in \mathcal{E}$ be a belief operator satisfying B-Decomposition, B-Monotonicity, B-Conjunction, and B-Necessitation (recall Remark 2). I define the possibility correspondence $\overline{\Pi}^{\uparrow}_{B_i} : \Omega \ni \omega \mapsto (\Pi^{\uparrow}_{B_i}(\omega), \sigma_{B_i}(\omega)) \in \mathcal{E}$ *induced* by the belief operator $\overline{B}_i$ as follows. Fix $\omega \in \Omega$.

First, define

$$\sigma_{B_i}(\omega) := \sup\{S \in \mathcal{S} \mid \omega \in B_i(S^{\uparrow})\}.$$

Note that the set $\{S \in \mathcal{S} \mid \omega \in B_i(S^{\uparrow})\}$ is not empty because $\omega \in B_i((\inf \mathcal{S})^{\uparrow})$ holds by B-Necessitation. Note also that, by B-Monotonicity,

$$\sigma_{B_i}(\omega) = \sup\{S \in \mathcal{S} \mid \omega \in B_i(E) \text{ for some } (E,S) \in \mathcal{E}\}.$$

Second, for each $\omega \in \Omega$, define

$$\Pi_{B_i}(\omega) := \bigcap \{(r^{\sigma_{B_i}(\omega)}_{S(D^{\uparrow})})^{-1}(D) \in \mathcal{D}_{\sigma_{B_i}(\omega)} \mid \overline{D}^{\uparrow} \in \mathcal{E} \text{ satisfies } \omega \in B_i(D^{\uparrow})\}.$$

Proposition 1 (1a) below shows that the belief operator $\overline{B}_{\overline{\Pi}_i}$ induced by the well-defined possibility correspondence $\overline{\Pi}^{\uparrow}_i$ satisfies $\overline{\Pi}_{B_{\overline{\Pi}_i}} = \overline{\Pi}_i$. Conversely, Proposition 1 (1b) shows that, for a belief operator $\overline{B}_i$ satisfying B-Decomposition, B-Monotonicity, B-Conjunction, and B-Necessitation, the possibility correspondence $\overline{\Pi}^{\uparrow}_{B_i}$ is well-defined and induces the original belief operator, i.e., $\overline{B}_i = \overline{B}_{\overline{\Pi}_{B_i}}$. In fact, $\overline{\Pi}_{B_i}$ is a unique well-defined possibility correspondence that induces the original belief operator.

Since A_{σ_i} depends only on an awareness function σ_i, the awareness operator A_{σ_i} can be equivalently characterized by σ_i.

Definition 4. Let $\overline{A}_i : \mathcal{E} \ni (E,S) \mapsto \overline{A}_i(E,S) := (A_i(E),S) \in \mathcal{E}$ be an awareness operator satisfying A-Decomposition, A-Independence, A-Subspace-Monotonicity, A-Subspace-Conjunction, and Awareness-at-the-Bottom. Given such $\overline{A}_i$, define the mapping $\sigma_{A_i}(\cdot) : \Omega \to \mathcal{S}$ by

$$\sigma_{A_i}(\omega) := \sup\{S \in \mathcal{S} \mid \omega \in A_i(E) \text{ for some } (E,S) \in \mathcal{E}\} \text{ for each } \omega \in \Omega.$$

Proposition 1 (2a) shows that the awareness operator $\overline{A}_{\sigma_i}$ induced by the well-defined awareness function σ_i satisfies $\sigma_{A_{\sigma_i}} = \sigma_i$. Conversely, Proposition 1 (2b) shows that, for an awareness operator $\overline{A}_i$ satisfying A-Decomposition, A-Independence, A-Subspace-Monotonicity, A-Subspace-Conjunction, and Awareness-at-the-Bottom, the awareness function σ_{A_i} is well-defined and induces the original awareness operator, i.e., $\overline{A}_i = \overline{A}_{\sigma_{A_i}}$. In fact, σ_{A_i} is a unique well-defined awareness function that induces the original awareness operator.

Proposition 1. *1. (a) Any well-defined possibility correspondence $\overline{\Pi}_i^{\uparrow}$ satisfies $\overline{\Pi}_i^{\uparrow} = \overline{\Pi}^{\uparrow}_{B_{\overline{\Pi}_i}}$.*

(b) Let $\overline{B}_i : \mathcal{E} \ni (E, S) \mapsto \overline{B}_i(E, S) := (B_i(E), S) \to \mathcal{E}$ satisfy B-Decomposition, B-Monotonicity, B-Conjunction, and B-Necessitation. Then, $\overline{\Pi}^{\uparrow}_{B_i} : \Omega \to \mathcal{E}$ is well-defined and satisfies $\overline{B}_i = \overline{B}_{\overline{\Pi}_{B_i}}$. It is a unique well-defined possibility correspondence $\overline{\Psi}_i$ such that $\overline{B}_i = \overline{B}_{\overline{\Psi}_i}$.

2. (a) Any well-defined awareness function σ_i satisfies $\sigma_i = \sigma_{A_{\sigma_i}}$.

(b) Let $\overline{A}_i : \mathcal{E} \ni (E, S) \mapsto \overline{A}_i(E, S) := (A_i(E), S) \in \mathcal{E}$ satisfy A-Decomposition, A-Independence, A-Subspace-Monotonicity, A-Subspace-Conjunction, and Awareness-at-the-Bottom. Then, $\sigma_{A_i} : \Omega \to \mathcal{S}$ is well-defined and satisfies $\overline{A}_i = \overline{A}_{\sigma_{A_i}}$. It is a unique well-defined awareness function τ_i such that $\overline{A}_i = \overline{A}_{\tau_i}$.

Proposition 1 (1b) generalizes the equivalence between a belief operator and a possibility correspondence on a standard-state-space model (e.g., Morris, 1996) in two respects. First, in a standard-state-space model, the conditions on the belief operator are B-Monotonicity, B-Conjunction, and B-Necessitation. In a generalized-state-space model, the conditions on the belief operator require B-Decomposition in addition to these three properties.

Second, the proposition generalizes the uniqueness of the possibility correspondence in a standard-state-space model (Fukuda, 2019, Remark 1): for a given belief operator $\overline{B}_i$, if there exists a possibility correspondence $\overline{\Pi}_i^{\uparrow}$ that induces $\overline{B}_i$, then it is the unique possibility correspondence that induces $\overline{B}_i$.

2.3.1 Possibility Operator

Before characterizing the properties of belief and awareness operators by the corresponding properties of a possibility correspondence, I study the possibility operator. Recalling that a state ω' is considered possible by i at $\omega \in \Omega$ if $\omega' \in \Pi_i(\omega)$, player i considers an event (E, S) *possible* at a state ω if $\Pi_i(\omega) \cap E \neq \emptyset$ (i.e., if there is $\omega' \in E$ considered possible at ω). In other words, i considers (E, S) possible at ω iff $\Pi_i^{\uparrow}(\omega) \cap E \neq \emptyset$ and $\sigma_i(\omega) \succeq S$.

For each $(E, S) \in \mathcal{E}$, define

$$M_{\overline{\Pi}_i}(E) :=\{\omega \in \Omega \mid \Pi_i(\omega) \cap E \neq \emptyset\}$$
$$(=\{\omega \in \Omega \mid \Pi_i^{\uparrow}(\omega) \cap E \neq \emptyset \text{ and } \sigma_i(\omega) \succeq S\}).$$

The possibility correspondence $\overline{\Pi}_i^{\uparrow}$ *induces* the *possibility operator* $\overline{M}_{\overline{\Pi}_i} : \mathcal{E} \to \mathcal{E}$ if

$$\overline{M}_{\overline{\Pi}_i}(E, S) := (M_{\overline{\Pi}_i}(E), S) \in \mathcal{E} \text{ for any } (E, S) \in \mathcal{E}.$$

In this case, player i considers $\overline{E}$ possible at ω if $\omega \in M_{\overline{\Pi}_i}(\overline{E})$. If i considers (E, S) possible at ω then she is aware of (E, S) at ω in the sense that $M_{\overline{\Pi}_i}(E) \subseteq A_{\sigma_i}(E)$.

Appendix B establishes the equivalence between a possibility operator and a possibility correspondence. Proposition 9 (in Appendix B) shows that a possibility correspondence $\overline{\Pi}_i^{\uparrow}$ is well-defined iff it induces $\overline{M}_{\overline{\Pi}_i}$. In this case, for each $\overline{E} = (E, S)$,

$$\overline{M}_{\overline{\Pi}_i}(\overline{E}) = (\neg \overline{B}_{\overline{\Pi}_i})(\neg \overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S^{\uparrow}}) = (\neg \overline{B}_{\overline{\Pi}_i})(\neg \overline{E}) \wedge \overline{A}_{\sigma_i}(\overline{E}).$$

Recall that, in a standard-state-space model, player i considers an event possible iff she does not believe its negation. In contrast, Modica and Rustichini (1999) suggest that, in order for a player to consider an event possible, she has to be aware of the event. In a generalized-state-space model, in fact, player i considers an event possible iff she does not believe its negation and she is A_{σ_i}-aware of the event.

2.4 Characterizations of Properties of Belief and Awareness

This subsection characterizes properties of the belief operator by the possibility correspondence. Section 2.4.1 characterizes properties of belief and awareness by the possibility correspondence. Section 2.4.2 studies properties of the possibility correspondence under which two definitions of awareness coincide.

2.4.1 Characterizations

I characterize properties of the belief and awareness operators in terms of properties of the possibility correspondence. Proposition 2 demonstrates that $\overline{B}_{\overline{\Pi}_i}$ (or $\overline{B}_i$) satisfies a property of beliefs (e.g., Consistency/No-Contradiction) iff $\overline{\Pi}_i^{\uparrow}$ (or $\overline{\Pi}_{\overline{B}_i}^{\uparrow}$) satisfies the corresponding property (e.g., Generalized Seriality).

Proposition 2. *Let $\overline{\Pi}_i^{\uparrow}$ be well-defined. Each of the following characterizes a given property of the belief operator $\overline{B}_{\overline{\Pi}_i}$.*

1. (a) *Consistency: $\overline{B}_{\overline{\Pi}_i}(\cdot) \leqslant \overline{M}_{\overline{\Pi}_i}(\cdot)$.*
   (b) *No-Contradiction: $\overline{B}_{\overline{\Pi}_i}(\overline{\emptyset}^S) = \overline{\emptyset}^S$ for all $S \in \mathcal{S}$.*

(c) $\overline{M}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) = \overline{A}_{\sigma_i}(E, S)$ *for all* $(E, S) \in \mathcal{E}$.

(d) $\overline{\Pi}_i^{\uparrow}$ *satisfies Generalized Seriality:* $\Pi_i^{\uparrow}(\cdot) \neq \emptyset$ *(equivalently,* $\Pi_i(\cdot) \neq \emptyset$*).*

2. (a) *Truth Axiom:* $\overline{B}_{\overline{\Pi}_i}(\overline{E}) \leqslant \overline{E}$ *for any* $\overline{E} \in \mathcal{E}$.

(b) $\overline{\Pi}_i^{\uparrow}$ *satisfies Generalized Reflexivity:* $\omega \in \Pi_i^{\uparrow}(\omega)$ *for all* $\omega \in \Omega$.

3. (a) *Positive Introspection:* $\overline{B}_{\overline{\Pi}_i}(\cdot) \leqslant \overline{B}_{\overline{\Pi}_i}\overline{B}_{\overline{\Pi}_i}(\cdot)$.

(b) $\overline{\Pi}_i^{\uparrow}$ *satisfies Generalized Transitivity:* $\omega' \in \Pi_i^{\uparrow}(\omega)$ *implies* $\overline{\Pi}_i^{\uparrow}(\omega') \leqslant \overline{\Pi}_i^{\uparrow}(\omega)$.

(c) *If* $\omega' \in \Pi_i(\omega)$*, then* $\sigma_i(\omega') = \sigma_i(\omega)$ *and* $\Pi_i(\omega') \subseteq \Pi_i(\omega)$.

4. (a) *Generalized Negative Introspection I:* $(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) \leqslant \overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E})$ *for any* $\overline{E} = (E, S) \in \mathcal{E}$.

(b) *Generalized Negative Introspection II:* $(\overline{M}_{\overline{\Pi}_i}\overline{B}_{\overline{\Pi}_i}(\overline{E}) =)(\neg\overline{B}_{\overline{\Pi}_i})^2(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) \leqslant \overline{B}_{\overline{\Pi}_i}(\overline{E})$ *for any* $\overline{E} = (E, S) \in \mathcal{E}$.

(c) $\overline{A}_{\overline{\Pi}_i}$*-Weak Necessitation:* $\overline{A}_{\overline{\Pi}_i}(\overline{E}) = \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow})$ *for all* $\overline{E} = (E, S) \in \mathcal{E}$.

(d) $\overline{A}_{\overline{\Pi}_i}$*-Independence:* $\overline{A}_{\overline{\Pi}_i}(E, S) = \overline{A}_{\overline{\Pi}_i}(E', S)$ *for any* $(E, S), (E', S) \in \mathcal{E}$.

(e) $\overline{\Pi}_i^{\uparrow}$ *satisfies Generalized Euclideanness: if* $\omega' \in \Pi_i(\omega)$ *then* $\Pi_i^{\uparrow}(\omega) \subseteq \Pi_i^{\uparrow}(\omega')$.

The possibility correspondence $\overline{\Pi}_i^{\uparrow}$ is *generalized serial*, *generalized reflexive*, *generalized transitive*, and *generalized Euclidean* if it satisfies Generalized Seriality, Generalized Reflexivity, Generalized Transitivity, and Generalized Euclideanness, respectively.

Seven remarks on Proposition 2 are in order. The first is implications of Generalized Seriality. Part (1a) states that if player i believes an event then she considers the event possible. By Monotonicity, Consistency is equivalent to:

$$\overline{B}_{\overline{\Pi}_i}(\overline{E}^{\uparrow}) \leqslant (\neg\overline{B}_{\overline{\Pi}_i})(\neg\overline{E}^{\uparrow}) \text{ for all } \overline{E} \in \mathcal{E}.$$

Part (1b) states that there is no state at which player i believes any form of a contradiction $\overline{\emptyset}^S$. Part (1c) states that player i is $\overline{A}_{\sigma_i}$-aware of an event (E, S) iff she considers the tautology $\overline{S}^{\uparrow}$ possible. One can show that Part (1d) is also equivalent to: $\Pi_i|_{\sup \mathcal{S}}(\cdot) \neq \emptyset$.

Second, Generalized Reflexivity implies Generalized Seriality. In other words, as in standard-state-space models, Truth Axiom and Monotonicity imply Consistency, as Truth Axiom implies $\overline{B}_{\overline{\Pi}_i}(\overline{E}) \leqslant \overline{E} \leqslant (\neg\overline{B}_{\overline{\Pi}_i})(\neg\overline{E})$. Also, one can show that $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Reflexivity iff $\Pi_i^{\uparrow}|_{\sup \mathcal{S}}$ satisfies Generalized Reflexivity (i.e., $\omega \in \Pi_i^{\uparrow}(\omega)$ for all $\omega \in \sup \mathcal{S}$).

Third, in the similar way to Grant et al. (2015, Lemma 1), it can be shown that, under Confinedness and PPB, Generalized Reflexivity is equivalent to the following property:

- Reflexivity Given Awareness (RGA): if $\omega \in \sigma_i(\omega)$ then $\omega \in \Pi_i(\omega)$.

Also, if $\overline{\Pi}_i^{\uparrow}$ is generalized reflexive, then $\overline{A}_{\sigma_i}(\cdot) = \overline{M}_{\overline{\Pi}_i}\overline{A}_{\sigma_i}(\cdot)$: player i is aware of an event iff she considers it possible that she is aware of it.

Fourth, Generalized Transitivity and Generalized Euclideanness jointly yield the following axiom studied by Heifetz, Meier, and Schipper (2006):

- Stationarity: $\omega' \in \Pi_i(\omega)$ implies $\overline{\Pi}_i(\omega) = \overline{\Pi}_i(\omega')$.

As in the literature on standard-state-space models, I have divided Stationarity into Generalized Transitivity and Generalized Euclideanness in that they characterize Positive Introspection and Generalized Negative Introspection, respectively.

Fifth, the generalized negative introspective properties indeed generalize Negative Introspection in a standard-state-space model. Generalized Negative Introspection I reduces to Negative Introspection when $\mathcal{S} = \{S\}$ because of Necessitation. Generalized Negative Introspection II states that if player i considers it possible that she believes an event $\overline{E}$ then she believes $\overline{E}$, i.e., $\overline{M}_{\overline{\Pi}_i}\overline{B}_{\overline{\Pi}_i}(\cdot) \leqslant \overline{B}_{\overline{\Pi}_i}(\cdot)$. In a standard-state-space model, Generalized Negative Introspection II and Negative Introspection are written in this same form (i.e., possibility of belief implies belief).

Sixth, Proposition 2 (4c) characterizes Weak Necessitation with respect to the awareness operator $\overline{A}_{\overline{\Pi}_i}$. At the same time, it also characterizes $\overline{A}_{\sigma_i}$-Plausibility with equality because

$$\overline{A}_{\sigma_i}(\overline{E}) = \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) = \overline{A}_{\overline{\Pi}_i}(\overline{E}) = \overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}).$$

Thus, it states that two notions of awareness coincide: $\overline{A}_{\sigma_i} = \overline{A}_{\overline{\Pi}_i}$. Also, Proposition 2 (4d) implies that the awareness operator $\overline{A}_{\overline{\Pi}_i}$ coincides with $\overline{A}_{\sigma_i}$ iff $\overline{A}_{\overline{\Pi}_i}$ satisfies Independence (i.e., it only depends on the base space of each event).

Seventh, the following remark contrasts the difference between Generalized Negative Introspection (I or II) and Negative Introspection. While Generalized Euclideanness characterizes Generalized Negative Introspection (I or II), Generalized Euclideanness and $S(\cdot) = \sigma_i(\cdot)$ characterize Negative Introspection, equivalently, trivial (full) awareness.[8] In other words, whether (un)awareness is trivial depends on (i) whether Generalized Euclideanness holds and (ii) whether $S(\cdot) = \sigma_i(\cdot)$. Formally:

Remark 6. The following are equivalent.

1. Negative Introspection: $(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) \leqslant \overline{B}_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E})$.
2. Trivial $\overline{A}_{\overline{\Pi}_i}$-Awareness: $\overline{A}_{\overline{\Pi}_i}(E, S) = \overline{S}^{\uparrow}$ for any $(E, S) \in \mathcal{E}$.
3. $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Euclideanness and $\sigma_i(\cdot) = S(\cdot)$.

[8]Note that $\sigma_i(\cdot) = S(\cdot)$ iff $\overline{A}_{\sigma_i}(E, S) = \overline{S}^{\uparrow}$ for all $(E, S) \in \mathcal{E}$. As a technical remark, one can also show: $\sigma_i(\cdot) = S(\cdot)$ iff $\sigma_i|_{\sup \mathcal{S}}(\cdot) = \sup \mathcal{S}$.

The proof of Remark 6 is in Appendix A.3.

Next, I characterize the joint properties of belief and unawareness by the possibility correspondence.

Proposition 3. *Let $\overline{\Pi}_i^{\uparrow}$ be well-defined. Each of the following characterizes a given property of the belief and unawareness operators $\overline{B}_{\overline{\Pi}_i}$ and $\overline{A}_{\sigma_i}$.*

1. (a) *$\overline{A}_{\sigma_i}$-Introspection: $\overline{A}_{\sigma_i}(\cdot) \leqslant \overline{B}_{\overline{\Pi}_i}\overline{A}_{\sigma_i}(\cdot)$.*[9]

   (b) *$\overline{\Pi}_i^{\uparrow}$ satisfies Belief-in-Awareness: $\omega' \in \Pi_i^{\uparrow}(\omega)$ implies $\sigma_i(\omega') \succeq \sigma_i(\omega)$.*

   (c) *If $\omega' \in \Pi_i(\omega)$ then $\sigma_i(\omega') \succeq \sigma_i(\omega)$ (or, $\sigma_i(\omega') = \sigma_i(\omega)$).*

2. (a) *$\overline{B}_{\overline{\Pi}_i}\overline{U}_{\sigma_i}$-Introspection: $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\sigma_i}(E, S) = \overline{\emptyset}^S$ for all $(E, S) \in \mathcal{E}$.*

   (b) *$\overline{\Pi}_i^{\uparrow}$ satisfies No-Belief-in-Unawareness: for every $\omega \in \Omega$, there is $\omega' \in \Pi_i^{\uparrow}(\omega)$ with $\sigma_i(\omega') \succeq \sigma_i(\omega)$.*

Proposition 3 studies two properties. First, $\overline{A}_{\sigma_i}$-Introspection states that, whenever player i is $\overline{A}_{\sigma_i}$-aware of an event, she believes that she is $\overline{A}_{\sigma_i}$-aware of the event. Also, $\overline{A}_{\sigma_i}$-Introspection gives the sense in which player i believes her awareness function: $\Pi_i(\omega) \subseteq \{\omega' \in \Omega \mid \sigma_i(\omega') = \sigma_i(\omega)\}$, or $\omega \in B_{\overline{\Pi}_i} A_{\sigma_i}(\sigma_i^{\uparrow}(\omega))$, at each $\omega \in \Omega$. Also, $\overline{A}_{\sigma_i}$-Introspection is related to Positive Introspection in that $\overline{A}_{\sigma_i}$-Introspection is Positive Introspection (Generalized Transitivity) with respect to each $\overline{S}^{\uparrow}$. The characterization of A-Introspection is related to one given in Board and Chung (2021).

Second, $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\sigma_i}$-Introspection is referred to as KU-Introspection for a model of knowledge and unawareness. It states that, for any given event $\overline{E}$, there is no state at which player i believes that she is $\overline{U}_{\sigma_i}$-unaware of the event $\overline{E}$.

Three remarks on Proposition 3 are in order. First, Remark 7 below studies $\overline{A}_{\sigma_i}$-Introspection (Part (1)) and $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection (Parts (2-4)). The proof of the remark is in Appendix A.3.

Remark 7.

1. If $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Transitivity, then it satisfies Belief-in-Awareness.

2. $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection implies $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\sigma_i}$-Introspection.

3. Suppose that $\overline{\Pi}_i^{\uparrow}$ satisfies Belief-in-Awareness. Then $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\sigma_i}$-Introspection obtains iff $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Seriality.

4. The Generalized Reflexivity of $\overline{\Pi}_i^{\uparrow}$ implies $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection.

5. If $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Seriality and Generalized Transitivity, then $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection obtains.

[9] The other part, Correct-Belief-in-Awareness, holds by Remark 5.

Second, Remark 8 below studies $\overline{A}_{\overline{\Pi}_i}$-Introspection: $\overline{A}_{\overline{\Pi}_i}(\cdot) \leqslant \overline{B}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_i}(\cdot)$. The proof of the remark is in Appendix A.3.

Remark 8. 1. If $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Transitivity, then $\overline{A}_{\overline{\Pi}_i}$-Introspection holds. If $\overline{\Pi}_i^{\uparrow}$ additionally satisfies Generalized Reflexivity, then $\overline{A}_{\overline{\Pi}_i}$-Introspection holds with equality.

2. If $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Reflexivity and Generalized Transitivity, then $\overline{A}_{\overline{\Pi}_i}\overline{B}_{\overline{\Pi}_i}$-Self Reflection holds: $\overline{A}_{\overline{\Pi}_i}\overline{B}_{\overline{\Pi}_i} = \overline{A}_{\overline{\Pi}_i}$.

3. $\overline{A}_{\overline{\Pi}_i}$-Introspection and $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection (which follow when $\overline{\Pi}_i^{\uparrow}$ is generalized serial and generalized transitive) imply $\overline{A}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_i}$-Self Reflection: $\overline{A}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_i} = \overline{A}_{\overline{\Pi}_i}$.

Third, in a standard-state-space model, Truth Axiom and Negative Introspection yield Positive Introspection (e.g., Aumann, 1999). In a generalized-state-space model, Truth Axiom, $\overline{A}_{\sigma_i}$-Introspection, and Generalized Negative Introspection (I or II) yield Positive Introspection. In other words, if $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Reflexivity, Generalized Euclideanness, and Belief-in-Awareness, then it satisfies Generalized Transitivity.[10]

2.4.2 Implications of Weak Necessitation

Weak Necessitation yields a wide variety of properties of (un)awareness. Recall that Weak Necessitation implies $\overline{A}_{\overline{\Pi}_i} = \overline{A}_{\sigma_i}$. That is, for each event (E, S),

$$A_{\overline{\Pi}_i}(E) = \{\omega \in S \mid S = \sigma_i(\omega)\}^{\uparrow} \text{ and } U_{\overline{\Pi}_i}(E) = \{\omega \in S \mid S \neq \sigma_i(\omega)\}^{\uparrow}.$$

In the context of interactive awareness, if each $\overline{A}_{\overline{\Pi}_i}$ satisfies Weak Necessitation, then

$$\overline{A}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_j}(\overline{E}) = \overline{A}_{\overline{\Pi}_i}(\overline{E}) \text{ for all } \overline{E} \in \mathcal{E}.$$

In words, player i is aware that player j is aware of an event $\overline{E}$ iff player i is aware of $\overline{E}$. Moreover,

$$\overline{A}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_j}(\overline{E}) = \overline{A}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_j}\overline{A}_{\overline{\Pi}_j}(\overline{E}) \text{ for all } \overline{E} \in \mathcal{E}.$$

In words, player i is aware that player j is aware of an event $\overline{E}$ iff player i is aware that player j is aware of being aware of $\overline{E}$. For each of the above statements, the "only if" part (e.g., if player i is aware that player j is aware of $\overline{E}$ then player i is

[10]The proof goes as follows. Let $\omega' \in \Pi_i(\omega)$. By Belief-in-Awareness, $\sigma_i(\omega) = \sigma_i(\omega')$. It remains to show $\Pi_i(\omega') \subseteq \Pi_i(\omega)$. By Generalized Euclideanness and Generalized Reflexivity, $\omega_{\sigma_i(\omega)} \in \Pi_i(\omega) \subseteq \Pi_i(\omega')$. By repeating Generalized Euclideanness, $\Pi_i(\omega') \subseteq \Pi_i(\omega_{\sigma_i(\omega)}) \subseteq \Pi_i(\omega)$, where the last set inclusion follows from PPB.

aware of $\overline{E}$) plays an important role, for instance, in Feinberg (2021, Conditions C2 and C3).

In the context of single-agent awareness, Weak Necessitation is also related to "awareness of unawareness" and the value of information under unawareness (e.g., Fukuda, 2021). Thus, I characterize Weak Necessitation.

To that end, $\overline{A}_{\overline{\Pi}_i}$-Weak Necessitation implies the following properties of (un)awareness.[11]

Remark 9. If $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Euclideanness, then $\overline{B}_{\overline{\Pi}_i}$ and $\overline{A}_{\overline{\Pi}_i}$ satisfy the following properties.

1. Symmetry, AU-Introspection, A-Conjunction with equality, AB-Self Reflection, AA-Self Reflection, and Correct-Belief-in-Awareness.

2. Weak Negative Introspection: $(\neg \overline{B}_{\overline{\Pi}_i})(\cdot) \wedge \overline{A}_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\cdot) \leqslant \overline{B}_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\cdot)$.

3. Negative Non-Introspection: $(\neg \overline{B}_{\overline{\Pi}_i})(\cdot) \wedge (\neg \overline{B}_{\overline{\Pi}_i})^2(\cdot) \leqslant (\neg \overline{B}_{\overline{\Pi}_i})^3(\cdot)$.

4. Strong Plausibility with equality: $\overline{U}_{\overline{\Pi}_i}(\cdot) = \bigwedge_{n \in \mathbb{N}} (\neg \overline{B}_{\overline{\Pi}_i})^n(\cdot)$.

Part (1) follows from $\overline{A}_{\overline{\Pi}_i} = \overline{A}_{\sigma_i}$ and Remark 5.

For Part (2), Weak Negative Introspection says that if player i does not believe an event $\overline{E}$ and if she is $\overline{A}_{\overline{\Pi}_i}$-aware of not believing $\overline{E}$, then she believes that she does not believe $\overline{E}$. It can be seen that Weak Negative Introspection follows from Proposition 2 (4).

For Part (3), Negative Non-Introspection says, in contrast, that if player i does not believe an event $\overline{E}$ and if she does not believe that she does not believe it, then the lack of beliefs continues at the next level. Negative Non-Introspection can be equivalently stated as $\overline{U}_{\overline{\Pi}_i}(\cdot) \leqslant \overline{U}_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\cdot)$: if player i is unaware of an event then she is unaware of not believing it.

For Part (4), Strong Plausibility holds with equality because, by Monotonicity and Weak Necessitation,

$$\overline{U}_{\overline{\Pi}_i}(\overline{E}) = (\neg \overline{B}_{\overline{\Pi}_i})(\overline{S}^{\uparrow}) \leqslant (\neg \overline{B}_{\overline{\Pi}_i})^n(\overline{E}) \text{ for all } n \in \mathbb{N}.$$

Negative Non-Introspection is a special case when $n = 3$ in the above expression, as the left-most side reduces to $(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge (\neg \overline{B}_{\overline{\Pi}_i})^2(\overline{E})$. Since Generalized Euclideanness implies $\overline{U}_{\sigma_i} = \overline{U}_{\overline{\Pi}_i}$, I obtain $\overline{U}_{\sigma_i}$-Strong Plausibility with equality:

$$\overline{U}_{\sigma_i}(\cdot) = \bigwedge_{n \in \mathbb{N}} (\neg \overline{B}_{\overline{\Pi}_i})^n(\cdot).$$

[11]Two remarks are in order. First, the implications of $\overline{A}_{\overline{\Pi}_i}$-Weak Necessitation in Remark 9 are related to Board, Chung, and Schipper (2011, Proposition 2.3), which study the properties of the awareness operator $\overline{A}_{\sigma_i}$. This is because, under $\overline{A}_{\overline{\Pi}_i}$-Weak Necessitation, one has $\overline{A}_{\overline{\Pi}_i} = \overline{A}_{\sigma_i}$. Second, for the origins of these properties, see Heifetz, Meier, and Schipper (2006) and the references therein.

Hence, all the above properties (plus Weak Necessitation and Generalized Negative Introspections themselves) follow solely from Generalized Euclideanness. Now, does any of Weak Negative Introspection, Negative Non-Introspection, or Strong Plausibility with equality characterize Generalized Euclideanness? Proposition 4 shows that, under fairly natural assumptions (i.e., Consistency and Positive Introspection), they are all equivalent to Weak Necessitation.[12] To scrutinize the interdependence of these properties, Proposition 4 make the dependence of assumptions more explicit.

Proposition 4. *Let $\overline{\Pi}_i^{\uparrow}$ be well-defined.*

1. *Weak Negative Introspection, Negative Non-Introspection, and Strong Plausibility with equality in Remark 9 are all equivalent.*
2. *Let $\overline{\Pi}_i^{\uparrow}$ be such that $\overline{A}_{\overline{\Pi}_i}$-Introspection and $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection hold. The following are equivalent to Generalized Euclideanness.*
   (a) *$\overline{A}_{\overline{\Pi}_i}$-Symmetry.*
   (b) *$\overline{A}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection, i.e., $\overline{U}_{\overline{\Pi}_i}(\cdot) \leqslant \overline{U}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}(\cdot)$.*[13]
   (c) *Negative Non-Introspection, and thus each property in Part (1).*
3. (a) *Suppose $\overline{\Pi}_i^{\uparrow}$ satisfies Belief-in-Awareness. Generalized Euclideanness is equivalent to Subjective Negative Introspection: for all $\overline{E} = (E, S) \in \mathcal{E}$, $(\neg B_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) \leqslant \overline{B}_{\overline{\Pi}_i}((\neg B_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}))$.*
   (b) *Let $\overline{\Pi}_i^{\uparrow}$ satisfy Generalized Seriality. Then, it satisfies Generalized Euclideanness iff $\overline{A}_{\overline{\Pi}_i}$ satisfies Monotonicity: $\overline{E} \leqslant \overline{F}$ implies $\overline{A}_{\overline{\Pi}_i}(\overline{E}) \leqslant \overline{A}_{\overline{\Pi}_i}(\overline{F})$.*

As depicted in the bottom rectangle in Figure 1, Proposition 4 (1) asserts that the three properties, Weak Negative Introspection, Negative Non-Introspection, and Strong Plausibility with equality, are all equivalent. Then, Proposition 4 (2) shows that, while they are weaker to characterize Generalized Euclideanness in general, they characterize Generalized Euclideanness when $\overline{A}_{\overline{\Pi}_i}$-Introspection and $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection obtain (e.g., when the given possibility correspondence satisfies Generalized Seriality and Generalized Transitivity as in Remarks 7 and 8).[14] The proposition also shows that they are equivalent to $\overline{A}_{\overline{\Pi}_i}$-Symmetry and $\overline{A}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection.

Proposition 4 (3) shows that Weak Necessitation, $\overline{A}_{\overline{\Pi}_i}$-Monotonicity, and Subjective Negative Introspection are equivalent under Generalized Seriality and Belief-in-Awareness, the latter of which follows from Generalized Transitivity (as in Remark 7).

[12] Sections 4.4.1 and 4.4.2, respectively, study the role of Consistency and Positive Introspection and that of Weak Necessitation in a game-theoretic context.

[13] It can be shown that, by $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection, the equality holds.

[14] Fukuda (2021) studies a model of knowledge and unawareness in which A-Introspection and KU-Introspection follow as knowledge satisfies Truth Axiom, Monotonicity, and Positive Introspection.

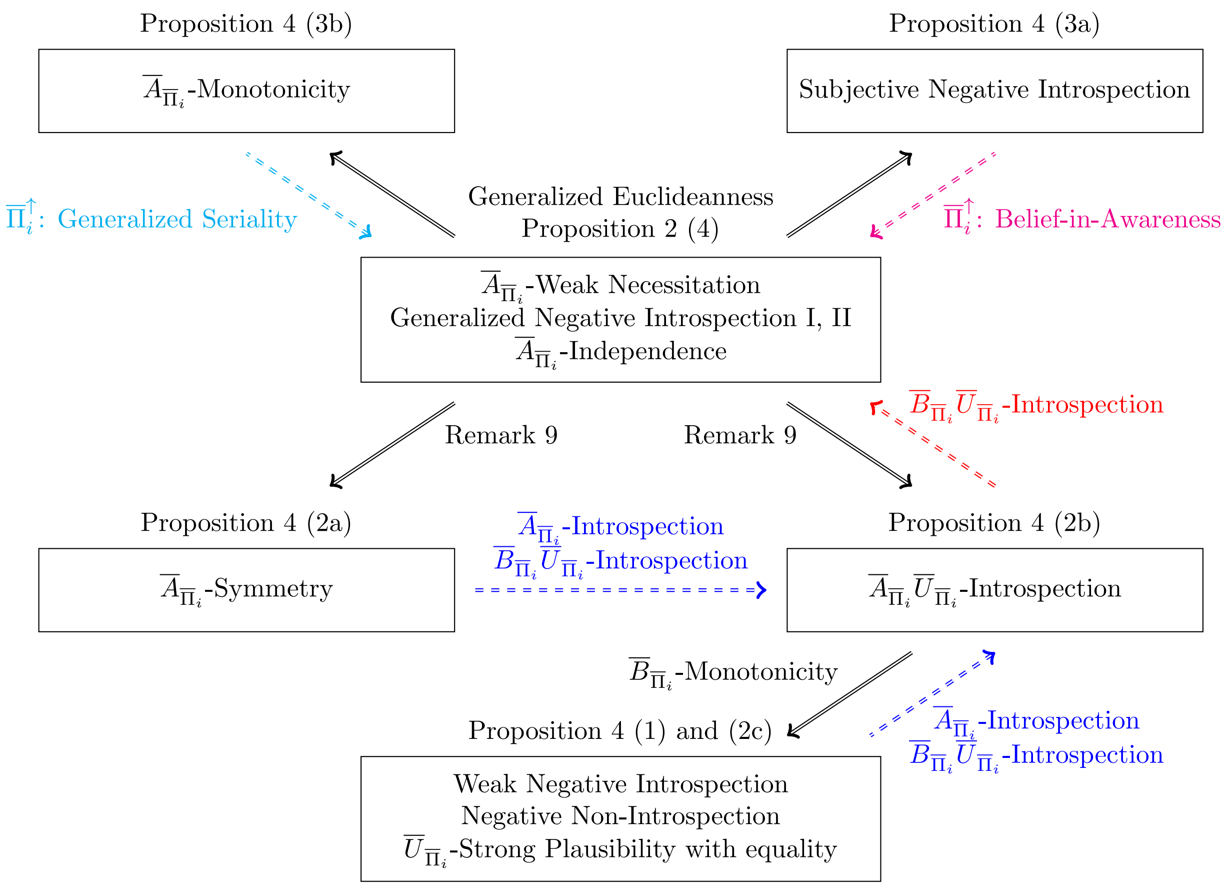

Figure 1: Proposition 4 (Generalized Euclideanness): each rectangle denotes a list of equivalent properties. Each rectangle, except for the one which characterizes Generalized Euclideanness, is associated with the corresponding part of Proposition 4. A solid arrow from one rectangle to another means that the former implies the latter without any additional conditions. The label "Remark 9" means that the implication has appeared in Remark 9. The label "$\overline{B}_{\overline{\Pi}_i}$-Monotonicity" means that the implication follows from $\overline{B}_{\overline{\Pi}_i}$-Monotonicity (which is satisfied as in Remark 2). A dashed arrow from one rectangle to another means that the former implies the latter under the preconditions specified next to the arrow.

Two additional remarks on Proposition 4 are in order. First:

Remark 10. If the possibility correspondence $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Seriality and Generalized Transitivity, then it also satisfies $\overline{A}_{\overline{\Pi}_i}$-Introspection, $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection, and Belief-in-Awareness. Then:

1. Each property in Proposition 4 characterizes Generalized Euclideanness.

2. Any of the properties of $\overline{B}_{\overline{\Pi}_i}$ in Proposition 4, together with Truth Axiom, Positive Introspection, and the properties in Remark 2 (i.e., Decomposition, Monotonicity, Conjunction, and Necessitation), characterizes the possibility-correspondence model of Heifetz, Meier, and Schipper (2006). Note that Truth Axiom implies Consistency.

3. In particular, Proposition 4 establishes that the properties of $\overline{B}_{\overline{\Pi}_i}$ mentioned in Heifetz, Meier, and Schipper (2006, Proposition 2) fully characterize their possibility-correspondence model.

Second, the proof of Proposition 4 (2) shows the following.

Remark 11. Under $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection, $\overline{A}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection is equivalent to Generalized Negative Introspection. This generalizes Chen, Ely, and Luo (2012, Proposition 1) in two sets of assumptions.

1. Assume that $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Seriality and Belief-in-Awareness. Then, Generalized Negative Introspection (I or II) is equivalent to $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection and $\overline{A}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection.

2. Let $\overline{\Pi}_i^{\uparrow}$ satisfy Generalized Reflexivity (thus, $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection obtains). Then, Generalized Negative Introspection (I or II) is equivalent to $\overline{A}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection.

3 Common Belief Operator and its Possibility Correspondence

This section formalizes the common belief operator, and identifies the possibility correspondence that induces the common belief operator. Section 3.1 introduces the common belief operator as the iteration of mutual beliefs as in the literature. To introduce the possibility correspondence that induces the common belief operator, Section 3.2 defines the possibility correspondence that induces the mutual belief operator. Then, Section 3.3 defines the possibility correspondence that induces the common belief operator.

3.1 Common Belief Operator

To define the common belief operator as the iteration of mutual beliefs, I start with defining the *mutual belief operator*:

$$\overline{B}_I(\cdot) := \bigwedge_{i \in I} \overline{B}_{\overline{\Pi}_i}(\cdot).$$

For each event $\overline{E}$, $\overline{B}_I(\overline{E})$ is the event that everybody believes $\overline{E}$. By $\overline{B}_{\overline{\Pi}_i}$-Decomposition for all $i \in I$, the mutual belief operator $\overline{B}_I$ satisfies Decomposition. I denote:

$$\overline{B}_I(\overline{E}) := (B_I(E), S(E)) \text{ for each } \overline{E} \in \mathcal{E}.$$

Then, define the *common belief operator* as:

$$\overline{C}_I(\cdot) := \bigwedge_{n \in \mathbb{N}} \overline{B}_I^n(\cdot).$$

Thus, for any event $\overline{E}$, $\overline{C}_I(\overline{E})$ is the event that everybody believes $\overline{E}$, everybody believes that everybody believes $\overline{E}$, and so forth *ad infinitum*. The common belief operator satisfies Decomposition: write

$$\overline{C}_I(\overline{E}) = (C_I(E), S(E)) \text{ for each } \overline{E} \in \mathcal{E}.$$

An event $\overline{E} \in \mathcal{E}$ is *common belief* (or *commonly believed*) among I at a state ω if $\omega \in C_I(E)$.

It can be seen that $\overline{C}_I$ satisfies Monotonicity, Conjunction, Necessitation in addition to Decomposition (recall Remark 2). Hence, by Remark 1 and Proposition 1, the common belief operator $\overline{C}_I$ is induced by a unique possibility correspondence.

Before defining the possibility correspondence that induces the common belief operator, I remark that the common belief operator admits the fixed-point characterization, generalizing Monderer and Samet (1989):

Remark 12. For each $\overline{E} \in \mathcal{E}$,

$$\overline{C}_I(\overline{E}) = \sup\{\overline{F} \in \mathcal{E} \mid \overline{F} \leqslant \overline{B}_I(\overline{F}) \text{ and } \overline{F} \leqslant \overline{B}_I(\overline{E})\}.$$

Note that the supremum is taken in $\langle \mathcal{E}, \leqslant \rangle$.

The proof of Remark 12 is in Appendix A.4. Since $\overline{C}_I(\overline{E})$ satisfies $\overline{C}_I(\overline{E}) \leqslant \overline{B}_I(\overline{E})$ and $\overline{C}_I(\overline{E}) \leqslant \overline{B}_I(\overline{C}_I(\overline{E}))$, it follows that Positive Introspection obtains: $\overline{C}_I(\cdot) \leqslant \overline{C}_I\overline{C}_I(\cdot)$. By Proposition 2, the possibility correspondence that induces $\overline{C}_I$ satisfies Generalized Transitivity. Also, an event $\overline{E}$ is common belief among I at ω iff there is an event $\overline{F}$ satisfying $\omega \in F$, $\overline{F} \leqslant \overline{B}_I(\overline{F})$, and $\overline{F} \leqslant \overline{B}_I(\overline{E})$. If some player i's belief operator $\overline{B}_{\overline{\Pi}_i}$ satisfies Truth Axiom, then the mutual belief operator $\overline{B}_I$ satisfies Truth Axiom. Then, the common belief operator $\overline{C}_I$ satisfies Truth Axiom, and common belief reduces to common knowledge.

3.2 Mutual-Belief Possibility Correspondence

To define the possibility correspondence that induces the common belief operator, I start with defining the possibility correspondence $\overline{\Pi}^{\uparrow}_{B_I} := (\Pi^{\uparrow}_{B_I}, \sigma_{B_I})$ that induces the mutual belief operator $\overline{B}_I : \mathcal{E} \to \mathcal{E}$ as:

$$\overline{\Pi}^{\uparrow}_{B_I}(\omega) := \bigwedge\{\overline{E} \in \mathcal{E} \mid \overline{\Pi}^{\uparrow}_i(\omega) \leqslant \overline{E} \text{ for all } i \in I\} \text{ for each } \omega \in \Omega.$$

By construction, $\overline{B}_I(\overline{E}) = (B_I(E), S(E))$, where $B_I(E) := \{\omega \in \Omega \mid \overline{\Pi}^{\uparrow}_{B_I}(\omega) \leqslant \overline{E}\}$. By Remark 1, the possibility correspondence $\overline{\Pi}^{\uparrow}_{B_I}$ is well-defined. Also,

$$\sigma_{B_I}(\omega) = \sup\{S \in \mathcal{S} \mid \overline{\Pi}^{\uparrow}_i(\omega) \leqslant \overline{S}^{\uparrow} \text{ for all } i \in I\} = \inf_{j \in I} \sigma_j(\omega).$$

As a technical remark, if each sub-space S is endowed with the power set $\mathcal{P}(S)$, then $\Pi^{\uparrow}_{B_I}$ is explicitly written as $\Pi^{\uparrow}_{B_I}(\omega) = \left(\bigcup_{i \in I}(\Pi_i(\omega))_{\sigma_{B_I}(\omega)}\right)^{\uparrow}$, that is, $\Pi_{B_I}(\omega)$ is the union of the projection of each $\Pi_i(\omega)$ onto the sub-space $\sigma_{B_I}(\omega)$. In a single state space, $\Pi_{B_I}(\omega)$ is the union of each $\Pi_i(\omega)$.

Next, I inductively define the possibility correspondence $\overline{\Pi}^{\uparrow}_{B^n_I} := (\Pi^{\uparrow}_{B^n_I}, \sigma_{B^n_I})$ that induces the n-th order mutual belief operator B^n_I. For $n = 1$, $\overline{\Pi}^{\uparrow}_{B^1_I} = \overline{\Pi}^{\uparrow}_{B_I}$. Given $\overline{\Pi}^{\uparrow}_{B^n_I}$ that induces $\overline{B}^n_I$, define

$$\overline{\Pi}^{\uparrow}_{B^{n+1}_I}(\omega) := \bigwedge\{\overline{E} \in \mathcal{E} \mid \overline{\Pi}^{\uparrow}_{B^n_I}(\omega') \leqslant \overline{E} \text{ for any } \omega' \in \Pi^{\uparrow}_{B_I}(\omega)\} \text{ for each } \omega \in \Omega.$$

Then, it can be seen that $\overline{\Pi}^{\uparrow}_{B^{n+1}_I}$ induces $\overline{B}^{n+1}_I$. Thus, for each $n \in \mathbb{N}$, $\overline{B}^n_I(\overline{E}) = (B^n_I(E), S(E))$, where $B^n_I(E) := \{\omega \in \Omega \mid \overline{\Pi}^{\uparrow}_{B^n_I}(\omega) \leqslant \overline{E}\}$.

3.3 Common-Belief Possibility Correspondence

I define the possibility correspondence $\overline{\Pi}^{\uparrow}_{C_I} := (\Pi^{\uparrow}_{C_I}, \sigma_{C_I})$ that induces the common belief operator as:

$$\overline{\Pi}^{\uparrow}_{C_I}(\omega) := \bigwedge\{\overline{E} \in \mathcal{E} \mid \overline{\Pi}^{\uparrow}_{B^n_I}(\omega) \leqslant \overline{E} \text{ for all } n \in \mathbb{N}\} \text{ for each } \omega \in \Omega.$$

By construction, for each $\overline{E} = (E, S) \in \mathcal{E}$,

$$(\{\omega \in \Omega \mid \overline{\Pi}^{\uparrow}_{C_I}(\omega) \leqslant \overline{E}\}, S) = \left(\bigcap_{n \in \mathbb{N}}\{\omega \in \Omega \mid \overline{\Pi}^{\uparrow}_{B^n_I}(\omega) \leqslant \overline{E}\}, S\right) = \bigwedge_{n \in \mathbb{N}} \overline{B}^n_I(\overline{E}) = \overline{C}_I(\overline{E}).$$

That is, $C_I(E) = \{\omega \in \Omega \mid \overline{\Pi}^{\uparrow}_{C_I}(\omega) \leqslant \overline{E}\}$. By Remark 1, the possibility correspondence $\overline{\Pi}^{\uparrow}_{C_I}$ is well-defined.

Remark 13. I remark that $\overline{\Pi}^{\uparrow}_{C_I}$ is the "generalized-transitive closure" of $\overline{\Pi}^{\uparrow}_{B_I}$: the finest generalized-transitive possibility correspondence as coarse as $\overline{\Pi}^{\uparrow}_{B_I}$, generalizing standard-state-space possibility-correspondence models (e.g., Aumann, 1976).

To formalize this, for two well-defined possibility correspondences $\overline{\Pi}^{\uparrow}_i$ and $\overline{\Pi}^{\uparrow}_j$, $\overline{\Pi}^{\uparrow}_i$ is *as refined as* $\overline{\Pi}^{\uparrow}_j$ (or $\overline{\Pi}^{\uparrow}_j$ is *as coarse as* $\overline{\Pi}^{\uparrow}_i$) if $\overline{\Pi}^{\uparrow}_i(\cdot) \leqslant \overline{\Pi}^{\uparrow}_j(\cdot)$. By Definition 3 and Proposition 1, $\overline{\Pi}^{\uparrow}_i$ is as refined as $\overline{\Pi}^{\uparrow}_j$ iff $\overline{B}_{\overline{\Pi}_j}(\cdot) \leqslant \overline{B}_{\overline{\Pi}_i}(\cdot)$.

First, a well-defined possibility correspondence $\overline{\Pi}^{\uparrow}_{C_I}$ satisfies Generalized Transitivity and is at least as coarse as $\overline{\Pi}^{\uparrow}_{B_I}$. This is because $\overline{C}_I$ satisfies Positive Introspection and $\overline{C}_I(\cdot) \leqslant \overline{B}_I(\cdot)$.

Second, if $\overline{\Pi}^{\uparrow}_x$ is a well-defined generalized-transitive possibility correspondence which is at least as coarse as $\overline{\Pi}^{\uparrow}_{B_I}$, then $\overline{\Pi}^{\uparrow}_{C_I}$ is at least as refined as $\overline{\Pi}^{\uparrow}_x$. This is because, it follows from $\overline{B}_{\overline{\Pi}_x}(\cdot) \leqslant \overline{B}_{\overline{\Pi}_x}\overline{B}_{\overline{\Pi}_x}(\cdot) \leqslant \overline{B}_I\overline{B}_{\overline{\Pi}_x}(\cdot)$ and $\overline{B}_{\overline{\Pi}_x}(\cdot) \leqslant \overline{B}_I(\cdot)$ that

$$\overline{B}_{\overline{\Pi}_x}(\overline{E}) \leqslant \sup\{\overline{F} \in \mathcal{E} \mid \overline{F} \leqslant \overline{B}_I(\overline{F}) \text{ and } \overline{F} \leqslant \overline{B}_I(\overline{E})\} = \overline{C}_I(\overline{E}) \text{ for all } \overline{E} \in \mathcal{E}.$$

4 Rationality and Common Belief in Rationality

This section studies implications of rationality and common belief in rationality in a strategic game with unawareness. Section 4.1 defines a strategic game with unawareness, a model of the game, and rationality of the players in the unawareness model. Section 4.2 defines the solution concept of iterated elimination of strictly dominated actions for a strategic game with unawareness, and studies implications of rationality and common belief in rationality. Section 4.3 provides examples. Section 4.4 studies the role of introspective properties. Section 4.5 briefly discusses other solution concepts.

4.1 A Strategic Game with Unawareness, a Model of the Game, and Rationality

4.1.1 A Strategic Game with Unawareness

A *(strategic) game with unawareness* is a non-empty collection $\mathcal{V} = (V_\lambda)_{\lambda \in \Lambda}$ with the following two ingredients. First, the collection contains a finite strategic game $G = \langle (C_i)_{i \in I}, (\succcurlyeq_i)_{i \in I} \rangle$, which I call the *underlying* game: I is a finite set of players, each C_i is a non-empty finite set of actions available to player i, and $\succcurlyeq_i$ is i's (complete and transitive) preference ordering on $C := \bigtimes_{i \in I} C_i$.[15] Denote by $\sim_i$ and $\succ_i$ the indifference and strict preference orderings, respectively.

[15]I avoid using A_i as the set of actions because $A_{\overline{\Pi}_i}$ or A_{σ_i} denotes the awareness operator. Note that the common belief operator $\overline{C}_I$ always has the subscript I.

Before moving on to the second ingredient, a few remarks are in order. The focus on finite games is for ease of analysis. Throughout examples, I represent the underlying game by the players' utility functions, i.e., $G = \langle (C_i)_{i \in I}, (u_i)_{i \in I} \rangle$, where each $u_i : C \to \mathbb{R}$ represents her underlying preferences $\succcurlyeq_i$. As I have not introduced players' unawareness at this point (the second ingredient captures players' unawareness), the definition of a strategic game is standard. One can interpret the underlying game G as an outside analysts' representation of the game if the players would be aware of every aspect of it.

Second, with the underlying game G defined, $\mathcal{V} = (V_\lambda)_{\lambda \in \Lambda}$ is a collection of "subgames" of G defined as follows. For each $\lambda \in \Lambda$, $V_\lambda = \langle (V_{\lambda,i})_{i \in I}, (\succcurlyeq_{\lambda,i})_{i \in I} \rangle$ is a "subgame" of G in the sense that, for each player $i \in I$, $V_{\lambda,i}$ is a non-empty subset of C_i and $\succcurlyeq_{\lambda,i}$ is given from her original preferences $\succcurlyeq_i$ on $\bigtimes_{i \in I} V_{\lambda,i}$ (or the utility function u_i that represents her original preferences). With some abuse of notation, I identify and denote $V_\lambda := \bigtimes_{i \in I} V_{\lambda,i}$. There exists a partial order $\succeq$ such that $\langle \mathcal{V}, \succeq \rangle$ is a complete lattice, with the additional property that, for all $\lambda, \mu \in \Lambda$, $V_\lambda \succeq V_\mu$ iff $V_{\lambda,i} \supseteq V_{\mu,i}$ for all $i \in I$. The top element of $\mathcal{V}$ is the underlying game G. I call each V_λ a *view* or a *subgame* depending on the contexts. Each V_λ is meant to capture a strategic situation in which some players may have a limited understanding of the underlying game G.

In a subgame V_λ, while some players may be unaware of certain actions in a subgame V_ν with $V_\lambda \not\succeq V_\nu$, they can envision actions in any V_μ with $V_\lambda \succeq V_\mu$. Thus, in the subgame V_λ, each player takes into account the behavior of the opponents in "lower" subgames V_μ with $V_\lambda \succeq V_\mu$.

Two remarks are in order. First, although the strategic game $\mathcal{V}$ with unawareness specifies possible ways in which players may entail limited understanding of the underlying game, the strategic game $\mathcal{V}$ with unawareness itself does not specifically represent how each player views the underlying game, how she views how another player views the underlying game, and so forth. Rather, such interactive reasoning about players' awareness is represented through the notion of a model, which is defined shortly. Differently put, in this paper, the description of the strategic situation with unawareness and that of players' specific interactive reasoning are separated as a game with unawareness and a model.[16]

Second, the definition of a strategic game with unawareness here does not consider the case in which players may be unaware of some other players. It would be possible to incorporate such consideration by associating, with each $\lambda \in \Lambda$, a strategic game $V_\lambda = \langle (V_{\lambda,i})_{i \in I_\lambda}, (\succcurlyeq_{\lambda,i})_{i \in I_\lambda} \rangle$ where I_λ is a non-empty subset of players. The order $\succeq$ is

[16]Yet, it would still be interesting to specify, within the definition of a strategic game with unawareness, how each player i views the underlying game (which would be given by $V_i \in \mathcal{V}$), how each player i views the view of player j (which would be given by $V_{(i,j)} \in \mathcal{V}$) and so on. For instance, Čopič and Galeotti (2006) and Feinberg (2021) take that route, as they study games with unawareness as is, without using the apparatus of an unawareness model. I leave this line of research as an interesting avenue for future research.

modified so that $V_\lambda \succeq V_\mu$ iff $I_\lambda \supseteq I_\mu$ and $V_{\lambda,i} \supseteq V_{\mu,i}$ for all $i \in I_\mu$.[17]

4.1.2 A Model of the Game with Unawareness

I incorporate, into the strategic game $\mathcal{V} = (V_\lambda)_{\lambda\in\Lambda}$ with unawareness, the descriptions of how the players interactively view the game and how they behave. Namely, a *model* of the game $\mathcal{V}$ with unawareness is a tuple

$$\mathcal{M} := \langle \langle (S_\lambda, \mathcal{D}_\lambda)_{\lambda\in\Lambda}, \succeq, r\rangle, (\overline{\Pi}_i^{\uparrow})_{i\in I}, (\mathcal{C}_i)_{i\in I}, (\delta_i)_{i\in I}\rangle$$

with the following four ingredients.[18] First, $\langle (S_\lambda, \mathcal{D}_\lambda)_{\lambda\in\Lambda}, \succeq, r\rangle$ is a generalized state space, where the index set Λ and the partial order $\succeq$ for the generalized state space are the same as those for the strategic game $\mathcal{V} = (V_\lambda)_{\lambda\in\Lambda}$ with unawareness: for each pair $(\lambda, \mu) \in \Lambda$, $S_\lambda \succeq S_\mu$ iff $V_\lambda \succeq V_\mu$.[19] Denote by $\mathcal{S} := \{S_\lambda\}_{\lambda\in\Lambda}$ the collection of subspaces. Let $\Omega := \bigcup_{\lambda\in\Lambda} S_\lambda$ be the set of states, and let $\mathcal{E}$ be the collection of events.

Second, each $\overline{\Pi}_i^{\uparrow} : \Omega \to \mathcal{E}$ is player i's (well-defined) possibility correspondence. Denote by $\overline{\Pi}_i^{\uparrow}(\cdot) = (\Pi_i^{\uparrow}(\cdot), \sigma_i(\cdot))$.

Third, each $\mathcal{C}_i : \Omega \to \mathcal{P}(C_i) \setminus \{\emptyset\}$ is a correspondence with the following property: for each $\omega \in \Omega$, $\mathcal{C}_i(\omega) = V_{\lambda,i}$ where λ is the unique element of Λ such that $S_\lambda = S(\omega)$. I call each $\mathcal{C}_i$ the action correspondence of player i. Denote by $\mathcal{C}(\omega) := \bigtimes_{i\in I} \mathcal{C}_i(\omega)$ for each $\omega \in \Omega$. Thus, $\mathcal{C}$ associates, with each state $\omega \in \Omega$, the corresponding subgame $\mathcal{C}(\omega)$.

While the action correspondences $(\mathcal{C}_i)_{i\in I}$ can be defined from the generalized state space, I have introduced them for ease of notation. For instance, the action correspondences make it easier to define the following objects. Take a state $\omega \in \Omega$. The subgame $\mathcal{C}(\omega)$ is the outside analysts' "view" of the players' feasible actions at ω. The subgame $\mathcal{C}(\omega_{\sigma_i(\omega)})$ is player i's "view" of the game at ω, where $\omega_{\sigma_i(\omega)}$ is the projection of ω on $\sigma_i(\omega)$. The subgame $\mathcal{C}((\omega_{\sigma_j(\omega)})_{\sigma_i(\omega_{\sigma_j(\omega)})})$ is player i's "view" of player j's "view" of the game at ω, where $(\omega_{\sigma_j(\omega)})_{\sigma_i(\omega_{\sigma_j(\omega)})}$ is the projection of $\omega_{\sigma_j(\omega)}$ on $\sigma_i(\omega_{\sigma_j(\omega)})$.

Note that each action correspondence $\mathcal{C}_i$ satisfies the following three properties. The first property is: for any $C_i' \in \mathcal{P}(C_i)\setminus\{\emptyset\}$, $\langle C_i'\rangle := \{\omega \in \Omega \mid C_i' \subseteq \mathcal{C}_i(\omega)\}$ forms an event (i.e., $(\langle C_i'\rangle, S(\langle C_i'\rangle)) \in \mathcal{E}$).[20] This property states that any (non-empty) subset C_i' of player i's actions C_i is an object of belief and unawareness. Similarly, the second property is: for any $C_i' \in \mathcal{P}(C_i) \setminus \{\emptyset\}$, $A_{\sigma_i}(\langle C_i'\rangle) = \{\omega \in \Omega \mid C_i' \subseteq \mathcal{C}_i(\omega_{\sigma_i(\omega)})\}$ forms an

[17]Papers such as Feinberg (2021), Halpern and Rêgo (2014), Meier and Schipper (2014), and Rêgo and Halpern (2012) consider the possibility that some players may be unaware of other players.

[18]Technically, I impose one more assumption on the model $\mathcal{M}$ in Section 4.1.3.

[19]It is a slight abuse of notation to use the same partial order $\succeq$ between the generalized state space and the strategic game with unawareness. However, one can interpret this in a way such that Λ itself is partially ordered by $\succeq$.

[20]In fact, this event is $\overline{S}_\lambda^{\uparrow}$, where $\lambda = \inf\{\mu \in \Lambda \mid C_i' \subseteq V_{\mu,i}\}$.

event. The third property is: for any $\omega \in \Omega$, if $\omega', \omega'' \in \{\tilde{\omega} \in \sigma_i(\omega) \mid \overline{\Pi}_i^{\uparrow}(\tilde{\omega}) = \overline{\Pi}_i^{\uparrow}(\omega)\}$ then $\mathcal{C}_i(\omega') = \mathcal{C}_i(\omega'')$. This property follows simply because ω' and ω'' reside in the same subspace $\sigma_i(\omega)$.[21]

Fourth, $\delta_i : \Omega \to C_i$ is a *strategy* (or a *decision function*) of player i with the following four properties. The first property is: $\delta_i(\omega) \in \mathcal{C}_i(\omega_{\sigma_i(\omega)})$ for all $\omega \in \Omega$. It states that, at each state ω, the action $\delta_i(\omega)$ that the strategy δ_i prescribes resides in the set of actions $\mathcal{C}_i(\omega_{\sigma_i(\omega)})$ associated with her awareness.

The second property is: $\delta_i(\omega) = \delta_i(\omega_{\sigma_i(\omega)})$ for any $\omega \in \Omega$. It states that the actions prescribed by the strategy δ_i at ω and its projection $\omega_{\sigma_i(\omega)}$ are consistent.

The third property is: $[\delta_i|_{\sigma_i(\omega)} = c_i] := \{\tilde{\omega} \in \sigma_i(\omega) \mid \delta_i(\tilde{\omega}) = c_i\}$ belongs to $\mathcal{D}_{\sigma_i(\omega)}$ for each $\omega \in \Omega$ and $c_i \in \mathcal{C}_i(\omega_{\sigma_i(\omega)})$. It states that, for any subspace $\sigma_i(\omega)$ and for any feasible action $c_i \in \mathcal{C}_i(\omega_{\sigma_i(\omega)})$ of player i, the set of states at which player i takes action c_i is an event. That is, $([\delta_i|_{\sigma_i(\omega)} = c_i]^{\uparrow}, \sigma_i(\omega)) \in \mathcal{E}$.

The fourth property is a joint property of the strategy profile $(\delta_j)_{j \in I \setminus \{i\}}$ of the opponents and the action correspondence $\mathcal{C}_i$: for every $\omega \in \Omega$, $S \in \mathcal{S}$ with $\sigma_i(\omega) \succeq S$, and $c_i, c_i' \in \mathcal{C}_i(\omega_S)$,

$$[\![c_i' \succcurlyeq_i c_i]\!]_S := \{\tilde{\omega} \in S \mid (c_i', \delta_{-i}(\tilde{\omega})) \succcurlyeq_i (c_i, \delta_{-i}(\tilde{\omega}))\}$$

is an event in S, i.e., $([\![c_i' \succcurlyeq_i c_i]\!]_S^{\uparrow}, S) \in \mathcal{E}$. This property states that, for any state ω, for any subspace S as "low" as the subspace $\sigma_i(\omega)$ of which player i is aware, and for any pair of actions c_i and c_i' of which player i is aware in the subspace S, the set of states in S at which taking c_i' would be as great as taking c_i provided that the other players follow the strategies $\delta_{-i}|_S$ is a well-defined event in S. The idea is that, at ω, player i takes into account the strategies of the opponents in any subspace S with $\sigma_i(\omega) \succeq S$.

The fifth property is that: for all $\omega \in \Omega$,

$$\overline{\Pi}_i^{\uparrow}(\omega) \leqslant ([\delta_i|_{\sigma_i(\omega)} = \delta_i(\omega)], \sigma_i(\omega)), \text{ i.e., } \Pi_i(\omega) \subseteq \{\tilde{\omega} \in \sigma_i(\omega) \mid \delta_i(\tilde{\omega}) = \delta_i(\omega)\}.$$

In words, the fifth property states that player i is *certain of her strategy* δ_i in the sense that she believes that her action is $\delta_i(\omega)$.

To conclude the definition of a model, I remark on the case in which players may be unaware of some other players (recall the remark at the end of Section 4.1.1). While this paper focuses on a model of a game that allows only for unawareness of actions, one could incorporate unawareness of players by specifying, for each player $i \in I$, an event $(\Xi_i, S(\Xi_i)) \in \mathcal{E}$ that she "exists." Player i's possibility correspondence, action correspondence, and strategy are defined on the set Ξ_i. For instance, Meier and Schipper (2014, Appendix A) consider such possibility.

[21] In fact, as long as two states ω' and ω'' reside in some subspace S_λ, then $\mathcal{C}_i(\omega') = V_{\lambda,i} = \mathcal{C}_i(\omega'')$.

4.1.3 Rationality

Given a strategic game $\mathcal{V}$ with unawareness and a model $\mathcal{M}$ of the game, player i is *rational* at $\omega \in S_\lambda$ in a subspace S_λ if, for any available action $c'_i \in \mathcal{C}_i(\omega_{\sigma_i(\omega)})$ at ω, she considers it possible that playing $\delta_i(\omega) \in \mathcal{C}_i(\omega_{\sigma_i(\omega)})$ is at least as good as playing c'_i given the opponents' strategies $\delta_{-i}|_{S_\mu}$ for some $S_\mu \in \mathcal{S}$ with $\sigma_i(\omega) \succeq S_\mu$. More formally, letting $\mathrm{RAT}_i(S_\lambda)$ be the set of states in S_λ at which player i is rational,

$$\begin{aligned}\mathrm{RAT}_i(S_\lambda) :=& \{\omega \in S_\lambda \mid \text{for all } c'_i \in \mathcal{C}_i(\omega_{\sigma_i(\omega)}), \text{ there exists } S_\mu \in \mathcal{S} \text{ with } \sigma_i(\omega) \succeq S_\mu \\ & \qquad \text{such that } \omega \in M_{\overline{\Pi}_i}([\![\delta_i(\omega) \succsim_i c'_i]\!]^{\uparrow}_{S_\mu})\} \\ =& \{\omega \in S_\lambda \mid \text{for all } c'_i \in \mathcal{C}_i(\omega_{\sigma_i(\omega)}), \text{ there exist } S_\mu \in \mathcal{S} \text{ and } \omega' \in \Pi_i(\omega) \\ & \qquad \text{such that } \sigma_i(\omega) \succeq S_\mu \text{ and } (\delta_i(\omega), \delta_{-i}(\omega'_{S_\mu}) \succsim_i (c'_i, \delta_{-i}(\omega'_{S_\mu})\}.\end{aligned}$$

I assume that (i.e., I restrict attention to models of the given strategic game in which) $\overline{\mathrm{RAT}}^{\uparrow}_i(S_\lambda) := (\mathrm{RAT}^{\uparrow}_i(S_\lambda), S_\lambda)$ is an event. Thus, it is the event that player i is rational in S_λ. Let

$$\mathrm{RAT}_I(S_\lambda) := \bigcap_{i \in I} \mathrm{RAT}_i(S_\lambda)$$

be the set of states in S_λ at which every player is rational. Denote by $\overline{\mathrm{RAT}}^{\uparrow}_I(S_\lambda) := (\mathrm{RAT}^{\uparrow}_I(S_\lambda), S_\lambda)$ the event that every player is rational in S_λ. The notion of rationality defined here is an extension of the standard notion of rationality in a standard state-space model without unawareness (see, e.g., Bonanno, 2008, 2015; Chen, Long, and Luo, 2007).

The reason that I define the rationality of a player for each subspace is that the set of states $\omega \in \Omega$ at which player i is rational may not simply be a well-defined event. To see this, suppose that, for a given generalized state space, the lowest subspace $S' = \inf \mathcal{S}$ satisfies $S' = \{\omega\}$. Suppose also that $\mathcal{C}_i(\omega)$ is a singleton action set and that $\Pi_i(\omega) = (\{\omega\}, S')$ for player i. Then, player i would be rational at ω. If the set of states at which player i is rational is an event, then this event coincides with (Ω, S') irrespective of player i's behavior on any other subspace, which is an implausible requirement.[22]

With this definition in mind, the players *commonly believe their rationality* at a state $\omega \in \Omega$ if

$$\omega \in C(\mathrm{RAT}^{\uparrow}_I(S(\omega))).$$

Denote by CBR the set of states at which the players commonly believe their rationality. Similarly, the players are *rational and commonly believe their rationality* at a

[22] In Example 1 (Prisoners' Dilemma) in Section 4.3, the lowest subspace S' satisfies $S' = \{\omega_3\}$, and in the subgame associated with ω_3, each player's action set is a singleton set (consisting of cooperation). Since each player's available action is a single action, she is rational at ω_3. If the set of states at which player i is rational is an event, then the event that player i is rational corresponds to the entire set of states.

state $\omega \in \Omega$ if

$$\omega \in \mathrm{RAT}_I^{\uparrow}(S(\omega)) \cap C(\mathrm{RAT}_I^{\uparrow}(S(\omega))).$$

Denote by RCBR the set of states at which the players are rational and commonly believe their rationality. I often refer to CBR as the set of states at which common belief in rationality holds. Likewise, I often refer to RCBR the set of states at which rationality and common belief in rationality holds.

4.2 Iterated Elimination of Strictly Dominated Actions as an Implications of Rationality and Common Belief in Rationality

To define iterated elimination of strictly dominated actions (IESDA) for a game with unawareness, fix an arbitrary strategic game with unawareness $\mathcal{V} = (V_\lambda)_{\lambda \in \Lambda}$ and, with some abuse of terminology, identify any subset $V'_\lambda = \bigtimes_{i \in I} V'_{\lambda,i}$ of V_λ with a (sub-)game.

For any subgame V'_λ of V_λ, an action $c_i \in V'_{\lambda,i}$ is *strictly dominated* given $V'_{\lambda,-i} := \bigtimes_{j \in I \setminus \{i\}} V'_{\lambda,j}$ if there exists $c'_i \in V'_{\lambda,i}$ such that $(c'_i, c_{-i}) \succ_i (c_i, c_{-i})$ for all $c_{-i} \in V'_{\lambda,-i}$. Otherwise, the action $c_i \in V'_{\lambda,i}$ is *not strictly dominated* given $V'_{\lambda,-i}$. Note that, since I focus on qualitative beliefs, the notion of domination calls only for pure actions, as in the literature (e.g., Bonanno, 2008, 2015).

Definition 5. A process of *iterated elimination of strictly dominated actions (IESDA)* is a sequence of $((V^k, W^k))_{k \in \mathbb{N} \cup \{0\}}$, where $V^k := (V^k_\lambda)_{\lambda \in \Lambda}$ and $V^k_\lambda := (V^k_{\lambda,i})_{i \in I}$ and $W^k := (W^k_\lambda)_{\lambda \in \Lambda}$ and $W^k_\lambda := (W^k_{\lambda,-i})_{i \in I}$ for each $k \in \mathbb{N} \cup \{0\}$ and $\lambda \in \Lambda$, inductively defined as follows.

For $k = 0$, $V^0_{\lambda,i} := V_{\lambda,i}$ and $W^0_{\lambda,i} := \bigtimes_{j \in I \setminus \{i\}} V_{\lambda,j}$ for all $(\lambda, i) \in \Lambda \times I$.

For $k > 0$, suppose that $V^{k-1} = (V^{k-1}_\lambda)_{\lambda \in \Lambda}$ and $W^{k-1} = (W^{k-1}_\lambda)_{\lambda \in \Lambda}$ are defined. For each $(\lambda, i) \in \Lambda \times I$, define:

$$W^k_{\lambda,-i} := \{(c_j)_{j \in I \setminus \{i\}} \in W^{k-1}_{\lambda,-i} \mid \text{for each } j \in I \setminus \{i\}, \text{ there exists } \mu \in \Lambda \text{ with } V_\lambda \succeq V_\mu \text{ such that } c_j \in V^{k-1}_{\mu,j}\}.$$

Then, if there exist $(\lambda, i) \in \Lambda \times I$ and $c_i \in V^{k-1}_{\lambda,i}$ such that c_i is strictly dominated given $W^k_{\lambda,-i}$, then define $(V^k_{\lambda,i})_{(\lambda,i) \in \Lambda \times I}$ by eliminating *at least one* such action $c_i \in V^{k-1}_{\lambda,i}$ for some $(\lambda, i) \in \Lambda \times I$. If there do not exist $(\lambda, i) \in \Lambda \times I$ and $c_i \in V^{k-1}_{\lambda,i}$ such that c_i is strictly dominated given $W^k_{\lambda,-i}$, then $V^k_{\lambda,i} = V^{k-1}_{\lambda,i}$ for all $(\lambda, i) \in \Lambda \times I$.

Since $((V^k, W^k))_{k \in \mathbb{N} \cup \{0\}}$ is a decreasing sequence of a finite collection of finite sets, there exists the smallest number k with $V^{k+1} = V^k$ and $W^{k+1} = W^k$. Define $V^{\mathrm{IESDA}} := V^k$.[23]

[23]If there exists k such that $V^{k+1} = V^k$, then $W^{k+2} = W^{k+1}$. Thus, it is enough to find the smallest number k such that $V^{k+1} = V^k$. With this k, $V^{\mathrm{IESDA}} = V^k$.

As discussed in the Introduction, the IESDA procedure is similar to the corresponding one in Perea (2022): at the k-th step, player i evaluates actions in $V_{\lambda,i}^{k-1}$ by taking into account the possibility that any of her opponents j takes an action c_j from $V_{\mu,j}^{k-1}$ with $V_\lambda \succeq V_\mu$.

I start with preliminary results. First, fix any strategic game $\mathcal{V} = (V_\lambda)_{\lambda\in\Lambda}$ with unawareness. For any process of IESDA, the set $V_\lambda^{\mathrm{IESDA}}$ is not empty (i.e., there exists an action profile c_λ such that $c_\lambda \in V_\lambda^{\mathrm{IESDA}}$) for every $\lambda \in \Lambda$. Second, IESDA is an order-independent procedure, i.e., any two processes of IESDA lead to the same profile of sets $(V_\lambda^{\mathrm{IESDA}})_{\lambda\in\Lambda}$.

Lemma 1. *Fix a strategic game $\mathcal{V} = (V_\lambda)_{\lambda\in\Lambda}$ with unawareness.*

1. *Fix a process of IESDA. Then, $V_\lambda^{\mathrm{IESDA}} \neq \emptyset$ for each $\lambda \in \Lambda$.*
2. *IESDA is an order-independent procedure.*

While a strategic game with unawareness consists of a *collection* of strategic games, the proof of the lemma turns out to be similar to the standard case well-known in the literature.

4.2.1 The "Bottom-Up" Elimination Procedure and Examples

To provide epistemic characterizations of IESDA, I consider the following "bottom-up" elimination procedure using Lemma 1.[24] Fix an arbitrary strategic game with unawareness $\mathcal{V} = (V_\lambda)_{\lambda\in\Lambda}$.

Definition 6. 1. Take $\underline{\lambda} \in \Lambda$ with $V_{\underline{\lambda}} = \inf_{\mu\in\Lambda} V_\mu$, i.e., let $V_{\underline{\lambda}} \in \mathcal{V}$ be the bottom element of $\mathcal{V}$. For the subgame $V_{\underline{\lambda}}$, apply the "standard" elimination procedure to the strategic game defined on $V_{\underline{\lambda}}$. Namely:

(a) Let $V_{\underline{\lambda}}^0 = (V_{\underline{\lambda},i})_{i\in I}$.

(b) If $V_{\underline{\lambda}}^{k-1} = (V_{\underline{\lambda},i}^{k-1})_{i\in I}$ is defined for some $k \in \mathbb{N}$, then define $V_{\underline{\lambda}}^{k} = (V_{\underline{\lambda},i}^{k})_{i\in I}$ by eliminating *at least one* action $c_i \in V_{\underline{\lambda},i}^{k-1}$ which is strictly dominated given $V_{\underline{\lambda},-i}^{k-1} := \bigtimes_{j\in I\setminus\{i\}} V_{\underline{\lambda},j}^{k-1}$ if such c_i exists for some $i \in I$. If there does not exist any (i, c_i) such that $c_i \in V_{\underline{\lambda},i}^{k-1}$ is strictly dominated given $V_{\underline{\lambda},-i}^{k-1}$, then let $V_{\underline{\lambda}}^{k} = V_{\underline{\lambda}}^{k-1}$.

Since $(V_{\underline{\lambda}}^{k})_{k\in\mathbb{N}\cup\{0\}}$ is a decreasing sequence of finite subgames, there exists the smallest number k with $V_{\underline{\lambda}}^{k+1} = V_{\underline{\lambda}}^{k}$. Define $V_{\underline{\lambda}}^{*} := V_{\underline{\lambda}}^{k}$.

[24] Perea (2022) considers such "bottom-up" elimination procedure. Heifetz, Meier, and Schipper (2013b) also propose what they call "upward induction" in their analysis of Bayesian equilibrium for games with unawareness.

2. Let $\mu \in \Lambda$ be such that V_λ^* is defined for all $\lambda \in \Lambda \setminus \{\mu\}$ with $V_\mu \succeq V_\lambda$. I define V_μ^* for such μ.

   (a) Let $V_{\mu,i}^0 := V_{\mu,i}$ and $W_{\mu,-i}^0 := V_{\mu,-i}$ for each $i \in I$.

   (b) Suppose that $V_\mu^{k-1} = (V_{\mu,i}^{k-1})_{i \in I}$ and $W_\mu^{k-1} = (W_{\mu,-i}^{k-1})_{i \in I}$ are defined for some $k \in \mathbb{N}$. For each $i \in I$, define:

$$W_{\mu,-i}^k := \{(c_j)_{j \in I \setminus \{i\}} \in W_{\mu,-i}^{k-1} \mid \text{for each } j \in I \setminus \{i\}, \text{ either } c_j \in V_{\mu,j}^{k-1} \text{ or } c_j \in V_{\nu,j}^* \text{ for some } \nu \in \Lambda \setminus \{\mu\} \text{ with } S_\mu \succeq S_\nu\}.$$

   Then, define $V_\mu^k = (V_{\mu,i}^k)_{i \in I}$ by eliminating *at least one* action $c_i \in V_{\mu,i}^{k-1}$ which is strictly dominated given $W_{\mu,-i}^{k-1}$ if such c_i exists. If there does not exist any (i, c_i) such that $c_i \in V_{\mu,i}^{k-1}$ is strictly dominated given $W_{\mu,-i}^{k-1}$, then let $V_\mu^k = V_\mu^{k-1}$.

   Since $(V_\mu^k)_{k \in \mathbb{N} \cup \{0\}}$ is a decreasing sequence of finite sets, there exists the smallest number k with $V_\mu^{k+1} = V_\mu^k$. Define $V_\mu^* := V_\mu^k$.

Inductively, one obtains V_λ^* for all $\lambda \in \Lambda$. By Lemma 1, $V_\lambda^* = V_\lambda^{\text{IESDA}}$ for all $\lambda \in \Lambda$. For the strategic games with unawareness represented by Tables 1 and 2 in the Introduction, one can show that IESDA leads to the action profiles mentioned in the Introduction.

4.2.2 The Main Results

With the definitions and the lemma in mind, I provide the main result.

Proposition 5. *Fix a strategic game $\mathcal{V} = (V_\lambda)_{\lambda \in \Lambda}$ with unawareness. Let $\mathcal{M}$ be a model of the game $\mathcal{V}$. For each $\lambda \in \Lambda$, if $\omega \in \text{RAT}_I(S_\lambda) \cap C(\text{RAT}_I^\uparrow(S_\lambda))$, then $\delta(\omega) \in V_\lambda^{\text{IESDA}}$.*

Proposition 5 states that if the players are rational and commonly believe their rationality at a state, then their actions $\delta(\omega)$ survive the given process of IESDA in the sense that actions $\delta(\omega)$ are in V_λ^{IESDA} with $V_\lambda = \mathcal{C}(\omega)$. The proposition extends the corresponding epistemic characterization of IESDA in a strategic game without unawareness (e.g., Bonanno, 2008, 2015).[25]

I also provide an alternative epistemic characterization as an implication of common belief in rationality when the players have correct common belief in rationality. Section 4.4.1 generalizes this result by providing conditions under which the players have correct common belief in rationality.

[25] In the context of standard state spaces, the literature also shows that the converse holds. This paper focuses only on the essential part of the epistemic characterization.

	$c_{2,1}$	$c_{2,2}$
$c_{1,1}$	$2,2$	$-1,3$
$c_{1,2}$	$3,-1$	$1,1$

	$c_{2,1}$
$c_{1,1}$	$2,2$

Table 3: Example 1: the strategic game with unawareness.

Proposition 6. *Fix a strategic game $\mathcal{V} = (V_\lambda)_{\lambda\in\Lambda}$ with unawareness. Let $\mathcal{M}$ be a model of the game $\mathcal{V}$ such that $\overline{C}(\overline{\mathrm{RAT}}^{\uparrow}_I(S)) \leqslant \overline{\mathrm{RAT}}^{\uparrow}_I(S)$ for all $S \in \mathcal{S}$. For each $\lambda \in \Lambda$ and $\omega \in S_\lambda$, if $\omega \in C(\mathrm{RAT}^{\uparrow}_I(S_\lambda))$, then $\delta(\omega) \in V^{\mathrm{IESDA}}_\lambda$.*

The proof of Proposition 6 is omitted as it follows from Proposition 5.

4.3 Examples

I provide examples of a model of a game. Throughout the examples, let $I = \{1, 2\}$, where player 1 is the row player and player 2 the column player.

Example 1. I consider Prisoners' Dilemma. The underlying game V is represented by the payoff table in the left panel of Table 1. The strategic game $\mathcal{V}$ consists of $\mathcal{V} = \{V, V'\}$, where V' is represented by the payoff table in the right panel of Table 1. In the subgame V', each player's available action is only $c_{i,1}$ (cooperation).

I start with the predictions under IESDA using the bottom-up procedure (Definition 6). In V', since each player's action set is a singleton, it follows that $V'^{*}_i = \{c_{i,1}\}$ for each $i \in I$. In V, for each $i \in I$, I have $V^0_i = \{c_{i,1}, c_{i,2}\}$ and $W^0_{-i} = W^1_{-i} = \{c_{-i,1}, c_{-i,2}\}$. Since $c_{i,1} \in V^0_i$ is strictly dominated by $c_{i,2} \in V^0_i$ given W^1_{-i}, it follows that $V^1_i = \{c_{i,2}\}$. As this is a singleton set, no further eliminations occur so that $V^*_i = \{c_{i,2}\}$ for each $i \in I$. Thus, I obtain:

$$V^{\mathrm{IESDA}}_1 = \{c_{1,2}\}, V^{\mathrm{IESDA}}_2 = \{c_{2,2}\},$$
$$V'^{\mathrm{IESDA}}_1 = \{c_{1,1}\}, \text{ and } V'^{\mathrm{IESDA}}_2 = \{c_{2,1}\}.$$

In words, each V^{IESDA}_i consists of a singleton set of the defection action, and each V'^{IESDA}_i consists of a singleton set of the cooperation action.

In the underlying game, the possibility that, for each player i, her opponent j may be unaware of defect action $c_{j,2}$ and thus she may end up taking cooperation action $c_{j,1}$ does not affect the prediction under IESDA because, irrespective of whether the opponent is unaware of defect action $c_{j,2}$, defect action $c_{i,2}$ strictly dominates cooperation action $c_{i,1}$.

Next, I consider a model of the game. There are two subspaces, $(S, \mathcal{D}) = (\{\omega_1, \omega_2\}, \mathcal{P}(S))$ and $(S', \mathcal{D}') = (\{\omega_3\}, \mathcal{P}(S'))$ with $S \succeq S'$. The left panel of Figure 2 depicts the projections (the identity projections and the composites are omitted).

For player 1, her possibility correspondence is defined so that $\Pi_i(\omega) = (\{\omega\}, S(\omega))$ for each $\omega \in \Omega$. For player 2, her possibility correspondence is defined so that:

$$\overline{\Pi}_2(\omega_1) = (\{\omega_1\}, S) \text{ and } \overline{\Pi}_2(\omega_2) = \overline{\Pi}_2(\omega_3) = (\{\omega_3\}, S').$$

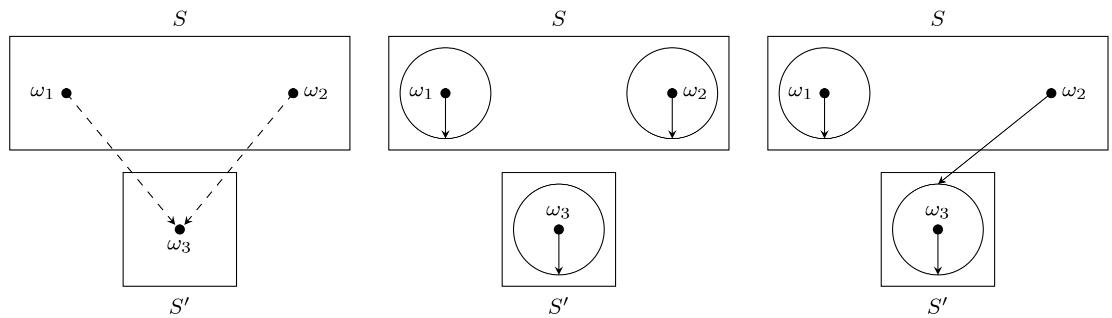

Figure 2: Example 1: the projections and the possibility correspondences.

The central and right panels of Figure 2, respectively, depict the possibility correspondences of players 1 and 2. An arrow from state ω to the corresponding circle depicts the correspondence $\omega \mapsto \Pi_i(\omega)$.

The players' action correspondences are defined as follows: for each $i \in I$,

$$\mathcal{C}_i(\omega_1) = \mathcal{C}_i(\omega_2) = \{c_{i,1}, c_{i,2}\} \text{ and } \mathcal{C}_i(\omega_3) = \{c_{i,1}\}.$$

Thus, at $\omega \in \{\omega_1, \omega_2\}$, the players' action correspondences yield the underlying game (the left panel of Table 3). At ω_3, in contrast, the players' action correspondences yield the subgame represented by the right panel of Table 3.

Finally, the strategies of the players are:

$$(\delta_1(\omega), \delta_2(\omega)) = \begin{cases} (c_{1,2}, c_{2,2}) & \text{if } \omega = \omega_1 \\ (c_{1,2}, c_{2,1}) & \text{if } \omega = \omega_2 \\ (c_{1,1}, c_{2,1}) & \text{if } \omega = \omega_3 \end{cases}.$$

I show that $\mathrm{RAT}_I(S) = S$ and $\mathrm{RAT}_I(S') = S'$. At ω_1, each player is aware of every player's actions. They commonly believe that each player is aware of all of every player's actions. At that state, each player, whose available actions are $\{c_{i,1}, c_{i,2}\}$ and who takes $c_{i,2}$ (defection), is rational in the subspace S.

Consider ω_2. Similarly to the previous argument, player 1, who takes $c_{1,2}$, is rational. Player 2 is unaware of action $c_{j,2}$ for each player j. Given that player 2's view at ω_2 is the subgame represented by the right panel of Figure 2, she is indeed rational at ω_2 in S. Similarly, each player i is rational at ω_3 in S'.

Hence, CBR = RCBR = $\{\omega_1, \omega_3\}$. At ω_1, players are (commonly) aware of the underlying game and their actions satisfy $\delta(\omega_1) = (c_{1,2}, c_{2,2}) \in V^{\mathrm{IESDA}}$. At ω_3, players are (commonly) aware of the game V' and their actions satisfy $\delta(\omega_3) = (c_{1,1}, c_{2,1}) \in V'^{\mathrm{IESDA}}$.

Example 2. I consider a variant of the strategic game originally studied by Feinberg (2021).[26] The underlying game is represented by the payoff table in the left panel of

[26] I analyze a "similar" but different strategic game because this paper does not consider domination by a mixed action.

	$c_{2,1}$	$c_{2,2}$	$c_{2,3}$
$c_{1,1}$	$0,2$	$3,3$	$0,2$
$c_{1,2}$	$2,2$	$2,1$	$2,0$
$c_{1,3}$	$1,1$	$4,0$	$1,1$

	$c_{2,1}$	$c_{2,2}$	$c_{2,3}$
$c_{1,1}$	$0,2$	$3,3$	$0,2$
$c_{1,2}$	$2,2$	$2,1$	$2,0$

Table 4: Example 2: the strategic game with unawareness.

Table 2. Note that this strategic game, without consideration of unawareness, has a unique prediction $(c_{1,2}, c_{2,1})$ under IESDA.

I start with the predictions under IESDA using the bottom-up procedure. Within V', the elimination procedure reduces to the standard one, and the elimination procedure terminates after eliminating $c_{2,3}$. Thus:

$$V_1'^* = \{c_{1,1}, c_{1,2}\} \text{ and } V_2'^* = \{c_{2,1}, c_{2,2}\}.$$

In V, for each $i \in I$, I have $V_i^0 = \{c_{i,1}, c_{i,2}, c_{i,3}\}$ and $W_{-i}^0 = W_{-i}^1 = \{c_{-i,1}, c_{-i,2}, c_{-i,3}\}$. Since $c_{1,1} \in V_1^0$ is strictly dominated by $c_{1,3} \in V_1^0$ given W_{-1}^1 and no other actions are strictly dominated, it follows that $V_1^1 = \{c_{1,2}, c_{1,3}\}$ and $V_2^1 = \{c_{2,1}, c_{2,2}, c_{2,3}\}$. Then, $W_{-1}^2 = \{c_{2,1}, c_{2,2}, c_{2,3}\}$ and $W_{-2}^2 = \{c_{1,1}, c_{1,2}, c_{1,3}\}$. I have $c_{1,1} \in W_{-2}^2$ because player 2 considers the possibility that player 1 may be unaware of V. This leads to a different prediction from the one without the consideration of unawareness. Because player 2 takes into account the possibility that player 1 takes $c_{1,1}$, it can be seen that no further elimination occurs: $V_1^2 = V_1^1$ and $V_2^2 = V_2^1$. Thus, I obtain:

$$\begin{aligned} V_1^{\text{IESDA}} &= \{c_{1,2}, c_{1,3}\}, V_2^{\text{IESDA}} = \{c_{2,1}, c_{2,2}, c_{2,3}\}, \\ V_1'^{\text{IESDA}} &= \{c_{1,1}, c_{1,2}\}, \text{ and } V_2'^{\text{IESDA}} = \{c_{2,1}, c_{2,2}\}. \end{aligned}$$

Next, I consider a model of the game. There are two subspaces, $(S, \mathcal{D}) = (\{\omega_1, \omega_2\}, \mathcal{P}(S))$ and $(S', \mathcal{D}') = (\{\omega_3\}, \mathcal{P}(S'))$ with $S \succeq S'$. The left panel of Figure 3 depicts the projections (the identity projections and the composites are omitted).

The players' possibility correspondences are defined so that:

$$\begin{aligned} &\overline{\Pi}_1(\omega_1) = \overline{\Pi}_1(\omega_2) = (\{\omega_2\}, S) \text{ and } \overline{\Pi}_1(\omega_3) = (\{\omega_3\}, S'); \text{ and} \\ &\overline{\Pi}_2(\omega_1) = (\{\omega_1\}, S) \text{ and } \overline{\Pi}_2(\omega_2) = \overline{\Pi}_2(\omega_3) = (\{\omega_3\}, S'). \end{aligned}$$

The central and right panels of Figure 3, respectively, depict the possibility correspondences of players 1 and 2. An arrow from state ω to the corresponding circle depicts the correspondence $\omega \mapsto \Pi_i(\omega)$.[27]

The players' action correspondences are defined as follows:

$$\begin{aligned} &\mathcal{C}_1(\omega_1) = \mathcal{C}_1(\omega_2) = \{c_{1,1}, c_{1,2}, c_{1,3}\} \text{ and } \mathcal{C}_1(\omega_3) = \{c_{1,1}, c_{1,2}\} \text{ and} \\ &\mathcal{C}_2(\omega_1) = \mathcal{C}_2(\omega_2) = \mathcal{C}_2(\omega_3) = \{c_{2,1}, c_{2,2}, c_{2,3}\}. \end{aligned}$$

[27] For the interpretation of the players' possibility correspondences in this model, see Feinberg (2021) and Meier and Schipper (2014).

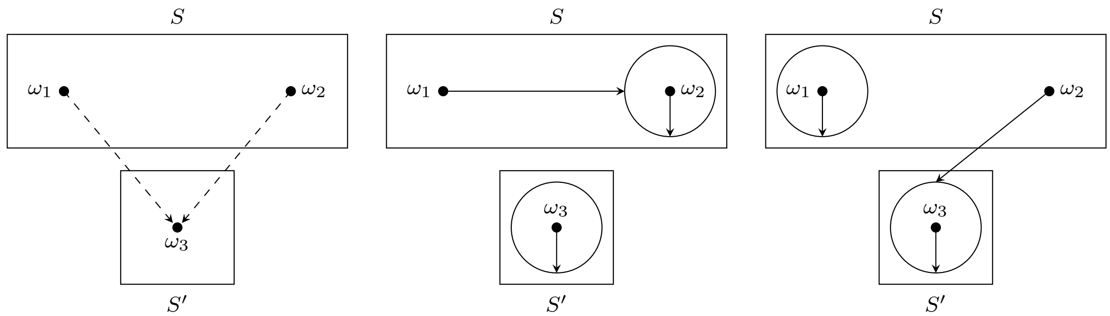

Figure 3: Example 2: the projections and the possibility correspondences.

Thus, at $\omega \in \{\omega_1, \omega_2\}$, the players' action correspondences yield the underlying game (the left panel of Table 4). At ω_3, in contrast, the players' action correspondences yield the strategic game represented by the right panel of Table 4.

Finally, the strategies of the players are:

$$(\delta_1(\omega), \delta_2(\omega)) = \begin{cases} (c_{1,3}, c_{2,3}) & \text{if } \omega = \omega_1 \\ (c_{1,3}, c_{2,2}) & \text{if } \omega = \omega_2 \\ (c_{1,1}, c_{2,2}) & \text{if } \omega = \omega_3 \end{cases}.$$

Consider the rationality of the players. I show that $\mathrm{RAT}_I(S) = S$ and $\mathrm{RAT}_I(S') = S'$. To that end, I start with the subspace S'. Consider ω_3. Player 1, who takes $c_{1,1}$, is rational at ω_3. This is roughly because player 1 finds it better to play $c_{1,2}$ only when player 2 (who takes $c_{2,2}$) would take $c_{2,1}$ or $c_{2,3}$. Player 2, who takes $c_{2,2}$, is rational at ω_3 roughly because she finds it better to play $c_{2,1}$ only when player 1 (who takes $c_{1,1}$) would take $c_{1,2}$.

Next, I consider the subspace S. Player 1 takes $c_{1,2}$ at $\omega \in \{\omega_1, \omega_2\}$. For action $c_{1,1}$, player 1 considers it possible that taking $c_{1,2}$ would be as good as taking $c_{1,1}$ in the subspace S at ω. Likewise, for action $c_{1,3}$, player 1 considers it possible that taking $c_{1,2}$ would be as good as taking $c_{1,3}$ in the subspace S at ω.

Player 2 takes $c_{2,2}$ at ω_1. While there is no state in S at which player 2 considers it possible that taking $c_{2,2}$ is as good as taking $c_{2,1}$, at ω_3 in S', player 2 considers it possible that taking $c_{2,2}$ is as good as taking $c_{2,1}$. For $c_{2,3}$, player 2 considers it possible that taking $c_{2,2}$ is as good as taking $c_{2,1}$ at any state. Player 2 takes $c_{2,1}$ at ω_2. Player 2 considers it possible that taking $c_{2,1}$ is as good as taking $c_{2,2}$ or $c_{2,3}$ at ω_2.

Since $\mathrm{RAT}_I(S) = S$ and $\mathrm{RAT}_I(S') = S'$, it can be seen that $C(\mathrm{RAT}_I(S)) = \emptyset$ and $C(\mathrm{RAT}_I^{\uparrow}(S')) = (S')^{\uparrow}$. Thus, CBR = RCBR = $\{\omega_3\}$.

4.4 The Role of Introspective Properties

4.4.1 Positive Introspection and Consistency

In the second epistemic characterization of IESDA, the important assumption is that the common belief in rationality is correct: $\overline{C}(\overline{\mathrm{RAT}}^{\uparrow}_{I}(S)) \leqslant \overline{\mathrm{RAT}}^{\uparrow}_{I}(S)$ for each $S \in \mathcal{S}$. In the standard state-space model, it is well-known that if a player's belief satisfies Positive Introspetcion and Consistency, then whenever she believes that she is rational she is indeed rational (e.g., Fukuda, 2023a). I extend this result to general state spaces.

Proposition 7. *Fix player $i \in I$. Suppose that $\overline{B}_i$ satisfies Positive Introspection and Consistency. Then, $\overline{B}_i(\overline{\mathrm{RAT}}^{\uparrow}_{i}(S)) \leqslant \overline{\mathrm{RAT}}^{\uparrow}_{i}(S)$ for all $S \in \mathcal{S}$.*

With this in mind, I restate Proposition 6.

Corollary 1. *Fix a strategic game $\mathcal{V}$ with unawareness.*

Fix also a model $\mathcal{M}$ of the game such that each player's belief operator satisfies Positive Introspection and Consistency. For each $\lambda \in \Lambda$ and $\omega \in S_\lambda$, if $\omega \in C(\mathrm{RAT}^{\uparrow}_{I}(S_\lambda))$, then $\delta(\omega) \in V^{\mathrm{IESDA}}_{\lambda}$.

Corollary 1 follows from Propositions 6 and 7.

4.4.2 Weak Necessitation

Here, I discuss the role of Weak Necessitation (or Generalized Euclideanness). In the context of standard state spaces, it is well-known that if a player's belief satisfies Negative Introspection, then whenever she is rational, she believes that she is rational (e.g., Fukuda, 2023a). I extend this result to generalized standard state spaces.

Proposition 8. *Fix player $i \in I$. Suppose that $\overline{B}_{\overline{\Pi}_i}$ satisfies Generalized Negative Introspection (I or II) and A_{σ_i}-Introspection. Then, $\overline{\mathrm{RAT}}^{\uparrow}_{i}(S) \wedge \overline{A}_{\sigma_i}(\overline{\mathrm{RAT}}^{\uparrow}_{i}(S)) \leqslant \overline{B}_{\overline{\Pi}_i}(\overline{\mathrm{RAT}}^{\uparrow}_{i}(S))$ for any $S \in \mathcal{S}$.*

Note that Proposition 8 requires the awareness of rationality. For instance, in Example 1, player 2, who takes the cooperation action $c_{i,2}$, is rational at ω_2 because she "thinks" that she faces the subgame consisting of a singleton action of the cooperation action. She does not believe her own rationality in the subspace S because she is not aware of the underlying game V.

I make the following two remarks. First, when player i's possibility correspondence violates Generalized Euclideanness, there may exists a state at which she is $\overline{U}_{\overline{\Pi}_i}$-unaware of her own rationality and she might take a strictly dominated action. Second, when Generalized Euclideanness fails, it may be the case that player i is rational, she is aware of her own rationality, and yet she does not believe her own rationality (i.e., a counterexample to Proposition 8). The following example illustrates these points.

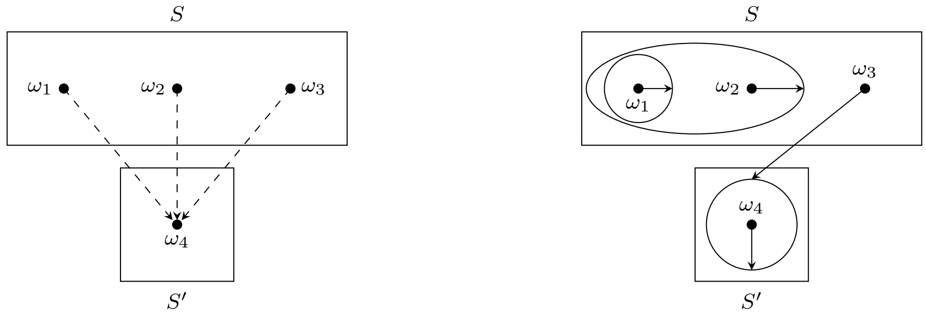

Figure 4: Example 3: the projections and the possibility correspondences.

Example 3. I revisit Example 1 (Prisoners' Dilemma). The strategic game with unawareness is given by Example 1. Next, I define a model of the game (technically, I consider two models, as I consider two strategy profiles). Let $\mathcal{S} = \{S, S'\}$, $(S, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(S))$, and $(S', \mathcal{D}') = (\{\omega_4\}, \mathcal{P}(S'))$. The projections are depicted in the left panel of Figure 4.

Define the possibility correspondence of each player $i \in I$ so that:

$$\Pi_i(\omega_1) = (\{\omega_1\}, S), \Pi_i(\omega_2) = (\{\omega_1, \omega_2\}, S), \text{ and}$$
$$\Pi_i(\omega_3) = \Pi_i(\omega_4) = (\{\omega_4\}, S').$$

It can be seen that while each player i's possibility correspondence satisfies Generalized Reflexivity and Generalized Transitivity, it violates Generalized Euclideanness (e.g., $\omega_1 \in \Pi_i(\omega_2)$ but $\Pi_i^{\uparrow}(\omega_2) \not\subseteq \overline{\Pi}_i^{\uparrow}(\omega_1)$).

Define the action correspondence of player $i \in I$ as follows:

$$\mathcal{C}_i(\omega) = \{c_{i,1}, c_{i,2}\} \text{ for each } \omega \in S \text{ and } \mathcal{C}_i(\omega_4) = \{c_{i,1}\}.$$

For the players' strategies, I consider the following two strategy profiles.

1. For each $i \in I$, let $\delta_i = (c_{i,2}, c_{i,1}, c_{i,1}, c_{i,1})$. Then, $\mathrm{RAT}_i(S) = \{\omega_1, \omega_3\}$. Thus, $\overline{U}_{\overline{\Pi}_i}(\mathrm{RAT}_i(S)) = \{\omega_2, \omega_3\}$ and $\overline{U}_{\sigma_i}(\mathrm{RAT}_i(S)) = \{\omega_3\}$. At ω_2, player i is $\overline{U}_{\overline{\Pi}_i}$-unaware of her own rationality, and takes a strictly dominated action $\delta_i(\omega_2) = c_{i,1} \in \mathcal{C}_i((\omega_2)_{\sigma_i(\omega_2)}) = \{c_{i,1}, c_{i,2}\}$.

2. For each $i \in I$, let $\delta_i = (c_{i,1}, c_{i,2}, c_{i,1}, c_{i,1})$. Then, $\mathrm{RAT}_i(S) = \{\omega_2, \omega_3\}$. Thus, $\overline{A}_{\overline{\Pi}_i}(\mathrm{RAT}_i(S)) = \{\omega_1, \omega_2\}$, $\overline{A}_{\sigma_i}(\mathrm{RAT}_i(S)) = \{\omega_1, \omega_2\}$, and $\overline{B}_{\overline{\Pi}_i}(\mathrm{RAT}_i(S)) = \{\omega_1\}$. At ω_2, she is rational and is aware of her own rationality. However, she does not believe her own rationality.

4.5 Other Solution Concepts

The framework of this paper can be applied to other epistemic characterizations of solution concepts of games, as the possibility correspondence model on a standard state space has been one of the canonical tools in the literature. Also, while this paper has focused on qualitative beliefs, I believe that one can incorporate probabilistic beliefs into the framework of this paper.[28]

For instance, Bonanno and Tsakas (2018) and Fukuda (2023b) characterize Börgers (1993)'s elimination procedure as an implication of common belief in "weak dominance" rationality, in the context of standard-state-space possibility correspondence models of qualitative beliefs. Interestingly, the epistemic characterization of Börgers (1993)'s elimination procedure may call for the failure of Truth Axiom. Thus, in extending the epistemic characterization of Börgers (1993)'s elimination procedure, the framework of this paper would be useful, as this paper is the first one that dispenses with Truth Axiom in the context of generalized-state-space possibility correspondence models.

In contrast, the elimination of inferior action profiles, the solution concept first studied by Stalnaker (1994) and then further studied by Bonanno (2008), Bonanno and Tsakas (2018), Fukuda (2023b), and Hillas and Samet (2020), is characterized as an implication of common knowledge of "weak-dominance" rationality. Again, the framework of this paper makes it possible to compare the role of Truth Axiom.[29]

5 Concluding Remarks

The first part of this paper has developed a possibility correspondence model of belief and unawareness on a generalized state space. The second part has applied the general framework to the epistemic characterization of iterated elimination of strictly dominated actions (IESDA) in a strategic game with unawareness as the implication of rationality and common belief in rationality.

In the first part, Proposition 1 identifies (i) the minimal set of conditions on a possibility correspondence under which it induces a belief operator and (ii) the minimal set of conditions on a belief operator under which the belief operator induces the corresponding possibility correspondence. Moreover, Propositions 2 and 3 establish the equivalence for a wide variety of assumptions on players' beliefs. The notion of possibility in the generalized state space is typically different from the one in a standard state space (Proposition 9 in Appendix B). Proposition 4 characterizes Weak Necessitation and shows that it is equivalent to various properties on unawareness (e.g., AU Introspection and Strong Plausibility with equality) under fairly natural conditions (i.e., Positive Introspection and Consistency). This paper also defines common belief in a possibility correspondence model of belief.

[28] For instance, this would make it possible to provide epistemic characterizations of (mixed-strategy) Nash and correlated equilibria. Aside from epistemic characterizations of solution concepts of games, this also opens up the applications to Agreement and No-Trade theorems.

[29] In the context of dynamic games, Samet (2013) compares common belief in rationality and common knowledge of rationality.

The characterizations of properties of a possibility correspondence could yield a wide range of unawareness structures that vary with assumptions on players' beliefs. It also clarifies different properties of unawareness that result from different assumptions on belief. A belief-operator-based unawareness structure may be easier to use in certain applications in that players' belief operators defined on events is a primitive. In such cases, the analysis in this paper provides a way to construct players' possibility correspondences that induce the given belief operators.

The second part of the paper, the main application of the first part, has formulated the epistemic characterization of iterated elimination of strictly dominated actions (IESDA) as an implication of rationality and common belief in rationality in the context of strategic games with unawareness. To the best of my knowledge, this paper is the first paper that connects a possibility correspondence model of belief and unawareness and epistemic analyses of games with unawareness.

A Proofs

A.1 Section 2.2

Proof of Remark 5. If one defines $\overline{A}_{\overline{B}_i}(E,S) := \overline{B}_i(\overline{S^{\uparrow}})$ from a belief operator satisfying B-Decomposition, B-Monotonicity, B-Conjunction, and B-Necessitation as in Remark 2, then the resulting operator $\overline{A}_{\overline{B}_i}$ satisfies the properties in Remark 4. Then, I show that any belief operator $\overline{B}_i$ satisfying the properties in Remark 2 and the awareness operator $\overline{A}_{\overline{B}_i}$ jointly satisfy the properties in Remark 5.

1. A-Symmetry follows because $\overline{A}_{\overline{B}_i}(\overline{E}) = \overline{B}_i(\overline{S^{\uparrow}}) = \overline{A}_{\overline{B}_i}(\neg\overline{E})$.

2. AU-Introspection follows because $\overline{U}_{\overline{B}_i}(E,S) = (\neg\overline{B}_i)(\overline{S^{\uparrow}}) = \overline{U}_{\overline{B}_i}\overline{U}_{\overline{B}_i}(E,S)$.

3. I have

$$\bigwedge_{x\in X} \overline{A}_{\overline{B}_i}(E_x,S_x) = \bigwedge_{x\in X} \overline{B}_i(\overline{S_x^{\uparrow}}) = \overline{B}_i\left(\bigwedge_{x\in X} \overline{S_x^{\uparrow}}\right) = \overline{B}_i\left(\overline{(\sup_{x\in X} S_x)^{\uparrow}}\right) = \overline{A}_{\overline{B}_i}\left(\bigwedge_{x\in X}(E_x,S_x)\right),$$

   where the first and the forth equality follow from the definition of $\overline{A}_{\overline{B}_i}$, the second from B-Monotonicity and B-Conjunction, and the third from the definition of the operation of taking conjunction.

4. AB-Self Reflection follows because $\overline{A}_{\overline{B}_i}\overline{B}_i(E,S) = \overline{B}_i(\overline{S^{\uparrow}}) = \overline{A}_{\overline{B}_i}(E,S)$.

5. AA-Self Reflection follows because $\overline{A}_{\overline{B}_i}\overline{A}_{\overline{B}_i}(E,S) = \overline{B}_i(\overline{S^{\uparrow}}) = \overline{A}_{\overline{B}_i}(E,S)$.

6. It follows from B-Monotonicity that $\overline{B}_i\overline{A}_{\overline{B}_i}(E,S) = \overline{B}_i\overline{B}_i(\overline{S^{\uparrow}}) \leqslant \overline{B}_i(\overline{S^{\uparrow}}) = \overline{A}_{\overline{B}_i}(E,S)$.

7. It follows from B-Monotonicity that $\overline{U}_{\overline{B}_i}(E,S) = (\neg B_i)(\overline{S}^{\uparrow}) \leqslant \bigwedge_{n\in\mathbb{N}}(\neg\overline{B}_i)^n(E,S)$.

□

A.2 Section 2.3

Proof of Proposition 1. I only prove Part (1), as the proof for Part (2) is similar (recall footnote 7).

Part (1a). Fix $\omega \in \Omega$. Applying Definition 3 to the belief operator $\overline{B}_{\overline{\Pi}_i}$, $\overline{\Pi}^{\uparrow}_{B_{\overline{\Pi}_i}}(\omega)$ satisfies the following:

$$\sigma_{B_{\overline{\Pi}_i}}(\omega) = \sup\{S(D^{\uparrow}) \in \mathcal{S} \mid \overline{D}^{\uparrow} \in \mathcal{E} \text{ satisfies } \overline{\Pi}^{\uparrow}_i(\omega) \leqslant \overline{D}^{\uparrow}\} \text{ and;}$$

$$\Pi_{B_{\overline{\Pi}_i}}(\omega) = \bigcap\{(r^{\sigma_{B_{\overline{\Pi}_i}}(\omega)}_{S(D)})^{-1}(D) \in \mathcal{D}_{\sigma_{B_{\overline{\Pi}_i}}(\omega)} \mid \overline{D}^{\uparrow} \in \mathcal{E} \text{ satisfies } \overline{\Pi}^{\uparrow}_i(\omega) \leqslant \overline{D}^{\uparrow}\}$$
$$= \bigcap\{D^{\uparrow} \in \mathcal{P}(\Omega) \mid \overline{D}^{\uparrow} \in \mathcal{E} \text{ satisfies } \overline{\Pi}^{\uparrow}_i(\omega) \leqslant \overline{D}^{\uparrow}\}.$$

First, I show $\sigma_{B_{\overline{\Pi}_i}}(\omega) = \sigma_i(\omega)$. Since $\overline{\Pi}^{\uparrow}_i(\omega) \leqslant \overline{\Pi}^{\uparrow}_i(\omega)$, it follows from the definition of $\sigma_{B_{\overline{\Pi}_i}}$ that $\sigma_i(\omega) \preceq \sigma_{B_{\overline{\Pi}_i}}(\omega)$. Conversely, $S(D) \preceq \sigma_i(\omega)$ for any $\overline{D}^{\uparrow}$ with $\overline{\Pi}^{\uparrow}_i(\omega) \leqslant \overline{D}^{\uparrow}$. Hence, $\sigma_{B_{\overline{\Pi}_i}}(\omega) \preceq \sigma_i(\omega)$.

Second, I show $\Pi_i(\omega) = \Pi_{B_i}(\omega)$. Since $\overline{\Pi}^{\uparrow}_i(\omega) \leqslant \overline{\Pi}^{\uparrow}_i(\omega)$, it follows $\Pi_{B_{\overline{\Pi}_i}}(\omega) \subseteq (r^{\sigma_{B_{\overline{\Pi}_i}}(\omega)}_{\sigma_i(\omega)})^{-1}(\Pi_i(\omega)) = \Pi_i(\omega)$. Conversely, $\Pi_i(\omega) \subseteq D^{\uparrow}$ for any $\overline{D}^{\uparrow}$ with $\overline{\Pi}^{\uparrow}_i(\omega) \leqslant \overline{D}^{\uparrow}$. Hence, $\Pi_i(\omega) \subseteq \bigcap\{D^{\uparrow} \in \mathcal{P}(\Omega) \mid \overline{\Pi}^{\uparrow}_i(\omega) \leqslant \overline{D}^{\uparrow}\} = \Pi_{B_i}(\omega)$.

Part (1b). The proof consists of three steps. The first step shows that $B_{\overline{\Pi}_{B_i}} = B_i$. To that end, observe first that

$$\overline{\Pi}^{\uparrow}_{B_i}(\omega) = \left(\left(\bigcap\{(r^{\sigma_{B_i}(\omega)}_{S(D)})^{-1}(D) \mid \omega \in B_i(D^{\uparrow})\}\right)^{\uparrow}, \sigma_{B_i}(\omega)\right) = \bigwedge\{\overline{D}^{\uparrow} \in \mathcal{E} \mid \omega \in B_i(D^{\uparrow})\}.$$

This is because, (i) $\omega \in B_i(\Pi^{\uparrow}_{B_i}(\omega))$ by Conjunction; and (ii) by Monotonicity, $\omega \in B_i(D^{\uparrow})$ for any $\overline{D}^{\uparrow} \in \mathcal{E}$ with $\overline{\Pi}^{\uparrow}_{B_i}(\omega) \leqslant \overline{D}^{\uparrow}$.

To show $B_{\overline{\Pi}_{B_i}} = B_i$, fix $\overline{D}^{\uparrow} \in \mathcal{E}$. If $\omega \in B_{\overline{\Pi}_{B_i}}(D^{\uparrow}) := \{\omega \in \Omega \mid \overline{\Pi}^{\uparrow}_{B_i}(\omega) \leqslant \overline{D}^{\uparrow}\}$ then $\omega \in B_i(D^{\uparrow})$. Conversely, if $\omega \in B_i(D^{\uparrow})$, then $\Pi_{B_i}(\omega) \subseteq (r^{\sigma_{B_i}(\omega)}_{S(D)})^{-1}(D) \subseteq D^{\uparrow}$ and $\sigma_{B_i}(\omega) \succeq S(D)$. Then, $\omega \in B_{\overline{\Pi}_{B_i}}(D^{\uparrow})$.

The second step shows that $\overline{\Pi}_{B_i}$ is well-defined. Since $(B_{\overline{\Pi}_{B_i}}(E), S(E)) = (B_i(E), S(E)) \in \mathcal{E}$ for all $(E, S(E)) \in \mathcal{E}$, it follows from Remark 1 that the possibility correspondence $\overline{\Pi}_{B_i}$ is well-defined, i.e., it satisfies Regularity, PPI, PPB, and Confinedness.

The third step proves uniqueness. If $\overline{\Psi}_i$ is well-defined and induces $\overline{B}_i$, then $\overline{B}_{\overline{\Psi}_i} = \overline{B}_i = \overline{B}_{\overline{\Pi}_{B_i}}$. Then, it follows from Part (1a) that $\overline{\Pi}_{B_i} = \overline{\Pi}_{B_{\overline{\Psi}_i}} = \overline{\Psi}_i$. □

A.3 Section 2.4

Proof of Proposition 2. 1. First, $\Pi_i(\omega) \neq \emptyset$ iff $\Pi_i^{\uparrow}(\omega) \neq \emptyset$. The "if" part follows because if there is $\omega' \in \Pi_i^{\uparrow}(\omega)$ then $w'_{\sigma_i(\omega)} \in \Pi_i(\omega)$.

Second, I establish that (1a) and (1d) are equivalent. Assume (1d). If $\omega \in B_{\overline{\Pi}_i}(E)$, then $\Pi_i(\omega) \subseteq E$. I have $\emptyset \neq \Pi_i(\omega) = \Pi_i(\omega) \cap E$, i.e., $\omega \in M_{\overline{\Pi}_i}(E)$. Conversely, assume (1a). Since $\omega \in B_{\overline{\Pi}_i}(\Pi_i^{\uparrow}(\omega)) \subseteq M_{\overline{\Pi}_i}(\Pi_i^{\uparrow}(\omega))$, I have $\Pi_i(\omega) = \Pi_i(\omega) \cap \Pi_i^{\uparrow}(\omega) \neq \emptyset$.

Third, I show that, under Conjunction and Monotonicity, (1a) and (1b) are equivalent. Consistency and Monotonicity imply $\overline{B}_{\overline{\Pi}_i}(\overline{\emptyset}^S) \leqslant (\neg \overline{B}_{\overline{\Pi}_i})(\overline{S}^{\uparrow}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) = \overline{\emptyset}^S$ for all $S \in \mathcal{S}$. Conversely, by Conjunction and No-Contradiction, $\overline{B}_{\overline{\Pi}_i}(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\neg \overline{E}) \leqslant \overline{B}_i(\overline{\emptyset}^S) = \overline{\emptyset}^S$. Thus, I have

$$\begin{aligned}\overline{B}_{\overline{\Pi}_i}(\overline{E}) &\leqslant (\overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee (\neg \overline{B}_{\overline{\Pi}_i})(\neg \overline{E})) \wedge (\overline{B}_{\overline{\Pi}_i}(\neg \overline{E}) \vee (\neg \overline{B}_{\overline{\Pi}_i})(\neg \overline{E})) \\ &= (\overline{B}_{\overline{\Pi}_i}(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\neg \overline{E})) \vee (\neg \overline{B}_{\overline{\Pi}_i})(\neg \overline{E}) = (\neg \overline{B}_{\overline{\Pi}_i})(\neg \overline{E}).\end{aligned}$$

By Monotonicity, $\overline{B}_{\overline{\Pi}_i}(\overline{E}) \leqslant (\neg \overline{B}_{\overline{\Pi}_i})(\neg \overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) = \overline{M}_{\overline{\Pi}_i}(\overline{E})$.

Fourth, I show that (1c) and (1d) are equivalent. Under (1d), $M_{\overline{\Pi}_i}(S^{\uparrow}) = \{\omega \in \Omega \mid \sigma_i(\omega) \succeq S\} = A_{\sigma_i}(E)$ for all $(E, S) \in \mathcal{E}$. Conversely, for each $\omega \in \Omega$, I have $\Pi_i(\omega) \neq \emptyset$ because

$$\omega \in A_{\sigma_i}(\sigma_i^{\uparrow}(\omega)) = M_{\overline{\Pi}_i}(\sigma_i^{\uparrow}(\omega)) = \{\omega' \in \Omega \mid \Pi_i(\omega') \cap \sigma_i^{\uparrow}(\omega) \neq \emptyset \text{ and } \sigma_i(\omega') \succeq \sigma_i(\omega)\}.$$

2. Generalized Reflexivity implies that if $\omega \in B_{\overline{\Pi}_i}(E)$ then $\omega \in \Pi_i^{\uparrow}(\omega) \subseteq E$. Conversely, Truth Axiom implies $\omega \in B_{\overline{\Pi}_i}(\Pi_i^{\uparrow}(\omega)) \subseteq \Pi_i^{\uparrow}(\omega)$.

3. First, I establish the equivalence between (3a) and (3b). Assume (3b). If $\omega \in B_{\overline{\Pi}_i}(E)$ then, for any $\omega' \in \Pi_i^{\uparrow}(\omega)$, I have $\overline{\Pi}_i^{\uparrow}(\omega') \leqslant \overline{\Pi}_i^{\uparrow}(\omega) \leqslant \overline{E}$, i.e., $\omega' \in B_{\overline{\Pi}_i}(E)$. Thus, $\Pi_i^{\uparrow}(\omega) \subseteq B_{\overline{\Pi}_i}(E)$. Together with $\sigma_i(\omega) \succeq S(E)$, I obtain $\omega \in B_{\overline{\Pi}_i} B_{\overline{\Pi}_i}(E)$. I get $\overline{B}_{\overline{\Pi}_i}(\overline{E}) \leqslant \overline{B}_{\overline{\Pi}_i} \overline{B}_{\overline{\Pi}_i}(\overline{E})$. Conversely, (3a) implies $\omega \in B_{\overline{\Pi}_i}(\Pi_i^{\uparrow}(\omega)) \subseteq B_{\overline{\Pi}_i} B_{\overline{\Pi}_i}(\Pi_i^{\uparrow}(\omega))$, and thus $\Pi_i^{\uparrow}(\omega) \subseteq B_{\overline{\Pi}_i}(\Pi_i^{\uparrow}(\omega))$. Now, if $\omega' \in \Pi_i^{\uparrow}(\omega)$ then $\omega' \in B_{\overline{\Pi}_i}(\Pi_i^{\uparrow}(\omega))$, i.e., $\overline{\Pi}_i^{\uparrow}(\omega') \leqslant \overline{\Pi}_i^{\uparrow}(\omega)$.

Second, I establish the equivalence between (3b) and (3c). Note that if $\omega' \in \Pi_i(\omega)$, then, by Confinedness, $\sigma_i(\omega) = S(\omega') \succeq \sigma_i(\omega')$. Suppose (3b). If $\omega' \in \Pi_i(\omega)$ then $\omega' \in \Pi_i(\omega) \subseteq \Pi_i^{\uparrow}(\omega)$. Then, $\Pi_i(\omega') \subseteq \Pi_i^{\uparrow}(\omega') \subseteq \Pi_i^{\uparrow}(\omega)$ and $\sigma_i(\omega') \succeq \sigma_i(\omega)$. Thus, $\sigma_i(\omega') = \sigma_i(\omega)$ and $\Pi_i(\omega') \subseteq \Pi_i(\omega)$. Conversely, suppose (3c). Let $\omega' \in \Pi_i^{\uparrow}(\omega)$, and let $S := \sigma_i(\omega) \preceq S(\omega')$. Then, $\omega' \in (r_S^{S(\omega')})^{-1}(\Pi_i(\omega))$. Since $\omega'_S \in \Pi_i(\omega)$, it follows from (3c) that $\Pi_i(\omega'_S) \subseteq \Pi_i(\omega)$ and $S = \sigma_i(\omega'_S)$. By PPI, I have $\Pi_i(\omega') \subseteq (r_S^{\sigma_i(\omega')})^{-1}(\Pi_i(\omega'_S)) \subseteq (r_S^{\sigma_i(\omega')})^{-1}(\Pi_i(\omega)) \subseteq \Pi_i^{\uparrow}(\omega)$. Thus, I obtain $\Pi_i^{\uparrow}(\omega') \subseteq \Pi_i^{\uparrow}(\omega)$.

4. First, the equivalence among (4b), (4c), and (4e) follows from Fukuda (2021). Second, I show (4c) and (4d) are equivalent. If $\overline{A}_{\overline{\Pi}_i}$ satisfies Weak Necessitation, then it satisfies A-Independence: $\overline{A}_{\overline{\Pi}_i}(E,S) = B_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) = \overline{A}_{\overline{\Pi}_i}(E',S)$ for any $(E,S),(E',S') \in \mathcal{E}$. Conversely, if $\overline{A}_{\overline{\Pi}_i}$ satisfies A-Independence, then

$$B_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) \geqslant \overline{A}_{\overline{\Pi}_i}(E,S) = \overline{A}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) = B_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) \vee B_{\overline{\Pi}_i}(\neg B_{\overline{\Pi}_i})(\overline{S}^{\uparrow}) \geqslant B_{\overline{\Pi}_i}(\overline{S}^{\uparrow}).$$

Third, I show (4c) and (4a) are equivalent. Assuming (4c), I have:

$$\begin{aligned}(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) &= (\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{A}_{\overline{\Pi}_i}(\overline{E}) \\ &= (\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge (\overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee B_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E})) \\ &= (\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) \leqslant \overline{B}_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}).\end{aligned}$$

The first equality follows from (4c). The second equality follows from the definition of $\overline{A}_{\overline{\Pi}_i}$. The third equality follows from the "De Morgan" law.

Conversely, assuming (4a),

$$\overline{A}_{\overline{\Pi}_i}(\overline{E}) = \overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee B_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) \geqslant \overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee ((\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge B_{\overline{\Pi}_i}(\overline{S}^{\uparrow})) = B_{\overline{\Pi}_i}(\overline{S}^{\uparrow}).$$

The first equality follows from the definition of $\overline{A}_{\overline{\Pi}_i}$. The inequality follows from (4a). The last equality follows from $\overline{B}_{\overline{\Pi}_i}$-Monotonicity. Since $\overline{A}_{\overline{\Pi}_i}(\overline{E}) \leqslant B_{\overline{\Pi}_i}(\overline{S}^{\uparrow})$ follows from $\overline{B}_{\overline{\Pi}_i}$-Monotonicity, I obtain (4c).

□

Proof of Remark 6. First, I show that (1) implies (2). It follows from (1) that

$$\overline{S}^{\uparrow} = \overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee (\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) \leqslant \overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) = \overline{A}_{\overline{\Pi}_i}(\overline{E}).$$

Second, I show that (2) implies (3). By (2), since $\overline{A}_{\overline{\Pi}_i}(\cdot) \leqslant \overline{A}_{\sigma_i}(\cdot)$, I obtain $\overline{A}_{\sigma_i}(E,S) = \overline{S}^{\uparrow}$ for all $(E,S) \in \mathcal{E}$, which is equivalent to $\sigma_i(\cdot) = S(\cdot)$. Also, Generalized Euclideanness follows from Proposition 2, as $\overline{A}_{\overline{\Pi}_i}$-Independence holds.

Third, I show that (3) implies (1). Since $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Euclideanness, it follows from Proposition 2 (4) that $\overline{B}_{\overline{\Pi}_i}$ satisfies Generalized Negative Introspection I, i.e., $(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) \leqslant \overline{B}_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E})$. Since $\sigma_i(\cdot) = S(\cdot)$ implies that $\overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) = \overline{S}^{\uparrow}$, it follows that $\overline{B}_{\overline{\Pi}_i}$ satisfies Negative Introspection, i.e., (1). □

Proof of Proposition 3. 1. I show that (1a) and (1b) are equivalent. Assume (1a). Since $\omega \in A_{\sigma_i}(\sigma_i^{\uparrow}(\omega)) \subseteq B_{\overline{\Pi}_i}(A_{\sigma_i}(\sigma_i^{\uparrow}(\omega)))$, it follows that $\Pi_i^{\uparrow}(\omega) \subseteq A_{\sigma_i}(\sigma_i^{\uparrow}(\omega))$. If $\omega' \in \Pi_i^{\uparrow}(\omega)$ then $\sigma_i(\omega') \succeq \sigma_i(\omega)$.

Conversely, assume (1b). Take $\overline{E} = (E,S) \in \mathcal{E}$. If $\omega \in A_{\sigma_i}(E)$ then $\sigma_i(\omega) \succeq S$. For any $\omega' \in \Pi_i^{\uparrow}(\omega)$, I have $S \preceq \sigma_i(\omega) \preceq \sigma_i(\omega')$ and thus $\omega' \in A_{\sigma_i}(E)$. This implies that $\Pi_i^{\uparrow}(\omega) \subseteq A_{\sigma_i}(E)$. Since $\sigma_i(\omega) \succeq S$, it follows that $\omega \in B_{\overline{\Pi}_i} A_{\sigma_i}(E)$. Thus, $A_{\sigma_i}(E) \subseteq B_{\overline{\Pi}_i} A_{\sigma_i}(E)$, and hence $\overline{A}_{\sigma_i}(\overline{E}) \leqslant \overline{B}_{\overline{\Pi}_i} \overline{A}_{\sigma_i}(\overline{E})$.

The equivalence between (1b) and (1c) is established in the similar way to the characterization of Positive Introspection in Proposition 2, and thus the proof is omitted.

2. Assume (2a). Fix $\omega \in \Omega$. Since $B_{\overline{\Pi}_i} U_{\sigma_i}(\Pi_i^{\uparrow}(\omega)) = \emptyset$, I have $\omega \notin B_{\overline{\Pi}_i} U_{\sigma_i}(\Pi_i^{\uparrow}(\omega))$. Thus, for any $\omega' \in \Pi_i^{\uparrow}(\omega)$, I must have $\sigma_i(\omega') \not\succeq \sigma_i(\omega)$. Thus, (2b) obtains.

   Conversely, assume (2b). Suppose to the contrary that there are $\omega \in \Omega$ and $\overline{E} \in \mathcal{E}$ such that $\omega \in B_{\overline{\Pi}_i} U_{\sigma_i}(E)$. Then, $\Pi_i^{\uparrow}(\omega) \subseteq U_{\sigma_i}(E)$ and $\sigma_i(\omega) \succeq S(E)$. By (2b), there is $\omega' \in \Pi_i^{\uparrow}(\omega) \subseteq U_{\sigma_i}(E)$ such that $\sigma_i(\omega') \succeq \sigma_i(\omega)$. This implies $\sigma_i(\omega') \succeq S(E)$ and $\sigma_i(\omega') \not\succeq S(E)$, a contradiction.

□

Proof of Remark 7. 1. The statement follows because the property in Proposition 2 (3c) implies the property in Proposition 3 (1b)

2. It follows from $\overline{U}_{\sigma_i}(\cdot) \leqslant \overline{U}_{\overline{\Pi}_i}(\cdot)$ and $\overline{B}_{\overline{\Pi}_i}$-Monotonicity that $\overline{B}_{\overline{\Pi}_i} \overline{U}_{\overline{\Pi}_i}$-Introspection implies $\overline{B}_{\overline{\Pi}_i} \overline{U}_{\sigma_i}$-Introspection.

3. If $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Seriality, then for any $\omega \in \Omega$, there is $\omega' \in \Pi_i(\omega)$. Then, Belief-in-Awareness implies $\sigma_i(\omega') \succeq \sigma_i(\omega)$. Thus, there is $\omega' \in \Pi_i^{\uparrow}(\omega)$ such that $\sigma_i(\omega') \succeq \sigma_i(\omega)$.

   Conversely, I have $\overline{B}_{\overline{\Pi}_i}(\emptyset, S) \leqslant \overline{B}_{\overline{\Pi}_i} \overline{U}_{\sigma_i}(\emptyset, S) = (\emptyset, S)$, where the inequality follows from $\overline{B}_{\overline{\Pi}_i}$-Monotonicity and the equality from $\overline{B}_{\overline{\Pi}_i} \overline{U}_{\sigma_i}$-Introspection. By Proposition 2, it follows that $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Seriality. Note that, for this "only if" part, Belief-in-Awareness is not used.

4. The assertion holds because Generalized Reflexivity implies No-Belief-in-Unawareness (i.e., Proposition 3 (2b)). One can also directly show that, under Monotonicity, Truth Axiom implies $\overline{B}_{\overline{\Pi}_i} \overline{U}_{\overline{\Pi}_i}$-Introspection.

5. The assertion holds because Generalized Seriality and Generalized Transitivity imply No-Belief-in-Unawareness (i.e., Proposition 3 (2b)). One can also directly show that, under Monotonicity, Consistency and Positive Introspection imply $\overline{B}_{\overline{\Pi}_i} \overline{U}_{\overline{\Pi}_i}$-Introspection.

□

Proof of Remark 8. 1. If $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Transitivity, then the induced belief operator satisfies Positive Introspection. Together with Monotonicity,

$$\overline{B}_{\overline{\Pi}_i} \overline{A}_{\overline{\Pi}_i}(\overline{E}) = \overline{B}_{\overline{\Pi}_i}(\overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E})) \geqslant \overline{B}_{\overline{\Pi}_i} \overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i} \overline{B}_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E})$$
$$\geqslant \overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i}(\neg \overline{B}_{\overline{\Pi}_i})(\overline{E}) = \overline{A}_{\overline{\Pi}_i}(\overline{E}).$$

If $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Reflexivity, then $\overline{A}_{\overline{\Pi}_i}(\cdot) \geqslant \overline{B}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_i}(\cdot)$ follows from Truth Axiom.

2. Since $\overline{B}_{\overline{\Pi}_i} = \overline{B}_{\overline{\Pi}_i}\overline{B}_{\overline{\Pi}_i}$ by Truth Axiom and Positive Introspection,

$$\overline{A}_{\overline{\Pi}_i}\overline{B}_{\overline{\Pi}_i}(\overline{E}) = \overline{B}_{\overline{\Pi}_i}\overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i}\overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) = \overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) = \overline{A}_{\overline{\Pi}_i}(\overline{E}).$$

3. By $\overline{A}_{\overline{\Pi}_i}$-Introspection and $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection,

$$\overline{A}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_i}(\overline{E}) = \overline{B}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})\overline{A}_{\overline{\Pi}_i}(\overline{E}) = \overline{A}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}(\overline{E}) = \overline{A}_{\overline{\Pi}_i}(\overline{E}).$$

□

Proof of Proposition 4. Part (1). First, I establish the equivalence between Weak Negative Introspection and Negative Non-Introspection. Assume Weak Negative Introspection. I have

$$\overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \geqslant (\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{A}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \geqslant (\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})^2(\overline{E}),$$

where the first inequality follows from Weak Negative Introspection and the second from the definition of the awareness operator $\overline{A}_{\overline{\Pi}_i}$. Since this implies

$$(\neg\overline{B}_{\overline{\Pi}_i})^2(\overline{E}) \leqslant \overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee (\neg\overline{B}_{\overline{\Pi}_i})^3(\overline{E}),$$

Negative Non-Introspection follows because

$$(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge (\neg\overline{B}_{\overline{\Pi}_i})^2(\overline{E}) \leqslant (\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge \left(\overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee (\neg\overline{B}_{\overline{\Pi}_i})^3(\overline{E})\right) \leqslant (\neg\overline{B}_{\overline{\Pi}_i})^3(\overline{E}).$$

Conversely, assume Negative Non-Introspection. Since it is equivalent to $\overline{U}_{\overline{\Pi}_i}(\cdot) = \overline{U}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\cdot)$, I have $\overline{A}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \leqslant \overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E})$. Then, I get Weak Negative Introspection as follows:

$$\begin{aligned}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{A}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) &\leqslant (\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge \left(\overline{B}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E})\right) \\ &= (\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \leqslant \overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}).\end{aligned}$$

Next, I show the equivalence between Negative Non-Introspection and Strong Plausibility with equality. Strong Plausibility with equality implies Negative Non-Introspection:

$$(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge (\neg\overline{B}_{\overline{\Pi}_i})^2(\overline{E}) = \overline{U}_{\overline{\Pi}_i}(\overline{E}) = \bigwedge_{n\in\mathbb{N}} (\neg\overline{B}_{\overline{\Pi}_i})^n(\overline{E}) \leqslant (\neg\overline{B}_{\overline{\Pi}_i})^3(\overline{E}).$$

Conversely, I show by induction on $n \in \mathbb{N} \cup \{0\}$ that $(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge (\neg\overline{B}_{\overline{\Pi}_i})^2(\overline{E}) \leqslant (\neg\overline{B}_{\overline{\Pi}_i})^{n+3}(\overline{E})$. If $n = 0$ then the assertion holds by Negative Non-Introspection. If the assertion holds for any $m \in \{0, \ldots, n\}$, then I get

$$\begin{aligned}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge (\neg\overline{B}_{\overline{\Pi}_i})^2(\overline{E}) &\leqslant (\neg\overline{B}_{\overline{\Pi}_i})^{n+2}(\overline{E}) \wedge (\neg\overline{B}_{\overline{\Pi}_i})^{n+3}(\overline{E}) \\ &= (\neg\overline{B}_{\overline{\Pi}_i})(\neg\overline{B}_{\overline{\Pi}_i})^{n+1}(\overline{E}) \wedge (\neg\overline{B}_{\overline{\Pi}_i})^2(\neg\overline{B}_{\overline{\Pi}_i})^{n+1}(\overline{E}) \\ &\leq (\neg\overline{B}_{\overline{\Pi}_i})^3(\neg\overline{B}_{\overline{\Pi}_i})^{n+1}(\overline{E}) = (\neg\overline{B}_{\overline{\Pi}_i})^{n+4}(\overline{E}).\end{aligned}$$

Thus, the induction is complete. Now, I establish Strong Plausibility with equality: $(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge (\neg\overline{B}_{\overline{\Pi}_i})^2(\overline{E}) \leqslant \bigwedge_{n\in\mathbb{N}}(\neg\overline{B}_{\overline{\Pi}_i})^n(\overline{E})$.

Part (2). First, as discussed in Remark 9, Generalized Euclideanness implies (2a), (2b), and (2c). Also, (2b) implies (2c) by $\overline{B}_{\overline{\Pi}_i}$-Monotonicity:

$$\begin{aligned}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge (\neg\overline{B}_{\overline{\Pi}_i})^2(\overline{E}) &= \overline{U}_{\overline{\Pi}_i}(\overline{E}) \leqslant \overline{U}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}(\overline{E}) \\ &\leqslant (\neg\overline{B}_{\overline{\Pi}_i})^2\left((\neg\overline{B}_{\overline{\Pi}_i})(\overline{E}) \wedge (\neg\overline{B}_{\overline{\Pi}_i})^2(\overline{E})\right) \leqslant (\neg\overline{B}_{\overline{\Pi}_i})^3(\overline{E}).\end{aligned}$$

Second, I show that, under $\overline{A}_{\overline{\Pi}_i}$-Introspection and $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection, (2a) implies (2b):

$$\overline{U}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}(\overline{E}) = \overline{U}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_i}(\overline{E}) = (\neg\overline{B}_{\overline{\Pi}_i})\overline{A}_{\overline{\Pi}_i}(\overline{E}) \wedge (\neg\overline{B}_{\overline{\Pi}_i})(\neg\overline{B}_{\overline{\Pi}_i})\overline{A}_{\overline{\Pi}_i}(\overline{E}) = \overline{U}_{\overline{\Pi}_i}(\overline{E}).$$

Third, I show that, under $\overline{A}_{\overline{\Pi}_i}$-Introspection and $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection, (2c) implies (2b). Assume (2c). As discussed, it is equivalent to $\overline{U}_{\overline{\Pi}_i} = \overline{U}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})$. By $\overline{A}_{\overline{\Pi}_i}$-Introspection and $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection, I get $\overline{A}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_i}$-Self Reflection: $\overline{A}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_i} = \overline{A}_{\overline{\Pi}_i}$. Now, $\overline{A}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection follows because

$$\begin{aligned}\overline{A}_{\overline{\Pi}_i}(\overline{E}) &= \overline{A}_{\overline{\Pi}_i}\overline{A}_{\overline{\Pi}_i}(\overline{E}) = (\neg\overline{U}_{\overline{\Pi}_i})\overline{A}_{\overline{\Pi}_i}(\overline{E}) = (\neg\overline{U}_{\overline{\Pi}_i})((\neg\overline{B}_{\overline{\Pi}_i})\overline{A}_{\overline{\Pi}_i}(\overline{E})) \\ &= (\neg\overline{U}_{\overline{\Pi}_i})\overline{U}_{\overline{\Pi}_i}(\overline{E}) = \overline{A}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}(\overline{E}).\end{aligned}$$

Fourth, I show that, under $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection, (2b) implies Weak Necessitation (i.e., Generalized Euclideanness). By $\overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}$-Introspection,

$$\overline{A}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}(\overline{E}) = \overline{B}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}(\overline{E}) \vee \overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})\overline{U}_{\overline{\Pi}_i}(\overline{E}) = \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}).$$

Thus, $(\neg\overline{B}_{\overline{\Pi}_i})(\overline{S}^{\uparrow}) \leqslant \overline{U}_{\overline{\Pi}_i}(\overline{E}) \leqslant \overline{U}_{\overline{\Pi}_i}\overline{U}_{\overline{\Pi}_i}(\overline{E}) = (\neg\overline{B}_{\overline{\Pi}_i})(\overline{S}^{\uparrow})$.

Part (3a). By Monotonicity, Subjective Negative Introspection implies Generalized Negative Introspection I:

$$(\neg B_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) \leqslant \overline{B}_{\overline{\Pi}_i}((\neg B_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow})) \leqslant \overline{B}_{\overline{\Pi}_i}(\neg B_{\overline{\Pi}_i})(\overline{E}).$$

Conversely, by Generalized Negative Introspection I and Conjunction, I obtain

$$\begin{aligned}(\neg B_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) &\leqslant \overline{B}_{\overline{\Pi}_i}(\neg B_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) \leqslant \overline{B}_{\overline{\Pi}_i}(\neg B_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}\overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) \\ &\leqslant \overline{B}_{\overline{\Pi}_i}((\neg B_{\overline{\Pi}_i})(\overline{E}) \wedge \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow})).\end{aligned}$$

The first inequality follows from Generalized Negative Introspection I. The second inequality follows from Belief-in-Awareness, as Belief-in-Awareness implies $\overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) \leqslant$

$\overline{B}_{\overline{\Pi}_i}\overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow})$. Then, the third inequality follows from Conjunction.

Part (3b). Suppose that $\overline{\Pi}_i^{\uparrow}$ satisfies Generalized Seriality. Thus, $\overline{B}_{\overline{\Pi}_i}$ satisfies No Contradiction. If $\overline{A}_{\overline{\Pi}_i}$ satisfies Monotonicity, then

$$\overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) = \overline{B}_{\overline{\Pi}_i}(\overline{\emptyset}^S) \vee \overline{B}_{\overline{\Pi}_i}(\neg\overline{B}_{\overline{\Pi}_i})(\overline{\emptyset}^S) = \overline{A}_{\overline{\Pi}_i}(\overline{\emptyset}^S) \leqslant \overline{A}_{\overline{\Pi}_i}(E,S) \leqslant \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}),$$

where the last equality follows from $\overline{B}_{\overline{\Pi}_i}$-Monotonicity.

Conversely, suppose that $\overline{A}_{\overline{\Pi}_i}$ satisfies Weak Necessitation. If $\overline{E} \leqslant \overline{F}$ then $\overline{A}_{\overline{\Pi}_i}(\overline{E}) = \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}(E)) \leqslant \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}(F)) = \overline{A}_{\overline{\Pi}_i}(\overline{F})$ by $\overline{B}_{\overline{\Pi}_i}$-Monotonicity. □

A.4 Section 3.1

Proof of Remark 12. First, $\overline{C}_I(\overline{E}) = \bigwedge_{n\in\mathbb{N}} \overline{B}_I^n(\overline{E}) \leqslant \overline{B}_I(\overline{E})$. Second, $\bigwedge_{n\in\mathbb{N}} B_I^n(\overline{E}) \leqslant \bigwedge_{n\in\mathbb{N}} B_I^{n+1}(\overline{E}) \leqslant \overline{B}_I\left(\bigwedge_{n\in\mathbb{N}} B_I^n(\overline{E})\right)$, where the last inequality follows because $\overline{B}_I$ satisfies Conjunction. Thus,

$$\overline{C}_I(\overline{E}) \leqslant \sup\{\overline{F} \in \mathcal{E} \mid \overline{F} \leqslant \overline{B}_I(\overline{F}) \text{ and } \overline{F} \leqslant \overline{B}_I(\overline{E})\}.$$

Conversely, take $\overline{F} \in \mathcal{E}$ with $\overline{F} \leqslant \overline{B}_I(\overline{F})$ and $\overline{F} \leqslant \overline{B}_I(\overline{E})$. I show $\overline{F} \leqslant \overline{B}_I^n(\overline{E})$ by induction on n. For $n = 1$, $\overline{F} \leqslant \overline{B}_I(\overline{E})$ holds by assumption. If $\overline{F} \leqslant \overline{B}_I^n(\overline{E})$, then it follows from Monotonicity and the assumption that $\overline{F} \leqslant \overline{B}_I(\overline{F}) \leqslant \overline{B}_I^{n+1}(\overline{E})$. Since $\overline{F}$ is arbitrary, it follows that

$$\sup\{\overline{F} \in \mathcal{E} \mid \overline{F} \leqslant \overline{B}_I(\overline{F}) \text{ and } \overline{F} \leqslant \overline{B}_I(\overline{E})\} \leqslant \overline{C}_I(\overline{E}).$$

The proof is complete. □

A.5 Section 4.2

Proof of Lemma 1. Part (1). Since there exists $k \in \mathbb{N} \cup \{0\}$ such that $V^k = V^{\text{IESDA}}$, it suffices to show that $V_{\lambda,i}^k \neq \emptyset$ for all (k, λ, i). I prove this assertion by induction on $k \in \mathbb{N} \cup \{0\}$. Let $k = 0$. By definition, $V_{\lambda,i}^k = V_{\lambda,i} \neq \emptyset$ for all (λ, i). Assume the induction hypothesis that there is some $k \in \mathbb{N}$ such that $V_{\lambda,i}^{k-1} \neq \emptyset$ for all (λ, i). Take $(\lambda, i) \in \Lambda \times I$. By definition, $W_{\lambda,-i}^k \neq \emptyset$. Now, suppose to the contrary that $V_{\lambda,i}^k = \emptyset$. Then, any action $c_i \in V_{\lambda,i}^{k-1}$ is strictly dominated given $W_{\lambda,-i}^k$. Thus, there exists $\{c_i^{(\ell)}\}_{\ell=1}^L$ in $V_{\lambda,i}^{k-1}$ such that each $c_i^{(\ell)}$ is strictly dominated by $c_i^{(\ell+1)}$ given $W_{\lambda,-i}^{k-1}$ for each $\ell \in \{1, \ldots, L-1\}$ and $c_i^{(L)}$ is strictly dominated by $c_i^{(1)}$ given $W_{\lambda,-i}^{k-1}$. However, by the definition of strict dominance, this is impossible. Thus, $V_{\lambda,i}^k \neq \emptyset$.

Part (2). Define an operator g that associates, with each

$$(X, Y) = (X_{\lambda,i}, Y_{\lambda,-i})_{(\lambda,i)\in\Lambda\times I}$$

where, for each $(\lambda, i) \in \Lambda \times I$, $X_{\lambda,i} \in \mathcal{P}(V_{\lambda,i}) \setminus \{\emptyset\}$ and $Y_{\lambda,-i} = \bigtimes_{j \in I \setminus \{i\}} Y_{\lambda,j}$ with $Y_{\lambda,j} \in \mathcal{P}(V_{\lambda,j}) \setminus \{\emptyset\}$ for all $j \in I \setminus \{i\}$,

$$g(X, Y) = \left(\tilde{X}, \tilde{Y}\right) = \left(\tilde{X}_{\lambda,i}, \tilde{Y}_{\lambda,-i}\right)_{(\lambda,i) \in \Lambda \times I},$$

where

$$\tilde{Y}_{\lambda,-i} := \{(c_j)_{j \in I \setminus \{i\}} \in Y_{\lambda,-i} \mid \text{for all } j \in I \setminus \{i\}, \text{ there exists } \mu \in \Lambda \text{ such that } V_\lambda \succeq V_\mu \text{ and } c_j \in X_{\mu,j}\}$$

and

$$\tilde{X}_{\lambda,i} := \{c_i \in X_{\lambda,i} \mid c_i \text{ is not strictly dominated given } \tilde{Y}_{\lambda,-i}\}.$$

For ease of presentation, I denote

$$\tilde{X} = \left(\tilde{X}_{\lambda,i}\right)_{(\lambda,i) \in \Lambda \times I} = g_1(X, Y) \text{ and } \tilde{Y} = \left(\tilde{Y}_{\lambda,-i}\right)_{(\lambda,i) \in \Lambda \times I} = g_2(X, Y).$$

Observe that g is monotone in the following sense: for any $(X, Y) = (X_{\lambda,i}, Y_{\lambda,-i})_{(\lambda,i) \in \Lambda \times I}$ and $(X', Y') = (X'_{\lambda,i}, Y'_{\lambda,i})_{(\lambda,i) \in \Lambda \times I}$, if $X_{\lambda,i} \subseteq X'_{\lambda,i}$ and $Y_{\lambda,-i} \subseteq Y'_{\lambda,-i}$ for all $(\lambda, i) \in \Lambda \times I$, then $\tilde{X}_{\lambda,i} \subseteq \tilde{X}'_{\lambda,i}$ and $\tilde{Y}_{\lambda,-i} \subseteq \tilde{Y}'_{\lambda,-i}$ for all $(\lambda, i) \in \Lambda \times I$. This is because if $c_i \in X_{\lambda,i}$ is not strictly dominated given $Y_{\lambda,-i}$ then $c_i \in X_{\lambda,i} \subseteq X'_{\lambda,i}$ is not strictly dominated given $Y'_{\lambda,-i} \supseteq Y_{\lambda,-i}$.

Let $(V^k, W^k)_{k \in \mathbb{N} \cup \{0\}}$ be a process of iterated elimination of strictly dominated actions. For ease of exposition, let $V^k := \bigtimes_{(\lambda,i) \in \Lambda \times I} V^k_{\lambda,i}$ and $W^k := \bigtimes_{(\lambda,i) \in \Lambda \times I} W^k_{\lambda,-i}$ for all $k \in \mathbb{N} \cup \{0\}$. Also, let $V := \bigtimes_{(\lambda,i) \in \Lambda \times I} V_{\lambda,i}$ and $W := \bigtimes_{(\lambda,i) \in \Lambda \times I} \bigtimes_{j \in I \setminus \{i\}} V_{\lambda,j}$.

I show by induction that, for any (T, S) with $g(T, S) = (T, S)$, $T \subseteq V^k$ and $S \subseteq W^k$ for all $k \in \mathbb{N} \cup \{0\}$. Fix such (T, S). For $k = 0$, $T \subseteq V = V^0$ and $S \subseteq W = W^0$ follows by definition. Suppose that $T \subseteq V^k$ and $S \subseteq W^k$ for some $k \in \mathbb{N} \cup \{0\}$. Since g is monotone, it follows that $T = g_1(T, S) \subseteq g_1(V^k, W^k) \subseteq V^{k+1}$ and $S = g_2(T, S) \subseteq g_2(V^k, W^k) \subseteq W^{k+1}$. Thus, terminal sets of different processes include each other so that they are all the same. □

Proof of Proposition 5. I show that, for each $\lambda \in \Lambda$, if $\omega \in \mathrm{RAT}_I(S_\lambda) \cap C(\mathrm{RAT}^\uparrow_I(S_\lambda))$ then $\delta(\omega) \in V^{\mathrm{IESDA}}_\lambda$ using the definition of the bottom-up procedure (Definition 6). For each $k \in \mathbb{N} \cup \{0\}$, $\lambda \in \lambda$, and $i \in I$ in the given process of IESDA, denote by $D^k_{\lambda,i}$ the set of actions in $V^k_{\lambda,i}$ which are eliminated by the process (note that $D^k_{\lambda,i}$ may be empty). Suppose $\omega \in \mathrm{RAT}_I(S_\lambda) \cap C(\mathrm{RAT}^\uparrow_I(S_\lambda))$ for some $\lambda \in \Lambda$.

Suppose that $\lambda = \underline{\lambda}$, where $S_{\underline{\lambda}} = \inf \mathcal{S}$. In this case, the IESDA procedure within $S_{\underline{\lambda}}$ and the definition of rationality in $S_{\underline{\lambda}}$ both reduce to the standard definitions (e.g., Bonanno, 2008, 2015).

I show that

$$\delta_i(\omega') \notin D^k_{\underline{\lambda},i} \text{ for each } i \in I, \omega' \in \Pi_{C_I}(\omega), \text{ and } k \in \mathbb{N} \cup \{0\}$$

by induction on k. Let $k = 0$. If $\delta_i(\omega') \in D^0_{\underline{\lambda},i}$ for some $(i, \omega') \in I \times \Pi_{C_I}(\omega)$, then there is $c'_i \in \mathcal{C}_i(\omega') = V_{\underline{\lambda},i}$ such that $\delta_i(\omega')$ is strictly dominated by c'_i given $W^0_{\underline{\lambda},-i} = V_{\underline{\lambda},-i}$. Thus, $[\![\delta_i(\omega') \succcurlyeq_i c'_i]\!]_{S_{\underline{\lambda}}} = \emptyset$. This implies that $\omega' \notin M_{\overline{\Pi}_i}([\![\delta_i(\omega') \succcurlyeq_i c'_i]\!]_{S_{\underline{\lambda}}})$. Hence, $\omega' \notin \mathrm{RAT}_i(S_\lambda)$. Since $\mathrm{RAT}_I(S_\lambda) \subseteq \mathrm{RAT}_i(S_\lambda)$, it follows that $\omega' \notin \mathrm{RAT}_I(S_\lambda)$, a contradiction to $\omega' \in \Pi_{C_I}(\omega) \subseteq \mathrm{RAT}_I(S_\lambda)$.

Assume that there exists $k \in \mathbb{N} \cup \{0\}$ such that

$$\delta_i(\omega') \notin D^\ell_{\underline{\lambda},i} \text{ for each } i \in I, \omega' \in \Pi_{C_I}(\omega), \text{ and } \ell \in \{0, \dots, k\}.$$

Thus, $\delta(\omega') \in V^{k+1}_{\underline{\lambda}}$ for all $\omega' \in \Pi_{C_I}(\omega)$. Now, suppose, to the contradiction,

$$\delta_i(\omega') \in D^{k+1}_{\underline{\lambda},i} \text{ for some } (i, \omega') \in I \times \Pi_{C_I}(\omega).$$

Then, there is $c'_i \in V^{k+1}_{\underline{\lambda},i}$ such that $(c'_i, c_{-i}) \succ_i (\delta_i(\omega'), c_{-i})$ for all $c_{-i} \in W^{k+1}_{\underline{\lambda},-i} = V^{k+1}_{\underline{\lambda},-i}$. Since $\Pi_{C_I}(\omega') \subseteq \Pi_{C_I}(\omega)$, it follows that $\Pi_{C_I}(\omega') \subseteq [\![c'_i \succ_i \delta_i(\omega')]\!]_{S_{\underline{\lambda}}}$. Since $\Pi_i(\omega') \subseteq \Pi_{C_I}(\omega') \subseteq [\![c'_i \succ_i \delta_i(\omega')]\!]_{S_{\underline{\lambda}}}$, it follows that $\omega' \notin M_{\overline{\Pi}_i}([\![\delta_i(\omega') \succcurlyeq_i c'_i]\!]_{S_{\underline{\lambda}}})$. Hence, I obtain $\omega' \notin \mathrm{RAT}_i(S_{\underline{\lambda}})$, which leads to a contradiction. The induction is complete. Hence, $\delta(\omega') \in V^{\mathrm{IESDA}}_\lambda$ for all $\omega' \in \Pi_{C_I}(\omega)$.

Now, I show that $\delta(\omega) \in V^{\mathrm{IESDA}}_\lambda$. To that end, since $\omega \in \mathrm{RAT}_I(S_{\underline{\lambda}})$, it follows that $\omega \in M_{\overline{\Pi}_i}([\![\delta_i(\omega) \succcurlyeq_i \delta_i(\omega)]\!]_{S_{\underline{\lambda}}})$ for each $i \in I$. This implies $\Pi_i(\omega) \neq \emptyset$ for each $i \in I$. On the one hand, for any $i \in I$ and $\omega' \in \Pi_i(\omega)$, I have $\delta_i(\omega) = \delta_i(\omega')$. On the other hand, for any $i \in I$ and $\omega' \in \Pi_i(\omega)$, I have $\omega' \in \Pi_{C_I}(\omega)$. Thus, $\delta(\omega) \in V^{\mathrm{IESDA}}_\lambda$, as desired.

Next, let $\mu \in \Lambda$ be such that, for all $\lambda \in \Lambda \setminus \{\mu\}$ with $V_\mu \succeq V_\lambda$, if $\omega \in \mathrm{RAT}_I(S_\lambda) \cap C(\mathrm{RAT}^{\uparrow}_I(S_\lambda))$ then $\delta(\omega) \in V^{\mathrm{IESDA}}_\lambda$. I show

$$\delta_i(\omega') \notin D^k_{\mu,i} \text{ for each } i \in I, \omega' \in \Pi_{C_I}(\omega), \text{ and } k \in \mathbb{N} \cup \{0\}$$

by induction on k. Let $k = 0$. If $\delta_i(\omega') \in D^0_{\mu,i}$ for some $(i, \omega') \in I \times \Pi_{C_I}(\omega)$, then there is $c'_i \in \mathcal{C}_i(\omega'_{\sigma_i(\omega')}) \subseteq V_{\mu,i}$ such that $\delta_i(\omega')$ is strictly dominated by c'_i given $W^0_{\mu,-i} = V_{\mu,-i}$. Thus, $[\![\sigma_i(\omega') \succcurlyeq_i c'_i]\!]_{S_\lambda} = \emptyset$ for all $\lambda \in \Lambda$ with $V_\mu \succeq V_\lambda$. Then, $\omega' \notin M_{\overline{\Pi}_i}([\![\delta_i(\omega') \succcurlyeq_i c'_i]\!]_{S_\lambda})$ for all $\lambda \in \Lambda$ with $V_\mu \succeq V_\lambda$. Hence, $\omega' \notin \mathrm{RAT}_i(S_\mu)$, which leads to a contradiction.

Assume that there exists $k \in \mathbb{N} \cup \{0\}$ such that

$$\delta_i(\omega') \notin D^\ell_{\mu,i} \text{ for each } i \in I, \omega' \in \Pi_{C_I}(\omega), \text{ and } \ell \in \{0, \dots, k\}.$$

Thus, $\delta(\omega') \in V^{k+1}_\mu$ for all $\omega' \in \Pi_{C_I}(\omega)$. Now, suppose, to the contradiction,

$$\delta_i(\omega') \in D^{k+1}_{\mu,i} \text{ for some } (i, \omega') \in I \times \Pi_{C_I}(\omega).$$

Then, there is $c'_i \in V^{k+1}_{\mu,i}$ such that $(c'_i, c_{-i}) \succ_i (\delta_i(\omega'), c_{-i})$ for all $c_{-i} \in W^{k+1}_{\lambda,-i}$. Thus, Thus, $[\![\delta_i(\omega') \succcurlyeq_i c'_i]\!]_{S_\lambda} = \emptyset$ for all $\lambda \in \Lambda$ with $V_\mu \succeq V_\lambda$. Then, $\omega' \notin M_{\overline{\Pi}_i}([\![\delta_i(\omega') \succcurlyeq_i c'_i]\!]_{S_\lambda})$ for all $\lambda \in \Lambda$ with $V_\mu \succeq V_\lambda$. Hence, $\omega' \notin \mathrm{RAT}_i(S_\mu)$, a contradiction. The induction is complete. Hence, $\delta(\omega') \in V^{\mathrm{IESDA}}_\mu$ for all $\omega' \in \Pi_{C_I}(\omega)$.

Now, I show $\delta(\omega) \in V^{\mathrm{IESDA}}_\mu$. To that end, since $\omega \in \mathrm{RAT}_I(S_\mu)$, it follows that, for each $i \in I$, there exists $\lambda \in \Lambda$ such that $\sigma_i(\omega) \succeq S_\lambda$ and $\omega \in M_{\overline{\Pi}_i}([\![\delta_i(\omega) \succcurlyeq_i \delta_i(\omega)]\!]_{S_\lambda})$. This implies $\Pi_i(\omega) \neq \emptyset$ for each $i \in I$. On the one hand, for any $i \in I$ and $\omega' \in \Pi_i(\omega)$, I have $\delta_i(\omega) = \delta_i(\omega')$. On the other hand, for any $i \in I$ and $\omega' \in \Pi_i(\omega)$, I have $\omega' \in \Pi_{C_I}(\omega)$. Thus, $\delta(\omega) \in V^{\mathrm{IESDA}}_\lambda$, as desired.

□

A.6 Section 4.4

Proof of Proposition 7. Let $\omega \in B_{\overline{\Pi}_i}(\mathrm{RAT}^{\uparrow}_i(S))$, where $S \in \mathcal{S}$. To show that $\omega \in \mathrm{RAT}^{\uparrow}_i(S)$, it suffices to show that $\omega_S \in \mathrm{RAT}_i(S)$.

Since $\omega \in B_{\overline{\Pi}_i}(\mathrm{RAT}^{\uparrow}_i(S))$, it follows that $\sigma_i(\omega) \succeq S$ and $\Pi_i(\omega) \subseteq \mathrm{RAT}^{\uparrow}_i(S)$. Since $S(\omega) \succeq S$, it follows from PPB that $\Pi_i(\omega_S) \subseteq \mathrm{RAT}^{\uparrow}_i(S)$. Thus, $\sigma_i(\omega_S) \succeq S$. Since $S \succeq \sigma_i(\omega_S)$ follows from Confinedness, it follows that $\sigma_i(\omega_S) = S$. Thus, $\Pi_i(\omega_S) \subseteq \mathrm{RAT}_i(S)$.

Since $\Pi_i(\omega_S) \neq \emptyset$ by Generalized Seriality, take $\omega' \in \Pi_i(\omega_S) \subseteq \mathrm{RAT}_i(S)$. By Generalized Transitivity, $\sigma_i(\omega') = \sigma_i(\omega_S) = S$ and $\Pi_i(\omega') \subseteq \Pi_i(\omega_S)$. It follows from $\sigma_i(\omega') = \sigma_i(\omega_S)$ that $\mathcal{C}_i(\omega'_{\sigma_i(\omega')}) = \mathcal{C}_i((\omega_S)_{\sigma_i(\omega_S)})$. Since player i is certain of her strategy δ_i, it follows from $\omega' \in \Pi_i(\omega_S)$ that $\delta_i(\omega_S) = \delta_i(\omega')$.

Since $\omega' \in \mathrm{RAT}_i(S)$, for any $c'_i \in \mathcal{C}_i(\omega'_{\sigma_i(\omega')})$, there are $S' \in \mathcal{S}$ and $\omega'' \in \Pi_i(\omega')$ such that $\sigma_i(\omega') \succeq S'$ and $(\delta_i(\omega'), \sigma_{-i}(\omega''_{S'})) \succcurlyeq_i (c'_i, \sigma_{-i}(\omega''_{S'}))$.

Now, for any $c'_i \in \mathcal{C}_i((\omega_S)_{\sigma_i(\omega_S)}) = \mathcal{C}_i(\omega'_{\sigma_i(\omega')})$, there are $S' \in \mathcal{S}$ and $\omega'' \in \Pi_i(\omega') \subseteq \Pi_i(\omega_S)$ such that $\sigma_i(\omega_S) \succeq S'$ and $(\delta_i(\omega_S), \sigma_{-i}(\omega''_{S'})) \succcurlyeq_i (c'_i, \sigma_{-i}(\omega''_{S'}))$. Hence, $\omega_S \in \mathrm{RAT}_i$, as desired. □

Proof of Proposition 8. Let $\omega \in \mathrm{RAT}^{\uparrow}_i(S) \cap A_{\sigma_i}(\mathrm{RAT}^{\uparrow}_i(S))$, where $S \in \mathcal{S}$. Then, $\omega_S \in \mathrm{RAT}_i(S)$ and $\sigma_i(\omega) \succeq S$. To show that $\omega \in B_i(\mathrm{RAT}^{\uparrow}_i(S))$, it is enough to show $\omega_S \in B_i(\mathrm{RAT}^{\uparrow}_i(S))$, i.e., $\overline{\Pi}^{\uparrow}_i(\omega_S) \leqslant (\mathrm{RAT}^{\uparrow}_i(S), S)$, i.e., $\Pi^{\uparrow}_i(\omega_S) \subseteq \mathrm{RAT}^{\uparrow}_i(S)$ and $\sigma_i(\omega_S)$.

By PPB under Confinedness, it follows from $S \preceq \sigma_i(\omega) \preceq S(\omega)$ that $\sigma_i(\omega_S) = S$. Thus, it suffices to show $\Pi_i(\omega_S) \subseteq \mathrm{RAT}_i(S)$.

Since $\omega_S \in \mathrm{RAT}_i(S)$, it follows that, for any $c'_i \in \mathcal{C}_i((\omega_S)_{\sigma_i(\omega_S)})$, there are $S' \in \mathcal{S}$ and $\omega' \in \Pi_i(\omega_S)$ such that $\sigma_i(\omega_S) \succeq S'$ and $(\delta_i(\omega_S), \delta_{-i}(\omega'_{S'})) \succcurlyeq_i (c'_i, \delta_{-i}(\omega'_{S'}))$.

Take any $\omega'' \in \Pi_i(\omega_S)$. Since player i is certain of her own strategy, $\delta_i(\omega'') = \delta_i(\omega_S)$. By Generalized Euclideanness, $\Pi^{\uparrow}_i(\omega_S) \subseteq \Pi^{\uparrow}_i(\omega'')$. By $\overline{A}_{\sigma_i}$-Introspection (i.e., Belief-in-Awareness), $\sigma_i(\omega_S) = \sigma_i(\omega'')$. Thus, $\Pi_i(\omega_S) \subseteq \Pi_i(\omega'')$. Also, $\mathcal{C}_i(\omega''_{\sigma_i(\omega'')}) = \mathcal{C}_i((\omega_S)_{\sigma_i(\omega_S)})$, as $\sigma_i(\omega_S) = \sigma_i(\omega'')$.

Then, for any $c'_i \in \mathcal{C}_i(\omega''_{\sigma_i(\omega'')})$, there are $S' \in \mathcal{S}$ and $\omega' \in \Pi_i(\omega_S) \subseteq \Pi_i(\omega'')$ such that $\sigma_i(\omega'') \succeq S'$ and $(\delta_i(\omega''), \delta_{-i}(\omega'_{S'})) \not\succ_i (c'_i, \delta_{-i}(\omega'_{S'}))$. Hence, $\omega'' \in \text{RAT}_i(S)$. Thus, $\Pi_i(\omega_S) \subseteq \text{RAT}_i(S)$, as desired. □

B Possibility Operator

As discussed in Section 2.3.1, I establish the equivalence between the possibility operator and the possibility correspondence.

Proposition 9. *1. Let σ_i be well-defined. The following are equivalent.*

(a) $\overline{\Pi}_i^{\uparrow} = (\Pi_i^{\uparrow}, \sigma_i)$ is well-defined.

(b) $M_{\overline{\Pi}_i}(D^{\uparrow}, S) = (\{\omega \in S \mid \sigma_i(\omega) = S \text{ and } \Pi_i(\omega) \cap D \neq \emptyset\})^{\uparrow}$ for any $(D^{\uparrow}, S) \in \mathcal{E}$.

(c) $\overline{\Pi}_i^{\uparrow}$ induces $\overline{M}_{\overline{\Pi}_i}$: $\overline{M}_{\overline{\Pi}_i}(D^{\uparrow}, S) = (M_{\overline{\Pi}_i}(D^{\uparrow}), S) \in \mathcal{E}$ for any $(D^{\uparrow}, S) \in \mathcal{E}$.

2. Assume that $\overline{\Pi}_i^{\uparrow} = (\Pi_i^{\uparrow}, \sigma_i)$ is well-defined. Fix $\overline{D}^{\uparrow} = (D^{\uparrow}, S) \in \mathcal{E}$.

(a) $M_{\overline{\Pi}_i}(D^{\uparrow}) = (\neg B_{\overline{\Pi}_i})(\neg D^{\uparrow}) \cap B_{\overline{\Pi}_i}(S^{\uparrow}) = (\neg B_{\overline{\Pi}_i})(\neg D^{\uparrow}) \cap A_{\sigma_i}(D^{\uparrow})$.

(b) $B_{\overline{\Pi}_i}(D^{\uparrow}) = (\neg M_{\overline{\Pi}_i})(\neg D^{\uparrow}) \cap B_{\overline{\Pi}_i}(S^{\uparrow}) = (\neg M_{\overline{\Pi}_i})(\neg D^{\uparrow}) \cap A_{\sigma_i}(D^{\uparrow})$.

The proof is in Appendix B.1. The first two parts of Proposition 9 require the awareness function to be given. This is because if, for instance, $\Pi_i(\cdot) = \emptyset$, then, for any function $\sigma_i : \Omega \to \mathcal{S}$ which may fail to be well-defined, the possibility operator satisfies $\overline{M}_{\overline{\Pi}_i}(E, S) = (M_{\overline{\Pi}_i}(E), S) = (\emptyset, S) \in \mathcal{E}$.

The second part of Proposition 9 also shows how the belief and possibility operators are related through the awareness operator $\overline{A}_{\sigma_i}$ in specific forms. This part follows because, if $\overline{\Pi}_i^{\uparrow} = (\Pi_i^{\uparrow}, \sigma_i)$ is well-defined, then

$$(\neg B_{\overline{\Pi}_i})(\neg D^{\uparrow}) = \{\omega \in S \mid \text{if } \sigma_i(\omega) = S \text{ then } \Pi_i(\omega) \cap D \neq \emptyset\}^{\uparrow} \text{ and}$$
$$(\neg M_{\overline{\Pi}_i})(\neg D^{\uparrow}) = \{\omega \in S \mid \text{if } \sigma_i(\omega) = S \text{ then } \Pi_i(\omega) \subseteq D\}^{\uparrow}.$$

Player i considers (E, S) possible iff she does not know the negation of (E, S) and she believes a "tautology" $(S^{\uparrow}, S)$ in S.[30] In other words, she considers (E, S) possible iff she does not know the negation of (E, S) and she is $\overline{A}_{\overline{\sigma}_i}$-aware of (E, S). If her awareness operator $\overline{A}_{\overline{\Pi}_i}$ satisfies Weak Necessitation ($\overline{A}_{\overline{\Pi}_i}(E, S) = \overline{B}_{\overline{\Pi}_i}(\overline{S}^{\uparrow}) = \overline{A}_{\sigma_i}(E, S)$), then the definition of possibility reduces that of Modica and Rustichini (1999). Moreover, unlike standard models, the possibility operator does not necessarily satisfy $\overline{M}_{\overline{\Pi}_i}(E, S) = (\neg \overline{B}_{\overline{\Pi}_i})(\neg(E, S))$.

[30] One could define another "possibility" operator $(\neg \overline{B}_{\overline{\Pi}_i})(\neg \overline{E})$.

To conclude, I provide the properties of the possibility operator $\overline{M}_{\overline{\Pi}_i}$ which corresponds to those in Remark 2. Then, I provide the equivalence between the belief and possibility operators in terms of those properties.

If $\overline{\Pi}_i^{\uparrow}$ induces $\overline{M}_{\overline{\Pi}_i}$, then one can show that $\overline{M}_{\overline{\Pi}_i}$ satisfies the following properties.

Remark 14. If $\overline{\Pi}_i^{\uparrow}$ induces $\overline{M}_{\overline{\Pi}_i}$, then $\overline{M}_{\overline{\Pi}_i}$ satisfies the following.

1. Decomposition: $S(\overline{M}_{\overline{\Pi}_i}(\overline{E})) = S(\overline{E})$ for all $\overline{E} \in \mathcal{E}$.

2. Monotonicity: $\overline{M}_{\overline{\Pi}_i}(\overline{E}) \leqslant \overline{M}_{\overline{\Pi}_i}(\overline{F})$ for any $\overline{E}, \overline{F} \in \mathcal{E}$ with $\overline{E} \leqslant \overline{F}$.

3. Disjunction: $\overline{M}_{\overline{\Pi}_i}\left(\bigvee_{x \in X} \overline{E}_x\right) \leqslant \bigvee_{x \in X} \overline{M}_{\overline{\Pi}_i}(\overline{E}_x)$.

4. Contradiction: $\overline{M}_{\overline{\Pi}_i}(\emptyset, \inf \mathcal{S}) = (\emptyset, \inf \mathcal{S})$.

When I stress that the possibility operator $\overline{M}_{\overline{\Pi}_i}$ satisfies some of the above properties, I append "$\overline{M}_{\overline{\Pi}_i}$-" or "M-" to each property (e.g., $\overline{M}_{\overline{\Pi}_i}$-Decomposition or M-Decomposition).

Three technical remarks are in order. First, Monotonicity and Disjunction are jointly equivalent to Disjunction with equality: $\overline{M}_{\overline{\Pi}_i}\left(\bigvee_{x \in X} \overline{E}_x\right) = \bigvee_{x \in X} \overline{M}_{\overline{\Pi}_i}(\overline{E}_x)$. Second, Contradiction can be seen as a special case of Disjunction with $X = \emptyset$. Third, $\overline{M}_{\overline{\Pi}_i}$-Disjunction is the dual of $\overline{B}_{\overline{\Pi}_i}$-Conjunction, and $\overline{M}_{\overline{\Pi}_i}$-Contradiction is the dual of $\overline{B}_{\overline{\Pi}_i}$-Necessitation.

Now, I establish the duality between belief and possibility operators.

Proposition 10. *Suppose that $\overline{A}_i : \mathcal{E} \to \mathcal{E}$ is an (awareness) operator satisfying the five properties in Remark 4. For a given $\overline{B}_i : \mathcal{E} \to \mathcal{E}$, define $\overline{M}_{\overline{B}_i}(\overline{E}) := (\neg \overline{B}_i)(\neg \overline{E}) \wedge \overline{A}_i(\overline{E})$. Likewise, for a given $\overline{M}_i : \mathcal{E} \to \mathcal{E}$, let $\overline{B}_{\overline{M}_i}(\overline{E}) := (\neg \overline{M}_i)(\neg \overline{E}) \wedge \overline{A}_i(\overline{E})$.*

1. *If $\overline{B}_i$ satisfies the four properties in Remark 2, then $\overline{M}_{\overline{B}_i}$ satisfies the four properties in Remark 14. Likewise, if $\overline{M}_i$ satisfies the four properties in Remark 14, then $\overline{B}_{\overline{M}_i}$ satisfies the four properties in Remark 2.*

2. *For any $\overline{B}_i$ with $\overline{B}_i(\cdot) \leqslant \overline{A}_i(\cdot)$, $\overline{B}_i = \overline{B}_{\overline{M}_{\overline{B}_i}}$. Likewise, for any $\overline{M}_i$ with $\overline{M}_i(\cdot) \leqslant \overline{A}_i(\cdot)$, $\overline{M}_i = \overline{M}_{\overline{B}_{\overline{M}_i}}$.*

The proof is in Appendix B.1.

B.1 Proofs

Proof of Proposition 9. I only show Part (1), as the proof for Part (2) is sketched in the main text.

First, I show that (1a) implies (1b). Take $\omega \in M_{\overline{\Pi}_i}(D^{\uparrow})$. Then, $\Pi_i(\omega) \cap D^{\uparrow} \neq \emptyset$ and $\sigma_i(\omega) \succeq S$. Thus, $\Pi_i(\omega) \cap (r_S^{\sigma_i(\omega)})^{-1}(D) \neq \emptyset$. By Confinedness, $S(\omega) \succeq \sigma_i(\omega)$, and thus $S(\omega) \succeq S$. By PPB and Confinedness, $S = \sigma_i(\omega_S)$. It follows from PPI that $\Pi_i^{\uparrow}(\omega) \subseteq \Pi_i^{\uparrow}(\omega_S)$. Then,

$$\emptyset \neq \Pi_i(\omega) \cap (r_S^{\sigma_i(\omega)})^{-1}(D) \subseteq (r_S^{\sigma_i(\omega)})^{-1}(\Pi_i(\omega_S)) \cap (r_S^{\sigma_i(\omega)})^{-1}(D) = (r_S^{\sigma_i(\omega)})^{-1}(\Pi_i(\omega_S) \cap D).$$

Thus, $\Pi_i(\omega_S) \cap D \neq \emptyset$. This implies that $\omega \in \{\omega' \in S \mid \sigma_i(\omega') = S \text{ and } \Pi_i(\omega') \cap D \neq \emptyset\}^{\uparrow}$.

To prove the converse set inclusion, suppose that $\omega \in \{\omega' \in S \mid \sigma_i(\omega') = S \text{ and } \Pi_i(\omega') \cap D \neq \emptyset\}^{\uparrow}$. Then, together with Confinedness and NIA, $\Pi_i(\omega_S) \cap D \neq \emptyset$ and $S(\omega) \succeq \sigma_i(\omega) \succeq \sigma_i(\omega_S) = S$. Now, suppose to the contrary that $\Pi_i(\omega) \cap D^{\uparrow} = \emptyset$. Then, $\Pi_i(\omega) \cap (r_S^{\sigma_i(\omega)})^{-1}(D) = \emptyset$, i.e., $\Pi_i(\omega) \subseteq (r_S^{\sigma_i(\omega)})^{-1}(S \setminus D)$. By PPB, $\Pi_i(\omega_S) \subseteq S \setminus D$, i.e., $\Pi_i(\omega_S) \cap D = \emptyset$, a contradiction. Hence, $\Pi_i(\omega) \cap D^{\uparrow} \neq \emptyset$. Since $\sigma_i(\omega) \succeq S$, it follows that $\omega \in M_{\overline{\Pi}_i}(D^{\uparrow})$.

Next, by inspection, (1b) and (1c) are equivalent. Thus, I show that (1c) implies (1a). First, I establish Regularity. By Confinedness (which holds because σ_i is well-defined),

$$M_{\overline{\Pi}_i}(D^{\uparrow}) \cap S = \{\omega \in S \mid \sigma_i(\omega) = S \text{ and } \Pi_i(\omega) \cap D \neq \emptyset\} \in \mathcal{D}.$$

Then, I have

$$\begin{aligned} D(\overline{M}_{\overline{\Pi}_i}(D^{\uparrow}, S)) &= \{\omega \in S \mid \sigma_i(\omega) = S \text{ and } \Pi_i(\omega) \cap D \neq \emptyset\} \\ &= S \setminus \{\omega \in S \mid \text{if } \sigma_i(\omega) = S \text{ then } \Pi_i(\omega) \subseteq S \setminus D\} \\ &= S \setminus (\{\omega \in S \mid \sigma_i(\omega) \neq S\} \cup \{\omega \in S \mid \sigma_i(\omega) = S \text{ and } \Pi_i(\omega) \subseteq (S \setminus D)\}) \in \mathcal{D}. \end{aligned}$$

By using A-Regularity (which holds because σ_i is well-defined),

$$\begin{aligned} \{\omega \in S \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant (D^{\uparrow}, S)\} &= \{\omega \in S \mid \sigma_i(\omega) = S \text{ and } \Pi_i(\omega) \subseteq D\} \\ &= D(\neg \overline{M}_{\overline{\Pi}_i}(\neg \overline{D}^{\uparrow})) \cap D(\overline{A}_{\sigma_i}(\neg \overline{D}^{\uparrow})) \in \mathcal{D}. \end{aligned}$$

Second, I show PPI. Let $S \preceq S(\omega)$. By (1c) and Confinedness (which holds because σ_i is well-defined), $\omega_S \in \neg M_{\overline{\Pi}_i}(\neg \overline{\Pi}_i^{\uparrow}(w_S))$. Then, $\omega_S \in (r^S_{\sigma_i(\omega_S)})^{-1}(D)$, where $D = D(\neg \overline{M}_{\overline{\Pi}_i}(\neg \overline{\Pi}_i^{\uparrow}(w_S)))$. Then, $\omega \in (r^{S(\omega)}_{\sigma_i(\omega_S)})^{-1}(D)$. Since $\sigma_i(\omega_S) \preceq \sigma_i(\omega)$ by NIA (which holds because σ_i is well-defined), I get $\Pi_i(\omega) \cap (\sigma_i(\omega_S) \setminus \Pi_i(\omega_S))^{\uparrow} = \emptyset$. Thus, $\Pi_i(\omega) \subseteq (r^{\sigma_i(\omega)}_{\sigma_i(\omega_S)})^{-1}(\Pi_i(\omega_S))$. Hence, $\overline{\Pi}_i^{\uparrow}(\omega) \leqslant \overline{\Pi}_i^{\uparrow}(\omega_S)$.

Third, I show PPB. Let $S \preceq S(\omega)$, and assume $\overline{\Pi}_i(\omega) \leqslant (D^{\uparrow}, S)$. By PPA (which holds because σ_i is well-defined), $S \preceq \sigma_i(\omega_S)$. Thus, $\Pi_i(\omega) \subseteq (r_S^{\sigma_i(\omega)})^{-1}(D)$. Then, $\omega \in \neg M_{\overline{\Pi}_i}(\neg D^{\uparrow})$. Since $\omega_S \in \neg M_{\overline{\Pi}_i}(\neg D^{\uparrow})$ and $S \preceq \sigma_i(\omega_S)$, it follows that $\Pi_i(\omega_S) \cap \neg D^{\uparrow} = \emptyset$. Thus, $\Pi_i(\omega_S) \subseteq (r_S^{\sigma_i(\omega_S)})^{-1}(D)$. Hence, $\overline{\Pi}_i^{\uparrow}(\omega_S) \leqslant (D^{\uparrow}, S)$. □

Proof of Proposition 10. Part (1). I only prove the first statement, as the second statement can be proved in the same way.

First, $\overline{B}_i$-Decomposition and $\overline{A}_i$-Decomposition yield $\overline{M}_{\overline{B}_i}$-Decomposition. Second, $\overline{B}_i$-Necessitation implies $\overline{M}_{\overline{B}_i}$-Contradiction because:

$$\begin{aligned}\overline{M}_{\overline{B}_i}(\emptyset, \inf \mathcal{S}) &= \neg \overline{B}_i(\neg\emptyset, \inf \mathcal{S}) \wedge \overline{A}_i(\emptyset, \inf \mathcal{S}) \\ &= \neg \overline{B}_i(\Omega, \inf \mathcal{S}) \wedge (\Omega, \inf \mathcal{S}) = \neg(\Omega, \inf \mathcal{S}) = (\emptyset, \inf \mathcal{S}).\end{aligned}$$

Third, I show that $\overline{B}_i$-Monotonicity and $\overline{B}_i$-Conjunction imply $\overline{M}_{\overline{B}_i}$-Monotonicity and $\overline{M}_{\overline{B}_i}$-Disjunction. To that end, I show that the combination of $\overline{B}_i$-Monotonicity and $\overline{B}_i$-Conjunction, i.e.,

$$\overline{B}_i\left(\bigwedge_{x\in X} \overline{E}_x\right) = \bigwedge_{x\in X} \overline{B}_i(\overline{E}_x),$$

implies the combination of $\overline{M}_{\overline{B}_i}$-Monotonicity and $\overline{M}_{\overline{B}_i}$-Disjunction, i.e.,

$$\overline{M}_{\overline{B}_i}\left(\bigvee_{x\in X} \overline{E}_x\right) = \bigvee_{x\in X} \overline{M}_{\overline{B}_i}(\overline{E}_x).$$

Since $\neg\overline{B}_i\left(\bigwedge_{x\in X} \neg\overline{E}_x\right) = \neg\bigwedge_{x\in X} \overline{B}_i(\neg\overline{E}_x)$, I have $\left(\neg\overline{B}_i\neg\right)\left(\bigvee_{x\in X} \overline{E}_x\right) = \bigvee_{x\in X}(\neg\overline{B}_i\neg)(\overline{E}_x)$. Hence, I get

$$\begin{aligned}\overline{M}_{\overline{B}_i}\left(\bigvee_{x\in X} \overline{E}_x\right) &= \left(\neg\overline{B}_i\neg\right)\left(\bigvee_{x\in X} \overline{E}_x\right) \wedge \overline{A}_i\left(\bigvee_{x\in X} \overline{E}_x\right) = \left(\bigvee_{x\in X} (\neg\overline{B}_i\neg)\overline{E}_x\right) \wedge \overline{A}_i\left(\bigvee_{x\in X} \overline{E}_x\right) \\ &= \bigvee_{x\in X}\left((\neg\overline{B}_i\neg)(\overline{E}_x) \wedge \overline{A}_i\left(\bigvee_{x\in X} \overline{E}_x\right)\right) \\ &\geqslant \bigvee_{x\in X}\left((\neg\overline{B}_i\neg)(\overline{E}_x) \wedge \overline{A}_i(\overline{E}_x)\right) = \bigvee_{x\in X} \overline{M}_{\overline{B}_i}(\overline{E}_x).\end{aligned}$$

At the same time, by A-Independence and A-Subspace-Monotonicity,

$$\begin{aligned}\overline{M}_{\overline{B}_i}\left(\bigvee_{x\in X} \overline{E}_x\right) &= \bigvee_{x\in X}\left((\neg\overline{B}_i\neg)(\overline{E}_x) \wedge \overline{A}_i\left(\bigwedge_{x\in X} (\overline{E}_x)\right)\right) \\ &\leqslant \bigvee_{x\in X}\left((\neg\overline{B}_i\neg)(\overline{E}_x) \wedge \overline{A}_i(\overline{E}_x)\right) = \bigvee_{x\in X} \overline{M}_{\overline{B}_i}(\overline{E}_x).\end{aligned}$$

Part (2). I have

$$\begin{aligned}\overline{B}_{\overline{M}_{\overline{B}_i}}(\overline{E}) &= (\neg\overline{M}_{\overline{B}_i})(\neg\overline{E}) \wedge \overline{A}_i(\overline{E}) = \neg\left((\neg\overline{B}_i)(\neg\neg\overline{E}) \wedge \overline{A}_i(\neg\overline{E})\right) \wedge \overline{A}_i(\overline{E}) \\ &= \left(\overline{B}_i(\overline{E}) \vee \overline{U}_i(\overline{E})\right) \wedge \overline{A}_i(\overline{E}) = \overline{B}_i(\overline{E}) \wedge \overline{A}_i(\overline{E}) = \overline{B}_i(\overline{E}).\end{aligned}$$

Likewise, $\overline{M}_{\overline{B}_{\overline{M}_i}} = \overline{M}_i$ by exchanging $\overline{B}_i$ and $\overline{M}_i$. □

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