Are the Players in an Interactive Belief Model Meta-certain of the Model Itself?

Online Supplementary Appendix

Satoshi Fukuda[†]

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This Online Appendix is structured as follows. Appendix B provides additional discussions on the main text (namely, Section 3). Appendix C discusses applications to probabilistic beliefs. Proofs are relegated to Appendix D.

B Informativeness, Possibility, and Certainty

This appendix provides an alternative characterization of the statement that player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ in terms of her reasoning of the signal. If she is certain of the signal x, then she would be able to rank the underlying states based on the collection of observational contents that hold at each state. Call a state ω *at least as informative as* a state ω' according to the signal x if, for any observational content F that holds at ω' under x, it holds at ω under x (Definition S.1). This appendix then characterizes properties of beliefs and the certainty of a mapping in terms of the notion of informativeness (Propositions S.1 and S.2).

Thus, I start with defining the informativeness of a signal.

Definition S.1. For states ω and ω' in Ω, ω is *at least as informative as ω' according to a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$* if

$$\{F \in \mathcal{X} \mid \omega' \in x^{-1}(F)\} \subseteq \{F \in \mathcal{X} \mid \omega \in x^{-1}(F)\}. \tag{S.1}$$

States ω and ω' are *equally informative according to $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$* if

$$\{F \in \mathcal{X} \mid \omega' \in x^{-1}(F)\} = \{F \in \mathcal{X} \mid \omega \in x^{-1}(F)\}. \tag{S.2}$$

The ideas behind Definition S.1 are (i) that the informational content of a signal mapping $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω is expressed as the collection of observational contents $\{F \in \mathcal{X} \mid x(\omega) \in F\}$ true at ω and (ii) that informational contents are ranked by the implication in the form of set inclusion.[S.1] While the notion of informativeness

[†]Department of Decision Sciences and IGIER, Bocconi University, Milan 20136, Italy; and Department of Economics, Leavey School of Business, Santa Clara University, Santa Clara, CA 95053, USA.

[S.1]The notion of informativeness is closely related to that of information studied by Bonanno (2002). Ghirardato (2001), Lipman (1995), and Mukerji (1997) also study information processing in which informational contents are ranked by the implication in the form of set inclusion.

(i.e., the relation induced by Expression (S.1)) is reflexive and transitive, the notion of equal informativeness (i.e., the relation induced by Expression (S.2)) is an equivalence relation.

Three remarks on Definition S.1 are in order. First, when $\mathcal{X}$ is not necessarily closed under complementation, Definition S.1 does not take into account the collection of observational contents $\{F \in \mathcal{X} \mid x(\omega) \notin F\}$ that do not hold at ω. In contrast, when $\mathcal{X}$ is closed under complementation, it can be seen that if ω is at least as informative as ω' according to a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$, then ω and ω' are equally informative.

Second, suppose that B_i satisfies Consistency, Positive Introspection, and Negative Introspection. If ω is at least as informative as ω' according to a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$, then ω and ω' are equally informative.[S.2]

Third, under the assumption that $x^{-1}(\{x(\omega)\}) \in \mathcal{D}$ for each $\omega \in \Omega$, the equivalence relation of equal informativeness coincides with the one induced by the partition $\{x^{-1}(\{x(\omega)\}) \mid \omega \in \Omega\}$: ω and ω' are equally informative iff $x(\omega) = x(\omega')$.

The following proposition characterizes the certainty of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ from informativeness. Namely, player i is certain of the signal x iff the notion of possibility derived from her beliefs is incorporated in the notion of informativeness derived from the signal in the following sense: whenever player i considers state ω' possible at state ω (i.e., $\omega' \in b_{B_i}(\omega)$), state ω' is at least as informative as state ω according to $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$.

Proposition S.1. *Assume the Kripke property for B_i. Player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff possibility implies informativeness (i.e., if $\omega' \in b_{B_i}(\omega)$ then ω' is at least as informative as ω according to $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$).*

Next, I apply the notion of informativeness to i's qualitative-type mapping $t_i : \Omega \to M(\Omega)$ with respect to $\{\beta_E \mid E \in \mathcal{D}\}$. That is, suppose that player i is reasoning about the underlying states based on her possession of beliefs. For states ω and ω' in Ω, ω is at least as informative as ω' to i (precisely, according to $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \{\beta_E \mid E \in \mathcal{D}\})$) iff $t_i(\omega')(\cdot) \leq t_i(\omega)(\cdot)$ (i.e., $t_i(\omega')(E) \leq t_i(\omega)(E)$ for all $E \in \mathcal{D}$). Likewise, states ω and ω' are equally informative according to i iff $t_i(\omega) = t_i(\omega')$.

Fix $\omega \in \Omega$, and let $(\uparrow t_i(\omega)) := \{\omega' \in \Omega \mid t_i(\omega)(\cdot) \leq t_i(\omega')(\cdot)\}$ be the set of states that are at least as informative to i as ω. Also, define $(\downarrow t_i(\omega)) := \{\omega' \in \Omega \mid t_i(\omega')(\cdot) \leq t_i(\omega)(\cdot)\}$ and $[t_i(\omega)] := \{\omega' \in \Omega \mid t_i(\omega) = t_i(\omega')\}$. If $\omega' \in [t_i(\omega)]$ then ω and ω' are indistinguishable to player i in that her qualitative-types (and thus the collections of events that she believes) are exactly the same at these states. Put differently, the equal informativeness is translated into the indistinguishability. Thus, the collection

[S.2] The proof goes as follows. Suppose to the contrary that there are $\omega, \omega' \in \Omega$ such that $\{F \in \mathcal{X} \mid \omega' \in B_i(x^{-1}(F))\} \subsetneq \{F \in \mathcal{X} \mid \omega \in B_i(x^{-1}(F))\}$. Then, there is $F \in \mathcal{X}$ with the following properties: $\omega \in B_i(x^{-1}(F)) \subseteq B_i B_i(x^{-1}(F))$ (by Positive Introspection) and $\omega' \in (\neg B_i)(x^{-1}(F)) \subseteq B_i(\neg B_i)(x^{-1}(F))$ (by Negative Introspection), a contradiction to Consistency.

$\{[t_i(\omega)] \mid \omega \in \Omega\}$ forms a partition of Ω generated by the qualitative-type mapping t_i. Note that $(\uparrow t_i(\omega))$, $(\downarrow t_i(\omega))$, and $[t_i(\omega)]$ may not necessarily be an event.

Now, I examine the sense in which a player is certain of her qualitative-type mapping by studying how introspective properties imply the relations between informativeness and possibility.

Proposition S.2. *Let $\overrightarrow{\Omega}$ be a belief model. Let $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \{\beta_E \mid E \in \mathcal{D}\})$ be player i's qualitative-type mapping.*

1. *(a) B_i satisfies Truth Axiom iff $(\uparrow t_{B_i}(\cdot)) \subseteq b_{B_i}(\cdot)$.*

   (b) If B_i satisfies Positive Introspection, then $b_{B_i}(\cdot) \subseteq (\uparrow t_{B_i}(\cdot))$. If B_i satisfies the Kripke property, the converse also holds.

   (c) If B_i satisfies Negative Introspection, then $b_{B_i}(\cdot) \subseteq (\downarrow t_{B_i}(\cdot))$. If B_i satisfies the Kripke property, the converse also holds.

2. *(a) If B_i satisfies Truth Axiom and Positive Introspection, then $(\uparrow t_{B_i}(\cdot)) = b_{B_i}(\cdot)$. If t_{B_i} satisfies the Kripke property, the converse also holds.*

   (b) If B_i satisfies Truth Axiom, (Positive Introspection), and Negative Introspection, then $(\uparrow t_{B_i}(\cdot)) = [t_{B_i}(\cdot)] = b_{B_i}(\cdot)$. If B_i satisfies the Kripke property, the converse also holds.

Part (1a) states that, under Truth Axiom, informativeness implies possibility. In Part (1b), since player i is certain of her qualitative-type mapping t_i with respect to $\{\beta_E \mid E \in \mathcal{D}\}$, the notion of possibility that comes from her beliefs is already encoded in the notion of informativeness. That is, Part (1b) states that possibility implies informativeness when player i is certain of her qualitative-type mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \{\beta_E \mid E \in \mathcal{D}\})$. Hence, when player i's qualitative-type mapping satisfies Truth Axiom and Positive Introspection as in a reflexive-and-transitive possibility correspondence model (see footnote 10), the notions of informativeness and possibility coincide: $b_{B_i}(\cdot) = (\uparrow t_{B_i}(\cdot))$.

Part (1b) and (1c) jointly state that, under Positive Introspection and Negative Introspection, if player i considers ω' possible at ω then the states ω and ω' are equally informative.

In a model of knowledge in which player i's qualitative-type mapping satisfies Truth Axiom, (Positive Introspection,) and Negative Introspection, either notion of informativeness or possibility induces the same partition $\{b_{B_i}(\omega) \mid \omega \in \Omega\} = \{[t_{B_i}(\omega)] \mid \omega \in \Omega\}$ of Ω with the following property: if $\omega' \in [t_i(\omega)] = b_{B_i}(\omega)$, then, for any event, she knows it at ω iff she knows it at ω'. In a model of qualitative belief in which player i's qualitative-type mapping satisfies Consistency, Positive Introspection, and Negative Introspection, $\emptyset \neq b_{B_i}(\cdot) \subseteq [t_{B_i}(\cdot)] = (\uparrow t_{B_i}(\cdot))$.

C Applications to Probabilistic Beliefs

While the main text analyzed the (meta-)certainty of a qualitative-belief model, this appendix demonstrates that the framework of this paper allows one to analyze the (meta-)certainty of a probabilistic-belief model. Appendix C.1 shows that a "type space" is accommodated as a belief model of the framework of this paper. Appendix C.2 studies the certainty of one's own probabilistic-type mapping, paralleling Section 3. Appendix C.3 introduces the notion of informativess as in Appendix B. Appendix C.4 studies the meta-common-certainty of a type space, paralleling Section 4. Appendix C.5 discusses epistemic characterizations of mixed-strategy Nash equilibria to study the role that the meta-certainty of a model plays, corresponding to Section 5.

C.1 A Type Space as a Belief Model

A (probabilistic-)type (on Ω) is a σ-additive probability measure $\nu \in \Delta(\Omega)$.[S.3] This appendix considers a type space of the form $\langle(\Omega, \mathcal{D}), (\tau_i)_{i \in I}\rangle$ with the following two ingredients. First, $(\Omega, \mathcal{D})$ is a measurable state space. Second, player i's (probabilistic-)type mapping τ_i is a measurable mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, where $\mathcal{D}_\Delta$ is the σ-algebra generated by $\beta_E^p := \{\nu \in \Delta(\Omega) \mid \nu(E) \geq p\}$ for all $(E, p) \in \mathcal{D} \times [0, 1]$ (Heifetz and Samet, 1998). It associates, with each state ω, the player's probabilistic beliefs $\tau_i(\omega) \in \Delta(\Omega)$ at that state.

The type mapping τ_i of player i induces her p-belief operator $B^p_{\tau_i}$ (Monderer and Samet, 1989): it associates, with each event E, the event that (i.e., the set of states at which) player i believes E with probability at least p (i.e., p-believes E). Formally, $B^p_{\tau_i}(E) := \tau_i^{-1}(\beta_E^p)$. Thus, $\omega \in B^p_{\tau_i}(E)$ iff $\tau_i(\omega)(E) \geq p$. As in Samet (2000), the type mapping τ_i and the collection of p-belief operators $(B^p_{\tau_i})_{p \in [0,1]}$ are equivalent, that is, a type space of the form $\langle(\Omega, \mathcal{D}), (\tau_i)_{i \in I}\rangle$ is equivalent to $\langle(\Omega, \mathcal{D}), (B^p_{\tau_i})_{(i,p) \in I \times [0,1]}\rangle$.[S.4]

Since each $B^1_{\tau_i}$ satisfies Monotonicity and Countable Conjunction, one can introduce the common 1-belief operator C^1. More generally, Proposition S.4 in Appendix C.2 shows that, when each player is certain of her own type mapping τ_i, the common p-belief operator C^p is well-defined and reduces to the chain of mutual p-beliefs.

C.2 Certainty of Own Probabilistic-Type Mapping

Next, I study when a player is certain of her own (probabilistic-)type mapping. Thus, I add to a belief model (in which $(B_i)_{i \in I}$ is a primitive) each player i's (probabilistic-)type mapping τ_i, which induces her p-belief operator $B^p_{\tau_i}$. Especially, the case with

[S.3] While one can analyze finitely-additive or non-additive beliefs, for ease of exposition I focus on σ-additive probabilistic beliefs when it comes to quantitative beliefs.

[S.4] This framework also enables one to analyze both qualitative and probabilistic beliefs at the same time (e.g., Fukuda, 2024): for example, in an extensive-form game with perfect information, each player has knowledge about players' past moves while she has beliefs about the future moves of the opponents.

$B_i = B^1_{\tau_i}$ studies whether player i is certain of her type mapping within the type space $\langle(\Omega, \mathcal{D}), (\tau_i)_{i\in I}\rangle$ itself. In the case in which B_i is either a knowledge or qualitative belief operator, the outside analysts consider players' knowledge or qualitative beliefs about their probabilistic beliefs.

As discussed in Section 3, a belief operator B_i satisfies *Positive Certainty* (with respect to $B^p_{\tau_i}$) if $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$. Likewise, B_i satisfies *Negative Certainty* (with respect to $B^p_{\tau_i}$) if $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$. With these in mind:

Proposition S.3. *Let $\overrightarrow{\Omega}$ be a belief model, and let $\tau_i : \Omega \to \Delta(\Omega)$ be player i's type mapping.*

1. (a) *Player i is certain of her type mapping τ_i with respect to $\{\beta^p_E \mid (p, E) \in [0,1] \times \mathcal{D}\}$ iff B_i satisfies Positive Certainty: $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$.*

   (b) *Player i is certain of her type mapping τ_i with respect to $\{\neg\beta^p_E \mid (p, E) \in [0,1] \times \mathcal{D}\}$ iff B_i satisfies Negative Certainty: $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$.*

   (c) *If player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then B_i satisfies Positive Certainty and Negative Certainty.*

2. (a) *Let B_i satisfy Truth Axiom and Negative Introspection. Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff B_i satisfies Positive Certainty iff B_i satisfies Negative Certainty.*

   (b) *Let B_i satisfy Consistency, Positive Introspection, and Negative Introspection. Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff B_i satisfies Positive Certainty and Negative Certainty.*

   (c) *Let B_i satisfy Entailment: $B_i(\cdot) \subseteq B^1_{\tau_i}$. Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff B_i satisfies Positive Certainty and Negative Certainty.*

Part (1) characterizes the statement that player i is certain of her type mapping τ_i with respect to the possession of p-beliefs $\{\beta^p_E \mid (E, p) \in \mathcal{D} \times [0,1]\}$ or the lack of p-beliefs $\{\neg\beta^p_E \mid (E, p) \in \mathcal{D} \times [0,1]\}$. It also states that if player i is certain of the type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then the belief operator B_i satisfies Positive Certainty and Negative Certainty: $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$.

Part (2a) corresponds to the case when B_i is a fully-introspective knowledge operator in addition to her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$. Part (2b) corresponds to the case in which B_i is a fully-introspective qualitative belief operator in addition to her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$. When probabilistic beliefs and knowledge (or qualitative belief) are present, the introspective properties of Positive Certainty $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ and Negative Certainty $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$ are the standard assumptions (e.g., Aumann, 1999). Whenever player i believes an event E with probability at least p, she knows that she believes E with probability at least p. Whenever player i does not believe an event E with probability at least p, she knows that she does not believe E with probability at least p. In this environment, Parts (2a) and (2b) formalize the sense in which player i is certain of her probabilistic beliefs (her type mapping).

Part (2c) sheds light on the certainty of a type mapping in the type space (i.e., purely probabilistic model) in the case in which B_i is taken as the probability 1-belief operator $B^1_{\tau_i}$. The introspective properties of probabilistic beliefs are now formulated in terms of probability-one belief about own beliefs: $B^p_{\tau_i}(\cdot) \subseteq B^1_{\tau_i} B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(\cdot)$. That is, if player i p-believes an event E, then she believes with probability one that she p-believes E; and if player i does not p-believe an event E, then she believes with probability one that she does not p-believe E. These two introspective properties are essential in the syntactic formulation of type spaces such as Heifetz and Mongin (2001) and Meier (2012). Part (2c) justifies the statement that player i is certain of her own type mapping in a type space.

I remark on two additional implications of Proposition S.3. First, Proposition S.3 and Remark 5 allow one to formalize the sense in which each player is certain of her "prior." Consider a model $\langle (\Omega, \mathcal{D}), (\tau_i)_{i \in I}, (\mu_i)_{i \in I} \rangle$ with the following properties: $(\Omega, \mathcal{D})$ is a measurable space, $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ is player i's (probabilistic-)type mapping, and $\mu_i \in \Delta(\Omega)$ is a *prior* satisfying

$$\mu_i(E) = \int_\Omega \tau_i(\omega)(E) \mu_i(d\omega) \text{ for all } E \in \mathcal{D}. \tag{S.3}$$

That is, the prior belief $\mu_i(E)$ is equal to the expectation of the posterior beliefs $t_i(\cdot)(E)$ with respect to the prior μ_i (e.g., Mertens and Zamir, 1985). The model admits a *common prior* if $\mu_i = \mu_j$ for all $i, j \in I$. If each μ_i would be identified as a constant mapping $\mu_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then

$$\mu_i^{-1}(\beta^p_E) = \begin{cases} \emptyset & \text{if } \mu_i(E) < p \\ \Omega & \text{if } \mu_i(E) \geq p \end{cases}.$$

Since $B^1_{\tau_i}$ satisfies Necessitation, each player i is certain of every player j's prior. In fact, the players are commonly certain of the priors.

Second, if the players are certain of their own type mappings, then the common p-belief operator reduces to the iteration of mutual p-beliefs. To that end, I formally define the common p-belief operator C^p following Monderer and Samet (1989). As in the main text, an event $E \in \mathcal{D}$ is *p-evident* to player i if $E \subseteq B^p_{\tau_i}(E)$: player i p-believes E whenever E is true. Denote by $\mathcal{J}_{B^p_{\tau_i}}$ the collection of p-evident events to player i. The event E is *publicly p-evident* if it is p-evident to every player $i \in I$. Then, the set of states at which E is common p-belief is defined as:

$$C^p(E) := \{\omega \in \Omega \mid \text{there is } F \in \bigcap_{i \in I} \mathcal{J}_{B^p_{\tau_i}} \text{ with } \omega \in F \subseteq B^p_I(E)\},$$

where $B^p_I : \mathcal{D} \to \mathcal{D}$ is the mutual p-belief operator defined by $B^p_I(\cdot) := \bigcap_{i \in I} B^p_{\tau_i}(\cdot)$. An event E is common p-belief at a state ω iff there is a publicly-p-evident event that is true at ω and that implies the mutual p-belief in E.

Proposition S.4. *Let $(\Omega, \mathcal{D})$ be a measurable space, and let $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ be player i's type mapping for each $i \in I$. Let $B^1_{\tau_i}$ be player i's probability-one belief operator. If each player i is certain of her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ in the model $\langle(\Omega, \mathcal{D}), (B^1_{\tau_i})_{i\in I}\rangle$, then the common p-belief operator reduces to the iteration of mutual p-beliefs: $C^p(\cdot) = \bigcap_{n\in\mathbb{N}}(B^p_I)^n(\cdot)$.*

C.3 Informativeness

Next, I apply the notion of informativeness to player i's probabilistic-type mapping $\tau_i : \Omega \to \Delta(\Omega)$ with respect to $\{\beta^p_E \mid (E, p) \in \mathcal{D} \times [0, 1]\}$. That is, player i is reasoning about the underlying states based on her possession of p-beliefs. Since the notion of possibility comes from qualitative beliefs, I start with a model that has both qualitative and probabilistic beliefs.

A state ω is at least as informative as a state ω' to i (precisely, according to $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\beta^p_E \mid (E, p) \in \mathcal{D} \times [0, 1]\})$) iff $\tau_i(\omega')(\cdot) \leq \tau_i(\omega)(\cdot)$. However, since each $\tau_i(\cdot)$ is (σ-)additive, it follows that ω is at least as informative as ω' to i iff ω and ω' are equally informative: $\omega' \in [\tau_i(\omega)] := \{\omega'' \in \Omega \mid \tau_i(\omega'') = \tau_i(\omega)\}$. If player i does not believe an event E with probability at least p at a state, then she does believe E^c with probability at least $1 - p$. Since player i is able to reason about the possession of beliefs for any event and any probability, she is also able to reason about the lack of beliefs when her probabilistic beliefs are (σ-)additive.[S.5]

Proposition S.5. *Let $\overrightarrow{\Omega}$ be a belief model, and let $\tau_i : \Omega \to \Delta(\Omega)$ be player i's type mapping.*

1. *Either Positive Certainty $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ or Negative Certainty $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$ yields $b_{B_i}(\cdot) \subseteq [\tau_i(\cdot)]$: possibility implies (equal) informativeness.*
2. *Under the Kripke property of B_i, conversely, $b_{B_i}(\cdot) \subseteq [\tau_i(\cdot)]$ implies Positive Certainty $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ and Negative Certainty $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$.*

C.3.1 Harsanyi Property

In the literature on type spaces (e.g., Meier, 2008, 2012; Mertens and Zamir, 1985), a player is "certain" of her own type if, at each state ω, she believes, with probability one, the set of states indistinguishable from (i.e., equally informative to) ω.[S.6] I show the sense in which such "Harsanyi" property holds iff the player is certain of her own type mapping. Thus, my characterization also formally captures the original idea behind Harsanyi (1967-1968) that each player "is certain of" her own type mapping.

[S.5]While one can obtain a nuanced understanding of the relation between the informativeness and certainty of a type mapping τ_i when each $\tau_i(\cdot)$ is a non-additive measure, I focus on studying the sense in which player i is certain of her σ-additive type mapping τ_i. Recall footnote S.3.

[S.6]In applications such as robust mechanism design, Bergemann and Morris (2005, Sections 2.2 and 2.5) for instance impose the condition that each player is certain of her own ("payoff-")type.

Formally, player i's type mapping $\tau_i : \Omega \to \Delta(\Omega)$ satisfies the *Harsanyi property* if $[\tau_i(\omega)] \subseteq E$ implies $\omega \in B^1_{\tau_i}(E)$ for any $(\omega, E) \in \Omega \times \mathcal{D}$. That is, whenever an event E is implied by the set of states $[\tau_i(\omega)]$ indistinguishable from ω, player i believes E with probability one at ω.

Under the regularity condition $[\tau_i(\cdot)] \in \mathcal{D}$, since each $\tau_i(\omega)$ satisfies Monotonicity, the Harsanyi property is equivalent to $\tau_i(\omega)([\tau_i(\omega)]) = 1$ for each $\omega \in \Omega$. It states that, at each state, player i assigns probability one to the set of states indistinguishable from that state. In fact, in the type-space literature, the informal assumption that each player is certain of her own type is formally represented as the condition on the type mapping to put probability one on the set of types indistinguishable from its own (Mertens and Zamir, 1985; Vassilakis and Zamir, 1993).

In a type space, I show that the Harsanyi property characterizes the idea that a player is certain of her own type mapping with respect to the beliefs that she could have been able to possess.

Proposition S.6. *Let $(\Omega, \mathcal{D})$ be a measurable space, and let $\tau_i : \Omega \to \Delta(\Omega)$ be player i's type mapping.*

1. *Suppose that $[\tau_i(\cdot)] \in \mathcal{D}$. The type mapping τ_i satisfies the Harsanyi property iff player i is certain of $\tau_i : \Omega \to \Delta(\Omega)$ with respect to her realized beliefs $\{\{\tau_i(\omega)\} \mid \omega \in \Omega\}$.*

2. *Let $\mathcal{D}$ be generated from a countable algebra. The following are all equivalent.*
   - *(a) The type mapping τ_i satisfies the Harsanyi property.*
   - *(b) Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\{\tau_i(\omega)\} \mid \omega \in \Omega\})$.*
   - *(c) Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$.*
   - *(d) Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\beta^p_E \mid (E, p) \in \mathcal{D} \times [0, 1]\})$.*
   - *(e) Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\neg\beta^p_E \mid (E, p) \in \mathcal{D} \times [0, 1]\})$.*

The proposition implies that, under the regularity condition $[\tau_i(\cdot)] \in \mathcal{D}$, the Harsanyi property is equivalent to $B^p_{\tau_i}(\cdot) \subseteq B^1_{\tau_i} B^p_{\tau_i}(\cdot)$ or $(\neg B^p_{\tau_i})(\cdot) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(\cdot)$.[S.7]

C.4 Meta-Common-Certainty of a Type Space

I examine when the players are commonly certain of their probabilistic-type mappings. Similarly to Proposition S.3, one can show: if player i is certain of player j's type mapping $\tau_j : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then $B^p_{\tau_j}(\cdot) \subseteq B_i B^p_{\tau_j}(\cdot)$ and $(\neg B^p_{\tau_j})(\cdot) \subseteq B_i(\neg B^p_{\tau_j})(\cdot)$ hold. Formally, similarly to Proposition S.3, the following can be shown:

[S.7] The Harsanyi property can also be applied to qualitative beliefs. The old working paper contains the qualitative-belief analogue of Proposition S.6 and additional results.

Remark S.1. Let $\overrightarrow{\Omega}$ be a belief model, and let $\tau_j : \Omega \to \Delta(\Omega)$ be player j's type mapping.

1. Player i is certain of τ_j with respect to $\{\beta_E^p \mid (E,p) \in \mathcal{D} \times [0,1]\}$ iff $B_{\tau_j}^p(\cdot) \subseteq B_i B_{\tau_j}^p(\cdot)$.
2. Player i is certain of τ_j with respect to $\{\neg\beta_E^p \mid (E,p) \in \mathcal{D} \times [0,1]\}$ iff $(\neg B_{\tau_j}^p)(\cdot) \subseteq B_i(\neg B_{\tau_j}^p)(\cdot)$.
3. If player i is certain of $\tau_j : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then $B_{\tau_j}^p(\cdot) \subseteq B_i B_{\tau_j}^p(\cdot)$ and $(\neg B_{\tau_j}^p)(\cdot) \subseteq B_i(\neg B_{\tau_j}^p)(\cdot)$. The converse holds when B_i satisfies Consistency, Positive Introspection, and Negative Introspection.

As in Remark 3, Remark S.1 roughly states that player i is certain of player j's probabilistic-type mapping iff (i) whenever player j p-believes an event E at ω, player i believes that player j p-believes the event E at ω; and (ii) whenever player j does not p-believe an event E at ω, player i believes that player j does not p-believe the event E at ω.

Now, I ask one of the main questions of this paper in the context of probabilistic-type mappings: when are the players in a belief model commonly certain of their probabilistic-type mappings?

Theorem S.1. *Let $\overrightarrow{\Omega}$ be a belief model, and let $\tau_i : \Omega \to \Delta(\Omega)$ be player i's type mapping for each $i \in I$. Assume Consistency, Positive Introspection, and Negative Introspection for every B_i. The players are commonly certain of the type mappings $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff $B_{\tau_i}^p(\cdot) \subseteq CB_{\tau_i}^p(\cdot)$ and $(\neg B_{\tau_i}^p)(\cdot) \subseteq C(\neg B_{\tau_i}^p)(\cdot)$ for every $(i,p) \in I \times [0,1]$. If $B_i = B_{\tau_i}^1$ is taken for every $i \in I$, then $C^1 = B_I^1$.*

Roughly, Theorem S.1 states: the players are commonly certain of their probabilistic-type mappings iff (i) for any event E which some player i p-believes at some state ω, it is commonly believed that player i p-believes E at ω; and (ii) for any event E which some player i does not p-believe at some state ω, it is commonly believed that player i does not p-believe E at ω. If each player's belief operator B_i in the belief model is taken as probability-one belief operator $B_{\tau_i}^1$, then the probability-one common belief operator reduces to the probability-one mutual belief operator.

Lastly, I present a consequence of the meta-common-certainty of a belief model in the context of probabilistic beliefs, paralleling Proposition 2.

Proposition S.7. *Let $\overrightarrow{\Omega}$ be a belief model such that each B_i satisfies Consistency. Let $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ be a signal mapping such that, for any $F \in \mathcal{X}$, there exists a sub-collection $(F_\lambda)_{\lambda \in \Lambda}$ of $\mathcal{X}$ with $F^c = \bigcup_{\lambda \in \Lambda} F_\lambda$.*

1. *Let $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ be player i's type mapping, and assume Entailment: $B_i(\cdot) \subseteq B_{\tau_i}^1(\cdot)$. If player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ and if player j is certain of player i's type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then player j is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$.*

2. *Suppose that the players are commonly certain of their type mappings* $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$. *Suppose Entailment for every player* i: $B_i(\cdot) \subseteq B^1_{\tau_i}(\cdot)$ *Then, player* i *is certain of* $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ *iff player* j *is certain of* $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$.

C.5 Mixed-Strategy Nash Equilibria

Here, I briefly consider an epistemic characterization of mixed-strategy Nash equilibria. A *(strategic) game* is a tuple $\Gamma = \langle (A_i)_{i \in I}, (u_i)_{i \in I} \rangle$, where each A_i is a non-empty finite set of player i's actions, and $u_i : A \to \mathbb{R}$ is player i's von-Neumann Morgenstern utility function. I simply focus on players' probabilistic beliefs. Thus, a (belief) *model* of the game Γ is a tuple $\langle (\Omega, \mathcal{D}), (\tau_i)_{i \in I}, (\sigma_i)_{i \in I} \rangle$: each $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ is player i's type mapping, and $\sigma_i : (\Omega, \mathcal{D}) \to (A_i, \mathcal{P}(A_i))$ is player i's (pure) strategy. In this context, player i is *rational* at ω if

$$\int u_i(\sigma(\tilde{\omega}))\tau_i(\omega)(d\tilde{\omega}) \geq \int u_i(a_i, \sigma_{-i}(\tilde{\omega}))\tau_i(\omega)(d\tilde{\omega}) \text{ for all } a_i \in A_i.$$

I start with the two-players case: $I = \{1, 2\}$. One of the well-known epistemic characterizations of mixed-strategy Nash equilibria is stated as follows (e.g., Aumann and Brandenburger, 1995; Stalnaker, 1994): if each player i is certain of the other's beliefs about i's strategy choice at ω (i.e., player i is certain of the conjecture $\tau_j(\omega) \circ \sigma_i^{-1} \in \Delta(A_i)$ where j is the opponent) and if each player i 1-believes that the other is rational at ω, then the resulting pair of conjectures $(\tau_2(\omega) \circ \sigma_1^{-1}, \tau_1(\omega) \circ \sigma_2^{-1})$ constitutes a mixed-strategy Nash equilibrium. In this statement, the common certainty of the model is not required.[S.8] In this epistemic characterization, each player i is certain not of the other's type mapping τ_j but of the conjecture (j's beliefs about i's actions).

Next, consider the case in which $I = \{1, 2, \dots, n\}$. Suppose that each player's type mapping τ_i is induced from a common prior μ (recall Expression (S.3) with respect to $\mu_i = \mu$). For ease of exposition, restrict attention to the case in which Ω is finite and the common prior puts positive probability to every state. If the players mutually 1-believe that they are rational at ω and if they are commonly certain of their conjectures at ω, then, for each player j, all the conjectures of players $i \in I \setminus \{j\}$ induce the same conjecture $\phi_j \in \Delta(A_j)$, and $(\phi_j)_{j \in I}$ is a mixed-strategy Nash equilibrium.

Now, I provide a simple example which illustrates that the common certainty of the model is not required for the above epistemic characterizations of mixed-strategy Nash equilibria.

Example S.1. Consider the following three-players coordination game depicted by Table S.1. Player 1 chooses a row, player 2 does a column, and player 3 does a matrix.

[S.8] I will provide an example in the context of $|I| \geq 3$, which requires a tighter condition on the players' beliefs.

	L	R
U	1, 1, 1	0, 0, 0
D	0, 0, 0	0, 0, 0

A

	L	R
U	0, 0, 0	0, 0, 0
D	0, 0, 0	1, 1, 1

B

Table S.1: Three-players Coordination Game in Example S.1

Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \ldots, \omega_8\}, \mathcal{P}(\Omega))$. Assume that there is a uniform common prior $\mu = (\frac{1}{8}, \frac{1}{8}, \ldots, \frac{1}{8})$. For player 1, let

$$\tau_1(\omega) = \begin{cases} (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0) & \text{if } \omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\} \\ (0, 0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) & \text{if } \omega \in \{\omega_5, \omega_6, \omega_7, \omega_8\} \end{cases}.$$

For player 2, let

$$\tau_2(\omega) = \begin{cases} (\frac{1}{4}, \frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, 0, 0) & \text{if } \omega \in \{\omega_1, \omega_2, \omega_5, \omega_6\} \\ (0, 0, \frac{1}{4}, \frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}) & \text{if } \omega \in \{\omega_3, \omega_4, \omega_7, \omega_8\} \end{cases}.$$

For player 3, let

$$\tau_3(\omega) = \begin{cases} (\frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}0) & \text{if } \omega \in \{\omega_1, \omega_3, \omega_5, \omega_7\} \\ (0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}) & \text{if } \omega \in \{\omega_2, \omega_4, \omega_6, \omega_8\} \end{cases}.$$

Thus, the players are not commonly certain of the model (i.e., their type mappings). Let

$$\begin{aligned} \sigma_1 &= (U, U, U, U, D, D, D, D), \\ \sigma_2 &= (L, L, R, R, L, L, R, R), \text{ and} \\ \sigma_3 &= (A, B, A, B, A, B, A, B). \end{aligned}$$

Now, it can be seen that the conditions for the above epistemic characterization are met, and their conjectures satisfy the following, which constitute a mixed-strategy Nash equilibrium in which every player is mixing with probability $\frac{1}{2}$:

$$\begin{aligned} \phi_1 &= \tau_j(\omega) \circ \sigma_1^{-1} = \left(\frac{1}{2}, \frac{1}{2}\right) \text{ on } (U, D) \text{ for each } j \neq 1, \\ \phi_2 &= \tau_j(\omega) \circ \sigma_2^{-1} = \left(\frac{1}{2}, \frac{1}{2}\right) \text{ on } (L, R) \text{ for each } j \neq 2, \text{ and} \\ \phi_3 &= \tau_j(\omega) \circ \sigma_3^{-1} = \left(\frac{1}{2}, \frac{1}{2}\right) \text{ on } (A, B) \text{ for each } j \neq 3. \end{aligned}$$

D Proofs

D.1 Appendix B

Proof of Proposition S.1. For the "only if" part, suppose that player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$. Take $\omega' \in b_{B_i}(\omega)$. For any $F \in \mathcal{X}$ with $\omega \in x^{-1}(F)$, it follows from the supposition that $\omega \in B_i(x^{-1}(F))$. By the definition of b_{B_i}, I have $\omega' \in b_{B_i}(\omega) \subseteq x^{-1}(F)$. For the "if" part, assume the Kripke property. Suppose that possibility implies informativeness. Take any $\omega \in \Omega$ and $F \in \mathcal{X}$ with $\omega \in x^{-1}(F)$. To show $\omega \in B_i(x^{-1}(F))$, it is enough to show $b_{B_i}(\omega) \subseteq x^{-1}(F)$. Now, if $\omega' \in b_{B_i}(\omega)$, then it follows from the supposition that $\omega' \in x^{-1}(F)$. □

Proof of Proposition S.2. Observe that if ω' is at least as informative to i as ω according to t_{B_i} (i.e., $\omega' \in (\uparrow t_{B_i}(\omega))$), then

$$b_{B_i}(\omega') = \bigcap\{E \in \mathcal{D} \mid t_{B_i}(\omega')(E) = 1\} \subseteq \bigcap\{E \in \mathcal{D} \mid t_{B_i}(\omega)(E) = 1\} = b_{B_i}(\omega).$$

Moreover, if t_{B_i} satisfies the Kripke property, then the converse also holds: $b_{B_i}(\omega') \subseteq b_{B_i}(\omega)$ implies $\omega' \in (\uparrow t_{B_i}(\omega))$. This is because, if $t_{B_i}(\omega)(E) = 1$ then $b_{B_i}(\omega') \subseteq b_{B_i}(\omega) \subseteq E$ and thus $t_{B_i}(\omega')(E) = 1$.

1. (a) Since Truth Axiom yields $\omega' \in b_{B_i}(\omega')$ for all $\omega' \in \Omega$, it follows that $\omega' \in b_{B_i}(\omega') \subseteq b_{B_i}(\omega)$ for all $\omega' \in (\uparrow t_{B_i}(\omega))$. Conversely, Truth Axiom follows from $\omega \in (\uparrow t_{B_i}(\omega)) \subseteq b_{B_i}(\omega)$ for all $\omega \in \Omega$.

   (b) Suppose $\omega' \in b_{B_i}(\omega)$. For any $F \in \mathcal{D}$ with $t_{B_i}(\omega)(F) = 1$, it follows from Positive Introspection that $t_{B_i}(\omega)(t_{B_i}^{-1}(\beta_F)) = 1$. By the supposition, $\omega' \in b_{B_i}(\omega) \subseteq t_{B_i}^{-1}(\beta_F)$, and hence $t_{B_i}(\omega')(F) = 1$. Thus, $\omega' \in (\uparrow t_{B_i}(\omega))$.

   Conversely, let B_i satisfy the Kripke property, and assume $b_{B_i}(\cdot) \subseteq (\uparrow t_{B_i}(\cdot))$. Suppose $\omega \in B_i(E)$. In order to show $\omega \in B_iB_i(E)$, it is enough to prove $\omega' \in B_i(E)$ for all $\omega' \in b_{B_i}(\omega)$. Take any $\omega' \in b_{B_i}(\omega)$. Since $\omega' \in (\uparrow t_{B_i}(\omega))$ and $\omega \in B_i(E)$, it follows that $\omega' \in B_i(E)$.

   (c) The proof is analogous to Part (1b). Suppose $\omega' \in b_{B_i}(\omega)$. Suppose to the contrary that $\omega' \notin (\downarrow t_{B_i}(\omega))$, i.e., $t_{B_i}(\omega)(F) = 0 < 1 = t_{B_i}(\omega')(F)$ for some $F \in \mathcal{D}$. By Negative Introspection, $t_{B_i}(\omega)(\neg t_{B_i}^{-1}(\beta_F)) = 1$, and thus $\omega' \in b_{B_i}(\omega) \subseteq \neg t_{B_i}^{-1}(\beta_F)$, and hence $t_{B_i}(\omega')(F) = 0$, a contradiction.

   Conversely, let B_i satisfy the Kripke property, and suppose $b_{B_i}(\cdot) \subseteq (\downarrow t_{B_i}(\cdot))$. If $\omega \notin B_i(E)$, then $b_{B_i}(\omega) \cap E^c \neq \emptyset$. In order to establish $\omega \in B_i(\neg B_i)(E)$, it is enough to show that $b_{B_i}(\omega') \cap E^c \neq \emptyset$ for all $\omega' \in b_{B_i}(\omega)$. Take any $\omega' \in b_{B_i}(\omega)$. Since $\omega' \in (\downarrow t_{B_i}(\omega))$ and since $t_{B_i}(\omega)(E) = 0$, it follows $t_{B_i}(\omega')(E) = 0$, i.e., $b_{B_i}(\omega') \cap E^c \neq \emptyset$.

2. The assertion follows from Part (1).

□

D.2 Appendix C

Proof of Proposition S.3. 1. For (1a), player i is certain of τ_i with respect to $\{\beta_E^p \mid (E,p) \in \mathcal{D} \times [0,1]\}$ iff $B_{\tau_i}^p(E) \subseteq B_i B_{\tau_i}^p(E)$, as $B_{\tau_i}^p(E) = \tau_i^{-1}(\beta_E^p)$.

For (1b), player i is certain of τ_i with respect to $\{\neg\beta_E^p \mid (E,p) \in \mathcal{D} \times [0,1]\}$ iff $(\neg B_{\tau_i}^p)(E) \subseteq B_i(\neg B_{\tau_i}^p)(E)$, as $(\neg B_{\tau_i}^p)(E) = \tau_i^{-1}(\neg\beta_E^p)$. Then, (1c) follows from the previous two parts.

2. (a) If player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ then B_i satisfies Positive Certainty. Conversely, let B_i satisfy Positive Certainty. By (1a), $\tau_i^{-1}(\{\beta_E^p \mid (E,p) \in \mathcal{D} \times [0,1]\}) \subseteq \mathcal{J}_{B_i}$. Since B_i satisfies Truth Axiom and Negative Introspection, $\mathcal{J}_{B_i}$ is a sub-σ-algebra of $\mathcal{D}$. Thus, $\tau_i^{-1}(\mathcal{D}_\Delta) \subseteq \mathcal{J}_{B_i}$. Hence, player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$.

   Next, I show that, since B_i satisfies Truth Axiom and Negative Introspection, Positive Certainty is equivalent to Negative Certainty. Assume Positive Certainty. Then, $(\neg B_{\tau_i}^p) = (\neg B_i) B_{\tau_i}^p = B_i(\neg B_i) B_{\tau_i}^p = B_i(\neg B_{\tau_i}^p)$. The first and third equalities follow from Positive Certainty and Truth Axiom, and the second from Negative Introspection and Truth Axiom.

   Conversely, assume Negative Certainty. Then, $B_{\tau_i}^p = (\neg B_i)(\neg B_{\tau_i}^p) = B_i(\neg B_i)(\neg B_{\tau_i}^p) = B_i B_{\tau_i}^p$. The first and third equalities follow from Negative Certainty and Truth Axiom, and the second from Negative Introspection and Truth Axiom.

   (b) It is sufficient to prove the "if" part. First, it follows from Lemma A.1 that, under the assumptions on B_i, $\mathcal{B}_i = \{B_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$ is a sub-σ-algebra of $\mathcal{D}$.

   Second, since B_i satisfies Positive Introspection, $\mathcal{B}_i \subseteq \mathcal{J}_{B_i}$. Third, I show that Positive Certainty, Negative Certainty, and Consistency of B_i imply $B_{\tau_i}^p(E) = B_i B_{\tau_i}^p(E)$. The "$\subseteq$" part is Positive Certainty. Conversely, it follows from Negative Certainty and Consistency that $(\neg B_{\tau_i}^p)(E) \subseteq B_i(\neg B_{\tau_i}^p)(E) \subseteq (\neg B_i) B_{\tau_i}^p(E)$. Then, $B_i B_{\tau_i}^p(E) \subseteq B_{\tau_i}^p(E)$.

   Fourth, since $\tau_i^{-1}(\beta_E^p) = B_{\tau_i}^p(E) = B_i B_{\tau_i}^p(E) \in \mathcal{B}_i$ and since $\mathcal{B}_i$ is a σ-algebra, $\tau_i^{-1}(\mathcal{D}_\Delta) = \sigma(\{\tau_i^{-1}(\beta_E^p) \in \mathcal{D} \mid (E,p) \in \mathcal{D} \times [0,1]\}) \subseteq \sigma(\mathcal{B}_i) = \mathcal{B}_i \subseteq \mathcal{J}_{B_i}$.

   (c) It suffices to prove the "if" part. First, applying Lemma A.1 to $B_{\tau_i}^1$, $\mathcal{B}_{\tau_i}^1 := \{B_{\tau_i}^1(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$ is a sub-σ-algebra of $\mathcal{D}$. Second, since B_i satisfies Positive Certainty, $\mathcal{B}_{\tau_i}^1 \subseteq \mathcal{J}_{B_i}$. Third, I show below that Positive Certainty, Negative Certainty, and Consistency of $B_{\tau_i}^1$ (i.e., $B_{\tau_i}^1(E) \subseteq (\neg B_{\tau_i}^1)(E^c)$) imply $B_{\tau_i}^p(E) = B_{\tau_i}^1 B_{\tau_i}^p(E)$. Fourth, since $\tau_i^{-1}(\beta_E^p) = B_{\tau_i}^p(E) = B_{\tau_i}^1 B_{\tau_i}^p(E) \in \mathcal{B}_{\tau_i}^1$ and since $\mathcal{B}_{\tau_i}^1$ is a σ-algebra, $\tau_i^{-1}(\mathcal{D}_\Delta) = \sigma(\{\tau_i^{-1}(\beta_E^p) \in \mathcal{D} \mid (E,p) \in \mathcal{D} \times [0,1]\}) \subseteq \sigma(\mathcal{B}_{\tau_i}^1) = \mathcal{B}_{\tau_i}^1 \subseteq \mathcal{J}_{B_i}$.

   Thus, I show the third statement $B_{\tau_i}^p(E) = B_{\tau_i}^1 B_{\tau_i}^p(E)$. It follows from Positive Certainty and Entailment that $B_{\tau_i}^p(E) \subseteq B_i B_{\tau_i}^p(E) \subseteq B_{\tau_i}^1 B_{\tau_i}^p(E)$. Conversely, it follows from Negative Certainty and Entailment that $(\neg B^p_{\tau_i})(E) \subseteq B_i(\neg B^p_{\tau_i})(E) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(E)$. Then, it follows from Consistency of $B^1_{\tau_i}$ that $B^1_{\tau_i} B^p_{\tau_i}(E) \subseteq (\neg B^1_{\tau_i})(\neg B^p_{\tau_i})(E) \subseteq B^p_{\tau_i}(E)$.

□

Proof of Proposition S.4. First, since each player i is certain of her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ within the model $\langle (\Omega, \mathcal{D}), (B^1_{\tau_i})_{i \in I} \rangle$, it follows from Proposition S.3 that Negative Certainty $(\neg B^p_{\tau_i})(\cdot) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(\cdot)$ holds. Second, I show below that $B^p_{\tau_i} B^p_{\tau_i}(\cdot) \subseteq B^p_{\tau_i}(\cdot)$. Third, I show that the mutual p-belief operator also satisfies $B^p_I B^p_I(\cdot) \subseteq B^p_I(\cdot)$. It means that the chain of mutual p-beliefs is decreasing. Fourth, since mutual p-beliefs are preserved for a decreasing sequence of events (i.e., if $E_n \downarrow E$ then $B^p_I(E_n) \downarrow B^p_I(E)$), the common p-belief operator C^p reduces to the iteration of mutual p-beliefs (see Monderer and Samet, 1989) and thus is well-defined.

Thus, it suffices to prove the second and third statements. I start with the second statement. If $p = 0$ then $B^p_{\tau_i} B^p_{\tau_i}(\cdot) = \Omega = B^p_{\tau_i}(\cdot)$. Thus, let $p > 0$. Let $\omega \in B^p_{\tau_i} B^p_{\tau_i}(E)$. Suppose to the contrary that $\omega \in (\neg B^p_{\tau_i})(E)$. Then, $\omega \in (\neg B^p_{\tau_i})(E) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(E)$. Then, $\tau_i(\omega)((\neg B^p_{\tau_i})(E)) = 1$ and $\tau_i(\omega)(B^p_{\tau_i}(E)) \geq p > 0$, and thus $1 = \tau_i(\omega)(B^p_{\tau_i}(E) \cup (\neg B^p_{\tau_i})(E)) = 1 + p > 1$, a contradiction.

Turning to the third statement, Monotonicity of $B^p_{\tau_i}$ implies that $B^p_{\tau_i} B^p_I(\cdot) \subseteq B^p_{\tau_i} B^p_{\tau_i}(\cdot) \subseteq B^p_{\tau_i}(\cdot)$. Taking the intersection over all $i \in I$, $B^p_I B^p_I(\cdot) \subseteq B^p_I(\cdot)$. □

Proof of Proposition S.5. 1. It can be seen that

$$[\tau_i(\omega)] = \bigcap_{(E,p) \in \mathcal{D} \times [0,1] : \omega \in B^p_{\tau_i}(E)} B^p_{\tau_i}(E) = \bigcap_{(E,p) \in \mathcal{D} \times [0,1] : \omega \in (\neg B^p_{\tau_i})(E)} (\neg B^p_{\tau_i})(E). \quad \text{(S.4)}$$

Now, $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ implies $b_{B_i}(\omega) \subseteq B^p_{\tau_i}(E)$ for any $(E, p) \in \mathcal{D} \times [0, 1]$ with $\omega \in B^p_{\tau_i}(E)$. Likewise, $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$ implies $b_{B_i}(\omega) \subseteq (\neg B^p_{\tau_i})(E)$ for any $(E, p) \in \mathcal{D} \times [0, 1]$ with $\omega \in (\neg B^p_{\tau_i})(E)$. In either case, $b_{B_i}(\omega) \subseteq [\tau_i(\omega)]$.

2. Take $(E, p) \in \mathcal{D} \times [0, 1]$. Since $b_{B_i}(\omega) \subseteq [\tau_i(\omega)] \subseteq B^p_{\tau_i}(E)$ for any $\omega \in B^p_{\tau_i}(E)$, it follows from the Kripke property that $B^p_{\tau_i}(E) \subseteq B_i B^p_{\tau_i}(E)$. Likewise, since $b_{B_i}(\omega) \subseteq [\tau_i(\omega)] \subseteq (\neg B^p_{\tau_i})(E)$ for any $\omega \in (\neg B^p_{\tau_i})(E)$, it follows from the Kripke property that $(\neg B^p_{\tau_i})(E) \subseteq B_i(\neg B^p_{\tau_i})(E)$.

□

Proof of Proposition S.6. 1. Let τ_i satisfy the Harsanyi property. For any $\omega \in \Omega$ and $\tau_i^{-1}(\{\tau_i(\omega)\}) = [\tau_i(\omega)] \subseteq E$, if $\omega' \in \tau_i^{-1}(\{\tau_i(\omega)\}) = [\tau_i(\omega)]$ then $\tau_i(\omega')(E) = \tau_i(\omega)(E) = 1$, i.e., $\omega' \in B^1_{\tau_i}(E)$. Thus, player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\{\tau_i(\omega)\} \mid \omega \in \Omega\})$. Conversely, for any $E \in \mathcal{D}$ with $\tau_i^{-1}(\{\tau_i(\omega)\}) = [\tau_i(\omega)] \subseteq E$, $\omega \in B^1_{\tau_i}(E)$, i.e., $\tau_i(\omega)(E) = 1$.

2. Let $\mathcal{D}$ be generated by a countable algebra $\mathcal{A}$. Let $[0,1]_{\mathbb{Q}} := [0,1] \cap \mathbb{Q}$. Similarly to Expression (S.4), one can show:

$$[\tau_i(\omega)] = \bigcap_{(E,p)\in\mathcal{A}\times[0,1]_{\mathbb{Q}}:\omega\in B^p_{\tau_i}(E)} B^p_{\tau_i}(E) = \bigcap_{(E,p)\in\mathcal{A}\times[0,1]_{\mathbb{Q}}:\omega\in(\neg B^p_{\tau_i})(E)} (\neg B^p_{\tau_i})(E) \in \mathcal{D}.$$

Then, it follows from Part (1) that (2a) and (2b) are equivalent. Part (2b) implies (2c), and (2c) implies (2d) and (2e).

Now, I show that (2d) implies (2a). Assume (2d). Fix $\omega \in \Omega$. For any $(E,p) \in \mathcal{A} \times [0,1]_{\mathbb{Q}}$ with $\omega \in B^p_{\tau_i}(E)$, I have $B^p_{\tau_i}(E) \subseteq B^1_{\tau_i} B^p_{\tau_i}(E)$. Since $\mathcal{A} \times [0,1]_{\mathbb{Q}}$ is countable, take the intersection over all $(E,p) \in \mathcal{A} \times [0,1]_{\mathbb{Q}}$ with $\omega \in B^p_{\tau_i}(E)$ to obtain:

$$[\tau_i(\omega)] = \bigcap_{(E,p)} B^p_{\tau_i}(E) \subseteq \bigcap_{(E,p)} B^1_{\tau_i} B^p_{\tau_i}(E) \subseteq B^1_{\tau_i}\left(\bigcap_{(E,p)} B^p_{\tau_i}(E)\right) = B^1_{\tau_i}([\tau_i(\omega)]).$$

Thus, (2a) holds.

Likewise, I show that (2e) implies (2a). Assume (2e). Fix $\omega \in \Omega$. For any $(E,p) \in \mathcal{A} \times [0,1]_{\mathbb{Q}}$ with $\omega \in (\neg B^p_{\tau_i})(E)$, I have $(\neg B^p_{\tau_i})(E) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(E)$. Since $\mathcal{A} \times [0,1]_{\mathbb{Q}}$ is countable, take the intersection over all $(E,p) \in \mathcal{A} \times [0,1]_{\mathbb{Q}}$ with $\omega \in (\neg B^p_{\tau_i})(E)$ to obtain:

$$[\tau_i(\omega)] = \bigcap_{(E,p)} (\neg B^p_{\tau_i})(E) \subseteq \bigcap_{(E,p)} B^1_{\tau_i}(\neg B^p_{\tau_i})(E) \subseteq B^1_{\tau_i}\left(\bigcap_{(E,p)} (\neg B^p_{\tau_i})(E)\right) = B^1_{\tau_i}([\tau_i(\omega)]).$$

Hence, (2a) holds.

□

Proof of Theorem S.1. Suppose that the players are commonly certain of their type mappings. Since player j is certain of player i's type mapping, it follows from Remark S.1 that $B^p_{\tau_i}(\cdot) \subseteq B_j B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B_j(\neg B^p_{\tau_i})(\cdot)$. Since j is arbitrary, $B^p_{\tau_i}(\cdot) \subseteq B_I B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B_I(\neg B^p_{\tau_i})(\cdot)$. Then, $B^p_{\tau_i}(\cdot) \subseteq C B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq C(\neg B^p_{\tau_i})(\cdot)$. Conversely, it follows from the supposition that $B^p_{\tau_i}(\cdot) \subseteq C B^p_{\tau_i}(\cdot) \subseteq B_j B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq C(\neg B^p_{\tau_i})(\cdot) \subseteq B_j(\neg B^p_{\tau_i})(\cdot)$. Thus, player j is certain of player i's type mapping.

Lastly, C^1 satisfies Countable Conjunction because each $B^1_{\tau_i}$ satisfies it. Since $B^1_I(\cdot) \subseteq B^1_{\tau_i}(\cdot) \subseteq C^1 B^1_{\tau_i}(\cdot)$ for each $i \in I$, $B_I(\cdot) \subseteq \bigcap_{i\in I} C^1 B_{\tau_i}(\cdot) \subseteq C^1 B^1_I(\cdot)$, where the last set inclusion follows because C^1 satisfies Countable Conjunction. Then, $B^1_I(\cdot)$ itself is a publicly-1-evident event implying the mutual 1-belief, and thus $C^1 = B^1_I$. □

Proof of Proposition S.7. The proof is similar to that of Proposition 2. It suffices to prove (1). Take $F \in \mathcal{X}$. By assumption, $x^{-1}(F) \subseteq B_i(x^{-1}(F))$. It is sufficient to show

$x^{-1}(F) \subseteq B_j(x^{-1}(F))$. It follows from Remark S.1 and Consistency of B_j that $B^p_{\tau_i} = B_j B^p_{\tau_i}$. Take $(F_\lambda)_{\lambda\in\Lambda}$ from $\mathcal{X}$ with $F^c = \bigcup_{\lambda\in\Lambda} F_\lambda$. Then, $\neg x^{-1}(F) = \bigcup_{\lambda\in\Lambda} x^{-1}(F_\lambda)$. Thus,

$$\begin{aligned}\neg x^{-1}(F) = \bigcup_{\lambda\in\Lambda} x^{-1}(F_\lambda) \subseteq \bigcup_{\lambda\in\Lambda} B_i(x^{-1}(F_\lambda)) \subseteq \bigcup_{\lambda\in\Lambda} B^1_{\tau_i}(x^{-1}(F_\lambda))\\ \subseteq B^1_{\tau_i}(x^{-1}(F^c)) \subseteq (\neg B^1_{\tau_i})(x^{-1}(F)),\end{aligned}$$

implying $x^{-1}(F) = B^1_{\tau_i}(x^{-1}(F))$. It follows that

$$x^{-1}(F) = B^1_{\tau_i}(x^{-1}(F)) = B_j B^1_{\tau_i}(x^{-1}(F)) = B_j(x^{-1}(F)).$$

□

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