On the Axiomatization of an Unawareness Structure from Knowing-Whether Operators*

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October 3, 2024

Abstract

This paper shows that, on a generalized state space model of unawareness, an agent's underlying knowledge is axiomatized from her knowing-whether operator if and only if her knowledge satisfies the Truth Axiom: whenever the agent knows an event, the event holds. The agent knows whether an event obtains if she knows it or knows its negation. Different knowledge operators lead to different knowing-whether operators if knowledge is truthful. Conversely, for any knowing-whether operator, there is a unique truthful knowledge operator that induces the given knowing-whether operator: the agent knows an event if and only if she knows whether the event holds and the event indeed holds. Qualitative or probabilistic beliefs may not be recovered from believing-whether. This paper then axiomatizes properties of knowledge and common knowledge, in terms of knowing-whether. The main contributions of the paper are as follows. First, conceptually, this paper provides a generalized-state-space model of knowledge and unawareness in which the only assumption on knowledge is the Truth Axiom. Second, practically, this paper may provide a simple way to construct a generalized-state-space model.

Journal of Economic Literature Classification Numbers: C70; D83
Keywords: Knowing-whether; Knowledge; Belief; Unawareness; Truth Axiom

*I am grateful to two anonymous referees and the handling co-editor for their helpful comments.
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1 Introduction

This paper concerns a situation in which agents reason about their knowledge (or more generally, beliefs) and unawareness. In a model that represents agents' knowledge and unawareness, each agent possesses knowledge about the environment in which they make decisions, and she may be unaware of certain aspects of the environment. Usually, the model specifies, with each object of knowledge (i.e., event), the event that she knows it. In contrast, when the primitive of the model specifies her knowledge in a way such that she knows whether each event obtains, can we, as outside analysts, recover or identify her underlying knowledge?

This paper provides the axiomatization of a model of knowledge and unawareness from knowing-whether. Given the notion of knowledge, an agent knows whether an event obtains if and only if she knows that the event obtains or knows that the event does not obtain. Given the notion of knowing-whether, the agent knows an event if and only if she knows whether the event obtains and the event indeed obtains. Proposition 1 in this paper shows that an agent's underlying knowledge can be axiomatized from knowing-whether if and only if the underlying notion of knowledge satisfies the Truth Axiom: if the agent knows an event, then the event indeed obtains.

Using Proposition 1 as the benchmark result, the main contributions of this paper are as follows. First, conceptually, this paper provides a generalized-state-space model of knowledge and unawareness where the sole assumption on knowledge is the Truth Axiom. Second, practically, this paper may provide a simple way to construct a model of knowledge and unawareness on a generalized state space.

In the paper, a model of knowledge and unawareness has three ingredients. The first is a generalized state space. The second is agents' knowledge operators. The third is agents' unawareness operators. To describe each ingredient of the model, I present a variant of the speculative-trade example of Heifetz, Meier, and Schipper (2006, Section 3), one of the pioneering papers on unawareness.

Consider an owner o of a firm (he) and a buyer b (she). The owner is aware of a possible lawsuit (denoted by ℓ) involving the firm, and he knows whether the lawsuit occurs or not. He is unaware of a potential innovation or novelty (denoted by n) increasing the value of the firm. The buyer, in contrast, is aware of the innovation, and she knows whether the innovation occurs or not. She is unaware of the lawsuit. Using the knowing-whether and unawareness operators as the primitives of a model, these statements are intuitively formalized.

As unawareness refers to the lack of conception rather than the lack of knowledge, the first ingredient of the model, a generalized state space, describes the ways in which the innovation n and the lawsuit ℓ can be objects of agents' knowledge and unawareness. Specifically, the generalized state space consists of a collection of disjoint subspaces $S_{n\ell}$, S_n, S_ℓ, and $S_\varnothing$.[1]

Different subspaces represent the different degrees to which the innovation n and

[1]Each space is called a subspace because the entire union is referred to as the state space.

the lawsuit ℓ can be described. The subspace $S_{n\ell} = \{\omega_{n\ell}, \omega_{n\neg\ell}, \omega_{\neg n\ell}, \omega_{\neg n\neg\ell}\}$ represents both aspects (n and ℓ). For instance, at the state $\omega_{n\ell}$, the innovation n occurs and the lawsuit ℓ occurs. At the state $\omega_{n\neg\ell}$, the innovation n occurs but the lawsuit ℓ does not occur.

In contrast, the subspace $S_n = \{\omega_n, \omega_{\neg n}\}$ represents only one aspect n, leaving the other aspect ℓ from the description of the states. At the state ω_n, the innovation n occurs. At the state $\omega_{\neg n}$, the innovation n does not occur. Similarly for $S_\ell = \{\omega_\ell, \omega_{\neg\ell}\}$. The subspace $S_\varnothing = \{\omega_\varnothing\}$ does not refer to the innovation n or the lawsuit ℓ.

Agents reason about events. In the context of standard state spaces where a state space consists of a single (sub-)space, each agent reasons about subsets of the state space. That is, the collection of events is a collection of subsets of the state space. In contrast, in a generalized state space, each event consists of a pair $\overline{E} = (E, S)$ where S is a subspace and E is a subset of the subspace S.[2] For the subspace $S_{n\ell}$, there are 2^4 events. For instance, the event $(\{\omega_{n\ell}, \omega_{\neg n\ell}\}, S_{n\ell})$ corresponds to the event in $S_{n\ell}$ that the lawsuit occurs. This is a different event from $(\{\omega_\ell\}, S_\ell)$, which is an event that belongs to the subspace S_ℓ.

Similarly to the standard-state model, agents' knowledge and unawareness operators are defined on the collection of events. An agent's knowledge operator associates, with each event $\overline{E}$, the event that she knows $\overline{E}$. Likewise, the agent's unawareness operator associates, with each event $\overline{E}$, the event that she is unaware of $\overline{E}$.

In the current literature on unawareness such as Heifetz, Meier, and Schipper (2006), each agent's possibility correspondence induces her knowledge and unawareness operators. In this paper, in contrast, each agent's knowledge and unawareness operators are a primitive of the model, and I study a condition under which the agent's knowledge operator can be replaced with her knowing-whether operator.

Through the speculative-trade example, I demonstrate that such approach would be quite intuitive and thus useful in practice. On the subspace $S_{n\ell}$, the owner and the buyer are unaware of any event $(E, S_{n\ell})$ and thus do not know whether the event $(E, S_{n\ell})$ obtains at any state $\omega \in S_{n\ell}$.

On the subspace S_n, the buyer knows whether any event (E, S_n) occurs and thus is aware (not unaware) of it at any state $\omega \in S_n$. In contrast, the owner is unaware of any event (E, S_n) and does not know whether (E, S_n) occurs at any state $\omega \in S_n$.

On the subspace S_ℓ, the owner knows whether any event (E, S_ℓ) occurs and thus is aware of it at any state $\omega \in S_\ell$. In contrast, the buyer is unaware of any event (E, S_ℓ) and does not know whether (E, S_ℓ) occurs at any state $\omega \in S_\ell$.

On the subspace $S_\varnothing$, the owner and the buyer know whether any event $(E, S_\varnothing)$ obtains at any state $\omega \in S_\varnothing$. Consequently, they are aware of any event $(E, S_\varnothing)$ at any state $\omega \in S_\varnothing$. This way, the specification of the unawareness structure in the speculative-trade example is complete. This example suggests that, when it comes to building a model of knowledge and unawareness, the model whose primitives are

[2]The formal definition is slightly more convoluted. See Section 2 for the general definition. I also revisit the speculative-trade example in Section 3.3.

agents' knowing-whether and unawareness operators might prove useful for further applications.

With the speculative-trade example in mind, the benchmark result of this paper (Proposition 1) roughly states that knowledge is recoverable from knowing-whether if and only if knowledge satisfies the Truth Axiom: if the agent knows an event at a state then the event holds true at that state. Moreover, the *unique* way that knowledge is recoverable from knowing-whether is that the agent knows an event at a state if and only if she knows whether the event holds or not and the event indeed holds at that state. I show through examples that the outside analysts may not be able to identify an agent's beliefs that may violate the Truth Axiom from her believing-whether operator.

I formally discuss Proposition 1. Fix a generalized state space. Let $\overline{K}$ be an agent's knowledge operator. Then the knowing-whether operator $\overline{J}_{\overline{K}}$ would map an event $\overline{E}$ to the event that she knows $\overline{E}$ or she knows its negation $\neg\overline{E}$: $\overline{J}_{\overline{K}}(\overline{E}) := \overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{E})$. I first examine a domain of knowledge operators under which the mapping $\overline{K} \mapsto \overline{J}_{\overline{K}}$ that transforms a knowledge operator into the knowing-whether operator is injective: different knowledge operators lead to different knowing-whether operators. Whenever knowledge operators $\overline{K}$ in the domain satisfy the Truth Axiom, the transformation is injective. Indeed, the collection of knowledge operators satisfying the Truth Axiom is a maximal domain under which the transformation $\overline{K} \mapsto \overline{J}_{\overline{K}}$ is injective.

The inverse maps each knowing-whether operator $\overline{J}$ to a truthful knowledge operator $\overline{K}_{\overline{J}}$ (i.e., the knowledge operator $\overline{K}_{\overline{J}}$ satisfying the Truth Axiom) defined by $\overline{K}_{\overline{J}}(\overline{E}) := \overline{J}(\overline{E}) \wedge \overline{E}$: the agent knows an event $\overline{E}$ if and only if she knows whether $\overline{E}$ is true and $\overline{E}$ holds true. To study possible ways in which a knowing-whether operator induces a knowledge operator, consider the transformation $\overline{J} \mapsto \overline{K}_{\overline{J}}$ that maps a knowing-whether operator $\overline{J}$ to the corresponding knowledge operator. In the domain in which any $\overline{J}$ is symmetric, i.e., the agent knows whether $\overline{E}$ if and only if she knows whether its negation $\neg\overline{E}$, the transformation $\overline{J} \mapsto \overline{K}_{\overline{J}}$ is injective. Again, the collection of symmetric knowing-whether operators is a maximal domain under which the transformation $\overline{J} \mapsto \overline{K}_{\overline{J}}$ is injective.

Combining these results, knowledge is identifiable from knowing-whether if and only if knowledge is truthful, and the unique way to identify underlying knowledge is: the agent knows an event $\overline{E}$ if and only if she knows whether $\overline{E}$ is true and $\overline{E}$ holds true.

This paper then axiomatizes models of knowledge and unawareness by properties of knowing-whether. Even in a model of this paper in which the only assumption on knowledge is the Truth Axiom, I show that one can readily introduce such notions as common knowledge.

This paper is organized as follows. The rest of this section discusses related literature. Section 2 sets up a model of knowledge and unawareness. Section 3 presents Proposition 1 stating that a knowing-whether operator can be the primitive of a model if and only if an underlying knowledge operator satisfies the Truth Axiom. The section also provides counterexamples when an underlying "knowledge" (in fact, belief) operator violates the Truth Axiom in the contexts of qualitative and probabilistic beliefs. The section also revisits the speculative-trade example to demonstrate that the framework of this paper may provide a simple way to construct a model of knowledge and unawareness. Section 4 discusses the applications of Proposition 1. Section 4.1 studies the axiomatization of a model by properties of knowing-whether. Section 4.2 shows that one can axiomatize a model of knowledge and unawareness from the notion of ignorance. Section 4.3 studies unawareness when it is directly defined from (not) knowing-whether. Section 4.4 introduces common knowledge in the framework of this paper. The proofs of the results in the main text are in Appendix A. Appendices B and C provide supplementary discussions.

Related Literature

I discuss related literature on (i) unawareness, (ii) knowing-whether, and (iii) the role of the Truth Axiom. First, this paper belongs to the broad literature on unawareness, especially the one on a representation of knowledge and unawareness in a richer structure than the standard state spaces. Dekel, Lipman, and Rustichini (1998) has provided an impossibility result to represent a non-trivial form of unawareness on a standard state space, and the burgeoning literature studies various models to capture a non-trivial form of unawareness in a richer structure. This paper adopts the formulation of a generalized state space by Heifetz, Meier, and Schipper (2006).[3]

At the same time, since the sole assumption on knowledge is the Truth Axiom, this paper also accommodates non-partitional possibility correspondences on a standard state space. Indeed, Proposition 3 (in Section 4.1) axiomatizes partitional and non-partitional possibility-correspondence models based on knowing-whether operators, irrespective of whether an underlying state space is a generalized one or a standard one. Standard-state-space non-partitional models can be used to study information processing violating Negative Introspection, its implications to solution concepts of games, and a certain form of unawareness.[4]

Second, there are strands of literature that study the notion of knowing-whether (or its dual notion of ignorance) in artificial intelligence and computer science, economics and game theory, and logic and philosophy. In artificial intelligence, computer science, logic, and philosophy, the notion of knowing-whether has been studied since Hintikka (1962, Section 1.7).

[3]There are other pioneering attempts to represent unawareness in a richer structure such as: Board and Chung (2021, 2022), Board, Chung, and Schipper (2011), Galanis (2011, 2013), Halpern (2001), Halpern and Rêgo (2008, 2009), Heifetz, Meier, and Schipper (2008, 2013), Heinsalu (2014), and Li (2009). See Schipper (2015) for an overview.

[4]See, for example, Bacharach (1985), Dekel and Gul (1997), Fagin and Halpern (1987), Geanakoplos (2021), Modica and Rustichini (1994, 1999), Morris (1996), Samet (1990), Schipper (2015), Shin (1993), and the references therein.

In the context of this paper, the expressive power of a model of knowing-whether or ignorance, in comparison with a standard-state-space possibility-correspondence model (precisely, a Kriple model), has been studied in the literature. Montgomery and Routley (1966, Theorem 3) show that a model of knowing-whether is equivalent to a model of knowledge when underlying knowledge satisfies the Truth Axiom in a standard-state-space possibility-correspondence model (in the context of this paper).[5] The transformations between knowledge and knowing-whether in this paper follow from theirs.[6] Yet, their result hinges on (i) certain logical assumptions on an agent's knowledge embodied by a possibility-correspondence model and (ii) the standard state space (i.e., a single state space) framework. Fan and van Ditmarsch (2015, Proposition 5) establish the equivalence by dropping an agent's logical reasoning properties within the framework of neighborhood systems (or Montague-Scott structures) on a standard state space.[7]

This paper provides an axiomatization of knowing-whether in the context of a generalized-state-space model of unawareness. As the modeling approaches themselves are technically quite different, this paper formally shows that the previous equivalence results in logic do not directly apply (Example 1 in Section 3.1). While the main overall contribution of this paper is to provide a tractable generalized-state-space model of unawareness from knowing-whether operator (i.e., Proposition 1 is a benchmark result for this purpose), this paper has a mild technical contribution to this literature in that the previous literature does not directly apply to establish the equivalence result on a generalized state space.[8]

[5]Two remarks are in order. First, while this paper uses the terminology of knowing-whether operation given its focus, the modal logic literature (as in Montgomery and Routley, 1966) refers to it as the non-contingency operation when the "knowledge" operation is taken as some other notion of necessity. Second, Montgomery and Routley (1966, Theorem 3) technically show the equivalence within the proof-theoretic (i.e., syntactic) framework. Since the logics they deal with are complete with respect to classes of standard-state-space possibility-correspondence (i.e., model-theoretic semantic) models, one can interpret their equivalence results between knowledge and knowing-whether in the context of standard-state-space possibility-correspondence models.

[6]That is, an agent knows whether an event holds if she knows the event or she knows its negation; and the agent knows an event if she knows whether the event holds and the event indeed holds.

[7]Two remarks are in order. First, Appendix B discusses neighborhood systems and formalizes neighborhood systems on a generalized state space. Second, for other related papers in the logic literature, see, for instance, Steinsvold (2008) and van der Hoek and Lomuscio (2004) for the context of ignorance and Demri (1997) for the equivalence between knowledge and knowing-whether (when knowledge is assumed to satisfy the Truth Axiom). Fan, Wang, and van Ditmarsch (2015) provide a general framework with which to study the equivalence between knowledge and knowing-whether (technically, the equivalence between the notions of necessity and non-contingency) within various logical systems.

[8]For the reader who is interested in seeing more formally how Proposition 1 can be connected to the logic literature, Appendix C studies an algebraic structure behind Proposition 1 in the context closer to the idea of Montgomery and Routley (1966, Theorem 3) and shows that the key algebraic structure is indeed simple. I would like to thank the referee for encouraging me to clarify this formal connection.

Turning to the literature in economics and game theory, the literature has studied the notion of knowing-whether in the context of standard state spaces. Hart, Heifetz, and Samet (1996) construct a rich knowledge model on a standard state space, utilizing the property that a knowing-whether operator is symmetric (i.e., an agent knows whether an event occurs if and only if she knows whether the negation of the event occurs). Given symmetry, if one considers a k-th order interactive knowledge of the form, agent 1 knowing-whether agent 2 (not) knowing-whether agent 1 (not) knowing-whether ... an event is true or not, then such sequences are never contradictory with each other. Lehrer and Samet (2011) study the "Agreement" theorem (Aumann, 1976) using the notion of ignorance (the dual notion of knowing-whether). In the literature looking at a set-algebraic representation of knowledge on a standard state space, the collection of events E such that the agent always knows whether E is true or not forms a σ-algebra (e.g., Hérves-Beloso and Monteiro, 2013; Lee, 2018; Wilson, 1978).[9] The exception in the economics and game theory literature is Heifetz, Meier, and Schipper (2006). While they do not formally use knowing-whether operators (on a generalized state space) or study how to introduce an unawareness structure from knowing-whether, they use the idea of knowing-whether to construct their examples (Examples 2 and 3), which suggest that the notion of knowing-whether may be useful to construct unawareness structures. This paper shows that indeed one can construct a generalized-state-space model of knowledge and unawareness from knowing-whether as long as the underlying notion of knowledge satisfies the Truth Axiom.

Third, moving on to the strands of literature that study the role of the Truth Axiom, recall that this paper contrasts knowledge and belief in the sense that qualitative or probabilistic beliefs that may violate the Truth Axiom may not necessarily be identified from believing-whether, while truthful knowledge can always be recovered from knowing-whether. Bonanno and Nehring (1998) study the role of the Truth Axiom on the agreement theorem and the absence of unbounded gains from betting. The literature on epistemic characterizations of solution concepts of games studies the differences in predictions between common belief in rationality and common knowledge of rationality.[10] These papers study epistemic characterizations and no-trade and agreement theorems in the context of standard state spaces. While this paper provides a model of knowledge and unawareness (where knowledge satisfies the Truth Axiom), it would still be possible to include probabilistic beliefs into the model.[11]

[9] The various strands of literature (in artificial intelligence and computer science, economics and game theory, and logic and philosophy) indeed represent an agent's knowledge as a set algebra such as a σ-algebra or a topology. In the economics and game theory literature, for instance, see, Dubra and Echenique (2004), Fukuda (2019), and Tóbiás (2021) in addition to the above-mentioned papers. When the agent's knowledge is represented as a topology, i.e., the agent's knowledge operator is an interior operator of the topology, the boundary operator turns out to be the complement of the knowing-whether operator (or the ignorance operator). See Section 4.2 for further discussions.

[10] See, for instance, Bonanno (2008), Bonanno and Tsakas (2018), Fukuda (2020, 2024), Guarino and Ziegler (2022), Hillas and Samet (2020), and Samet (2013).

[11] It would be an interesting avenue for future research to include probabilistic beliefs, in addition

In this sense, one would still be able to analyze a model of knowledge, probabilistic beliefs, and unawareness where knowledge is represented from knowing-whether.

2 Model

Unless otherwise stated, I focus on a single agent. A model consists of a generalized state space and the agent's knowledge and unawareness operators. To that end, Section 2.1 defines a generalized state space. Since the agent's knowledge and unawareness operators are defined on the collection of events, Section 2.2 defines events. Section 2.3 formally defines a model. Section 2.4 specifies properties of knowledge and unawareness.

2.1 A Generalized State Space

A *generalized state space* $\langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}, \succeq, r \rangle$ consists of three primitives. First, $(S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}$ is a non-empty collection of non-empty *subspaces*. Denote by $\mathcal{S} := \{S_\alpha\}_{\alpha \in \mathcal{A}}$ the collection of subspaces. Without loss, assume that the sets in $\mathcal{S}$ (i.e., the subspaces) are pairwise disjoint. Each S_α is endowed with a complete algebra $\mathcal{D}_\alpha$. That is, $\mathcal{D}_\alpha$ is a collection of subsets of S_α (i.e., it is a subset of the power set $\mathcal{P}(S_\alpha)$) that contains S_α and that is closed under complementation, arbitrary union, and arbitrary intersection. Call the entire union $\Omega := \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ the set of *states of the world*.

Second, $\langle \mathcal{S}, \succeq \rangle$ is a complete lattice. As in Heifetz, Meier, and Schipper (2006), a subspace S' is intended to be interpreted as being at least as "expressive" as a subspace S if $S' \succeq S$. Thus, the partial order $\succeq$ is intended to be interpreted as ranking subspaces $\mathcal{S}$ by amounts of "concepts" or "expressive power."

Third, $r := (r_S^{S'})_{S' \succeq S}$ is a collection of surjective projections $r_S^{S'} : (S', \mathcal{D}') \to (S, \mathcal{D})$ for each pair $(S, S') \in \mathcal{S}^2$ with $S' \succeq S$. I assume: (i) measurability: $(r_S^{S'})^{-1}(B) \in \mathcal{D}'$ for all $B \in \mathcal{D}$; (ii) each r_S^S is the identity mapping on S; and (iii) commutativity: $S'' \succeq S' \succeq S$ implies $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$.

The generalized state space is *standard* if $\mathcal{S}$ is a singleton, i.e., $\mathcal{S} = \{\Omega\}$.

2.2 Events

On the generalized state space, an event is defined as a pair consisting of a set of states and the subspace to which the event belongs. Formally, an *event* is a pair $(B^\uparrow, S) \in \mathcal{P}(\Omega) \times \mathcal{S}$, where

$$B^\uparrow := \bigcup \{ (r_S^{S'})^{-1}(B) \in \mathcal{P}(\Omega) \mid S' \succeq S \text{ for some } S' \in \mathcal{S} \} \text{ and } B \in \mathcal{D}.$$

to knowledge, into the model of this paper. Indeed, the literature has focused on either a model of knowledge and unawareness (e.g., Heifetz, Meier, and Schipper, 2006) or a model of probabilistic beliefs and unawareness (e.g., Heifetz, Meier, and Schipper, 2013).

Define the *domain* $\mathcal{E}$ by the collection of events.

Fix an event $(B^{\uparrow}, S_{\alpha})$. By denoting $(\emptyset^{S}, S) := (\emptyset, S)$, the set $B^{\uparrow}$ alone determines the subspace S_{α} to which the event $(B^{\uparrow}, S_{\alpha})$ belongs.

As stated, the event $(B^{\uparrow}, S_{\alpha})$ has two components. First, call S_{α} the *base space* of $(B^{\uparrow}, S_{\alpha})$ (or simply $B^{\uparrow}$), and denote $S(B^{\uparrow}, S_{\alpha}) = S_{\alpha}$ (or simply $S(B^{\uparrow}) = S_{\alpha}$). Second, call B the *basis* of $B^{\uparrow}$. For any event (E, S), the basis B of E satisfies $E = B^{\uparrow}$.

I introduce the following four operations on the domain $\mathcal{E}$. The first is a partial order $\leqslant$: for any events $(E, S(E))$ and $(F, S(F))$, define $(E, S(E)) \leqslant (F, S(F))$ as $E \subseteq F$ and $S(E) \succeq S(F)$. The greatest element is $(\Omega, \inf \mathcal{S}) = ((\inf \mathcal{S})^{\uparrow}, \inf \mathcal{S})$ while the least element is $(\emptyset, \sup \mathcal{S})$.

Second, for any collection of events $(B_{\lambda}^{\uparrow}, S_{\lambda})_{\lambda \in \Lambda}$, define its *conjunction* as

$$\bigwedge_{\lambda \in \Lambda} (B_{\lambda}^{\uparrow}, S_{\lambda}) := \left(\bigcap_{\lambda \in \Lambda} B_{\lambda}^{\uparrow}, \sup_{\lambda \in \Lambda} S_{\lambda} \right) = \left(\left(\bigcap_{\lambda \in \Lambda} (r_{S_{\lambda}}^{\sup_{\lambda \in \Lambda} S_{\lambda}})^{-1}(B_{\lambda}) \right)^{\uparrow}, \sup_{\lambda \in \Lambda} S_{\lambda} \right) \in \mathcal{E}.$$

Since $\bigwedge_{\lambda \in \Lambda}(B_{\lambda}^{\uparrow}, S_{\lambda})$ is the infimum of events $(B_{\lambda}^{\uparrow}, S_{\lambda})_{\lambda \in \Lambda}$ in a partially ordered set $\langle \mathcal{E}, \leqslant \rangle$, it forms a complete lattice.

Third, define the *negation* of an event $(B^{\uparrow}, S)$ by $\neg(B^{\uparrow}, S) := (\neg B^{\uparrow}, S) := ((S \setminus B)^{\uparrow}, S) \in \mathcal{E}$. By definition, $\neg\neg(B^{\uparrow}, S) = (B^{\uparrow}, S)$. By letting $\neg\emptyset^{S} := S^{\uparrow}$ and $\neg S^{\uparrow} := \emptyset^{S}$, I can unambiguously write $\neg\neg B^{\uparrow} = B^{\uparrow}$ for any $(B^{\uparrow}, S) \in \mathcal{E}$. Since the negation of $(B^{\uparrow}, S)$ is taken within the subspace S, the operation of negation is generally different from the operation of taking the set- or lattice-theoretical complement. As in Heifetz, Meier, and Schipper (2006), if $S(E) = S(F)$, then

$$(E, S(E)) \leqslant (F, S(F)) \text{ if and only if } \neg(F, S(F)) \leqslant \neg(E, S(E)).$$

Fourth, define the *disjunction* of $(B_{\lambda}^{\uparrow}, S_{\lambda})_{\lambda \in \Lambda}$ as $\bigvee_{\lambda \in \Lambda}(B_{\lambda}^{\uparrow}, S_{\lambda}) := \neg(\bigwedge_{\lambda \in \Lambda} \neg(B_{\lambda}^{\uparrow}, S_{\lambda})) \in \mathcal{E}$. The disjunction $\bigvee_{\lambda \in \Lambda}(B_{\lambda}^{\uparrow}, S_{\lambda})$ is generally different from the supremum of $(B_{\lambda}^{\uparrow}, S_{\lambda})_{\lambda \in \Lambda}$ in $\langle \mathcal{E}, \leqslant \rangle$. As in Heifetz, Meier, and Schipper (2006), if $S = S_{\lambda}$ for all $\lambda \in \Lambda$, then

$$\bigvee_{\lambda \in \Lambda} (B_{\lambda}^{\uparrow}, S_{\lambda}) = \left(\bigcup_{\lambda \in \Lambda} B_{\lambda}^{\uparrow}, S \right).$$

In particular, $(B^{\uparrow}, S) \vee (\neg B^{\uparrow}, S) = (S^{\uparrow}, S)$. It can be seen that the distributive laws hold: $(E, S) \wedge \bigvee_{\lambda \in \Lambda}(E_{\lambda}, S_{\lambda}) = \bigvee_{\lambda \in \Lambda} ((E, S) \wedge (E_{\lambda}, S_{\lambda}))$ and $(E, S) \vee \bigwedge_{\lambda \in \Lambda}(E_{\lambda}, S_{\lambda}) = \bigwedge_{\lambda \in \Lambda} ((E, S) \vee (E_{\lambda}, S_{\lambda}))$.

To simplify the notation of an event $(B^{\uparrow}, S(B^{\uparrow}))$, observe that the set $B^{\uparrow}$ determines its base space $S(B^{\uparrow})$. Thus, I add the over-line to the set $B^{\uparrow}$ to denote the event

$$\overline{B}^{\uparrow} := (B^{\uparrow}, S(B^{\uparrow})).$$

That is, I denote any event $(E, S(E))$ by $\overline{E} = (E, S(E))$.

To conclude this subsection, note that if the state space is standard, i.e., $\mathcal{S} = \{\Omega\}$, then one can identify a complete lattice $\langle \mathcal{E}, \leqslant \rangle$ with $\langle \mathcal{D}, \subseteq \rangle$. Also, one can identify the operations of conjunction $\bigwedge$, negation $\neg$, and disjunction $\bigvee$ as those of intersection, complementation, and union. Thus, in order to stress the fact that I am dealing with the standard state space, I often use the underlying complete algebra $\mathcal{D}$ (instead of $\mathcal{E}$) to denote the collection of events.

2.3 Model

A *model* is a tuple $\mathcal{M} := \langle \langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}, \succeq, r \rangle, (\overline{K}, \overline{U}) \rangle$ with the following three ingredients.[12] First, $\langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}, \succeq, r \rangle$ is a generalized state space. Let $\mathcal{E}$ be the collection of events, i.e., the domain.

Second, $\overline{K} : \mathcal{E} \ni (E, S(E)) = \overline{E} \mapsto \overline{K}(\overline{E}) = (K(E), S(K(E))) \in \mathcal{E}$ is the agent's *knowledge operator* satisfying (at least) the following condition:

$$S(K(E)) = S(E) \text{ for any } (E, S(E)) \in \mathcal{E}.$$

Since each subspace S is supposed to represent some aspects of the world, it would be natural to assume that an agent's knowledge about events in the subspace S is also an event in S.

Third, similarly, $\overline{U} : \mathcal{E} \ni (E, S(E)) = \overline{E} \mapsto \overline{U}(\overline{E}) = (U(E), S(U(E))) \in \mathcal{E}$ is the agent's *unawareness operator* satisfying (at least) the following condition:

$$S(U(E)) = S(E) \text{ for any } (E, S(E)) \in \mathcal{E}.$$

The next subsection specifies properties of the knowledge and unawareness operators.

For any event $\overline{E} = (E, S(E))$, $\overline{K}(\overline{E}) = (K(E), S(K(E)))$ is the event that the agent knows $\overline{E}$. The set $K(E)$ constitutes a part of the event $\overline{K}(\overline{E})$, and is interpreted as the set of states at which the agent knows $\overline{E}$. The base space of the event $\overline{K}(\overline{E})$ is assumed to be $S(E)$: $S(K(E)) = S(E)$. The same holds for the unawareness operator $\overline{U}$.

2.4 Properties of Knowledge and Unawareness

This subsection defines properties of knowledge and unawareness. First, I define the Truth Axiom. The knowledge operator $\overline{K}$ satisfies the *Truth Axiom* if

$$\overline{K}(\overline{E}) \leqslant \overline{E} \text{ for all } \overline{E} \in \mathcal{E}.$$

The Truth Axiom states that if the agent knows an event at a state then the event has to hold at that state. The Truth Axiom distinguishes knowledge from beliefs in that if the agent knows an event at a state then the event has to occur at that state.

[12] For the case in which there are multiple agents, letting I be a non-empty set of agents, a model is extended to a tuple $\langle \langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}, \succeq, r \rangle, (\overline{K}_i, \overline{U}_i)_{i \in I} \rangle$ such that each $\langle \langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}, \succeq, r \rangle, (\overline{K}_i, \overline{U}_i) \rangle$ is a (single-agent) model.

2.4.1 Possibility Correspondence Models on a Generalized State Space

The framework accommodates generalized-state-space models of knowledge and unawareness (e.g., Fukuda, 2021, 2023; Heifetz, Meier, and Schipper, 2006). First, Fukuda (2021) studies a model of knowledge and unawareness where the knowledge operator satisfies, in addition to the Truth Axiom, the following two properties. The first is *Monotonicity*:

$$\overline{E} \leqslant \overline{F} \text{ implies } \overline{K}(\overline{E}) \leqslant \overline{K}(\overline{F}).$$

Monotonicity states that if event $\overline{E}$ implies event $\overline{F}$ then the knowledge of $\overline{E}$ implies that of $\overline{F}$. The second is *Positive Introspection*:

$$\overline{K}(\cdot) \leqslant \overline{KK}(\cdot), \text{ that is, } \overline{K}(\overline{E}) \leqslant \overline{KK}(\overline{E}) \text{ for all } \overline{E} \in \mathcal{E}.$$

Positive Introspection states that if the agent knows an event then she knows that she knows the event.

Next, the framework of this paper subsumes possibility-correspondence models of knowledge and unawareness on a generalized state space. Heifetz, Meier, and Schipper (2006) propose a generalized-state-space possibility-correspondence model that reduces to the "partitional" possibility-correspondence model when the underlying generalized state space reduces to the standard state space.[13]

More generally, the framework of this paper subsumes any generalized-state-space possibility-correspondence model if the resulting knowledge operator satisfies the Truth Axiom. Thus, one can say more generally that the knowledge operator $\overline{K} : \mathcal{E} \ni \overline{E} \mapsto (K(E), S(E)) \in \mathcal{E}$ is *induced by* a possibility correspondence $\overline{\Pi}^{\uparrow} : \Omega \to \mathcal{E}$ if

$$\overline{K}(\overline{E}) = (\{\omega \in \Omega \mid \overline{\Pi}^{\uparrow}(\omega) \leqslant \overline{E}\}, S(E)) \text{ for all } \overline{E} \in \mathcal{E}. \tag{1}$$

In this regard, Fukuda (2023) characterizes Condition (1) in terms of the possibility correspondence $\overline{\Pi}^{\uparrow}$ or the knowledge operator $\overline{K}$. In terms of the knowledge operator, the characterization extends the well-known condition on the standard-state-space case to the generalized-state-space case.[14] Namely, under the following three conditions, the agent's knowledge operator $\overline{K} : \mathcal{E} \ni \overline{E} \mapsto (K(E), S(E)) \in \mathcal{E}$ is induced by a possibility correspondence on the generalized state space.[15] The first

[13]The more precise discussion on the standard-state-space "partitional" possibility-correspondence model is given in Section 2.4.3.

[14]As mentioned in Footnote 13, Section 2.4.3 discusses standard-state-space possibility-correspondence models.

[15]See Fukuda (2023) for the construction of the possibility correspondence $\overline{\Pi}^{\uparrow}$ that satisfies $\overline{K}(\overline{E}) = (\{\omega \in \Omega \mid \overline{\Pi}^{\uparrow}(\omega) \leqslant \overline{E}\}, S(E))$ from a given knowledge operator $\overline{K}$ that satisfies the three properties. Although the construction of the possibility correspondence $\overline{\Pi}^{\uparrow}$ on a generalized state space is convoluted and different from the corresponding well-known construction in a standard state space, I omit the detailed discussions on possibility correspondences because the primitive of a

condition is Monotonicity. The second condition is *Conjunction*: for any non-empty index set X and a collection of events $(\overline{E}_x)_{x\in X}$,

$$\bigwedge_{x\in X}\overline{K}(\overline{E}_x)\leqslant\overline{K}\left(\bigwedge_{x\in X}\overline{E}_x\right).$$

Conjunction states that if the agent knows each of a collection of events then she knows its conjunction. The third condition is *Necessitation*:

$$\overline{K}(\Omega,\inf\mathcal{S})=(\Omega,\inf\mathcal{S}).$$

Necessitation states that the agent knows the entire state space.[16]

Suppose that a given knowledge operator is induced by a possibility correspondence, i.e., the induced knowledge operator satisfies Monotonicity, Conjunction, and Necessitation. Given this assumption, I summarize two results.

First, the possibility correspondence that induces the knowledge operator is "non-partitional" (precisely, satisfies generalized reflexivity and generalized transitivity) if and only if (hereafter, often abbreviated as iff) the knowledge operator satisfies the Truth Axiom and Positive Introspection.[17]

Second, Fukuda (2023) shows that the possibility correspondence that induces the knowledge operator is "partitional" (precisely, satisfies generalized reflexivity, generalized transitivity, and generalized Euclideanness) iff the knowledge operator satisfies the Truth Axiom, Positive Introspection, and *Generalized Negative Introspection*:

$$(\neg\overline{K})(\overline{E})\wedge\overline{K}(\overline{S}^{\uparrow})\leqslant\overline{K}(\neg\overline{K})(\overline{E}).$$

Generalized Negative Introspection states that if the agent does not know an event $\overline{E}$ but she knows the subspace to which it belongs then she knows that she does not know $\overline{E}$.[18] In fact, in the context of possibility-correspondence models, Generalized

model of this paper is a knowledge (or knowing-whether) operator. In fact, the main contribution of this paper is to provide an abstract generalized-state-space model of unawareness in a more tractable manner through operators. For completeness, I mention formal definitions in footnotes (Footnotes 17 and 18 for possibility correspondences on a generalized state space, and Footnotes 20, 22, and 23 for those on a standard state space). Appendix A.3 also presents a supplementary result using possibility correspondences.

[16]Technically, Necessitation can be interpreted as Conjunction when $X=\emptyset$.

[17]Technically, a possibility correspondence $\overline{\Pi}^{\uparrow}:\Omega\to\mathcal{E}$ satisfies generalized reflexivity if $\omega\in\Pi^{\uparrow}(\omega)$ for all $\omega\in\Omega$. The possibility correspondence $\overline{\Pi}^{\uparrow}:\Omega\to\mathcal{E}$ satisfies generalized transitivity if $\omega'\in\Pi^{\uparrow}(\omega)$ implies $\overline{\Pi}^{\uparrow}(\omega')\leqslant\overline{\Pi}^{\uparrow}(\omega)$. Note that the adjective "generalized" is added to each property because the property refers to the set $\Pi^{\uparrow}(\cdot)$ instead of the possibility set $\Pi(\cdot)$. This will be clear (in Footnotes 22 and 23) when the corresponding properties are defined for possibility correspondences on a standard state space.

[18]Technically, a possibility correspondence $\overline{\Pi}^{\uparrow}:\Omega\to\mathcal{E}$ satisfies generalized Euclideanness if $\omega'\in\Pi(\omega)$ implies $\Pi^{\uparrow}(\omega)\subseteq\Pi^{\uparrow}(\omega')$.

Negative Introspection is equivalent to *Weak Necessitation*:

$$\overline{U}(\overline{E}) = (\neg\overline{K})(\overline{S^{\uparrow}}) \text{ for all } \overline{E} = (E, S) \in \mathcal{E}.$$

Weak Necessitation states that the agent is unaware of an event iff she does not know its subspace.[19]

2.4.2 Properties of Unawareness and Knowledge

I briefly discuss three joint properties of knowledge and unawareness first studied by Dekel, Lipman, and Rustichini (1998). First, the model satisfies *Plausibility* if

$$\overline{U}(\overline{E}) \leqslant (\neg\overline{K})(\overline{E}) \wedge (\neg\overline{K})(\neg\overline{K})(\overline{E}) \text{ for all } \overline{E} \in \mathcal{E}.$$

Plausibility states that if the agent is unaware of an event then she does not know it and she does not know that she does not know it. I mention two settings in which Plausibility holds. Firstly, when the unawareness operator is defined as

$$\overline{U}_{\overline{K}}(\cdot) := (\neg\overline{K})(\cdot) \wedge (\neg\overline{K})^2(\cdot), \tag{2}$$

by construction the model satisfies Plausibility. The same is true when the unawareness operator is defined as $\bigwedge_{n\in\mathbb{N}}(\neg\overline{K})^n(\cdot)$. Secondly, when the model satisfies Monotonicity (on the knowledge operator) and Weak Necessitation, it can be seen that the model satisfies Plausibility.

Second, *KU Introspection* refers to:

$$\overline{KU}(\overline{E}) = (\emptyset, S(E)) \text{ for all } \overline{E} \in \mathcal{E}.$$

KU Introspection states that, for any given event, there is no state at which the agent knows that she is unaware of the event. It can be seen that KU Introspection holds under Plausibility and the Truth Axiom and Monotonicity (of $\overline{K}$).

Third, *AU Introspection* refers to:

$$\overline{U}(\cdot) \leqslant \overline{UU}(\cdot).$$

AU Introspection states that if the agent is unaware of an event then she is unaware of being unaware of it. Fukuda (2021) shows that, when the knowledge operator

[19] Three remarks are in order. First, note that while Generalized Negative Introspection is a property of the knowledge operator $\overline{K}$, Weak Necessitation is a joint property of the knowledge operator and the unawareness operator $\overline{U}$. This holds because the possibility correspondence jointly specifies the agent's knowledge and unawareness. When the knowledge operator and the unawareness operator are separately defined, the equivalence between Generalized Negative Introspection and Weak Necessitation does not necessarily hold. Second, Fukuda (2023) provides other properties on knowledge which are equivalent to Generalized Negative Introspection. Third, when the knowledge operator is induced by a possibility correspondence, generalized reflexivity, generalized transitivity, and generalized Euclideanness, respectively, characterize the Truth Axiom, Positive Introspection, and Generalized Negative Introspection.

satisfies the Truth Axiom, Positive Introspection, and Monotonicity and when the unawareness operator is defined as the two levels of lack of knowledge (i.e., through Expression (2)), the axiom of AU Introspection is equivalent to Generalized Negative Introspection. Fukuda (2023) shows that, in a possibility-correspondence model on a generalized state space where the unawareness operator is defined through the possibility correspondence, the axiom of AU Introspection is characterized by generalized Euclideanness.

2.4.3 Possibility Correspondence Models on a Standard State Space

As a special case, the framework also accommodates partitional and non-partitional possibility-correspondence models on a standard state space. As in Condition (1), a knowledge operator $K : \mathcal{D} \to \mathcal{D}$ is *induced by* a possibility correspondence $\Pi : \Omega \to \mathcal{D}$ if $K(E) = \{\omega \in \Omega \mid \Pi(\omega) \subseteq E\}$ for all $E \in \mathcal{D}$.[20]

As is well-known in such fields as logic and game theory, the knowledge operator K in a model $\langle(\Omega, \mathcal{D}), K\rangle$ is induced by a possibility correspondence iff K satisfies: (i) Monotonicity: $E \subseteq F$ implies $K(E) \subseteq K(F)$; (ii) Necessitation: $K(\Omega) = \Omega$; and (iii) Conjunction: $\bigcap_{x \in X} K(E_x) \subseteq K(\bigcap_{x \in X} E_x)$ for any non-empty index set X.[21] The knowledge operator satisfies the Truth Axiom (i.e., $K(E) \subseteq E$ for all $E \in \mathcal{D}$) iff the possibility correspondence is reflexive.[22]

The knowledge operator K is induced by a reflexive-and-transitive (non-partitional) possibility correspondence if K additionally satisfies Positive Introspection: $K(E) \subseteq KK(E)$. The operator K is induced by a partitional (i.e., reflexive, transitive, and Euclidean) possibility correspondence if K additionally satisfies *Negative Introspection*:

$$(\neg K)(E) \subseteq K(\neg K)(E) \text{ for all } E \in \mathcal{D}.$$

Note that Negative Introspection and the Truth Axiom imply Positive Introspection (e.g., Aumann, 1999).[23]

[20]If this is the case, then one can always replace Π with $\Pi_K : \Omega \to \mathcal{D}$ defined by $\Pi_K(\omega) := \bigcap_{E \in \mathcal{D}: \omega \in K(E)} E$ for all $\omega \in \Omega$ (Fukuda, 2019).

[21]See, for instance, Chellas (1980, Section 7.3) in logic and Morris (1996) in game theory. Note that, while this characterization is often shown in the context where the collection of events is the power set $\mathcal{D} = \mathcal{P}(\Omega)$, since the collection $\mathcal{D}$ reduces to a complete algebra in the framework of this paper, the characterization continues to hold.

[22]Technically, a possibility correspondence Π is reflexive if $\omega \in \Pi(\omega)$ for all $\omega \in \Omega$. As discussed in Footnote 15, I omit the detailed discussions on possibility correspondences as this paper uses a knowledge (or knowing-whether) operator as the primitive of the model. Yet, there is a vast logic literature studying the correspondence between properties of knowledge and properties of a possibility correspondence (more precisely, an accessibility relation in the context of the logic literature). See, for instance, van Benthem (2001).

[23]Technically, a possibility correspondence Π is transitive if $\omega' \in \Pi(\omega)$ implies $\Pi(\omega') \subseteq \Pi(\omega)$. It is Euclidean if $\omega' \in \Pi(\omega)$ implies $\Pi(\omega) \subseteq \Pi(\omega')$.

3 Equivalence between Knowledge and Knowing-whether

Section 3.1 provides the benchmark result, Proposition 1, which studies conditions on knowledge (or belief that may violate the Truth Axiom, if at all possible) under which a knowing-whether operator can be a primitive of the model. Section 3.2 shows that Proposition 1 does not necessarily hold for beliefs that may violate the Truth Axiom. Section 3.3 provides an example of a model which is defined through the agent's knowing-whether and unawareness operator. The subsection demonstrates that the framework of this paper may provide a simple way to construct a model.

3.1 The Benchmark Result

Fix a generalized state space $\langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha\in\mathcal{A}}, \succeq, r\rangle$. Let $\mathcal{E}$ be the collection of events. Let $\mathbb{O}_\mathcal{E}$ be the set of operators $\overline{O} : \mathcal{E} \to \mathcal{E}$ such that

$$\overline{O}(E, S(E)) := (O(E), S(E)) \text{ for all } (E, S(E)) \in \mathcal{E}. \tag{3}$$

Consider a mapping that associates, with each operator $\overline{K} \in \mathbb{O}_\mathcal{E}$, the associated "knowing-whether" operator $\overline{J}_{\overline{K}} : \mathcal{E} \to \mathcal{E}$ defined by

$$\overline{J}_{\overline{K}}(\overline{E}) := \overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{E}) \text{ for each } \overline{E} \in \mathcal{E}.$$

By construction, $\overline{J}_{\overline{K}}$ satisfies *Symmetry*:

$$\overline{J}_{\overline{K}}(\overline{E}) = \overline{J}_{\overline{K}}(\neg\overline{E}) \text{ for each } \overline{E} \in \mathcal{E}.$$

I call the mapping $\overline{K} \mapsto \overline{J}_{\overline{K}}$ the *KJ-transformation* (i.e., the transformation that maps $\overline{K}$ to $\overline{J}_{\overline{K}}$).

I study a condition on a domain of the KJ-transformation $\overline{K} \mapsto \overline{J}_{\overline{K}}$ (i.e., a subset of operators $\mathbb{O}_\mathcal{E}$) under which different knowledge operators lead to different knowing-whether operators and vice versa. I also study possible ways in which an underlying knowledge operator $\overline{K}$ is identified from the corresponding knowing-whether operator $\overline{J}_{\overline{K}}$.

To that end, I define the following. First, for any operator $\overline{J} \in \mathbb{O}_\mathcal{E}$, I define the *inverse KJ-transformation* $\overline{J} \mapsto \overline{K}_{\overline{J}}$ by

$$\overline{K}_{\overline{J}}(\overline{E}) := \overline{J}(\overline{E}) \wedge \overline{E} \text{ for each } \overline{E} \in \mathcal{E}.$$

By construction, $\overline{K}_{\overline{J}}$ satisfies the Truth Axiom.

Second, I define $\mathbb{O}_\mathcal{E}^{\text{TA}} := \{\overline{K} \in \mathbb{O}_\mathcal{E} \mid \overline{K} \text{ satisfies the Truth Axiom}\}$ and $\mathbb{O}_\mathcal{E}^{\text{Sym}} := \{\overline{J} \in \mathbb{O}_\mathcal{E} \mid \overline{J} \text{ satisfies Symmetry}\}$. The definitions are self-explanatory.

Third, a subset $\mathbb{K} \subseteq \mathbb{O}_\mathcal{E}$ is *K-closed* if, $\overline{K} \in \mathbb{K}$ implies $\overline{K}' \in \mathbb{K}$, where

$$\overline{K}'(\overline{E}) := (\overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{E})) \wedge \overline{E} \text{ for each } \overline{E} \in \mathcal{E}.$$

In other words, $\mathbb{K}$ is K-closed if $\overline{K} \in \mathbb{K}$ implies $\overline{K}_{\overline{J}_{\overline{K}}} \in \mathbb{K}$. Similarly, a subset $\mathbb{J} \subseteq \mathbb{O}_{\mathcal{E}}$ is *J-closed* if, $\overline{J} \in \mathbb{J}$ implies $\overline{J}' \in \mathbb{J}$, where

$$\overline{J}'(\overline{E}) := (\overline{J}(\overline{E}) \wedge \overline{E}) \vee (\overline{J}(\neg\overline{E}) \wedge (\neg\overline{E})) \text{ for each } \overline{E} \in \mathcal{E}.$$

In other words, $\mathbb{J}$ is J-closed if $\overline{J} \in \mathbb{J}$ implies $\overline{J}_{\overline{K}_{\overline{J}}} \in \mathbb{J}$.

With these in mind, I establish the benchmark result:

Proposition 1. *Fix a generalized state space* $\langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}, \succeq, r \rangle$.

1. *For any* $\mathbb{K} \subseteq \mathbb{O}_{\mathcal{E}}^{\mathrm{TA}}$*, the KJ-transformation* $\mathbb{K} \ni \overline{K} \mapsto \overline{J}_{\overline{K}} \in \mathbb{O}_{\mathcal{E}}^{\mathrm{Sym}}$ *is injective. Conversely, let* $\mathbb{K} \subseteq \mathbb{O}_{\mathcal{E}}$ *be K-closed. If* $\mathbb{K} \ni \overline{K} \mapsto \overline{J}_{\overline{K}} \in \mathbb{O}_{\mathcal{E}}^{\mathrm{Sym}}$ *is injective, then* $\mathbb{K} \subseteq \mathbb{O}_{\mathcal{E}}^{\mathrm{TA}}$.

2. *For any* $\mathbb{J} \subseteq \mathbb{O}_{\mathcal{E}}^{\mathrm{Sym}}$*, the inverse KJ-transformation* $\mathbb{J} \ni \overline{J} \mapsto \overline{K}_{\overline{J}} \in \mathbb{O}_{\mathcal{E}}^{\mathrm{TA}}$ *is injective. Conversely, let* $\mathbb{J} \subseteq \mathbb{O}_{\mathcal{E}}$ *be J-closed. If* $\mathbb{J} \ni \overline{J} \mapsto \overline{K}_{\overline{J}} \in \mathbb{O}_{\mathcal{E}}^{\mathrm{TA}}$ *is injective, then* $\mathbb{J} \subseteq \mathbb{O}_{\mathcal{E}}^{\mathrm{Sym}}$.

3. *The KJ-transformation* $\mathbb{O}_{\mathcal{E}}^{\mathrm{TA}} \ni \overline{K} \mapsto \overline{J}_{\overline{K}} \in \mathbb{O}_{\mathcal{E}}^{\mathrm{Sym}}$ *is bijective. The inverse is* $\mathbb{O}_{\mathcal{E}}^{\mathrm{Sym}} \ni \overline{J} \mapsto \overline{K}_{\overline{J}} \in \mathbb{O}_{\mathcal{E}}^{\mathrm{TA}}$*. Consequently, if* $\overline{K} \in \mathbb{O}_{\mathcal{E}}^{\mathrm{TA}}$ *then* $\overline{K} = \overline{K}_{\overline{J}_{\overline{K}}}$*. Also, if* $\overline{J} \in \mathbb{O}_{\mathcal{E}}^{\mathrm{Sym}}$ *then* $\overline{J} = \overline{J}_{\overline{K}_{\overline{J}}}$.

The first part of Proposition 1 states that different knowledge operators lead to different knowing-whether operators under the Truth Axiom on knowledge. The second part states that different knowing-whether operators lead to different knowledge operators under Symmetry on knowing-whether operators. Thus, it states the need for restricting attention to symmetric knowing-whether operators.

The third part justifies the definition of the knowledge operator $\overline{K}_{\overline{J}}$ from a knowing-whether operator $\overline{J}$ as the inverse of the KJ-transformation. Let $\overline{K}^{\circ}$ be a knowledge operator satisfying the Truth Axiom generated from a symmetric operator $\overline{J}$, not necessarily $\overline{K}_{\overline{J}}$. However, if the knowing-whether operator $\overline{J}_{\overline{K}^{\circ}}$ generated from $\overline{K}^{\circ}$ reduces to $\overline{J}$ (i.e., $\overline{J}(\overline{E}) = \overline{K}^{\circ}(\overline{E}) \vee \overline{K}^{\circ}(\neg\overline{E})$ for all $\overline{E} \in \mathcal{E}$), then it must be the case that $\overline{K}^{\circ}(\overline{E}) = \overline{J}(\overline{E}) \wedge \overline{E}$ for all $\overline{E} \in \mathcal{E}$. Hence, given a symmetric knowing-whether operator $\overline{J}$, there is only one way to define the knowledge operator $\overline{K}^{\circ} = \overline{K}_{\overline{J}}$ whose knowing-whether operator reduces to the original operator $\overline{J}$.

The rest of this subsection discusses Proposition 1 with respect to the previous related results in logic. As discussed in the Introduction, interpreted in the context of the standard-state-space possibility-correspondence models, Montgomery and Routley (1966, Theorem 3) show the equivalence between a knowledge operator and a knowing-whether operator given the Truth Axiom of knowledge (recall also the second remark in Footnote 5). The transformations between knowledge and knowing-whether in this paper follow from theirs. Fan and van Ditmarsch (2015, Proposition 5) generalize the equivalence by dropping the three conditions of Monotonicity, Necessitation, and Conjunction within the framework of neighborhood systems on a standard state space.[24]

One may think that their results directly apply to the generalized-state-space framework of this paper because knowledge and knowing-whether operators are operators on the collection of events and the collection of events $\mathcal{E}$ forms a lattice (with respect to the partial order $\leqslant$). However, as discussed in Section 2.2, since each event $\overline{E}$ consists of the set of states at which the event holds and the subspace to which the event belongs, the following subtle considerations exist: (i) the lattice $\langle \mathcal{E}, \leqslant \rangle$ is not a complemented lattice with respect to the operation of negation $\neg$; and (ii) the operation of disjunction is generally different from the supremum in the lattice $\langle \mathcal{E}, \leqslant \rangle$.

As the knowing-whether operator is defined using the operation of disjunction, I show below through a simple counterexample that, without the assumptions that knowledge and knowing-whether operators map an event $(E, S(E))$ to another event which resides in the same subspace $S(E)$, Proposition 1 may not hold.[25] Thus, the prior papers in the logic literature do not directly apply to Proposition 1.

Example 1. Let $\mathcal{S} = \{S, S'\}$ with $S \succeq S'$. Let $\mathcal{D} = \mathcal{P}(S)$ and $\mathcal{D}' = \mathcal{P}(S')$. Let $r : S \to S'$ be a surjection. While not necessary, for ease of exposition, let $S = \{\omega_1, \omega_2\}$ and $S' = \{\omega_3\}$.

Define $\overline{K} : \mathcal{E} \to \mathcal{E}$ as follows: $\overline{K}(E, S) := (E, S)$ for any $E \in \mathcal{D}$; and

$$\overline{K}(B^{\uparrow}, S') := \begin{cases} (\emptyset, S) & \text{if } B = \emptyset \\ (B^{\uparrow}, S') & \text{if } B \in \mathcal{D}' \setminus \{\emptyset\} \end{cases}.$$

By construction, $\overline{K}$ satisfies the Truth Axiom. However, the subspace to which $\overline{K}(\emptyset, S')$ belongs is S, not S'. For the event $((S')^{\uparrow}, S')$, the induced knowing-whether operator $\overline{J}_{\overline{K}}$ satisfies:

$$\begin{aligned} \overline{J}_{\overline{K}}((S')^{\uparrow}, S') &= \overline{K}((S')^{\uparrow}, S') \vee \overline{K}(\emptyset, S') \\ &= ((S')^{\uparrow}, S') \vee (\emptyset, S) \\ &= (S, S). \end{aligned}$$

Consequently, $\overline{K}_{\overline{J}_{\overline{K}}} \neq \overline{K}$, as:

$$\overline{K}_{\overline{J}_{\overline{K}}}((S')^{\uparrow}, S') = (S, S) \wedge ((S')^{\uparrow}, S') = (S, S) \neq ((S')^{\uparrow}, S') = \overline{K}((S')^{\uparrow}, S').$$

For the other direction, define $\overline{J} : \mathcal{E} \to \mathcal{E}$ as follows: $\overline{J}(B^{\uparrow}, S') = ((S')^{\uparrow}, S')$ for all $B \in \mathcal{D}'$; and for all $E \in \mathcal{D}$,

$$\overline{J}(E, S) = \begin{cases} ((S')^{\uparrow}, S') & \text{if } E \in \{\emptyset, S\} \\ (S, S) & \text{otherwise} \end{cases}.$$

[24]Appendix B discusses neighborhood systems and extends the framework of neighborhood systems to a generalized state space.

[25]The proof of Proposition 1 in the Appendix explicitly states where these assumptions are used.

By construction, $\overline{J}$ satisfies Symmetry. However, for $E \in \{\emptyset, S\}$ in the subspace S, the subspace to which $\overline{J}(E, S)$ belongs is S', not S. One has:

$$\overline{K}_{\overline{J}}(S, S) = ((S')^{\uparrow}, S') \wedge (S, S) = (S, S) \text{ and } \overline{K}_{\overline{J}}(\emptyset, S) = ((S')^{\uparrow}, S') \wedge (\emptyset, S) = (\emptyset, S).$$

Thus, $\overline{J}_{\overline{K}_{\overline{J}}} \neq \overline{J}$, as:

$$\overline{J}_{\overline{K}_{\overline{J}}}(S, S) = \overline{K}_{\overline{J}}(S, S) \vee \overline{K}_{\overline{J}}(\emptyset, S) = (S, S) \neq ((S')^{\uparrow}, S') = \overline{J}(S, S).$$

In sum, Condition (3) is crucial in establishing Proposition 1, and the proposition is not a direct implication of the previous literature. On the one hand, at a conceptual level, it would be a natural requirement that an agent's knowledge about events in a subspace S is described within the subspace S. On the other hand, at a technical level, Condition (3) is important in establishing the equivalence between knowledge and knowing-whether. □

To conclude this subsection, note that, as discussed in Footnote 8, Appendix C studies an algebraic structure behind Proposition 1 in the context closer to the logic literature, specifically, Montgomery and Routley (1966). While the underlying algebraic structure (i.e., Lemma 1 in Appendix C) turns out to be rather simple, in order to apply Lemma 1 to Proposition 1, one eventually needs to utilize the structure of the generalized state space (outlined in Section 2) and Condition (3) as in the proof of Proposition 1.

3.2 The Role of the Truth Axiom

Proposition 1 suggests the importance of the Truth Axiom for the equivalence between knowledge and knowing-whether. Thus, given the "knowing-"whether operator $\overline{J}_{\overline{K}}$ such that an underlying operator $\overline{K}$ violates the Truth Axiom, it may not necessarily be possible to recover the underlying operator $\overline{K}$. I show that simple examples exist even in the context of a standard state space.[26]

3.2.1 The Case with Qualitative Belief

Let $\Omega = \{\omega_1, \omega_2\}$, and let $\mathcal{D} = \mathcal{P}(\Omega)$. Suppose that the agent's qualitative belief is given by the following possibility correspondence: $\Pi(\cdot) = \{\omega_1\}$. Denoting by B the corresponding qualitative belief operator, i.e., $B(E) = \{\omega \in \Omega \mid \Pi(\omega) \subseteq E\}$,

$$B(E) = \begin{cases} \Omega & \text{if } E \in \{\{\omega_1\}, \Omega\} \\ \emptyset & \text{otherwise} \end{cases}.$$

[26]While it would be possible to axiomatize a model of probabilistic beliefs and unawareness in terms of probability p-belief operators, it would call for a large amount of new notations. Fortunately, examples exist in the context of standard state spaces. Moreover, to the best of my knowledge, such axiomatization does not pre-exist in the literature on unawareness.

It can be seen that B satisfies Positive Introspection (i.e., $B(\cdot) \subseteq BB(\cdot)$), Negative Introspection (i.e., $(\neg B)(\cdot) \subseteq B(\neg B)(\cdot)$), and Consistency (i.e., $B(E) \cap B(E^c) = \emptyset$ for all $E \in \mathcal{D}$).[27] Now, the operator B', which is defined by $B'(E) = (B(E) \cup B(E^c)) \cap E$, is induced by a partitional possibility correspondence Π': $\Pi'(\omega) = \{\omega\}$. That is, $B'(E) = E$ for all $E \in \mathcal{D}$. Hence, an underlying qualitative belief operator B is not identified from the believing-whether operator J_B (i.e., $J_B(E) := B(E) \cup B(E^c)$). A similar example is presented in Fan, Wang, and van Ditmarsch (2015) in their logical framework.[28]

3.2.2 The Case with Probabilistic Belief

Likewise, if the outside analysts consider p-belief operators $(B^p)_{p \in [0,1]}$ (Monderer and Samet, 1989) that dictate probabilistic beliefs, then the analysts may not be able to recover the underlying operator B^p from J_{B^p}. In the context of a standard state space, letting $(\Omega, \mathcal{D})$ be a measurable space, the agent's probabilistic beliefs are represented by a type mapping. The type mapping t associates, with each state $\omega \in \Omega$, a probability distribution $t(\omega) \in \Delta(\Omega)$ on $(\Omega, \mathcal{D})$ that dictates the agent's probabilistic beliefs at that state. Given the type mapping t, for each probability $p \in [0,1]$, p-belief operator associates, with each event E, the event that the agent believes E with probability at least $p \in [0,1]$:

$$B^p(E) := \{\omega \in \Omega \mid t(\omega)(E) \geq p\} \text{ for each } E \in \mathcal{D}.^{29}$$

Under a well-known measurability condition on t (e.g., Heifetz and Samet, 1998b), each $B^p : \mathcal{D} \to \mathcal{D}$ is well-defined.

As a simple example, let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2\}, \mathcal{P}(\Omega))$, and let each $t(\omega) \in \Delta(\Omega)$ be given as $t(\omega_j)(\{\omega_j\}) = \frac{2}{3}$ for each $j \in \{1, 2\}$. Then,

$$J_{B^p}(E) = \begin{cases} \Omega & \text{if } p \in [0, \frac{2}{3}] \text{ and } E \in \mathcal{D} \\ \Omega & \text{if } p \in (\frac{2}{3}, 1] \text{ and } E \in \{\emptyset, \Omega\} \\ \emptyset & \text{if } p \in (\frac{2}{3}, 1] \text{ and } E \in \{\{\omega_1\}, \{\omega_2\}\} \end{cases}.$$

It can be seen that the operators $(J_{B_p})_{p \in [0,1]}$ are induced by another type mapping t' defined as $t'(\omega)(\{\omega_1\}) = \frac{2}{3}$ for each $\omega \in \Omega$.

3.3 The Speculative-Trade Example

This subsection demonstrates that the framework of this paper may provide a simple way to construct a model of knowledge and unawareness. To that end, I revisit the

[27] In other words, the possibility correspondence Π is transitive (if $\omega' \in \Pi(\omega)$ then $\Pi(\omega') \subseteq \Pi(\omega)$), Euclidean (if $\omega' \in \Pi(\omega)$ then $\Pi(\omega) \subseteq \Pi(\omega')$), and serial ($\Pi(\cdot) \neq \emptyset$).

[28] See also Fan and van Ditmarsch (2015, Propositions 2, 3, and 4) for the comparison of expressive power between models with a belief operator and a "believing-whether" operator.

[29] As Samet (2000) shows, a belief (type) space can be introduced solely in terms of p-belief operators $(B^p)_{p \in [0,1]}$ instead of a type mapping t.

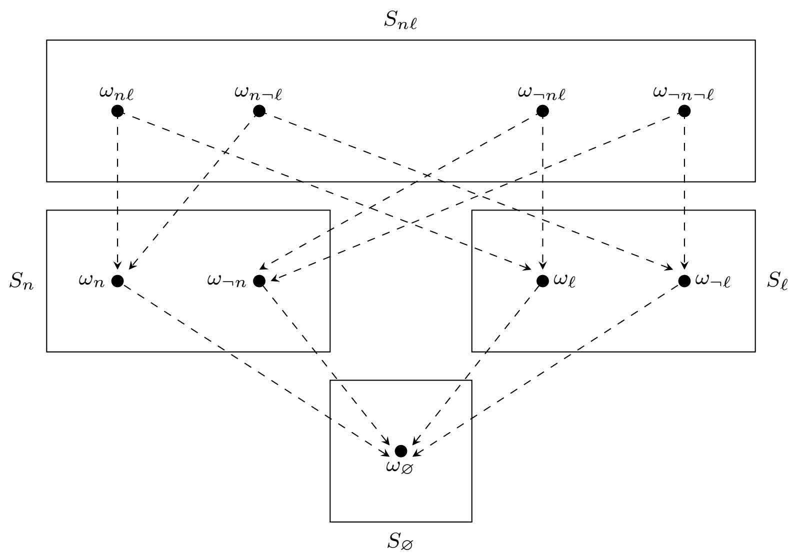

Figure 1: Subspaces and Projections in the Speculative-Trade Example

speculative-trade example in the Introduction.

As discussed in the Introduction, there are two agents, the owner o and the buyer b. There are four subspaces: $\mathcal{S} = \{S_{n\ell}, S_n, S_\ell, S_\varnothing\}$, where $S_{n\ell} = \{\omega_{n\ell}, \omega_{n\neg\ell}, \omega_{\neg n\ell}, \omega_{\neg n\neg\ell}\}$, $S_n = \{\omega_n, \omega_{\neg n}\}$, $S_\ell = \{\omega_\ell, \omega_{\neg\ell}\}$, and $S_\varnothing = \{\omega_\varnothing\}$. Each subspace $S_\alpha \in \mathcal{S}$ is endowed with the power set $\mathcal{D}_\alpha = \mathcal{P}(S_\alpha)$. The subspaces are ranked in a way such that the dashed arrows in Figure 1 depict the projections (note that the compositions of projections and the identity projections are omitted).

I define the knowing-whether and unawareness operators of both agents. Consider the owner. The idea is that, while the owner is aware of a possible lawsuit ℓ, he is unaware of a potential innovation n. On the subspace $S_{n\ell}$, since the owner is unaware and consequently ignorant of n, for any event $(E, S_{n\ell}) \in \mathcal{E}$,

$$\overline{J}_o(E, S_{n\ell}) = (\emptyset, S_{n\ell}) \text{ and } \overline{U}_o(E, S_{n\ell}) = (S_{n\ell}, S_{n\ell}).$$

Likewise, on the subspace S_n, since the owner is unaware and consequently ignorant of n, for any event $(E, S_n) \in \mathcal{E}$,

$$\overline{J}_o(E, S_n) = (\emptyset, S_n) \text{ and } \overline{U}_o(E, S_n) = (S_n^\uparrow, S_n).$$

In contrast, on the subspace S_ℓ, since the owner knows whether ℓ occurs and consequently he is aware of ℓ, for any event $(E, S_\ell) \in \mathcal{E}$,

$$\overline{J}_o(E, S_\ell) = (S_\ell^\uparrow, S_\ell) \text{ and } \overline{U}_o(E, S_\ell) = (\emptyset, S_\ell).$$

On the subspace $S_\varnothing$, since the owner knows whether ℓ occurs and consequently he is aware of ℓ, for any event $(E, S_\varnothing) \in \mathcal{E}$,

$$\overline{J}_o(E, S_\varnothing) = (S_\varnothing^\uparrow, S_\varnothing) \text{ and } \overline{U}_o(E, S_\varnothing) = (\emptyset, S_\varnothing).$$

The buyer, in contrast, is aware of the innovation but unaware of the lawsuit. Similarly to the case of the owner, for any event $(E, S) \in \mathcal{E}$ with $S \in \{S_{n\ell}, S_\ell\}$,

$$\overline{J}_b(E, S) = (\emptyset, S) \text{ and } \overline{U}_b(E, S) = \overline{S}^{\uparrow}.$$

For any event $(E, S) \in \mathcal{E}$ with $S \in \{S_n, S_\varnothing\}$,

$$\overline{J}_b(E, S) = \overline{S}^{\uparrow} \text{ and } \overline{U}_b(E, S) = (\emptyset, S).$$

Two remarks are in order. First, for the interested reader, Appendix A.3 presents the agents' possibility correspondences that induce their knowing-whether and unawareness operators of this example. Second, I briefly and informally mention that this is an example of speculative trades.[30] As in Heifetz, Meier, and Schipper (2006), consider the value of the firm. Suppose that the value is normalized by 1 at state $\omega_\varnothing$. Suppose further that if an innovation obtains then it increases the value of the firm by 0.1, while if a lawsuit obtains then it decreases the value of the firm by 0.1. Informally, at any state on the subspace $S_{n\ell}$, the awareness of the owner, who is unaware of the innovation but who knows whether the lawsuit obtains, is represented on the subspace S_ℓ. That is, whenever the lawsuit occurs, the owner values the firm at 0.9, and otherwise he values the firm at 1. In contrast, the awareness of the buyer, who is unaware of the lawsuit but who knows whether the innovation obtains, is represented on the subspace S_n. That is, whenever the innovation occurs, the buyer values the firm at 1.1, and otherwise she values the firm at 1. Thus, at any state in $S_{n\ell}$, they would strictly prefer to trade. Also, at any state in any subspace, both agents are willing to trade, and thus there is common knowledge of willingness to trade. While the main focus of this paper is not speculative trades, the point of this paper is to provide a general and intuitive framework to represent agents' knowledge and unawareness.

4 Applications

This section discusses applications of Proposition 1. Section 4.1 summarizes the main implication of Proposition 1, i.e., the main implication of the axiomatization of an unawareness structure by knowing-whether operators. Section 4.2 shows that the same analyses can be carried out from the ignorance operator instead of the knowing-whether operator. Section 4.3 discusses properties of unawareness when unawareness is defined through (not) knowing-whether. Section 4.4 shows that one can introduce common knowledge in the framework of this paper.

[30] See Galanis (2016, 2018) and Heifetz, Meier, and Schipper (2013) for speculative trades in an unawareness structure.

4.1 An Axiomatization of a Model by a Knowing-whether Operator

Proposition 1 implies that one can introduce a model by a symmetric knowing-whether operator $\overline{J}$: $\langle\langle(S_\alpha, \mathcal{D}_\alpha)_{\alpha\in\mathcal{A}}, \succeq, r\rangle, (\overline{J}, \overline{U})\rangle$. The resulting knowledge operator $\overline{K}_{\overline{J}}$, by construction, satisfies the Truth Axiom. By abusing the terminologies, the following represent the other properties of the knowledge operator $\overline{K}_{\overline{J}}$ in terms of $\overline{J}$.

1. $\overline{K}_{\overline{J}}$-Monotonicity: $\overline{E} \leqslant \overline{F}$ implies $\overline{J}(\overline{E}) \wedge \overline{E} \leqslant \overline{J}(\overline{F})$.
2. $\overline{K}_{\overline{J}}$-Conjunction: $\bigwedge_{x\in X}(\overline{J}_{\overline{K}}(\overline{E}_x) \wedge \overline{E}_x) \leqslant \overline{J}_{\overline{K}}(\bigwedge_{x\in X} \overline{E}_x)$.
3. $\overline{K}_{\overline{J}}$-Necessitation: $\overline{J}(\Omega, \sup \mathcal{S}) = (\Omega, \sup \mathcal{S})$.
4. $\overline{K}_{\overline{J}}$-Positive Introspection: $\overline{J}(\overline{E}) \wedge \overline{E} \leqslant \overline{J}(\overline{J}(\overline{E}) \wedge \overline{E})$.
5. $\overline{K}_J$-Generalized Negative Introspection: $\neg(\overline{J}(\overline{E}) \wedge \overline{E}) \wedge J(\overline{S}^{\uparrow}) \leqslant \overline{J}(\overline{J}(\overline{E}) \wedge \overline{E})$.

These properties are readily interpreted given that the agent knows an event $\overline{E}$ iff she knows whether $\overline{E}$ occurs and indeed $\overline{E}$ occurs. For instance, Monotonicity is rewritten in a way such that, for any events $\overline{E}$ and $\overline{F}$ such that $\overline{E}$ implies $\overline{F}$, if the agent knows whether an event $\overline{E}$ occurs and the event $\overline{E}$ occurs then she knows whether $\overline{F}$ occurs.

The joint properties of knowledge and unawareness can be expressed in terms of the knowing-whether operator $\overline{J}$ as follows.

Proposition 2. *Suppose that $\overline{J}$ satisfies Symmetry so that a given model can be introduced by $\overline{J}$. Then, the following hold.*

1. *Weak Necessitation is equivalent to: $\overline{U}(\overline{E}) = (\neg\overline{J})(\overline{S}^{\uparrow})$.*
2. *Under $\overline{K}_{\overline{J}}$-Positive Introspection, Plausibility is equivalent to: $\overline{U}(\cdot) \leqslant (\neg\overline{J})\overline{K}_{\overline{J}}(\cdot)$.*
3. *KU Introspection is equivalent to JU Introspection: $\overline{U}(\cdot) \leqslant (\neg\overline{J})\overline{U}(\cdot)$.*

The first part of Proposition 2 restates Weak Necessitation as follows: the agent is unaware of an event iff she does not know whether the subspace (to which the event belongs) occurs.

The second part states that, when $\overline{K}_{\overline{J}}$ satisfies Positive Introspection, Plausibility holds iff the following holds: if the agent is unaware of an event then she does not know whether she knows the event. Section 4.3 studies properties of unawareness when it is defined from (not) knowing-whether.

The third part states that KU Introspection holds iff the following holds: if the agent is unaware of an event then she does not know whether she is unaware of the event. The next subsection studies the notion of not knowing-whether.

4.1.1 Axiomatizing Possibility Correspondence Models

Here, I axiomatize properties of possibility correspondences on a generalized state space. The next proposition axiomatizes possibility-correspondence models based on a knowing-whether operator $\overline{J}$. To obtain axiomatizations associated with the knowing-whether operator $\overline{J}$ instead of the derived knowledge operator $\overline{K}_{\overline{J}}$, the proposition replaces $\overline{K}_{\overline{J}}$-Conjunction, $\overline{K}_{\overline{J}}$-Positive Introspection, and $\overline{K}_{\overline{J}}$-Generalized Negative Introspection with Conjunction, $\overline{J}$-Introspection, and $\overline{J}$-Reflection, respectively.

Proposition 3. *The following axioms on $\overline{J}$ characterize a possibility-correspondence model such that the induced knowledge operator satisfies the Truth Axiom and Positive Introspection (i.e., the possibility correspondence satisfies generalized reflexivity and generalized transitivity):*

1. *Symmetry;*
2. *$\overline{K}_{\overline{J}}$-Monotonicity;*
3. *$\overline{K}_{\overline{J}}$-Necessitation;*
4. *Conjunction: $\bigwedge_{x \in X} \overline{J}(\overline{E}_x) \leqslant \overline{J}(\bigwedge_{x \in X} \overline{E}_x)$; and*
5. *$\overline{J}$-Introspection: $\overline{J}(\overline{E}) \leqslant \overline{J}\overline{J}(\overline{E})$.*

Moreover, a possibility-correspondence model such that the induced knowledge operator additionally satisfies Generalized Negative Introspection is characterized by $\overline{J}$-Reflection in addition to the properties above:

6. *$\overline{J}$-Reflection: $\overline{J}\overline{J}(E) = \overline{J}(\overline{S}^{\uparrow})$.*

I briefly discuss the (new) properties of the knowing-whether operator $\overline{J}$ in Proposition 3. Conjunction (of $\overline{J}$) states that if the agent knows whether $\overline{E}_x$ obtains for each $x \in X$, then she knows whether its conjunction $\bigwedge_{x \in X} \overline{E}_x$ obtains. Under $\overline{K}_{\overline{J}}$-Monotonicity, Conjunction and $\overline{K}_{\overline{J}}$-Conjunction are equivalent.

Next, $\overline{J}$-Reflection corresponds to Weak Necessitation. $\overline{J}$-Reflection states that, for any event $\overline{E} = (E, S)$, the agent knows whether she knows whether $\overline{E}$ obtains iff she knows whether the subspace $\overline{S}^{\uparrow}$ obtains. It can be seen that, under $\overline{K}_{\overline{J}}$-Monotonicity, $\overline{J}$-Reflection implies $\overline{J}$-Introspection.

Next, $\overline{J}$-Introspection (i.e., Positive Introspection for $\overline{J}$) states that if the agent knows whether $\overline{E}$ obtains then she knows whether she knows whether $\overline{E}$ obtains. Two remarks are in order. First, $\overline{J}$-Introspection is generally weaker than $\overline{K}_{\overline{J}}$-Positive Introspection. Remark 1 below formalizes this point.

Remark 1. To formalize the sense in which $\overline{J}$-Introspection is generally weaker than $\overline{K}_{\overline{J}}$-Positive Introspection, assume that $\overline{J}$ satisfies Symmetry and K_J-Monotonicity.

E	$J(E)$	$J^n(E)(n \geq 2)$	$J(E) \cap E$	$J(J(E) \cap E)$
$\emptyset$	Ω	Ω	$\emptyset$	Ω
$\{\omega_1\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\emptyset$	Ω
$\{\omega_2\}$	Ω	Ω	$\{\omega_2\}$	Ω
$\{\omega_3\}$	$\{\omega_1, \omega_3\}$	Ω	$\{\omega_3\}$	$\{\omega_1, \omega_3\}$
$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	Ω	$\{\omega_1\}$	$\{\omega_2, \omega_3\}$
$\{\omega_1, \omega_3\}$	Ω	Ω	$\{\omega_1, \omega_3\}$	Ω
$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$
Ω	Ω	Ω	Ω	Ω

Table 1: J-Introspection does not necessairly imply K_J-Positive Introspection (Remark 1).

1. Under this assumption, $\overline{K}_{\overline{J}}$-Positive Introspection implies $\overline{J}$-Introspection. The proof is in Appendix A.2.

2. For the converse, I provide an example in the context of a standard state space in which $\overline{J}$-Introspection does not imply $\overline{K}_{\overline{J}}$-Positive Introspection.

   Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$, and let J be as in Table 1. By inspection, it can be seen from Table 1 that J satisfies Symmetry and K_J-Monotonicity. However, J fails K_J-Conjunction.[31] By inspection (i.e., by comparing the second and third columns of the table), J-Introspection holds. However, K_J-Positive Introspection fails: for $E = \{\omega_1, \omega_2\}$, $J(E) \cap E = \{\omega_1\} \not\subseteq \{\omega_2, \omega_3\} = J(J(E) \cap E)$.

Second, $\overline{J}$-Introspection does not necessarily hold with equality. If the agent knows whether she knows whether $\overline{E}$ obtains, then it is not necessarily the case that she knows whether $\overline{E}$ obtains. The following proposition states (i) that $\overline{J}^2 = \overline{J}^n$ for any $n \geq 2$ and (ii) that $\overline{J} = \overline{JJ}$ generically fails in that $\overline{J} = \overline{JJ}$ iff the agent knows everything true: $\overline{K}_{\overline{J}}(\overline{E}) = \overline{E}$ for all $\overline{E} \in \mathcal{E}$. The first statement implies that if $\overline{K}$ satisfies the Truth Axiom, Monotonicity, and Positive Introspection, then $\overline{J}_{\overline{K}}^2 = \overline{J}_{\overline{K}}^n$ for any $n \geq 2$.

Proposition 4. *Suppose that $\overline{J}$ satisfies Symmetry so that a given model can be introduced by $\overline{J}$. Moreover, suppose $\overline{K}_{\overline{J}}$-Monotonicity and $\overline{K}_{\overline{J}}$-Positive Introspection. Then, the following hold.*

1. $\overline{JJ} = \overline{JJJ}$.

2. $\overline{JJ} = \overline{J}$ *iff* $\overline{E} = \overline{K}_{\overline{J}}(\overline{E})$ *for every* $\overline{E} \in \mathcal{E}$.

[31] Consider $E = \{\omega_1, \omega_2\}$ and $F = \{\omega_1, \omega_3\}$. While $K_J(E) \cap K_J(F) = \{\omega_1\}$, $K_J(E \cap F) = \emptyset$.

4.2 Ignorance

The agent is *ignorant* of an event $\overline{E}$ if she does not know whether $\overline{E}$ holds. Denote by $\overline{\partial} := (\neg\overline{J})$ the ignorance operator. Since the notion of ignorance is mathematically the dual of that of knowing-whether, the analyses in this paper simply extend to the case in which the ignorance operator instead of a knowing-whether operator is a primitive.

By definition, the ignorance operator $\overline{\partial}$ satisfies Symmetry. Proposition 1 suggests knowledge is recoverable from ignorance. A knowledge operator $\overline{K}$ satisfying the Truth Axiom induces

$$\overline{\partial}_{\overline{K}}(\overline{E}) := (\neg\overline{K})(\overline{E}) \wedge (\neg\overline{K})(\neg\overline{E}).$$

Conversely, a symmetric operator $\overline{\partial}$ induces

$$\overline{K}_{\overline{\partial}}(\overline{E}) := \overline{E} \wedge (\neg\overline{\partial})(\overline{E}).$$

The (other) properties in Proposition 3 can be expressed as follows:

1. $\overline{K}_{\overline{\partial}}$-Monotonicity: If $\overline{E} \leqslant \overline{F}$ then $\overline{\partial}(\overline{F}) \leqslant (\neg\overline{E}) \vee \overline{\partial}(\overline{E})$.
2. $\overline{K}_{\overline{\partial}}$-Necessitation: $\overline{\partial}(\Omega, \inf \mathcal{S}) = (\emptyset, \inf \mathcal{S})$.
3. Disjunction: $\overline{\partial}(\bigvee_{x\in X} \overline{E}_x) \leq \bigvee_{x\in X} \overline{\partial}(\overline{E}_x)$.
4. $\overline{\partial}$-Introspection: $\overline{\partial}\overline{\partial}(\overline{E}) \leqslant \overline{\partial}(\overline{E})$.
5. $\overline{\partial}$-Reflection: $\overline{\partial}\overline{\partial}(\overline{E}) = \overline{\partial}(\overline{S}^{\uparrow})$.

Note that the equivalence between properties of the knowing-whether and ignorance operator hinges on the assumptions that these operators satisfy Symmetry. The proof for this result is omitted.[32]

In contrast to $\overline{J}$-Introspection, $\overline{\partial}$-Introspection states that if the agent is ignorant of being ignorant of an event then she is ignorant of the event. Next, $\overline{\partial}$-Reflection states that the agent is ignorant of being ignorant of an event $\overline{E}$ iff she is ignorant of the subspace $\overline{S}^{\uparrow}$.

In the context of standard state spaces where the collection of events is the power set of an underlying state space, it is well-known, especially in the field of logic,

[32]Here I provide the proof of the fact that Conjunction (of $\overline{J}$) implies Disjunction (of $\overline{\partial}$). Let $\overline{F}_x = (\neg\overline{E}_x)$. By Conjunction, $\overline{J}\left(\bigwedge_{x\in X}\overline{F}_x\right) \leqslant \bigwedge_{x\in X}\overline{F}_x$. Taking the operation of negation on both sides, $\bigvee_{x\in X}\overline{E}_x \leqslant (\neg\overline{J})\left(\bigwedge_{x\in X}\overline{F}_x\right)$. Since $\overline{J}$ (or $\overline{\partial}$) satisfies Symmetry,

$$(\neg\overline{J})\left(\bigwedge_{x\in X}\overline{F}_x\right) = (\neg\overline{J})\left(\neg\bigwedge_{x\in X}\overline{F}_x\right) = (\neg\overline{J})\left(\bigvee_{x\in X}\overline{E}_x\right) = \overline{\partial}\left(\bigvee_{x\in X}\overline{E}_x\right).$$

The converse is similarly shown.

that the properties of a knowledge operator K are closely related to the properties of an interior operator associated with a topology. The topology is the collection of self-evident events $\{E \in \mathcal{P}(\Omega) \mid E \subseteq K(E)\}$.[33] Then, the ignorance operator ∂_K defined by $\partial_K(E) := (\neg K)(E) \cap (\neg K)(\neg E)$ is the boundary operator associated with the topology. Thus, the axioms that characterize the ignorance operator correspond to those that characterize the boundary operator associated with the topology (that characterizes the agent's knowledge). Thus, Symmetry, K_∂-Necessitation, Disjunction (when $X = \{1, 2\}$), K_∂-Monotonicity, and ∂-Introspection characterize a boundary operator.[34] In the context of standard state spaces, these axioms can be written as follows:

0. Symmetry: $\partial(E) = \partial(E^c)$.
1. K_∂-Monotonicity: If $E \subseteq F$ then $\partial(F) \subseteq E^c \cup \partial(E)$.
2. K_∂-Necessitation: $\partial(\Omega) = \emptyset$.
3. Disjunction: $\partial(E_1 \cup E_2) \subseteq \partial(E_1) \cup \partial(E_2)$.
4. ∂-Introspection: $\partial\partial(E) \subseteq \partial(E)$.

4.3 Unawareness Derived from (Not) Knowing-whether

When the notion of unawareness is defined as the lack of knowledge instead of the lack of conception, i.e., the agent is unaware of an event if she does not know it and she does not know that she does not know it, one may define unawareness from (not) knowing-whether as follows: the agent is unaware of an event when she does not know whether she knows the event. Indeed, when the knowledge operator $\overline{K}$ satisfies the Truth Axiom and Positive Introspection,

$$(\neg\overline{J})(\overline{K}(\overline{E})) = (\neg\overline{K})\overline{K}(\overline{E}) \wedge (\neg\overline{K})(\neg\overline{K})(\overline{E}) = (\neg\overline{K})(\overline{E}) \wedge (\neg\overline{K})(\neg\overline{K})(\overline{E}).$$

Proposition 1 shows that a knowing-whether model $\langle\langle(S_\alpha, \mathcal{D}_\alpha)_{\alpha\in\mathcal{A}}, \succeq, r\rangle, \overline{J}\rangle$ can describe the knowledge and unawareness of the agent: the knowledge operator is represented by $\overline{K}_{\overline{J}}(\overline{E}) = \overline{J}(\overline{E}) \wedge \overline{E}$ and the unawareness operator by $\overline{U}_{\overline{J}}(\overline{E}) = (\neg\overline{J})(\overline{J}(\overline{E}) \wedge \overline{E})$.

Unlike knowing-whether, however, an unawareness operator cannot necessarily identify the underlying knowledge operator. As a simple example, if K is derived

[33] Consider a knowledge operator K that is induced by a reflexive-and-transitive possibility correspondence. Then, the knowledge operator satisfies such properties as: (i) Necessitation $K(\Omega) = \Omega$; (ii) Monotonicity $E \subseteq F$ implies $K(E) \subseteq K(F)$; (iii) the Truth Axiom: $K(E) \subseteq E$; (iv) Positive Introspection: $K(E) \subseteq KK(E)$; and Finite Conjunction: $K(E) \cap K(F) \subseteq K(E \cap F)$. (Technically, the knowledge operator satisfies the axiom of Conjunction: $\bigcap_{x\in X} K(E_x) \subseteq K\left(\bigcap_{x\in X} E_x\right)$.)

[34] See, for example, Gabai (1964) and Pervin (1964). As mentioned in Willard (2004), the characterization of a topology by a boundary operator seems to be less-known.

from a partition on a standard state space, then K satisfies Negative Introspection. Then, $U_K(\cdot) = (\neg K)(\cdot) \cap (\neg K)^2(\cdot) = \emptyset$. Thus, any knowledge operator induced by a partition leads to the trivial unawareness operator.

I make a remark on a non-trivial form of unawareness when unawareness is defined as the lack of knowledge as in this subsection. In a standard state space model, Chen, Ely, and Luo (2012) show that AU Introspection is equivalent to Negative Introspection. Using the notations of this paper, AU Introspection is written as $JK_J U_{K_J}(\cdot) \subseteq JK_J(\cdot)$. Thus, it is equivalent to Negative Introspection, $JK_J(\cdot) = \Omega$, under KU Introspection $K_J U_{K_J}(\cdot) = \emptyset$ (which follows from, for instance, the Truth Axiom and Monotonicity of K_J) and $J(\emptyset) = \Omega$ (which follows from Symmetry and Necessitation).

4.4 Common Knowledge

As the notion of knowing-whether is also useful in the multi-agent context, this subsection shows how to introduce the notion of common knowledge on a generalized-state-space model when the individual agents' knowledge operators only satisfy the Truth Axiom.[35] Thus, for instance, agents' knowledge may fail Monotonicity.

Let I be a non-empty set of agents, and consider a model $\mathcal{M} := \langle\langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}, \succeq, r\rangle, (\overline{J}_i)_{i \in I}\rangle$, where each $\overline{J}_i$ is agent i's knowing-whether operator on the generalized state space.

Call an event $\overline{E}$ a *common basis* among I if $\overline{E} \leqslant \overline{F}$ implies $\overline{E} \leqslant \overline{J}_i(\overline{F})$ for all $i \in I$. That is, the event $\overline{E}$ is a common basis if every agent knows whether $\overline{F}$ occurs as long as $\overline{E}$ implies $\overline{F}$.[36] Denote by $\mathcal{B}_I$ the collection of common bases.

When each agent's knowing-whether operator $\overline{J}_i$ satisfies $\overline{K}_{\overline{J}_i}$-Monotonicity (i.e., $\overline{E} \leqslant \overline{F}$ implies $\overline{J}_i(\overline{E}) \wedge \overline{E} \leqslant \overline{J}_i(\overline{F})$), the notion of a common basis reduces to that of self-evidence. Namely, call an event E *self-evident* to agent i if $\overline{E} \leqslant \overline{J}_i(\overline{E})$. In words, the event $\overline{E}$ is self-evident to agent i if she knows-whether $\overline{E}$ occurs whenever $\overline{E}$ is true.[37]

Now, define the *common knowledge operator* $\overline{C} : \mathcal{E} \to \mathcal{E}$ (among I) as

$$\overline{C}(\overline{E}) = \sup\{\overline{F} \in \mathcal{E} \mid \overline{F} \in \mathcal{B}_I \text{ and } \overline{F} \leqslant \overline{E}\} \text{ for each } \overline{E} \in \mathcal{E}.$$

Note that, since the collection of events $\mathcal{E}$ is a complete lattice with respect to $\leqslant$, $\overline{C} : \mathcal{E} \to \mathcal{E}$ is well-defined. Letting $\overline{C}(\overline{E}) = (C(E), S(E))$, the event $\overline{E}$ is *common*

[35] Another interesting application of knowing-whether is to study the cardinality of a state space. See Hart, Heifetz, and Samet (1996) and Heifetz and Samet (1998a).

[36] Fukuda (2020) defines the notion of a common basis on a standard state space model without assuming any properties of agents' beliefs.

[37] The idea of self-evidence has been formulated and studied by, for instance, Aumann (1976), Friedell (1969), Milgrom (1981), and Monderer and Samet (1989). In the context of a knowledge model, an event E is self-evident to agent i if $\overline{E} \leqslant \overline{K}_i(\overline{E})$, i.e., agent i knows $\overline{E}$ whenever $\overline{E}$ is true. Fukuda (2021) formalizes common knowledge on a generalized state space from the notion of self-evidence when each agent's knowledge operator satisfies the Truth Axiom, Monotonicity, and Positive Introspection.

knowledge among I at a state ω if $\omega \in C(E)$. It can be seen that the resulting common knowledge operator satisfies the Truth Axiom, Positive Introspection, and Monotonicity.

One can also define the *common knowing-whether operator* (the knowing-whether operator associated with common knowledge) $\overline{J}^* : \mathcal{E} \to \mathcal{E}$ as: $\overline{J}^*(\overline{E}) := \overline{C}(\overline{E}) \vee \overline{C}(\neg \overline{E})$ for each $\overline{E} \in \mathcal{E}$.

To sum up, I remark that it would be an interesting avenue for future research to explore interactive knowledge (or knowing-whether) in the framework of this paper in more detail. For instance, there may be strategic situations in which one agent knows which game is played but she does not know whether her opponent knows which game is played.

5 Conclusion

This paper studies conditions under which a generalized-state-space model of unawareness can be axiomatized from knowing-whether operators. Proposition 1 shows that an agent's underlying knowledge is identified from her knowing-whether operator if and only if her knowledge is truthful. The agent knows whether an event obtains if she knows it or knows its negation. Different knowledge operators lead to different knowing-whether operators if knowledge is truthful. Conversely, for any knowing-whether operator, there is a unique truthful knowledge operator that induces the given knowing-whether operator: the agent knows an event if and only if she knows whether the event is true and the event indeed holds. The Truth Axiom is indispensable in that qualitative or probabilistic beliefs may not be recovered from believing-whether operators.

This paper then axiomatizes properties of knowledge and common knowledge, in terms of knowing-whether operators. This paper also studies unawareness defined from (not) knowing-whether (or ignorance). The approach in this paper may provide a simple way to construct models of knowledge and unawareness on a generalized state space. It would be interesting to identify and then study a class of unawareness structures (e.g., the class that contains the examples in Heifetz, Meier, and Schipper (2006)) that can be modeled through knowing-whether operators in a simple way. That would facilitate the use of unawareness structures in practice.

This paper thus provides a model of knowledge and unawareness on a generalized state space in which the only assumption made on knowledge is the Truth Axiom. As this paper is the first paper that dispenses with Monotonicity (which requires an agent to be a perfect logical reasoner in that the agent knows any event that is implied by what she already knows), it would be interesting to explore implications of the violation of Monotonicity in a generalized-state-space model of unawareness.

A Appendix

A.1 Proofs for Section 3.1

Proof of Proposition 1. 1. For the first statement, suppose $\overline{J}_{\overline{K}_1} = \overline{J}_{\overline{K}_2}$. Fix $\overline{E} \in \mathcal{E}$. I show:

$$\overline{K}_1(\overline{E}) = (\overline{K}_1(\overline{E}) \vee \overline{K}_1(\neg\overline{E})) \wedge \overline{E} = (\overline{K}_2(\overline{E}) \vee \overline{K}_2(\neg\overline{E})) \wedge \overline{E} = \overline{K}_2(\overline{E}). \quad (4)$$

The first and third equalities in Expression (4) follow from the Truth Axiom and Condition (3). To see this, it follows from the distributive law and the Truth Axiom that

$$\begin{aligned}(\overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg\overline{E})) \wedge \overline{E} &= (\overline{K}_i(\overline{E}) \wedge \overline{E}) \vee (\overline{K}_i(\neg\overline{E}) \wedge \overline{E}) \\ &= \overline{K}_i(\overline{E}) \vee (\overline{K}_i(\neg\overline{E}) \wedge \overline{E}). \end{aligned} \quad (5)$$

For the second term of the last expression, it follows from Condition (3) and the Truth Axiom that

$$\overline{\emptyset}^{S(E)} \leqslant \overline{K}_i(\neg\overline{E}) \wedge \overline{E} \leqslant (\neg\overline{E}) \wedge \overline{E} = \overline{\emptyset}^{S(E)}.$$

Thus, it follows from Condition (3) that

$$\overline{K}_i(\overline{E}) \vee (\overline{K}_i(\neg\overline{E}) \wedge \overline{E}) = \overline{K}_i(\overline{E}) \vee \overline{\emptyset}^{S(E)} = \overline{K}_i(\overline{E}). \quad (6)$$

The second equality in Expression (4) follows from the supposition, as $\overline{J}_{\overline{K}_i} = \overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg\overline{E})$.

Conversely, for the second statement, let $\mathbb{K} \ni \overline{K} \mapsto \overline{J}_{\overline{K}} \in \mathbb{O}_{\mathcal{E}}^{\mathrm{Sym}}$ be injective. Take any $\overline{K} \in \mathbb{K}$. Since $\mathbb{K}$ is K-closed, take $\overline{K}'$ such that, for all $\overline{E} \in \mathcal{E}$,

$$\overline{K}'(\overline{E}) = (\overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{E})) \wedge \overline{E}. \quad (7)$$

I show $\overline{J}_{\overline{K}} = \overline{J}_{\overline{K}'}$. To that end, for any $\overline{E} \in \mathcal{E}$,

$$\overline{J}_{\overline{K}}(\overline{E}) = \overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{E}) \text{ and } \overline{J}_{\overline{K}'}(\overline{E}) = \overline{K}'(\overline{E}) \vee \overline{K}'(\neg\overline{E}).$$

Then,

$$\begin{aligned}\overline{K}'(\overline{E}) \vee \overline{K}'(\neg\overline{E}) &= \big((\overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{E})) \wedge \overline{E}\big) \vee \big((\overline{K}(\neg\overline{E}) \vee \overline{K}(\neg\neg\overline{E})) \wedge (\neg\overline{E})\big) \\ &= \big((\overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{E})) \wedge \overline{E}\big) \vee \big((\overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{E})) \wedge (\neg\overline{E})\big) \\ &= \big(\overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{E})\big) \wedge \big(\overline{E} \vee (\neg\overline{E})\big) \\ &= \big(\overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{E})\big) \wedge \overline{S(E)} \\ &= \overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{E}),\end{aligned}$$

where the first equality follows from Expression (7) and the last from Condition (3). Thus, $\overline{J}_{\overline{K}} = \overline{J}_{\overline{K}'}$. Since the KJ-transformation is assumed to be injective, it follows that $\overline{K} = \overline{K}'$. Since $\overline{K}'$ satisfies the Truth Axiom, so does $\overline{K}$.

2. For the first statement, suppose $\overline{K}_{\overline{J}_1} = \overline{K}_{\overline{J}_2}$. Fix $\overline{E} \in \mathcal{E}$. I show

$$\overline{J}_1(\overline{E}) = (\overline{K}_{\overline{J}_1}(\overline{E}) \vee \overline{K}_{\overline{J}_1}(\neg\overline{E})) = (\overline{K}_{\overline{J}_2}(\overline{E}) \vee \overline{K}_{\overline{J}_2}(\neg\overline{E})) = \overline{J}_2(\overline{E}). \tag{8}$$

To show the first and third equalities in Expression (8),

$$\begin{aligned}\overline{K}_{\overline{J}_i}(\overline{E}) \vee \overline{K}_{\overline{J}_i}(\neg\overline{E}) &= (\overline{J}_i(\overline{E}) \wedge \overline{E}) \vee (\overline{J}_i(\neg\overline{E}) \wedge (\neg\overline{E})) \\ &= (\overline{J}_i(\overline{E}) \wedge \overline{E}) \vee (\overline{J}_i(\overline{E}) \wedge (\neg\overline{E})) \\ &= \overline{J}_i(\overline{E}) \wedge (\overline{E} \vee (\neg\overline{E})) \\ &= \overline{J}_i(\overline{E}) \wedge \overline{S(E)} \\ &= \overline{J}_i(\overline{E}),\end{aligned} \tag{9}$$

where the second equality follows from the symmetry of $\overline{J}_i$ and the last from Condition (3). The second equality in Expression (8) follows from the supposition $\overline{K}_{\overline{J}_1} = \overline{K}_{\overline{J}_2}$. Thus, the inverse KJ-transformation is injective.

Conversely, for the second statement, let $\mathbb{J} \ni \overline{J} \mapsto \overline{K}_{\overline{J}} \in \mathbb{O}_{\mathcal{E}}^{\mathrm{TA}}$ be injective. Take any $\overline{J} \in \mathbb{J}$. Since $\mathbb{J}$ is J-closed, take $\overline{J}'$ such that, for any $\overline{E} \in \mathcal{E}$,

$$\overline{J}'(\overline{E}) = (\overline{J}(\overline{E}) \wedge \overline{E}) \vee (\overline{J}(\neg\overline{E}) \wedge (\neg\overline{E})). \tag{10}$$

I show $\overline{K}_{\overline{J}} = \overline{K}_{\overline{J}'}$. To that end, recall

$$\overline{K}_{\overline{J}}(\overline{E}) = \overline{J}(\overline{E}) \wedge \overline{E} \text{ and } \overline{K}_{\overline{J}'}(\overline{E}) = \overline{J}'(\overline{E}) \wedge \overline{E}.$$

Then,

$$\begin{aligned}\overline{J}'(\overline{E}) \wedge \overline{E} &= \big((\overline{J}(\overline{E}) \wedge \overline{E}) \vee (\overline{J}(\neg\overline{E}) \wedge (\neg\overline{E}))\big) \wedge \overline{E} \\ &= \big((\overline{J}(\overline{E}) \wedge \overline{E}) \wedge \overline{E}\big) \vee \big((\overline{J}(\neg\overline{E}) \wedge (\neg\overline{E})) \wedge \overline{E}\big) \\ &= (\overline{J}(\overline{E}) \wedge \overline{E}) \vee \overline{\emptyset}^{S(E)} \\ &= \overline{J}(\overline{E}) \wedge \overline{E},\end{aligned}$$

where the first equality follows from Expression (10) and the third and fourth equalities follow from Condition (3). Hence, $\overline{K}_{\overline{J}} = \overline{K}_{\overline{J}'}$. Since the inverse KJ-transformation is assumed to be injective, it follows that $\overline{J} = \overline{J}'$. Since $\overline{J}'$ satisfies Symmetry, so does $\overline{J}$.

3. This assertion follows from the previous assertions.

□

A.2 Proofs for Section 4.1

Proof of Proposition 2. 1. First, assume Weak Necessitation. For any $\overline{E} \in \mathcal{E}$,

$$\overline{U}(\overline{E}) = (\neg \overline{K}_{\overline{J}})(\overline{S}^{\uparrow}) = \neg(\overline{J}(\overline{S}^{\uparrow}) \wedge \overline{S}^{\uparrow}) = (\neg \overline{J})(\overline{S}^{\uparrow}) \vee \overline{\emptyset}^{S(E)} = (\neg \overline{J})(\overline{S}^{\uparrow}).$$

Conversely, assume $\overline{U}(\overline{E}) = (\neg \overline{J})(\overline{S}^{\uparrow})$ for all $\overline{E} \in \mathcal{E}$. Fix $\overline{E} \in \mathcal{E}$. Then,

$$\overline{U}(\overline{E}) = (\neg \overline{J})(\overline{S}^{\uparrow}) = \neg(\overline{K}_{\overline{J}}(\overline{S}^{\uparrow}) \vee \overline{K}_{\overline{J}}(\neg \overline{S}^{\uparrow})) = \neg(\overline{K}_{\overline{J}}(\overline{S}^{\uparrow}) \vee \overline{K}_{\overline{J}}(\overline{\emptyset}^{S(E)})).$$

By the Truth Axiom, $\overline{K}_{\overline{J}}(\overline{\emptyset}^{S(E)}) = \overline{\emptyset}^{S(E)}$, and thus

$$\neg(\overline{K}_{\overline{J}}(\overline{S}^{\uparrow}) \vee \overline{K}_{\overline{J}}(\overline{\emptyset}^{S(E)})) = (\neg \overline{K}_{\overline{J}})(\overline{S}^{\uparrow}).$$

2. First, assume Plausibility. Fix $\overline{E} \in \mathcal{E}$. By Plausibility,

$$\overline{U}(\overline{E}) \leqslant (\neg \overline{K}_{\overline{J}})(\overline{E}) \wedge (\neg \overline{K}_{\overline{J}})(\neg \overline{K}_{\overline{J}})(\overline{E}).$$

By the Truth Axiom of $\overline{K}_{\overline{J}}$, $(\neg \overline{K}_{\overline{J}})(\overline{E}) \leqslant (\neg \overline{K}_{\overline{J}})\overline{K}_{\overline{J}}(\overline{E})$. Thus,

$$\begin{aligned}\overline{U}(\overline{E}) &\leqslant (\neg \overline{K}_{\overline{J}})(\overline{E}) \wedge (\neg \overline{K}_{\overline{J}})(\neg \overline{K}_{\overline{J}})(\overline{E}) \\ &\leqslant (\neg \overline{K}_{\overline{J}})\overline{K}_{\overline{J}}(\overline{E}) \wedge (\neg \overline{K}_{\overline{J}})(\neg \overline{K}_{\overline{J}})(\overline{E}) \\ &= (\neg \overline{J})\overline{K}_{\overline{J}}(\overline{E}),\end{aligned}$$

as desired.

Conversely, assume $\overline{U}(\cdot) \leqslant (\neg \overline{J})\overline{K}_{\overline{J}}(\cdot)$. Also, assume $\overline{K}_{\overline{J}}$-Positive Introspection. Fix $\overline{E} \in \mathcal{E}$. Then,

$$\begin{aligned}\overline{U}(\overline{E}) \leqslant (\neg \overline{J})\overline{K}_{\overline{J}}(\overline{E}) &= (\neg \overline{K}_{\overline{J}})\overline{K}_{\overline{J}}(\overline{E}) \wedge (\neg \overline{K}_{\overline{J}})(\neg \overline{K}_{\overline{J}})(\overline{E}) \\ &\leqslant (\neg \overline{K}_{\overline{J}})(\overline{E}) \wedge (\neg \overline{K}_{\overline{J}})(\neg \overline{K}_{\overline{J}})(\overline{E}),\end{aligned}$$

where the second inequality follows because, by $\overline{K}_{\overline{J}}$-Positive Introspection, $(\neg \overline{K}_{\overline{J}})\overline{K}_{\overline{J}}(\overline{E}) \leqslant (\neg \overline{K}_{\overline{J}})(\overline{E})$.

3. First, assume KU Introspection. Fix $\overline{E} \in \mathcal{E}$. Then,

$$\begin{aligned}\overline{J}(\neg \overline{U})(\overline{E}) &= \overline{K}_{\overline{J}}(\neg \overline{U})(\overline{E}) \vee \overline{K}_{\overline{J}}\overline{U}(\overline{E}) \\ &= \overline{K}_{\overline{J}}(\neg \overline{U})(\overline{E}) \\ &\leqslant (\neg \overline{U})(\overline{E}).\end{aligned}$$

The second equality follows from KU Introspection, and the first inequality follows from the Truth Axiom of $\overline{K}_{\overline{J}}$.

Conversely, assume JU Introspection. Thus, $\overline{JU}(\cdot) \leqslant (\neg\overline{U})(\cdot)$. Fix $\overline{E} \in \mathcal{E}$. Then,

$$\begin{aligned}
\overline{K}_{\overline{J}}\overline{U}(\overline{E}) &= \overline{JU}(\overline{E}) \wedge \overline{U}(\overline{E}) \\
&= \overline{J}(\neg\overline{U})(\overline{E}) \wedge \overline{U}(\overline{E}) \\
&\leqslant (\neg\overline{U})(\overline{E}) \wedge \overline{U}(\overline{E}) = (\emptyset, S(E)).
\end{aligned}$$

The second equality follows from Symmetry of $\overline{J}$. The first inequality follows from JU Introspection (in the form of $\overline{JU}(\cdot) \leqslant (\neg\overline{U})(\cdot)$).

□

Proof of Proposition 3. Let an operator $\overline{J}$ be given. I show that, when the operator $\overline{J}$ satisfies the properties specified in the statement of the proposition, the corresponding knowledge operator $\overline{K}_{\overline{J}}$ satisfies the conditions under which it induces a possibility correspondence.

First, by construction, $\overline{K}_{\overline{J}}$ satisfies the Truth Axiom. Second, by $\overline{K}_{\overline{J}}$-Monotonicity, $\overline{K}_{\overline{J}}$ satisfies Monotonicity. Third, by $\overline{K}_{\overline{J}}$-Necessitation, $\overline{K}_{\overline{J}}$ satisfies Necessitation.

Fourth, Conjunction implies $\overline{K}_{\overline{J}}$-Conjunction because

$$\begin{aligned}
\bigwedge_{x\in X} \overline{K}_{\overline{J}}(\overline{E}_x) = \bigwedge_{x\in X} (\overline{J}(\overline{E}_x) \wedge \overline{E}_x) &\leqslant \bigwedge_{x\in X} \overline{J}(\overline{E}_x) \wedge \left(\bigwedge_{x\in X} \overline{E}_x\right) \\
&\leqslant \overline{J}\left(\bigwedge_{x\in X} \overline{E}_x\right) \wedge \left(\bigwedge_{x\in X} \overline{E}_x\right) = \overline{K}_{\overline{J}}\left(\bigwedge_{x\in X} \overline{E}_x\right).
\end{aligned}$$

The first inequality follows from the operation of taking the conjunction/infimum, and the second inequality follows from Conjunction.

Fifth, Conjunction and $\overline{J}$-Introspection imply $\overline{K}_{\overline{J}}$-Positive Introspection because

$$\overline{K}_{\overline{J}}(\overline{E}) = \overline{E} \wedge \overline{J}(\overline{E}) = \overline{E} \wedge \overline{J}(\overline{E}) \wedge \overline{JJ}(\overline{E}) \leqslant \overline{E} \wedge \overline{J}(\overline{E}) \wedge \overline{J}(\overline{E} \wedge \overline{J}(\overline{E})) = \overline{K}_{\overline{J}}\overline{K}_{\overline{J}}(\overline{E}).$$

The second equality follows from $\overline{J}$-Introspection, and the inequality follows from Conjunction.

Sixth, I show in four steps that $\overline{K}_{\overline{J}}$-Monotonicity, $\overline{K}_{\overline{J}}$-Positive Introspection, and $\overline{J}$-Reflection imply $\overline{K}_{\overline{J}}$-Negative Introspection. The first step invokes $\overline{K}_{\overline{J}}$-Negative Introspection:

$$\overline{K}_{\overline{J}}(\overline{S}^{\uparrow}) = \overline{JJ}(\overline{E}).$$

The second and third steps establish:

$$\overline{JJ}(\overline{E}) \leqslant \overline{K}(\overline{E}) \vee \overline{K}(\neg\overline{K})(\overline{E}). \tag{11}$$

To that end, the second step shows that the Truth Axiom and $\overline{K}_{\overline{J}}$-Monotonicity imply $\overline{K}(\neg\overline{E}) \leqslant \overline{K}(\neg\overline{K})(\overline{E})$. Indeed, since $\overline{K}(\overline{E}) \leqslant \overline{E}$ by the Truth Axiom, it follows from operating the negation that $(\neg\overline{E}) \leqslant (\neg\overline{K})(\overline{E})$. Operating the knowledge operator, it follows from $\overline{K}_{\overline{J}}$-Monotonicity that $\overline{K}_{\overline{J}}(\neg\overline{E}) \leqslant \overline{K}_{\overline{J}}(\neg\overline{K}_{\overline{J}})(\overline{E})$.

The third step then shows:

$$\begin{aligned}\overline{JJ}(\overline{E}) &= \overline{J}\left(\overline{K}_{\overline{J}}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{E})\right) \\ &= \overline{K}_{\overline{J}}\left(\overline{K}_{\overline{J}}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{E})\right) \vee \overline{K}_{\overline{J}}\left((\neg\overline{K}_{\overline{J}})(\overline{E}) \wedge (\neg\overline{K}_{\overline{J}})(\neg\overline{E})\right) \\ &\leqslant \overline{K}_{\overline{J}}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{K}_{\overline{J}})(\overline{E}) \\ &= \overline{K}_{\overline{J}}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{K}_{\overline{J}})(\overline{E}).\end{aligned}$$

To see that the first inequality holds, it follows from the Truth Axiom that

$$\overline{K}_{\overline{J}}\left(\overline{K}_{\overline{J}}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{E})\right) \leqslant \overline{K}_{\overline{J}}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{E});$$

and it follows from $\overline{K}_{\overline{J}}$-Monotonicity that

$$\overline{K}_{\overline{J}}\left((\neg\overline{K}_{\overline{J}})(\overline{E}) \wedge (\neg\overline{K}_{\overline{J}})(\neg\overline{E})\right) \leqslant \overline{K}_{\overline{J}}(\neg\overline{K}_{\overline{J}})(\overline{E}).$$

The last equality follows from the second step.

In the fourth step, it follows from Expression (11) that

$$\overline{K}(\overline{S}^{\uparrow}) \wedge (\neg\overline{K}_{\overline{J}})(\overline{E}) \leqslant \left(\overline{K}_{\overline{J}}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{K}_{\overline{J}})(\overline{E})\right) \wedge (\neg\overline{K}_{\overline{J}})(\overline{E}).$$

The right-hand side can be rewritten as:

$$\begin{aligned}\left(\overline{K}_{\overline{J}}(\overline{E}) \wedge (\neg\overline{K}_{\overline{J}})(\overline{E})\right) \vee \left(\overline{K}_{\overline{J}}(\neg\overline{K}_{\overline{J}})(\overline{E}) \wedge (\neg\overline{K}_{\overline{J}})(\overline{E})\right) &= \overline{\emptyset}^{S(E)} \vee \overline{K}_{\overline{J}}(\neg\overline{K}_{\overline{J}})(\overline{E}) \\ &= \overline{K}_{\overline{J}}(\neg\overline{K}_{\overline{J}})(\overline{E}),\end{aligned}$$

where the first equality follows from the Truth Axiom. Thus, I have obtained $\overline{K}_{\overline{J}}$-Generalized Negative Introspection.

Conversely, let $\overline{K}$ be a knowledge operator that satisfies the conditions under which it induces a possibility correspondence. I show that the knowing-whether operator $\overline{J}_{\overline{K}}$ satisfies the properties in the statement of Proposition 3.

First, by construction, $\overline{J}_{\overline{K}}$ satisfies Symmetry. Second, I show that, under the Truth Axiom of $\overline{K}$, $\overline{K}_{\overline{J}_{\overline{K}}}$-Monotonicity follows from Monotonicity of $\overline{K}$. If $\overline{E} \leqslant \overline{F}$, then

$$\overline{E} \wedge \overline{J}_{\overline{K}}(\overline{E}) = \overline{K}(\overline{E}) \leqslant \overline{K}(\overline{F}) = \overline{J}_{\overline{K}}(\overline{F}) \wedge \overline{F} \leqslant \overline{J}_{\overline{K}}(\overline{F}).$$

The first and third equalities follow because, by Proposition 1, $\overline{K}_{\overline{J}_{\overline{K}}} = \overline{K}$.

Third, Necessitation of $\overline{K}$ implies $\overline{K}_{\overline{J}_{\overline{K}}}$-Necessitation:

$$(\Omega, \inf \mathcal{S}) = \overline{K}(\Omega, \inf \mathcal{S}) \leqslant \overline{J}_{\overline{K}}(\Omega, \inf \mathcal{S}) \leqslant (\Omega, \inf \mathcal{S}).$$

The first equality follows because $\overline{K}$ satisfies Necessitation. The first inequality follows because $\overline{K}(\cdot) \leqslant \overline{J}_{\overline{K}}(\cdot)$. The second inequality follows because $J_{\overline{K}}(\Omega) \subseteq \Omega$.

Fourth, I show in two steps that, under Monotonicity of $\overline{K}$, Conjunction of $\overline{K}$ implies Conjunction of $\overline{J}_{\overline{K}}$. In the first step,

$$\begin{aligned}
\bigwedge_{x\in X} \overline{J}_{\overline{K}}(\overline{E}_x) &= \bigwedge_{x\in X} (\overline{K}(\overline{E}_x) \vee \overline{K}(\neg \overline{E}_x)) \\
&\leqslant \bigwedge_{x\in X} \left(\overline{K}(\overline{E}_x) \vee \left(\bigvee_{y\in X} \overline{K}(\neg \overline{E}_y) \right) \right) \\
&= \bigwedge_{x\in X} \overline{K}(E_x) \vee \left(\bigvee_{y\in X} \overline{K}(\neg \overline{E}_y) \right).
\end{aligned}$$

In the second step, since $\bigwedge_{y\in X} \overline{E}_y \leqslant \overline{E}_x$, it follows that $(\neg \overline{E}_x) \leqslant \neg \left(\bigwedge_{y\in X} \overline{E}_y \right)$. It follows from Monotonicity of $\overline{K}$ that $\overline{K}(\neg \overline{E}_x) \leqslant \overline{K} \left(\neg \left(\bigwedge_{y\in X} \overline{E}_y \right) \right)$. Then, I obtain

$$\bigvee_{x\in X} \overline{K}(\neg \overline{E}_x) \leqslant \overline{K} \left(\neg \left(\bigwedge_{y\in X} \overline{E}_y \right) \right).$$

By Conjunction of $\overline{K}$, I also have

$$\bigwedge_{x\in X} \overline{K}(\overline{E}_x) \leqslant \overline{K} \left(\bigwedge_{x\in X} \overline{E}_x \right).$$

Thus, I get

$$\begin{aligned}
\bigwedge_{x\in X} \overline{K}(E_x) \vee \left(\bigvee_{y\in X} \overline{K}(\neg \overline{E}_y) \right) &\leqslant \overline{K} \left(\bigwedge_{x\in X} \overline{E}_x \right) \vee \overline{K} \left(\neg \left(\bigwedge_{y\in X} \overline{E}_y \right) \right) \\
&= \overline{K} \left(\bigwedge_{x\in X} \overline{E}_x \right) \vee \overline{K} \left(\neg \left(\bigwedge_{x\in X} \overline{E}_x \right) \right) = \overline{J}_{\overline{K}} \left(\bigwedge_{x\in X} \overline{E}_x \right),
\end{aligned}$$

which proves

$$\bigwedge_{x\in X} \overline{J}_{\overline{K}}(\overline{E}_x) \leqslant \overline{J}_{\overline{K}} \left(\bigwedge_{x\in X} \overline{E}_x \right),$$

as desired.

Fifth, I show that Positive Introspection and Monotonicity of $\overline{K}$ imply $\overline{J}_{\overline{K}}$-Introspection. To see this, I start with showing that Positive Introspection and Monotonicity of $\overline{K}$ imply $\overline{J}_{\overline{K}}(\overline{E}) \leqslant \overline{KJ}_{\overline{K}}(\overline{E})$. Indeed,

$$\overline{J}_{\overline{K}}(\overline{E}) = \overline{K}(\overline{E}) \vee \overline{K}(\neg \overline{E}) \leqslant \overline{KK}(\overline{E}) \vee \overline{KK}(\neg \overline{E}) \leqslant \overline{K}(\overline{K}(\overline{E}) \vee \overline{K}(\neg \overline{E})) = \overline{KJ}_{\overline{K}}(\overline{E}).$$

The first inequality follows from Positive Introspection of $\overline{K}$, and the second inequality from Monotonicity of $\overline{K}$. Then, $\overline{J}_{\overline{K}}$-Introspection follows because

$$\overline{J}_{\overline{K}}(E) \leqslant \overline{K}\overline{J}_{\overline{K}}(\overline{E}) \leqslant \overline{J}_{\overline{K}}\overline{J}_{\overline{K}}(\overline{E}),$$

where the second inequality follows because $\overline{K}(\cdot) \leqslant \overline{J}_{\overline{K}}(\cdot)$.

Sixth, I show in five steps that $\overline{K}_{\overline{J}}$-Monotonicity, $\overline{K}_{\overline{J}}$-Positive Introspection, $\overline{K}_{\overline{J}}$-Generalized Negative Introspection, and Conjunction (of $\overline{J}$) imply J-Reflection. The first step establishes $\overline{J}\overline{J}(\overline{E}) \leqslant \overline{K}_{\overline{J}}(\overline{S}^{\uparrow})$:

$$\overline{J}\overline{J}(\overline{E}) = \overline{K}_{\overline{J}}\overline{J}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{J})(\overline{E}) \leqslant \overline{K}(\overline{S}),$$

where the inequality follows from $\overline{K}_{\overline{J}}$-Monotonicity.

The second to the fifth steps establish the converse inequality. In the second step, it follows from $\overline{K}_{\overline{J}}$-Generalized Negative Introspection that

$$\left(\overline{K}_{\overline{J}}(\overline{S}^{\uparrow}) \wedge (\neg\overline{K}_{\overline{J}})(\overline{E})\right) \vee \overline{K}_{\overline{J}}(\overline{E}) \leqslant \overline{K}_{\overline{J}}(\neg\overline{K}_{\overline{J}})(\overline{E}) \vee \overline{K}_{\overline{J}}(\overline{E}).$$

The left-hand side reduces to:

$$\begin{aligned}\left(\overline{K}_{\overline{J}}(\overline{S}^{\uparrow}) \wedge (\neg\overline{K}_{\overline{J}})(\overline{E})\right) \vee \overline{K}_{\overline{J}}(\overline{E}) &= (\overline{K}_{\overline{J}}(\overline{S}^{\uparrow}) \vee \overline{K}_{\overline{J}}(\overline{E})) \wedge ((\neg\overline{K}_{\overline{J}})(\overline{E}) \vee \overline{K}_{\overline{J}}(\overline{E}))\\ &= \overline{K}_{\overline{J}}(\overline{S}^{\uparrow}) \vee \overline{S}^{\uparrow}\\ &= \overline{K}_{\overline{J}}(\overline{S}^{\uparrow}),\end{aligned}$$

where the second equality follows from $\overline{K}_{\overline{J}}$-Monotonicity.

In the third step,

$$\begin{aligned}\overline{K}_{\overline{J}}(\overline{S}^{\uparrow}) &\leqslant \overline{K}_{\overline{J}}(\neg\overline{K}_{\overline{J}})(\overline{E}) \vee \overline{K}_{\overline{J}}(\overline{E}) = \overline{K}_{\overline{J}}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{K}_{\overline{J}})(\overline{E})\\ &\leqslant \overline{K}_{\overline{J}}\overline{K}_{\overline{J}}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{K}_{\overline{J}})(\overline{E})\\ &= \overline{J}\overline{K}_{\overline{J}}(\overline{E})\\ &= \overline{J}(\neg\overline{K}_{\overline{J}})(\overline{E}),\end{aligned}$$

where the second inequality follows from $\overline{K}_{\overline{J}}$-Positive Introspection and the last equality follows from Symmetry of $\overline{J}$.

In the fourth step, considering $(\neg\overline{E})$ instead of $\overline{E}$, I also have

$$\overline{K}_{\overline{J}}(\overline{S}^{\uparrow}) \leqslant \overline{J}(\neg\overline{K}_{\overline{J}})(\neg\overline{E}).$$

In the fifth step,

$$\begin{aligned}\overline{K}_{\overline{J}}(\overline{S}^{\uparrow}) &\leqslant \overline{J}(\neg\overline{K}_{\overline{J}})(\overline{E}) \wedge \overline{J}(\neg\overline{K}_{\overline{J}})(\neg\overline{E})\\ &\leqslant \overline{J}\left((\neg\overline{K}_{\overline{J}})(\overline{E}) \wedge (\neg\overline{K}_{\overline{J}})(\neg\overline{E})\right)\\ &= \overline{J}(\neg\overline{J})(\overline{E})\\ &= \overline{J}\overline{J}(\overline{E}),\end{aligned}$$

where the second inequality follows from Conjunction (of $\overline{J}$) and the the last equality from Symmetry (of $\overline{J}$). $\square$

Proof of Remark 1. By $\overline{K}_{\overline{J}}$-Positive Introspection, $\overline{J}(\overline{E}) \wedge \overline{E} \leqslant \overline{J}(\overline{J}(\overline{E}) \wedge \overline{E}) \wedge \overline{E}$. Applying $\overline{K}_{\overline{J}}$-Monotonicity to $\overline{J}(\overline{E}) \wedge \overline{E} \leqslant \overline{J}(\overline{E})$, $\overline{J}(\overline{J}(\overline{E}) \wedge \overline{E}) \wedge \overline{J}(\overline{E}) \wedge \overline{E} \leqslant \overline{JJ}(\overline{E})$. Then, $\overline{J}(\overline{E}) \wedge \overline{E} \leqslant \overline{JJ}(\overline{E})$. Replacing $\overline{E}$ with $\neg\overline{E}$ and using Symmetry, $\overline{J}(\overline{E}) \wedge (\neg\overline{E}) \leqslant \overline{JJ}(\overline{E})$. Thus, $\overline{J}(\overline{E}) = (\overline{J}(\overline{E}) \wedge \overline{E}) \vee (\overline{J}(\overline{E}) \wedge \neg\overline{E}) \leqslant \overline{JJ}(\overline{E})$. $\square$

Proof of Proposition 4.

1. I prove $\overline{J}^2(\overline{E}) = \overline{J}^3(\overline{E})$ for all $\overline{E} \in \mathcal{E}$ in two steps. Fix $\overline{E} \in \mathcal{E}$. The first step establishes $\overline{J}^2(\overline{E}) \leqslant \overline{J}^3(\overline{E})$. To that end, since $\overline{J}$-Introspection (i.e., $\overline{J}(\overline{F}) \leqslant \overline{J}^2(\overline{F})$ for all $\overline{F} \in \mathcal{E}$) follows from $\overline{K}_{\overline{J}}$-Positive Introspection (see Remark 1), substituting $\overline{F} = \overline{J}(\overline{E})$ yields $\overline{J}^2(\overline{E}) \leqslant \overline{J}^3(\overline{E})$.

   The second step establishes $\overline{J}^3(\overline{E}) \leqslant \overline{J}^2(\overline{E})$. I have:

$$
\begin{aligned}
\overline{J}^3(\overline{E}) &= \overline{JJ}^2(\overline{E}) \\
&= \overline{K}_{\overline{J}}\overline{J}^2(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{J}^2)(\overline{E}) \\
&= (\overline{J}^3(\overline{E}) \wedge \overline{J}^2(\overline{E})) \vee (\overline{J}(\neg\overline{J}^2)(\overline{E}) \wedge (\neg\overline{J}^2)(\overline{E})) \\
&= \overline{J}^2(\overline{E}) \vee (\overline{J}^3(\overline{E}) \wedge (\neg\overline{J}^2)(\overline{E})),
\end{aligned}
$$

   where the last equality follows from the first step (i.e., $\overline{J}^2(\overline{E}) \leqslant \overline{J}^3(\overline{E})$) and Symmetry of $\overline{J}$.

   Since it follows from $\overline{J}$-Introspection that $(\neg\overline{J}^2)(\overline{E}) \leqslant (\neg\overline{J})(\overline{E})$, it follows from K_J-Monotonicity and Symmetry that

$$
\begin{aligned}
\overline{J}^3(\overline{E}) \wedge (\neg\overline{J}^2)(\overline{E}) &= \overline{J}(\neg\overline{J}^2)(\overline{E}) \wedge (\neg\overline{J}^2)(\overline{E}) \\
&= \overline{K}_{\overline{J}}(\neg\overline{J}^2)(\overline{E}) \\
&\leqslant \overline{K}_{\overline{J}}(\neg\overline{J})(\overline{E}) \\
&\leqslant \overline{J}(\neg\overline{J})(\overline{E}) = \overline{J}^2(\overline{E}).
\end{aligned}
$$

   Thus, $\overline{J}^3(\overline{E}) \leqslant \overline{J}^2(\overline{E}) \vee \overline{J}^2(\overline{E}) = \overline{J}^2(\overline{E})$.

2. If $\overline{K}_{\overline{J}}(\overline{E}) = \overline{E}$ for all $\overline{E} \in \mathcal{E}$, then $\overline{J}(\overline{E}) = \overline{S}^{\uparrow} = \overline{JJ}(\overline{E})$ for all $\overline{E} \in \mathcal{E}$. Conversely, suppose $\overline{J} = \overline{JJ}$. Fix $\overline{E} \in \mathcal{E}$. Then,

$$
\overline{J}(\overline{E}) = \overline{JJ}(\overline{E}) = \overline{K}_{\overline{J}}\overline{J}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{J})(\overline{E}).
$$

   Thus, I get

$$
\overline{J}(\overline{E}) = \overline{J}(\overline{E}) \vee \overline{K}_{\overline{J}}\overline{J}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{J})(\overline{E}) = \overline{J}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{J})(\overline{E}),
$$

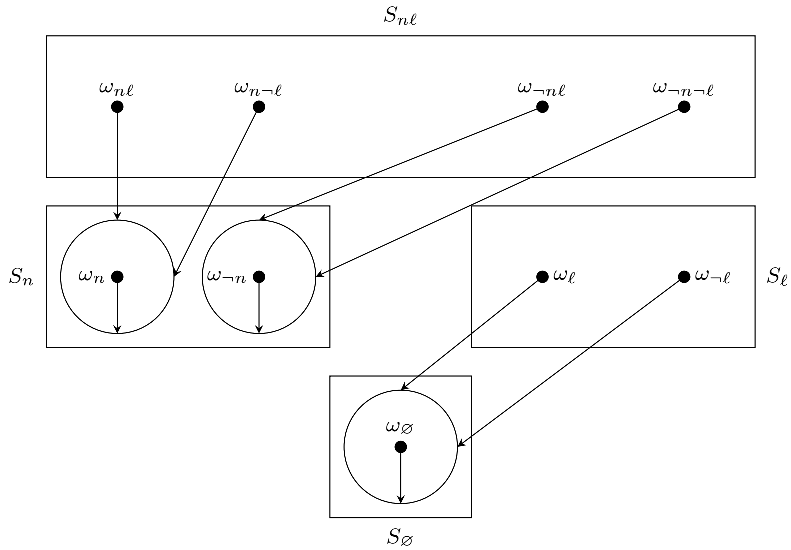

Figure 2: The Buyer's Possibility Correspondence in the Speculative-Trade Example

where the last equality follows from the Truth Axiom. Thus, $\overline{K}_{\overline{J}}(\neg\overline{J})(\overline{E}) \leqslant \overline{J}(\overline{E})$, and hence

$$(\neg\overline{J})(\overline{E}) \leqslant (\neg\overline{K}_{\overline{J}})(\neg\overline{J})(\overline{E}) \leqslant \overline{J}(\neg\overline{J})(\overline{E}) = \overline{J}\overline{J}(\overline{E}) = \overline{J}(\overline{E}).$$

Hence, I obtain

$$\overline{S}^{\uparrow} \leqslant \overline{J}(\overline{E}) \vee (\neg\overline{J})(\overline{E}) \leqslant \overline{J}(\overline{E}) = \overline{K}_{\overline{J}}(\overline{E}) \vee \overline{K}_{\overline{J}}(\neg\overline{E}).$$

This implies $\overline{K}_{\overline{J}}(\overline{E}) = \overline{E}$.

□

A.3 Possibility Correspondences for the Speculative-Trade Example

For completeness, I present the agents' possibility correspondences in the speculative-trade example of Section 3.3. As in Footnote 15, the possibility correspondence of an agent $i \in \{b, o\}$ is a mapping $\overline{\Pi}_i^{\uparrow} : \Omega \to \mathcal{E}$ such that her knowledge operator satisfies

$$\overline{K}_i(\overline{E}) = (\{\omega \in \Omega \mid \overline{\Pi}_i^{\uparrow}(\omega) \leqslant \overline{E}\}, S(E)) \text{ for each } \overline{E} \in \mathcal{E}.$$

Thus, I define each agent's possibility correspondence $\overline{\Pi}_i^{\uparrow}$ which induces her knowing-whether and unawareness operators $\overline{J}_i$ and $\overline{U}_i$ defined in Section 3.3.

While Figure 2 illustrates the buyer's possibility correspondence $\overline{\Pi}_b^{\uparrow}$, Figure 3 the owner's possibility correspondence $\overline{\Pi}_o^{\uparrow}$. In each figure, an arrow from a point to a

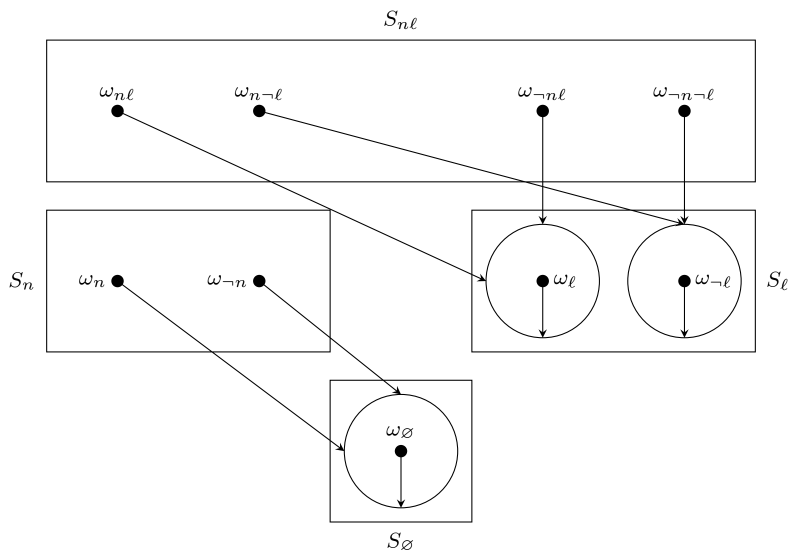

Figure 3: The Owner's Possibility Correspondence in the Speculative-Trade Example

circle means that the possibility correspondence maps the point ω to an event $\overline{\Pi}_i^{\uparrow}(\omega)$ which is depicted as the corresponding circle.

Consider the buyer. On the subspace $S_{n\ell}$, each state is mapped to the corresponding state in S_n because the buyer is unaware of the lawsuit. Since the buyer knows whether the innovation occurs, for instance, $\overline{\Pi}_b^{\uparrow}(\omega_{n\ell}) = (\{\omega_n\}, S_n)$ and $\overline{\Pi}_b^{\uparrow}(\omega_{\neg n\neg\ell}) = (\{\omega_{\neg n}\}, S_n)$. Similarly, on the subspace S_ℓ, each state is mapped to the corresponding state in $S_\varnothing$ because the buyer is unaware of the lawsuit. Thus, $\overline{\Pi}_b^{\uparrow}(\omega) = (\{\omega_\varnothing\}, S_\varnothing)$ for each $\omega \in S_\ell$.

On the subspace S_ℓ, in contrast, since the buyer knows whether the innovation occurs, $\overline{\Pi}_b^{\uparrow}(\omega) = (\{\omega\}, S_n)$ for each $\omega \in S_n$. Similarly, on $S_\varnothing$ (i.e., at $\omega_\varnothing$), the buyer knows whether the state is $\omega_\varnothing$: $\overline{\Pi}_b^{\uparrow}(\omega_\varnothing) = (\{\omega_\varnothing\}, S_\varnothing)$. The possibility correspondence of the owner is similarly defined.

B Neighborhood Systems on a Generalized State Space

This paper uses knowledge and unawareness operators defined on the collection of events as a primitive of a model and shows that one can axiomatize the model by a knowing-whether operator. One of the reasons that this paper takes such approach is that, as demonstrated through the speculative trade example in Section 3.3, such approach may allow for an easier construction and presentation of an economic example. At a technical level, another reason is that such approach allows one to add or drop properties of knowledge and unawareness one by one. Indeed, this paper shows that the only assumption is the Truth Axiom, and thus this paper is able to dispense with, for instance, agents' logical properties such as Monotonicity, Conjunction, and Necessitation.

There exists another general approach in the context of standard state spaces, namely, the framework called neighborhood systems.[38] In a standard state space model, it is well-known that using a neighborhood system is equivalent to using an operator when it comes to representing an agent's knowledge. This appendix extends the framework of neighborhood systems to a generalized state space.[39]

To that end, on a standard state space $(\Omega, \mathcal{D})$ where $\mathcal{D}$ is a complete algebra of sets on Ω, a *neighborhood system* is a map $\mathcal{N} : \Omega \to \mathcal{P}(\mathcal{D})$ with the "measurability" condition that $\{\omega \in \Omega \mid E \in \mathcal{N}(\omega)\} \in \mathcal{D}$ for any $E \in \mathcal{D}$.[40]

Given a (knowledge) operator $K : \mathcal{D} \to \mathcal{D}$, one can define the corresponding neighborhood system $\mathcal{N}_K : \Omega \to \mathcal{P}(\mathcal{D})$ by

$$\mathcal{N}_K(\omega) := \{E \in \mathcal{D} \mid \omega \in K(E)\} \text{ for all } \omega \in \Omega.$$

Then, $\mathcal{N}_K$ satisfies the measurability condition because $\{\omega \in \Omega \mid E \in \mathcal{N}_K(\omega)\} = K(E) \in \mathcal{D}$. Conversely, given a neighborhood system $\mathcal{N}$, define $K_{\mathcal{N}} : \mathcal{D} \to \mathcal{D}$ by

$$K_{\mathcal{N}}(E) := \{\omega \in \Omega \mid E \in \mathcal{N}_K\} \in \mathcal{D} \text{ for all } E \in \mathcal{D}.$$

The above argument implies that assigning K and $\mathcal{N}$ are equivalent in the sense that $K = K_{\mathcal{N}_K}$ and $\mathcal{N} = \mathcal{N}_{K_{\mathcal{N}}}$.

Now, I define a neighborhood system on a generalized state space $\langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha \in \mathcal{A}}, \succeq, r \rangle$. As in Section 2.1, denote by $\Omega = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ the set of states of the world. Also, denote by $\mathcal{E}$ the collection of events. With these notations in mind, a *neighborhood system* is a map $\mathcal{N} : \Omega \to \mathcal{P}(\mathcal{E})$ with the condition that, for any event $\overline{E} = (E, S_\alpha) \in \mathcal{E}$, $\{\omega \in S_\alpha \mid \overline{E} \in \mathcal{N}(\omega)\} \in \mathcal{D}_\alpha$.[41]

I show the equivalence between a knowledge operator and a neighborhood system. Given a knowledge operator $\overline{K} : \mathcal{E} \to \mathcal{E}$, define $\mathcal{N}_{\overline{K}} : \Omega \to \mathcal{E}$ by

$$\mathcal{N}_{\overline{K}}(\omega) := \{(E, S(E)) \in \mathcal{E} \mid \omega \in K(E)\} \text{ for each } \omega \in \Omega.^{42}$$

[38]See, for example, Chellas (1980, Part III) and Pacuit (2017) in the logic literature. In the economics and game-theory literature, Heifetz (1999) and Lismont and Mongin (1994a,b, 1995) use neighborhood systems in formalizing common belief and common knowledge.

[39]While the extension is straightforward, to the best of my knowledge there does not exist a paper that studies neighborhood systems on a generalized state space of knowledge and unawareness.

[40]This measurability condition is imposed because $\mathcal{D}$ may not necessarily be the power set of Ω. When $\mathcal{D} = \mathcal{P}(\Omega)$, the measurability condition is trivially satisfied.

[41]Note that $\mathcal{D}_\alpha$ is the complete algebra associated with the sub-space S_α.

[42]Recall that, for any event $\overline{E} = (E, S(E)) \in \mathcal{E}$, the knowledge operator $\overline{K}$ satisfies the property that $\overline{K}(\overline{E}) = (K(E), S(E))$.

Conversely, given a neighborhood system $\mathcal{N} : \Omega \to \mathcal{P}(\mathcal{E})$, define the corresponding knowledge operator $\overline{K}_{\mathcal{N}} : \mathcal{E} \to \mathcal{E}$ by $\overline{K}_{\mathcal{N}}(E, S(E)) := (K_{\mathcal{N}}(E), S(E))$ where

$$K_{\mathcal{N}}(E) := \{\omega \in S(E) \mid (E, S(E)) \in \mathcal{N}(\omega)\}^{\uparrow}.$$

It can be seen that $\overline{K} = \overline{K}_{\mathcal{N}_{\overline{K}}}$ and $\mathcal{N} = \mathcal{N}_{\overline{K}_{\mathcal{N}}}$.

Given the equivalence between a knowledge operator and a neighborhood system, one can define a model as a tuple $\langle\langle(S_\alpha, \mathcal{D}_\alpha)_{\alpha\in\mathcal{A}}, \succeq, r\rangle, (\mathcal{N}_{\overline{K}}, \mathcal{N}_{\overline{U}})\rangle$ where $\mathcal{N}_{\overline{K}}$ and $\mathcal{N}_{\overline{U}}$ are respectively a neighborhood system that corresponds to a knowledge operator $\overline{K}$ and an unawareness operator $\overline{U}$. When, for instance, unawareness is defined as the lack of knowledge as in Expression (2), one has:

$$(E, S) \in \mathcal{N}_{\overline{U}}(\omega) \text{ iff } (E, S) \notin \mathcal{N}_{\overline{K}}(\omega) \text{ and } (\{\tilde{\omega} \in S \mid (E, S) \notin \mathcal{N}_{\overline{K}}(\tilde{\omega})\}^{\uparrow}, S) \notin \mathcal{N}_{\overline{K}}(\omega).$$

One can thus, in principle, study properties of knowledge and unawareness on a generalized state space using neighborhood systems. The Truth Axiom can be expressed as

$$(\{\omega \in S \mid (E, S) \in \mathcal{N}(\omega)\}^{\uparrow}, S) \leqslant (E, S) \text{ for any } (E, S) \in \mathcal{E}.$$

Hence, the arguments in this paper (especially, Proposition 1 and Example 1) naturally extend to the framework of neighborhood systems on a generalized state space.[43]

While I do not represent any other properties of knowledge and unawareness due to the space constraint, one could represent properties of knowledge and unawareness in terms of neighborhood systems. One can also study common knowledge on a generalized state space using neighborhood systems, as Section 4.4 formulates common knowledge assuming that agents' knowledge operator are only required to satisfy the Truth Axiom.

C An Underlying Algebraic Structure behind Proposition 1

This Appendix discusses the underlying mathematical idea behind Proposition 1 in the context closer to the logic literature, especially the pioneering result by Montgomery and Routley (1966, Theorem 3). While Proposition 1 is not a direct application of the previous results in logic (recall Example 1 in Section 3.1), this Appendix

[43] Technically, throughout the discussion, I defined a neighborhood system $\mathcal{N}$ in a way such that the corresponding knowledge operator $\overline{K}_{\mathcal{N}}$ satisfies the property that $\overline{K}_{\mathcal{N}}(E, S) = (K_{\mathcal{N}}(E), S)$ for all $(E, S) \in \mathcal{E}$. To replicate Example 1 in the context of neighborhood systems, one needs to relax this assumption, for instance, as follows: for any $\overline{E} \in \mathcal{E}$, there exists $S' \in \mathcal{S}$ such that $(\{\omega \in \Omega \mid \overline{E} \mid \omega \in \mathcal{N}(\omega)\}, S') \in \mathcal{E}$, i.e., for any event $\overline{E}$, the set of states at which $\overline{E}$ is known forms an event (the subspace of which may not necessarily equal the subspace associated with $\overline{E}$).

demonstrates that one can extract the underlying algebraic structure behind Proposition 1 as a simple generalization of Montgomery and Routley (1966, Theorem 3).

Specifically, I will take a look at the results that the KJ-transformation $\mathbb{K} \ni \overline{K} \mapsto \overline{J}_{\overline{K}} \in \mathbb{O}_{\varepsilon}^{\text{Sym}}$ and the inverse KJ-transformation $\mathbb{J} \ni \overline{J} \mapsto \overline{K}_{\overline{J}} \in \mathbb{O}_{\varepsilon}^{\text{TA}}$ are injective.

The discussions are in three steps. The first step provides a preliminary definition. Namely, a partially-ordered set $\langle L, \leqslant_L \rangle$ forms a *meet-semilattice* if, for any pair of elements a and b in L, the meet (i.e., the greatest lower bound) $a \wedge_L b$ is well-defined in L. The goal is to apply $\langle L, \leqslant_L \rangle = \langle \mathcal{E}, \leqslant \rangle$, where $\mathcal{E}$ is the collection of events and $\leqslant$ is the partial order in a given generalized state space defined in Section 2. For ease of notation, henceforth I drop the subscript L on the operations on L (e.g., $\wedge$ instead of $\wedge_L$).

The second step is to formalize the underlying algebraic structure behind the injectivity of the KJ-transformation and the inverse KJ-transformation.

Lemma 1. *Let $\langle L, \leqslant \rangle$ be a meet-semilattice.*

1. *Let $\Box_1$, $\Box_2$, and $\neg$ be a unary operation on L such that, for all $a \in L$,*

$$\neg((\neg\Box_1 a) \wedge (\neg\Box_1 \neg a)) = \neg((\neg\Box_2 a) \wedge (\neg\Box_2 \neg a)) \text{ and} \tag{12}$$
$$\Box_i a = (\neg((\neg\Box_i a) \wedge (\neg\Box_i \neg a))) \wedge a \text{ for each } i \in \{1, 2\}. \tag{13}$$

   Then, $\Box_1 = \Box_2$.

2. *Let Δ_1 and Δ_2 be a unary operation on L such that, for all $b \in L$,*

$$(\Delta_1 b) \wedge b = (\Delta_2 b) \wedge b. \tag{14}$$

   Let $\neg$ be a unary operation on L such that, for all $a \in L$,

$$\Delta_i a = \neg\left(\neg((\Delta_i a) \wedge a) \wedge \neg((\Delta_i \neg a) \wedge \neg a)\right) \text{ for each } i \in \{1, 2\}. \tag{15}$$

   Then, $\Delta_1 = \Delta_2$.

The notations in Lemma 1 follow those of logic: the necessity operation $\Box$, which will be interpreted as a knowledge operator, and the non-contingency operation Δ, which will be interpreted as a knowing-whether operator.

Expression (12) is a hypothesis: roughly, it supposes that the necessity operations ("knowledge operators") $\Box_1$ and $\Box_2$ lead to the same corresponding non-contingency operation ("knowing-whether" operator). Then, under Expression (13) that allows for the restatement of $\Box_i$ by its right-hand side, the given necessity operations $\Box_1$ and $\Box_2$ coincide.[44]

[44] Similarly, Expression (14) is a hypothesis: roughly, it supposes that the non-contingency operations ("knowing-whether operators") Δ_1 and Δ_2 lead to the same corresponding necessity operation ("knowledge" operator). Then, under Expression (15) that allows for the restatement of Δ_i by its right-hand side, the given non-contingency operations Δ_1 and Δ_2 coincide.

Proof of Lemma 1. 1. Fix $a \in L$. Then,

$$\begin{aligned}\Box_1 a &= (\neg((\neg\Box_1 a) \wedge (\neg\Box_1 \neg a))) \wedge a && \text{by (13) with } i = 1\\ &= (\neg((\neg\Box_2 a) \wedge (\neg\Box_2 \neg a))) \wedge a && \text{by (12)}\\ &= \Box_2 a && \text{by (13) with } i = 2.\end{aligned}$$

2. Fix $a \in L$. Then,

$$\begin{aligned}\Delta_1 a &= \neg\left(\neg((\Delta_1 a) \wedge a) \wedge \neg((\Delta_1 \neg a) \wedge \neg a)\right) && \text{by (15) with } i = 1\\ &= \neg\left(\neg((\Delta_2 a) \wedge a) \wedge \neg((\Delta_2 \neg a) \wedge \neg a)\right) && \text{by applying (14) to } b \in \{a, \neg a\}\\ &= \Delta_2 a && \text{by (15) with } i = 2.\end{aligned}$$

□

The third step provides an alternative proof of the injectivity of the KJ-transformation and the inverse KJ-transformation. To apply Lemma 1, let $\langle\langle (S_\alpha, \mathcal{D}_\alpha)_{\alpha\in\mathcal{A}}, \succeq, r\rangle, (\overline{K}, \overline{U})\rangle$ be a model, let $\mathcal{E}$ be the collection of events, and let $\leqslant$ be the partial order on $\mathcal{E}$. Since $\langle \mathcal{E}, \leqslant\rangle$ is a complete lattice (recall Section 2), it is a meet-semilattice.

I start with the KJ-transformation.

Proof of the Injectivity of the KJ-Transformation. The proof applies Lemma 1 (1). Fix $\overline{E} \in \mathcal{E}$. Take $\Box_i = \overline{K}_i$ for each $i \in \{1, 2\}$. To show the injectivity of the KJ-transformation, assume $\overline{J}_{\overline{K}_1} = \overline{J}_{\overline{K}_2}$. Thus, it follows from the definitions of each $\overline{J}_{\overline{K}_i}$ and the disjunction $\vee$ that

$$\neg\left((\neg\overline{K}_1)(\overline{E}) \wedge (\neg\overline{K}_1)(\neg\overline{E})\right) = \neg\left((\neg\overline{K}_2)(\overline{E}) \wedge (\neg\overline{K}_2)(\neg\overline{E})\right),$$

that is, Expression (12) holds. Next, for each $i \in \{1, 2\}$,

$$\begin{aligned}\left(\neg\left((\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)(\neg\overline{E})\right)\right) \wedge \overline{E} &= \left(\overline{K}_i(\overline{E}) \vee \overline{K}_i(\neg\overline{E})\right) \wedge \overline{E}\\ &= (\overline{K}_i(\overline{E}) \wedge \overline{E}) \vee (\overline{K}_i(\neg\overline{E}) \wedge \overline{E})\\ &= \overline{K}_i(\overline{E}) \vee (\overline{K}_i(\neg\overline{E}) \wedge \overline{E}),\end{aligned}$$

where the first equality follows from the definition of the disjunction $\vee$, the second from the distributive law, and the third from the Truth Axiom of $\overline{K}_i$. Observe that the right-most side of the above expression is equal to the right-most side of Expression (5) in the proof of Proposition 1. By Expression (6) in the proof of Proposition 1, one obtains

$$\left(\neg\left((\neg\overline{K}_i)(\overline{E}) \wedge (\neg\overline{K}_i)(\neg\overline{E})\right)\right) \wedge \overline{E} = \overline{K}_i(\overline{E}),$$

that is, Expression (13). Thus, applying Lemma 1 (1), $\overline{K}_1(\overline{E}) = \overline{K}_2(\overline{E})$. Since $\overline{E} \in \mathcal{E}$ is arbitrary, the injectivity follows. □

Finally, I show that the inverse KJ-transformation is injective.

Proof of the Injectivity of the Inverse KJ-Transformation. The proof applies Lemma 1 (2). Fix $\overline{E} \in \mathcal{E}$. Take $\Delta_i = \overline{J}_i$ for each $i \in \{1, 2\}$. To show the injectivity of the inverse KJ-transformation, assume $\overline{K}_{\overline{J}_1} = \overline{K}_{\overline{J}_2}$. Thus, it follows from the definition of each $\overline{K}_{\overline{J}_i}$ that

$$\overline{J}_1(\overline{E}) \wedge \overline{E} = \overline{J}_2(\overline{E}) \wedge \overline{E},$$

that is, Expression (14) holds. Next, for each $i \in \{1, 2\}$,

$$\begin{aligned} \neg\left(\neg(\overline{J}_i(\overline{E}) \wedge \overline{E}) \wedge \neg(\overline{J}_i(\neg\overline{E}) \wedge (\neg\overline{E}))\right) &= (\overline{J}_i(\overline{E}) \wedge \overline{E}) \vee (\overline{J}_i(\neg\overline{E}) \wedge (\neg\overline{E})) \\ &= \overline{K}_{\overline{J}_i}(\overline{E}) \vee \overline{K}_{\overline{J}_i}(\neg\overline{E}), \end{aligned}$$

where the first equality follows from the definition of the disjunction $\vee$, and the second from the definition of $\overline{K}_{\overline{J}_i}$. Observe that the right-most side of the above expression is equal to the left-most side of Expression (9) in the proof of Proposition 1. By Expression (9) in the proof of Proposition 1, one obtains

$$\neg\left(\neg(\overline{J}_i(\overline{E}) \wedge \overline{E}) \wedge \neg(\overline{J}_i(\neg\overline{E}) \wedge (\neg\overline{E}))\right) = \overline{J}_i(\overline{E}),$$

that is, Expression (15). Thus, applying Lemma 1 (2), $\overline{J}_1(\overline{E}) = \overline{J}_2(\overline{E})$. Since $\overline{E} \in \mathcal{E}$ is arbitrary, the injectivity follows. □

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