Representing Higher-Order Non-Additive Beliefs through p-Belief Operators

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June 9, 2025

Abstract

A p-belief operator is a convenient tool in representing agents' higher-order beliefs. It maps an event E to the event that an agent believes E with probability at least p. By iterating agents' p-belief operators, the analysts can unfold one's beliefs about another's without explicitly constructing beliefs over the space of beliefs. This paper first provides the conditions under which an agent's p-belief operators induce her underlying beliefs at each state of the world, i.e., her type mapping, without any underlying assumption on beliefs. Then, the paper shows that p-belief operators alone can be a primitive of an interactive belief model for a wide variety of non-additive beliefs. The representations include Choquet and Dempster-Shafer beliefs. Finally, since this paper allows for a wide variety of interactive non-additive belief models, the paper discusses possible applications such as: common p-beliefs, the existence of a terminal non-additive belief space, and non-additive conditional beliefs.

JEL Classification: C70; D83
Keywords: p-Belief Operators; Non-additive Beliefs; Dempster-Shafer Beliefs; Terminal Belief Space; Epistemic Game Theory

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1 Introduction

In economic theory, the outside analysts often represent agents' quantitative beliefs by the notion of a type mapping. Consider an agent, Alice, who faces uncertainty about underlying states of the world. At each state, her quantitative beliefs about the underlying states are represented by a set function (usually, but not necessarily, a countably-additive probability measure) defined on a collection of events (i.e., subsets of states of the world). Call the set function the *type* at the prevailing state. It assigns a degree of her belief (for ease of terminology, call it probability) to each event. Alice's type mapping dictates her beliefs at every state: it associates, with each state, her type (her probabilistic beliefs) at the prevailing state.

Now, suppose that Alice and Bob are interactively reasoning about what they believe at each state. In such an interactive situation, a notion of p-belief operators (Friedell, 1969; Monderer and Samet, 1989) is a convenient tool to analyze interactive beliefs of such form as: Alice p-believes (i.e., she believes with probability at least p) that Bob q-believes (i.e., he believes with probability at least q) that some event obtains at a state. Alice's p-belief operator assigns, with each event E, the event that she p-believes E.

By iterating agents' p-belief operators, the outside analysts can unpack hierarchies of interactive beliefs of the above form without explicitly constructing higher-order beliefs such as Alice's beliefs on the space of Bob's beliefs. Especially, unlike the type-mapping approach, p-belief operators represent, in a tractable way, an approximate notion of common knowledge (Aumann, 1976; Friedell, 1969) referred to as common p-belief (or common certainty as a special case of $p = 1$) (Brandenburger and Dekel, 1987; Monderer and Samet, 1989): Alice and Bob p-believes an event E, they p-believe that they p-believe E, and so forth *ad infinitum*.[1] For example, common p-beliefs play a crucial role in the agreeing-to-disagree and no-trade theorems and the existence of a common prior.[2]

Now, if the analysts start with agents' p-belief operators as a primitive, then can they recover the agents' type mappings? Samet (2000) establishes the equivalence between an agent's type mapping and her collection of p-belief operators when her beliefs are countably additive (Gaifman (1988) also establishes a related result). Zhou (2010) provides the equivalence when agents' beliefs are finitely additive.

The purpose of this paper is to demonstrate that, for an arbitrary notion of

[1] Generally, there are two channels through which common knowledge is approximated. One is the number of iteration of reasoning. Rubinstein (1989) shows that strategic behavior under an arbitrarily long finite level of mutual knowledge may be different from that under common knowledge. The other is the approximation of knowledge by probabilistic beliefs. In a standard model in which agents possess countably-additive introspective beliefs, the approximate notion of common p-beliefs converges to common knowledge when probability p tends to one. As discussed, this paper relaxes countable additivity of beliefs.

[2] Pioneering papers include: Aumann (1976), Heifetz (2006), Milgrom and Stokey (1982), Monderer and Samet (1989), Morris (1994), Neeman (1996a,b), and Sonsino (1995).

beliefs that can be represented by a type mapping on an underlying state space, p-belief operators can in fact be a primitive of an interactive belief model (Theorem 1 in Section 4.1). Having the p-belief operator representation of an interactive belief model opens up a wide range of applications in epistemic game theory including epistemic analyses of game-theoretic solution concepts and the agreeing-to-disagree and no-trade theorems. One of the applications of this paper (Proposition 5 in Section 5.2) is to assert the existence of a terminal Harsanyi (1967-1968) belief space (e.g., Armbruster and Böge, 1979; Böge and Eisele, 1979; Brandenburger and Dekel, 1993; Heifetz and Samet, 1998; Mertens and Zamir, 1985) when beliefs are non-additive (not-necessarily-additive): a terminal belief space is a belief space to which, for any given belief space, there exists a unique structure-preserving map from the given space.

To obtain the main result of the paper on the foundation side, Proposition 2 (in Section 3.3) provides conditions on p-belief operators under which they induce a given type mapping. Then, Theorem 1 characterizes various non-additive beliefs in terms of p-belief operators property by property. This paper allows the outside analysts to scrutinize agents' interactive beliefs through p-belief operators based on their choice of agents' logical abilities.

Specifically, for given properties of beliefs (that are represented as the corresponding properties on a type mapping), I provide the corresponding conditions on a collection of p-belief operators under which the underlying type mapping is recovered. Examples of non-additive beliefs include: general non-additive measures (Choquet (1954) capacities), Dempster-Shafer beliefs (Dempster, 1967; Shafer, 1976), and possibility measures (Dubois and Prade, 1988; Zadeh, 1978).[3] Since I analyze properties of quantitative beliefs property by property, I also allow non-monotonic beliefs (i.e., an agent can fail to believe some of the consequences of her beliefs). In fact, the main results go through when the agent has conditional beliefs, a set of beliefs based on conditioning events (Section 5.3). Thus, the paper also extends the equivalence of conditional type mappings and conditional p-belief operators by Di Tillio, Halpern, and Samet (2014).

On the applications side, the main result of the paper provides foundations for epistemic characterizations of game-theoretic solution concepts when agents' beliefs are non-additive and the possibility of agreeing-to-disagree and speculative trade. Since each of these topics may require a separate paper, this paper instead formulates the notion of common p-belief (Section 5.1) and shows the existence of a terminal non-additive belief space (Section 5.2). Also, Section 2 studies a variant of Rubinstein (1989)'s e-mail game with non-additive beliefs.

This paper is organized as follows. The rest of the Introduction discusses the re-

[3]See also Halpern (2017) and Wang and Klir (2010) for surveys on general set functions. In decision and game theory, the seminal papers on the use of Choquet capacity are Schmeidler (1986, 1989). Ghirardato (2001) and Mukerji (1997) link Dempster-Shafer beliefs and probabilistic ignorance in their decision theoretic frameworks.

lated literature. Section 2 studies Rubinstein (1989)'s e-mail game with non-additive beliefs. Section 3 sets up the model, and presents the main result (Theorem 1) on the one-to-one correspondence between a type mapping and p-belief operators. With the main result in mind, Section 4 studies various properties of probabilistic beliefs. Section 5 discusses applications. Specifically, Section 5.1 incorporates common p-beliefs. Section 5.2 constructs a terminal belief space. Section 5.3 extends the analyses to conditional beliefs. Section 6 provides concluding remarks. The proofs are mostly relegated to Appendix A.

Related Literature

This paper is related to the following three strands of literature: (i) the construction of a terminal belief space; (ii) representations of higher-order beliefs through belief operators; and (iii) applications of non-additive beliefs to game theory (e.g., epistemic analyses of solution concepts and the agreeing-to-disagree and no-trade theorems).

First, capturing agents' beliefs by p-belief operators plays important roles when the outside analysts represent agents' infinite regress in their beliefs. I have already discussed approximate notions of common knowledge. A terminal belief space, which contains all conceivable hierarchies of interactive beliefs, may also be used to provide epistemic characterizations of game-theoretic solution concepts or to study strategic impacts of higher-order beliefs.

Heifetz and Samet (1998) construct, using p-belief operators, a terminal type space that contains any conceivable form of agents' hierarchies of countably-additive beliefs. Meier (2006) extends their result to the case where agents posses finitely-additive beliefs. This paper entirely relaxes the standard assumptions that individuals' beliefs are countably (or finitely) additive. As Fukuda (2024b) constructs a terminal space when agents' qualitative beliefs (or knowledge) are given by arbitrary belief operators, this paper establishes the existence of a terminal belief space when agents' beliefs take a specific form of non-additive beliefs such as Choquet capacities and Dempster-Shafer beliefs.[4] Chen (2010), Di Tillio (2008), Epstein and Wang (1996), and Ganguli, Hiefetz, and Lee (2016) construct a terminal preference space, i.e., a canonical representation of interactive beliefs where agents are non-expected-utility maximizers, by formulating a type mapping that generates interactive preferences.[5]

Second, agents' beliefs are syntactically represented in a logical system in interdisciplinary literature ranging in computer science and artificial intelligence, economics and game theory, and logic and philosophy. There, agents' beliefs surrounding a basic

[4]When the analysts work explicitly on the hierarchies of higher-order beliefs, the existence of a terminal space is non-trivial especially because the analysts often utilize a measure-theoretic and topological apparatus such as Kolmogorov Extension Theorem (see, for example, Brandenburger and Dekel, 1993; Pintér, 2005, 2012). The logical construction pioneered by Heifetz and Samet (1998) does not utilize a topological structure on an underlying uncertainty space or spaces dictating higher-order beliefs.

[5]Ahn (2007) constructs a terminal ambiguous-belief space.

uncertainty space, their beliefs about their beliefs, and so on, are explicitly modeled as logical formulas. One such syntactic representation of interactive beliefs incorporates statements of the form "Alice p-believes a proposition e." Such papers as Fagin and Halpern (1994), Heifetz and Mongin (2001), and Meier (2012) study sound-and-complete axiomatizations of probability logics in which agents' beliefs are countably additive. Zhou (2010) studies finitely-additive beliefs. Fagin, Halpern, and Megiddo (1990) consider a probability logic where an agent's beliefs are dictated by the inner measure induced from a probability measure.[6] While this paper takes a purely semantic (i.e., set-theoretical) approach whereby agents' beliefs are represented over a set of states of the world, it provides p-belief operator representations of various properties of agents' quantitative beliefs property by property, and thereby sheds light on a logical representation of agents' non-additive beliefs.

Third, while the literature on epistemic analyses of game-theoretic solution concepts such as various forms of rationalizability and Nash and correlated equilibria mainly focuses on the case in which agents' beliefs are countably additive, there are innovative papers at the intersection of decision and game theory that study the role of additivity. While it would be impossible to cite all papers that study the role of additivity of beliefs in epistemic game theory, earlier papers that study game-theoretic solution concepts under non-additive beliefs broadly construed include Dow and Werlang (1994), Eichberger and Kelsey (2000) (see also Eichberger and Kelsey, 2014), Epstein (1997), Groes et al. (1998), Haller (2000), Lo (1996, 1999, 2002), Marinacci (2000), and Salo and Weber (1995). Earlier papers that show that agreeing-to-disagree or speculative trades are possible under non-additive beliefs broadly construed include: Billot et al. (2000), Ganguli (2007), and Kajii and Ui (2005, 2006). Dominiak and Lefort (2015) analyze the agreement and no-trade theorems using particular forms of non-additivity (i.e., neo-additivity, which stands for "non-extreme-outcome-"additivity) developed by Chateauneuf, Eichberger, and Grant (2007) and Eichberger, Grant, and Kelsey (2010) and studied by Dominiak, Eichberger, and Lefort (2012) and Dominiak and Lefort (2013). Section 2 applies neo-additive beliefs to Rubinstein (1989)'s e-mail game. As to epistemic characterizations of game-theoretic solution concepts, Dominiak and Schipper (2021) consider rationality and common belief in rationality among Choquet-expected-utility maximizers. This paper axiomatizes a wide variety of properties of non-additive beliefs (which enable one to focus on a role of particular properties of non-additive beliefs), formulates the notion of common belief, and asserts the existence of a terminal belief space.

[6]Fagin and Halpern (1991) study the sense in which the inner measure can be seen as a non-additive Dempster-Shafer belief.

	A	B
A	M, M	1, $-L$
B	$-L$, 1	0, 0

Γ_a (probability $1-p$)

	A	B
A	0, 0	1, $-L$
B	$-L$, 1	M, M

Γ_b (probability p)

Figure 1: The Component Games: The parameters satisfy $L > M > 1$ and $p \in (0, \frac{1}{2})$.

2 An E-Mail Game with Non-Additive Beliefs

As a motivation to study non-additive beliefs in (epistemic) game theory, this section considers a variant of Rubinstein (1989)'s e-mail game. I follow his setup except for the agents' beliefs: as it will be clear, I introduce a form of optimism that overturns the striking result of Rubinstein (1989).

Setup of Rubinstein (1989). As in Figure 1, each of two agents has to choose one of the binary actions A or B. With probability $p \in (0, \frac{1}{2})$, the agents play Γ_b; with probability $1-p$, they play Γ_a. It is mutually beneficial to coordinate in both component games, but which action to coordinate on depends on the component game: namely, (A, A) in Γ_a and (B, B) in Γ_b. Assume $L > M > 1$.

Initially, while agent 1 is informed of the true game, agent 2 is not. They communicate through computers under the following protocol. If the game is Γ_b then agent 1's computer *automatically* sends a message to agent 2's computer; if the game is Γ_a then no message is sent. If a computer receives a message then it *automatically* sends a confirmation, including the confirmation of the confirmation, and so on. With a probability $\varepsilon \in (0, 1)$, any given message does not arrive at its intended destination. If a message does not arrive then the communication stops. Each agent's computer records the number of messages that it has sent.

The state space consists of a pair (q_1, q_2) where q_i is the number of messages that agent i's computer has sent:

$$\Omega = \{(q_1, q_2) \in (\mathbb{N} \cup \{0\})^2 \mid q_1 = q_2 \text{ or } q_1 = q_2 + 1\}.$$

At state (q, q), agent 1 sends q messages, all of which arrive at agent 2, and the q-th message sent by agent 2 is lost. At state $(q + 1, q)$ agent 1 sends $q + 1$ messages, and all but the last arrive at agent 2.

The prior belief μ on the state space is defined from the technological constraints of this environment: μ is the countably-additive probability measure such that $\mu(\omega) := \mu(\{\omega\})$ satisfies:

$$\mu(0, 0) = 1 - p;$$
$$\mu(q + 1, q) = p\varepsilon(1 - \varepsilon)^{2q} \text{ for any } q \geq 0 \text{ and;}$$
$$\mu(q + 1, q + 1) = p\varepsilon(1 - \varepsilon)^{2q+1} \text{ for any } q \geq 0.$$

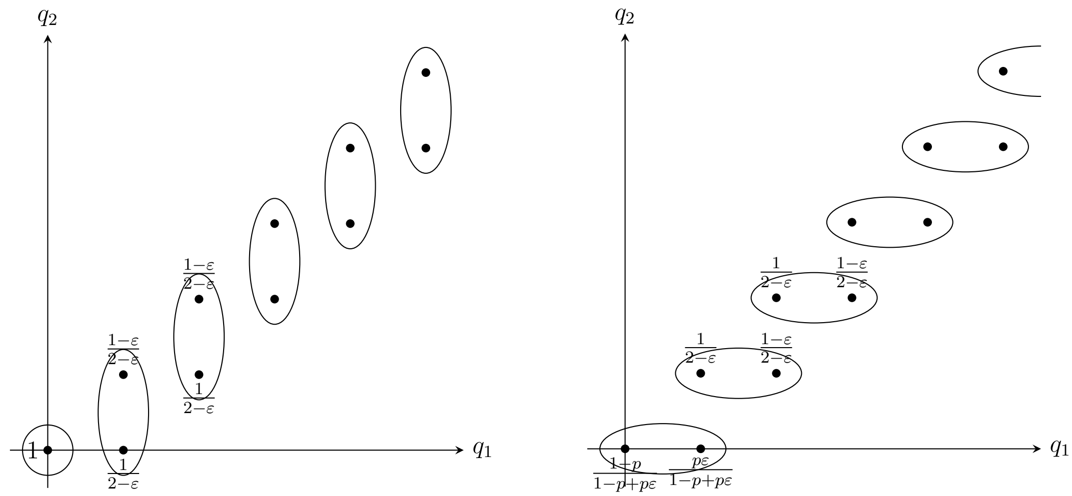

Figure 2: The Agents' Partitions and Type Mappings: the left panel depicts agent1's partition and type mapping; and the right panel depicts agent 2's partition and type mapping.

At each state, each agent has her ("posterior") belief over the states. I start by defining the information set of each agent at each state within the original setup of Rubinstein (1989). Denote by $P_i(q_1, q_2)$ the information set of agent i at (q_1, q_2), where $\{P_i(q_1, q_2)\}_{(q_1,q_2)\in\Omega}$ forms a partition of the state space Ω.

Agent 1 cannot distinguish between $(q_1, q_1 - 1)$ and (q_1, q_1) at a state (q_1, q_2) with $q_1 \geq 1$. Together with $P_1(0, 0) = \{(0, 0)\}$, agent 1's partition is:

$$\{\{(0, 0)\}\} \cup \{\{(q, q), (q, q - 1)\}\}_{q\geq 1}.$$

The left panel of Figure 2 illustrates P_1: each cell depicts the partition cell $P_1(q_1, q_2)$ that contains (q_1, q_2).

Agent 2 cannot distinguish between (q_2, q_2) and $(q_2 + 1, q_2)$ at a state (q_1, q_2) with $q_2 \geq 0$. Thus, agent 2's partition is:

$$\{\{(q, q), (q + 1, q)\}\}_{q\geq 0}.$$

The right panel of Figure 2 illustrates P_2: each cell depicts the partition cell $P_2(q_1, q_2)$ that contains (q_1, q_2).

Letting $\Delta(\Omega)$ be the set of countably-additive probability measures on Ω, denote by

$$\tau_i : \Omega \to \Delta(\Omega)$$

agent i's *type mapping* that associates, with each state ω, her beliefs at ω:

$$\tau_i(\omega)(\cdot) := \mu(\cdot \mid P_i(\omega)).$$

Call $\tau_i(\omega)$ agent i's *type* at ω. Note that each type $\tau_i(q_1, q_2)$ assigns probability 1 to the information set $P_i(q_1, q_2)$.

For agent 1, her type at state $(0,0)$ is a degenerate probability measure on $P_1(0,0) = \{(0,0)\}$, as depicted in the left panel of Figure 2. At any other state, as depicted the left panel of Figure 2, her type satisfies

$$\tau_1(q,q)(\{\omega\}) = \tau_1(q,q-1)(\{\omega\}) = \begin{cases} \frac{1-\varepsilon}{2-\varepsilon} & \text{if } \omega = (q,q) \\ \frac{1}{2-\varepsilon} & \text{if } \omega = (q,q-1) \end{cases}.$$

For agent 2, at any state $(0,0)$ or $(1,0)$, as depicted in the right panel of Figure 2, his type satisfies

$$\tau_2(0,0)(\{\omega\}) = \tau_2(1,0)(\{\omega\}) = \begin{cases} \frac{1-p}{1-p+p\varepsilon} & \text{if } \omega = (0,0) \\ \frac{p\varepsilon}{1-p+p\varepsilon} & \text{if } \omega = (1,0) \end{cases}.$$

At any other state, as depicted the right panel of Figure 2, his type satisfies

$$\tau_2(q,q)(\{\omega\}) = \tau_2(q+1,q)(\{\omega\}) = \begin{cases} \frac{1}{2-\varepsilon} & \text{if } \omega = (q,q) \\ \frac{1-\varepsilon}{2-\varepsilon} & \text{if } \omega = (q+1,q) \end{cases}.$$

For any subset E of Ω, the set of states at which agent i believes E with probability 1 (i.e., agent i is certain that E occurs) is:

$$B^1_{\tau_i}(E) := \{\omega \in \Omega \mid \tau_i(\omega)(E) \geq 1\}.$$

Call $B^1_{\tau_i}$ agent i's (probability) 1-belief operator. Section 3 provides an interactive belief model in which each agent's collection of (probability) p-belief operators is a primitive. As will be seen, agents' belief operators unpack higher-order reasoning.

Since the agents interactively reason about the game that they play, denote by $G : \Omega \to \{\Gamma_a, \Gamma_b\}$ the function that assigns the game that is played:

$$G(0,0) = \Gamma_a \text{ and } G(q_1, q_2) = \Gamma_b \text{ otherwise.}$$

Then, denote by

$$G_a := G^{-1}(\{\Gamma_a\}) = \{(0,0)\}$$

the event that the true game is Γ_a. Likewise, denote by

$$G_b := G^{-1}(\{\Gamma_b\}) = \Omega \setminus \{(0,0)\}$$

the event that the true game is Γ_b. Figure 3 illustrates G_a and G_b.

With these notations in mind, at any state $\omega = (q_1, q_2)$ at which the true game is Γ_a, i.e., at $\omega = (0,0)$, agent 1 is certain (i.e., believes with probability 1) that the true game is Γ_a (i.e., the event G_a occurs) because $\tau_1(\omega)(G_a) = 1$, as $P_1(\omega) \subseteq G_a$. At any other state, i.e., at $\omega \in G_b$, agent 1 is certain that the true game is Γ_b (i.e.,

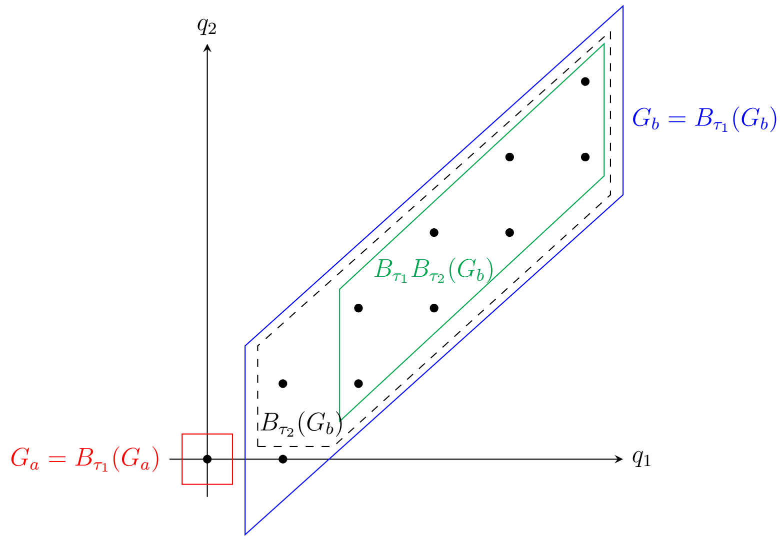

Figure 3: Events G_a and G_b and Agents' Interactive Reasoning through 1-Belief Operators

the event G_b occurs) because $\tau_1(\omega)(G_b) = 1$, as $P_1(\omega) \subseteq G_b$. In sum, as illustrated in Figure 3,

$$B^1_{\tau_1}(G_a) = G_a \text{ and } B^1_{\tau_1}(G_b) = G_b.$$

For agent 2, the event that he is certain that the game is Γ_b (i.e., the event G_b occurs) is, as illustrated in Figure 3,

$$B^1_{\tau_2}(G_b) = \Omega \setminus \{(0,0), (1,0)\}.$$

Thus, at each $\omega \in \{(1,0), (1,1)\}$, agent 1 is not certain that agent 2 is certain of G_b. This is because, as illustrated in Figure 3,

$$B^1_{\tau_1}B^1_{\tau_2}(G_b) = \Omega \setminus \{(0,0), (1,0), (1,1)\}.$$

Similarly, in each of the states $(1,1)$ and $(2,1)$, agent 2 is certain that the game is Γ_b but is not certain that agent 1 is certain that agent 2 is certain that the game is Γ_b. In this way, one can unpack higher-order beliefs through belief operators.

A strategy of agent i is defined as a mapping $\sigma_i : \Omega \to \{A, B\}$ which is measurable with respect to her partition. That is,

$$\sigma_i(\omega) = \sigma_i(\omega') \text{ if } \omega' \in P_i(\omega).$$

Since $P_i(\omega) = \{\tilde{\omega} \in \Omega \mid \tau_i(\tilde{\omega}) = \tau_i(\omega)\}$, the measurability condition states that agent i plays the same action at states she cannot distinguish based on her beliefs τ_i. Denote by $\sigma = (\sigma_1, \sigma_2)$ a strategy profile.

A strategy profile σ^* is an equilibrium if, for any agent i and any strategy σ_i of agent i,

$$U_{\tau_i}(\sigma^* \mid \omega) \geq U_{\tau_i}(\sigma_i, \sigma^*_{-i} \mid \omega) \text{ for all } \omega \in \Omega,$$

where, denoting by $u_i(\cdot \mid \tilde{\omega})$ agent i's component-game payoff function at state $\tilde{\omega}$, $U_{\tau_i}(\sigma \mid \omega)$ is agent i's expected utility from the strategy profile σ with respect to her belief $\tau_i(\omega)$:

$$U_{\tau_i}(\sigma \mid \omega) := \sum_{\tilde{\omega} \in \Omega} u_i(\sigma(\tilde{\omega}) \mid \tilde{\omega}) \tau_i(\omega)(\{\tilde{\omega}\}).$$

So far, while the setup is identical to that of Rubinstein (1989), I have introduced the agents' type mappings and 1-belief operators to see that iterating the agents' 1-belief operators represents their interactive reasoning. Rubinstein (1989) shows that the e-mail game has a unique equilibrium in which both agents always choose A.

Departure from Rubinstein (1989). Here instead, I apply "neo-additive" capacities (see Chateauneuf, Eichberger, and Grant, 2007; Dominiak and Lefort, 2013; Eichberger, Grant, and Kelsey, 2010) to modify the type mapping τ_i. To that end, first, I introduce the following "possibility measure" π_i: for each state $\omega \in \Omega$ and any subset E of Ω,

$$\pi_i(\omega)(E) := \begin{cases} 0 & \text{if } E \cap P_i(\omega) = \emptyset \\ 1 & \text{if } E \cap P_i(\omega) \neq \emptyset \end{cases}.$$

At each state ω, $\pi_i(\omega)(E) = 1$ if and only if (hereafter, iff) agent i considers E possible in the sense that she assigns a positive $\tau_i(\omega)$-probability to E. Section 4.1 studies (general) possibility measures. Then, letting $\delta \in [0, 1)$, I consider a new type mapping t_i, where

$$t_i(\omega)(E) := \delta\pi_i(\omega)(E) + (1 - \delta)\tau_i(\omega)(E).$$

Three remarks are in order. First, the case with $\delta = 0$ corresponds to the original Rubinstein (1989) e-mail game. Second, letting

$$B^1_{t_i}(E) = \{\omega \in \Omega \mid t_i(\omega)(E) \geq 1\} \text{ for each subset } E \text{ of } \Omega,$$

one has

$$B^1_{t_i} = B^1_{\tau_i}.$$

Third, $t_i(\omega)(\cdot) := \delta\pi_i(\omega)(\cdot) + (1 - \delta)\tau_i(\omega)(\cdot)$ conforms to a "neo-additive" capacity in the sense of Chateauneuf, Eichberger, and Grant (2007, Definition 3.3).

Agent i's strategy $\sigma_i : \Omega \to \{A, B\}$ is defined as before: $\sigma_i(\omega) = \sigma_i(\omega')$ if $\omega' \in P_i(\omega)$. Since $P_i(\omega) = \{\tilde{\omega} \in \Omega \mid t_i(\tilde{\omega}) = t_i(\omega)\}$, the measurability condition states that agent i plays the same action at states she cannot distinguish based on her beliefs t_i.

A strategy profile σ^* is an equilibrium if, for any agent i and any strategy σ_i of agent i,

$$U_{t_i}(\sigma^* \mid \omega) \geq U_{t_i}(\sigma_i, \sigma^*_{-i} \mid \omega) \text{ for all } \omega \in \Omega,$$

where $U_{t_i}(\sigma \mid \omega)$ is agent i's Choquet expected utility from the strategy profile σ with respect to her belief $t_i(\omega)$:

$$U_{t_i}(\sigma \mid \omega) := \delta \max_{\tilde{\omega} \in P_i(\omega)} u_i(\sigma(\tilde{\omega}) \mid \tilde{\omega}) + (1-\delta) \sum_{\tilde{\omega} \in \Omega} u_i(\sigma(\tilde{\omega}) \mid \tilde{\omega}) \tau_i(\omega)(\{\tilde{\omega}\}). \tag{1}$$

As it can be seen from the first term, while δ measures the degree of ambiguity, each agent is ambiguity-loving. Rubinstein (1989)'s original strategy profile is still an equilibrium for any δ because if the opponent always plays A then always taking A also maximizes the first term of agent i's (Choquet) expected utility function. Yet, I assert below that if δ is high enough then there exists another equilibrium in which the agents succeed in playing B as long as both receive at least one message. Formally:

Proposition 1. *If $\delta \geq \frac{1+L}{M+L}$, then the following strategy profile σ is an equilibrium irrespective of ε: agent i plays $\sigma_i(q_1, q_2) = B$ iff $q_i \geq 1$.*

Before proving the claim, in this equilibrium, agent 1 plays A (resp. B) whenever she is certain that the game is Γ_a (resp. Γ_b), and agent 2 plays A if and only if he does not receive the first message by agent 1. In other words, the claim states that if the agents are sufficiently optimistic then there exists an equilibrium in which each agent i plays B iff at least one message has been received.

Proof of Proposition 1. Consider state $\omega = (0,0)$. Observe $P_1(\omega) = \{\omega\}$. Given $\sigma_2(0,0) = A$, following σ_1 yields the best payoff M to agent 1 at $\omega = (0,0)$.

If agent 2 gets no message, then he is certain that either agent 1 did not send a message or the message that agent 1 sent did not arrive. If agent 2 chooses A, then, since agent 1 chooses A in the state $(0,0)$, agent 2's Choquet expected utility is at least

$$\left(\delta + (1-\delta)\frac{1-p}{1-p+p\varepsilon}\right) M$$

whatever agent 1 chooses in the state $(1,0)$. If agent 2 chooses B, then his payoff is at most

$$\delta M + (1-\delta)\left(\frac{1-p}{1-p+p\varepsilon}(-L) + \frac{p\varepsilon}{1-p+p\varepsilon} M\right),$$

where the first term comes from the fact that agent 1 plays B at the state $(1,0)$. Since $\delta < 1$, $p\varepsilon < \frac{1}{2} < 1-p$, and $L > M > 1$, it is strictly optimal for agent 2 to choose A.

Consider $\omega = (q_1, q_2)$ with $q_1 \geq 1$. Consider agent 1's decision when she sends q_1 messages. In this case, agent 1 is uncertain whether $q_2 = q_1$ or $q_2 = q_1 - 1$. If she chooses B, then, since agent 2 plays B when $q_2 = q_1$, agent 1's expected payoff is at least

$$\delta M + (1-\delta)\left(\frac{1}{2-\varepsilon}(-L) + \frac{1-\varepsilon}{2-\varepsilon} M\right) \geq \delta M + (1-\delta)(-L).$$

If she chooses A, then, since agent 1 is certain of G_b, her payoff is at most 1. If $\delta \geq \frac{1+L}{M+L}$, then playing B by following σ_1 is a best response.[7]

Consider $\omega = (q_1, q_2)$ with $q_2 \geq 1$. Consider agent 2's decision when he sends q_2 messages. In this case, agent 2 is uncertain whether $q_1 = q_2$ or $q_1 = q_2 + 1$. Agent 2 is certain that the game is G_b and agent 1 plays B at ω. Thus, playing B by following σ_2 is a best response. The proof is complete. □

This example shows that the consideration of non-additive beliefs may bring a novel insight into epistemic game theory. In Rubinstein (1989), each agent i is not certain whether her own message or the confirmation by the opponent gets astray. Since agent i assigns a higher probability to the event that her own message has been lost, she ends up playing A. In contrast, when agent i's type is modulated by the possibility measure π_i, she considers it possible that her own message has been received by the opponent. Thus, she takes into account the possibility that the opponent, who has received a message, takes action B. When δ is high enough, this becomes of the first-order effect.

The above argument is one possible departure from the standard expected-utility maximization framework and there are many possible ways in which non-additivity yields new insights. To develop an interactive belief model with non-additivity, the rest of the paper develops an interactive belief model in which agents' beliefs are non-additive.

3 Representations of Non-Additive Beliefs

This section lays out the framework of the rest of the paper. Section 3.1 defines a state space on which to represent agents' quantitative beliefs. Sections 3.2 and 3.3 provide two representations of agents' beliefs: a type mapping and p-belief operators. Proposition 2 is the benchmark result establishing the equivalence between these two representations without imposing any property on agents' beliefs. Specifically, it identifies the conditions on p-belief operators which recover an underlying type mapping.

3.1 A State Space

This subsection defines a state space on which agents' beliefs are represented. A *state space* is a pair $(\Omega, \mathcal{D})$ where Ω is a set of *states* of the world and $\mathcal{D}$ is a subcollection

[7] In fact, σ_1 is a best response if

$$\delta \geq \frac{L + (2 - M) + (M - 1)\varepsilon}{M + L}.$$

Since $M > 1$, the right-hand side is increasing in ε. Thus, if $\delta \geq \frac{1+L}{M+L}$ (as in the statement of Proposition 1), then the above condition holds irrespective of ε.

of the power set $\mathcal{P}(\Omega)$ about which agents reason. Call each $E \in \mathcal{D}$ an *event*. I assume that $\mathcal{D}$ forms an algebra on Ω: (i) $\{\emptyset, \Omega\} \subseteq \mathcal{D}$; (ii) $E \in \mathcal{D}$ implies $E^c \in \mathcal{D}$; and (iii) $\{E, F\} \subseteq \mathcal{D}$ implies $\{E \cap F, E \cup F\} \subseteq \mathcal{D}$. First, the tautology in the form of the entire set and the contradiction in the form of the empty set are an object of agents' interactive beliefs. Second, if E is an object of agents' beliefs, then so is its complement (negation) E^c. For ease of notation, I sometimes denote the complement of E also by $\neg E$. Third, if E and F are an object of interactive beliefs, then so are its intersection (conjunction) $E \cap F$ and union (disjunction) $E \cup F$. Special cases are when $\mathcal{D}$ is a σ-algebra or $\mathcal{D} = \mathcal{P}(\Omega)$. For example, some literature works with the power set algebra (e.g., Dempster-Shafer beliefs and possibility beliefs).

With the state space formally defined, I move on to defining the framework for representing agents' interactive beliefs on a state space. For the rest of this section, fix a state space $(\Omega, \mathcal{D})$. Agents are reasoning about some aspects of states Ω, and their objects of reasoning are represented by $\mathcal{D}$. In order to focus on the representation of beliefs itself, the rest of this section focuses on a single agent.

3.2 A Type Mapping

This paper considers two representations of the agent's beliefs. This subsection studies the first representation: a type mapping. The type mapping t associates, with each state $\omega \in \Omega$, her type $t(\omega)$. Her type $t(\omega)$ is a set function from $\mathcal{D}$ into $[0,1]$. Denote by $[0,1]^{\mathcal{D}}$ the collection of set functions from $\mathcal{D}$ into $[0,1]$.

Let $M(\Omega)$ be a generic subset of $[0,1]^{\mathcal{D}}$ which captures properties of the agent's beliefs. Thus, the specific feature of $M(\Omega)$ depends on the properties of beliefs imposed by the analysts. For example, if the analysts assume that the agent possesses the standard countably-additive probabilistic beliefs on $(\Omega, \mathcal{D})$, then $M(\Omega)$ is the set of countably-additive probability measures on $(\Omega, \mathcal{D})$. If, in contrast, the analysts consider non-additive probability measures (precisely, capacities) $\mu : \mathcal{D} \to [0,1]$ such that

$$\mu(\emptyset) = 0 \leq \mu(E) \leq \mu(F) \leq 1 = \mu(\Omega) \text{ for any } E, F \in \mathcal{D} \text{ with } E \subseteq F,$$

then $M(\Omega)$ is the set of capacities. Thus, at this point I do not explicitly add any particular property (Section 4 studies particular properties). Instead, I consider a generic subset $M(\Omega)$ of $[0,1]^{\mathcal{D}}$.

Next, given $M(\Omega)$, let $\mathcal{D}_M$ be the smallest algebra on $M(\Omega)$ including

$$\{\{\mu \in M(\Omega) \mid \mu(E) \geq p\} \in \mathcal{P}(M(\Omega)) \mid (E, p) \in \mathcal{D} \times [0,1]\} \text{ and} \tag{2}$$

$$\{\{\mu \in M(\Omega) \mid \mu(E) \leq p\} \in \mathcal{P}(M(\Omega)) \mid (E, p) \in \mathcal{D} \times [0,1]\}. \tag{3}$$

While $\{\mu \in M(\Omega) \mid \mu(E) \geq p\}$ is a collection of beliefs (set functions) such that the belief in E is at least p, $\{\mu \in M(\Omega) \mid \mu(E) \leq p\}$ is a collection of beliefs such that the belief in E is at most p. Thus, $\mathcal{D}_M$ is endowed with the structure which makes it possible to examine whether the agent's belief in an event $E \in \mathcal{D}$ is at least $p \in [0,1]$ and whether her belief in E is at most p. Note that $\mathcal{D}_M$ is equivalently generated by sets of the form

$$\{\mu \in M(\Omega) \mid \mu(E) \geq p\} \text{ and } \{\mu \in M(\Omega) \mid \mu(E) > p\} \text{ for some } (E,p) \in \mathcal{D} \times [0,1].$$

Given $t : \Omega \to M(\Omega)$, define its dual $\bar{t} : \Omega \to [0,1]^{\mathcal{D}}$ by

$$\bar{t}(\cdot)(E) := 1 - t(\cdot)(E^c) \text{ for each } E \in \mathcal{D}.$$

If $\bar{t} : \Omega \to M(\Omega)$, then $t : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ is measurable iff so is $\bar{t} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$. This is because

$$\bar{t}^{-1}(\{\mu \in M(\Omega) \mid \mu(E) \geq p\}) = t^{-1}(\{\mu \in M(\Omega) \mid \mu(E^c) \leq 1-p\}) \in \mathcal{D} \text{ and}$$
$$\bar{t}^{-1}(\{\mu \in M(\Omega) \mid \mu(E) \leq p\}) = t^{-1}(\{\mu \in M(\Omega) \mid \mu(E^c) \geq 1-p\}) \in \mathcal{D}.$$

Remark 1. The algebra $\mathcal{D}_M$ differs from the standard case in which the type mapping $t : \Omega \to M(\Omega)$ associates, with each state, the corresponding countably-additive probability measure on $(\Omega, \mathcal{D})$ (i.e., $M(\Omega)$ is the set of countably-additive probability measures on $(\Omega, \mathcal{D})$) as in Heifetz and Samet (1998): they introduce a σ-algebra on $M(\Omega)$ by the one that is generated from Expression (2). Indeed, if $\mathcal{D}$ is a σ-algebra, then the smallest σ-algebra that includes Expression (2) also includes Expression (3) because

$$\{\mu \in M(\Omega) \mid \mu(E) \leq p\} = \bigcap_{n \in \mathbb{N}} \{\mu \in M(\Omega) \mid \mu(E) \geq p + \frac{1}{n}\}^c.$$

In contrast, in the case of the algebra $\mathcal{D}_M$, one needs to consider open measurability (i.e., $t^{-1}(\{\mu \in M(\Omega) \mid \mu(E) > p\}) \in \mathcal{D}$) in addition to closed measurability (i.e., $t^{-1}(\{\mu \in M(\Omega) \mid \mu(E) \geq p\}) \in \mathcal{D}$).

With these definitions in mind, a mapping $t : \Omega \to M(\Omega)$ is the agent's *type mapping* if $t : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ is measurable. For each state $\omega \in \Omega$, call $t(\omega) \in M(\Omega)$ the agent's *type* at ω. The agent *p-believes* an event E at a state ω if

$$t(\omega)(E) \geq p,$$

i.e., she assigns probability at least p to E according to her type at ω.

Thus, a mapping $t : \Omega \to M(\Omega)$ is a type mapping if, for each event $E \in \mathcal{D}$ and probability $p \in [0,1]$, the set of states at which her type $t(\omega)$ at ω assigns probability at least p and at most p both form an event. For each $(p, E) \in [0,1] \times \mathcal{D}$, define

$$B_t^p(E) := t^{-1}(\{\mu \in M(\Omega) \mid \mu(E) \geq p\}) = \{\omega \in \Omega \mid t(\omega)(E) \geq p\} \in \mathcal{D} \text{ and} \tag{4}$$
$$L_t^p(E) := t^{-1}(\{\mu \in M(\Omega) \mid \mu(E) \leq p\}) = \{\omega \in \Omega \mid t(\omega)(E) \leq p\} \in \mathcal{D}. \tag{5}$$

Thus, $B_t^p(E)$ is the event that the agent p-believes E according to her type mapping t, and $L_t^p(E)$ is the event that the agent assigns probability at most p to E according to her type mapping t. Note that p-belief and dual p-belief operators B_t^p and L_t^p are each a well-defined operator from $\mathcal{D}$ into itself by the measurability of t.

Three remarks are in order. First, the agent's type mapping t *induces* her p-belief operators

$$\overrightarrow{B_t^p} := (B_t^p)_{p \in [0,1]}$$

such that each B_t^p is defined by Expression (4) through t. I will shortly introduce the agent's p-belief operators $\overrightarrow{B^p} := (B^p)_{p\in[0,1]}$ as a primitive of the belief model in such a way that the p-belief operators generate the type mapping.

Second, the agent's type mapping t *induces* her *dual* p-belief operators

$$\overrightarrow{L_t^p} := (L_t^p)_{p \in [0,1]}$$

(as in Heifetz and Mongin, 2001), where L_t^p is defined by Expression (5) through t.

Third, the dual p-belief operators $\overrightarrow{L_t^p}$ can simply be expressed from $\overrightarrow{B_t^p}$ in situations (i) in which the agent's beliefs are additive or (ii) in which the underlying $\mathcal{D}$ is a σ-algebra. Firstly, if every $t(\omega)$ is additive (i.e., $t(\omega)(E) + t(\omega)(E^c) = 1$ for each $E \in \mathcal{D}$ or, more concisely, $t = \overline{t}$) as in Heifetz and Mongin (2001), then, since

$$t(\omega)(E^c) \geq 1 - p \text{ iff } t(\omega)(E) \leq p,$$

it follows that

$$L_t^p(E) = B_t^{1-p}(E^c) \text{ for each } (p, E) \in [0,1] \times \mathcal{D}.$$

Secondly, if $\mathcal{D}$ is a σ-algebra, then for each $(p, E) \in [0,1] \times \mathcal{D}$,

$$L_t^p(E) = \begin{cases} \Omega & \text{if } p = 1 \\ \bigcap_{n\in\mathbb{N}} (\neg B_t^{p+\frac{1}{n}})(E) & \text{if } p \in [0,1) \end{cases}.[8] \tag{6}$$

I will set up a framework in a way such that, under the condition that one form of belief operators induces a type mapping, the other form of belief operators is always well-defined. In particular, if $\overrightarrow{B^p}$ induces the type mapping (which, in turn, recovers the original p-belief operators), then $\overrightarrow{L^p}$ is also induced from $\overrightarrow{B^p}$. Thus, I can express some properties of beliefs that involve reasoning of the form, the agent's belief in an event is at most p, just from the p-belief operators $\overrightarrow{B^p}$.

[8] For a given $p \in [0,1)$, let $n_0 \in \mathbb{N}$ be such that $p + \frac{1}{n_0} \in [0,1)$. Then, $L_t^p(E) = \bigcap_{n \geq n_0} (\neg B_t^{p+\frac{1}{n}})(E)$. If one defines $B_t^p(\cdot) = \emptyset$ for each $p > 1$, then one could write $L_t^p(E) = \bigcap_{n\in\mathbb{N}} (\neg B_t^{p+\frac{1}{n}})(E)$ for all $p \in [0,1]$.

3.3 A collection of p-Belief Operators

I move on to the second representation, p-belief operators: I define a collection of p-belief operators as a primitive. The collection $\overrightarrow{B^p} := (B^p)_{p\in[0,1]}$ of (the agent's p-belief) operators $B^p : \mathcal{D} \to \mathcal{D}$ is *regular* if it satisfies the following three conditions:

1. *Non-negativity*: $B^0(E) = \Omega$;
2. *p-Continuity from Below*: $p_n \uparrow p$ implies $B^{p_n}(E) \downarrow B^p(E)$;
3. *Limit Measurability*: $\bigcap_{n\in\mathbb{N}:p+\frac{1}{n}\leq 1} (\neg B^{p+\frac{1}{n}})(E) \in \mathcal{D}$.

For each $p \in [0, 1]$, call B^p the agent's *p-belief operator*.

The agent *p-believes* an event E at a state ω if

$$\omega \in B^p(E).$$

For each event $E \in \mathcal{D}$, the set $B^p(E)$ denotes the event that the agent p-believes E.

Non-negativity states that the agent always believes, with probability at least zero, any event E at any state ω. The axiom of p-Continuity from Below states that, for any increasing sequence $(p_n)_{n\in\mathbb{N}}$ with $p_n \uparrow p$, if the agent p_n-believes an event E at ω for all $n \in \mathbb{N}$ then she p-believes E at ω. Note that p-Continuity from Below presupposes:

- *p-Anti-Monotonicity*: if $p \leq q$ then $B^q(\cdot) \subseteq B^p(\cdot)$,

stating that B^p is non-increasing in p. Limit Measurability is a condition which allows for capturing the dual p-belief (i.e., the agent believes an event with probability at most p) from p-beliefs. If $\mathcal{D}$ is a σ-algebra, then Limit Measurability trivially holds.

For ease of exposition, let

$$B^p(\cdot) := \emptyset \text{ with } p > 1 \text{ and } B^p(\cdot) := \Omega \text{ with } p < 0.$$

Then, Limit Measurability is written as: $\bigcap_{n\in\mathbb{N}} (\neg B^{p+\frac{1}{n}})(E) \in \mathcal{D}$.

Similarly, the collection $\overrightarrow{L^p} := (L^p)_{p\in[0,1]}$ of (the agent's dual p-belief) operators $L^p : \mathcal{D} \to \mathcal{D}$ is *regular* if it satisfies the following three conditions:

1. *Unit*: $L^1(E) = \Omega$;
2. *p-Continuity from Above*: $p_n \downarrow p$ implies $L^{p_n} \downarrow L^p$;
3. *Dual Limit Measurability*: $\bigcap_{n\in\mathbb{N}:p-\frac{1}{n}\geq 0} (\neg L^{p-\frac{1}{n}})(E) \in \mathcal{D}$.

For each $p \in [0,1]$, call L^p the agent's *dual p-belief operator.*

Unit states that the agent always believes, with probability at most one, any event E at any state ω. The axiom of p-Continuity from Above states that, for any non-increasing sequence $(p_n)_{n\in\mathbb{N}}$ with $p_n \uparrow p$, if the agent believes, with probability at most p_n, an event E at ω for all $n \in \mathbb{N}$ then she believes, with probability at most p, the event E at ω. Note that $\overrightarrow{L^p}$ presupposes:

- *p-Monotonicity*: $p \leq q$ implies $L^p(\cdot) \subseteq L^q(\cdot)$,

stating that L^p is non-decreasing in p. Dual Limit Measurability is a condition which allows for capturing the p-belief (i.e., the agent believes an event with probability at least p) from dual p-beliefs. As for Limit Measurability, if $\mathcal{D}$ is a σ-algebra, then Dual Limit Measurability trivially holds.

For ease of exposition, let

$$L^p(\cdot) := \Omega \text{ with } p > 1 \text{ and } L^p(\cdot) := \emptyset \text{ with } p < 0.$$

Then, Dual Limit Measurability is written as: $\bigcap_{n\in\mathbb{N}}(\neg L^{p-\frac{1}{n}})(E) \in \mathcal{D}$.

The main purpose of this subsection is to establish the following benchmark result by which I can characterize various forms of non-additive beliefs using p-belief operators in Section 4. The benchmark result establishes the equivalence among representing the agent's beliefs by a type mapping, a regular collection of p-belief operators, and a regular collection of dual p-belief operators.

To that end, recall that a type mapping $t : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ induces the p-belief operators $\overrightarrow{B^p_t}$ through Expression (4). Conversely, a regular collection of p-belief operators $\overrightarrow{B^p}$ *induces* a mapping $t_B : \Omega \to M(\Omega)$ defined as follows:

$$t_B(\omega)(E) := \max\{p \in [0,1] \mid \omega \in B^p(E)\} \text{ for all } (p, E) \in [0,1] \times \mathcal{D}. \tag{7}$$

I show that the regularity conditions are the conditions that induce a well-defined type mapping.

Proposition 2. *1. Given a type mapping $t : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$, the collection of p-belief operators $\overrightarrow{B^p_t}$ is well-defined and regular.*

2. Conversely, given a regular collection $\overrightarrow{B^p}$ of p-belief operators, the mapping $t_B : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ is a well-defined type mapping.

3. Furthermore, $t = t_{B_t}$ and $\overrightarrow{B^p} = \overrightarrow{B^p_{t_B}}$.

Proposition 2 states that the regularity conditions (i.e., Non-negativity, p-Continuity from Below, and Limit Measurability) are the conditions on p-belief operators under which they can be a primitive of a belief model in that they induce a type mapping: a belief model in the form of $\langle(\Omega, \mathcal{D}), \overrightarrow{B^p}\rangle$ (where $\overrightarrow{B^p}$ is regular) and a belief model in the form of $\langle(\Omega, \mathcal{D}), t\rangle$ (where $t : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ is measurable) are equivalent. Also, Proposition 2 implies that regular p-belief operators induce a unique type mapping: for any type mappings t and t', Proposition 2 implies that

$$t = t' \text{ iff } \overrightarrow{B^p} = \overrightarrow{B'^p},$$

where $\overrightarrow{B'^p}$ corresponds to t'.

In fact, one can establish the one-to-one correspondence between a type mapping and dual p-belief operators and the one between p-belief operators and dual p-belief operators. To see this point, a regular collection of dual p-belief operators $\overrightarrow{L^p}$ *induces* a mapping $t_L : \Omega \to M(\Omega)$ defined as follows:

$$t_L(\omega)(E) := \min\{p \in [0,1] \mid \omega \in L^p(E)\} \text{ for all } (p, E) \in [0,1] \times \mathcal{D}. \tag{8}$$

Also, I define the dual p-belief operators $\overrightarrow{L^p_B}$ induced by a regular collection of p-belief operators $\overrightarrow{B^p}$ by combining Expressions (5) and (7):

$$L^p_B(E) := \{\omega \in \Omega \mid \max\{q \in [0,1] \mid \omega \in B^q(E)\} \leq p\} \text{ for each } (p, E) \in [0,1] \times \mathcal{D}. \tag{9}$$

Finally, I define the p-belief operators $\overrightarrow{B^p_L}$ induced by a regular collection of dual p-belief operators $\overrightarrow{L^p}$ by combining Expressions (4) and (8):

$$B^p_L(E) := \{\omega \in \Omega \mid \min\{q \in [0,1] \mid \omega \in L^q(E)\} \geq p\} \text{ for each } (p, E) \in [0,1] \times \mathcal{D}. \tag{10}$$

The following proposition establishes the one-to-one-correspondences.

Proposition 3.

1. (a) *Given a type mapping $t : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$, the collection of dual p-belief operators $\overrightarrow{L^p_t}$ is well-defined and regular.*
   (b) *Conversely, given a regular collection $\overrightarrow{L^p}$ of dual p-belief operators, the mapping t_L is a well-defined type mapping.*
   (c) *Furthermore, $t = t_{L_t}$ and $\overrightarrow{L^p} = \overrightarrow{L^p_{t_L}}$.*
2. (a) *Let $\overrightarrow{B^p}$ be a regular collection of p-belief operators. Then, $\overrightarrow{L^p_B}$ is a well-defined regular collection of dual p-belief operators.*
   (b) *Conversely, let $\overrightarrow{L^p}$ be a regular collection of dual p-belief operators. Then, $\overrightarrow{B^p_L}$ is a well-defined regular collection of p-belief operators.*
   (c) *Moreover, $L^p_{B_L} = L^p$ and $B^p_{L_B} = B^p$ for all $p \in [0,1]$.*

Figure 4 illustrates Propositions 2 and 3. The figure depicts the equivalence among a type mapping $t : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ (recall that a mapping $t : \Omega \to M(\Omega)$ is a type mapping if it is measurable), a regular collection of p-belief operators $\overrightarrow{B^p}$, and

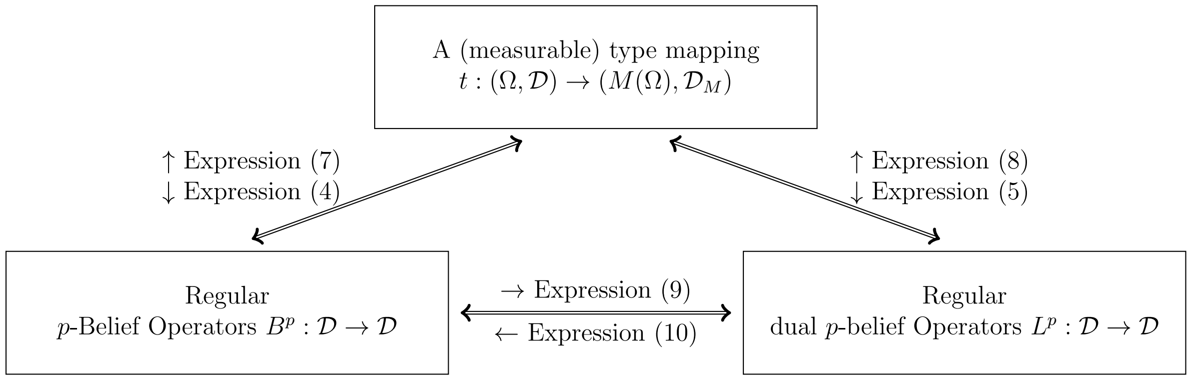

Figure 4: An Illustration of Propositions 2 and 3.

a regular collection of dual p-belief operators $\overrightarrow{L^p}$. The figure also illustrates how one representation induces another.

If a collection of p-belief operators $\overrightarrow{B^p}$ is regular, then the dual collection $\overrightarrow{L^p}$ is well-defined and regular. Likewise, if a collection of dual p-belief operators $\overrightarrow{L^p}$ is regular, then the collection $\overrightarrow{B^p}$ is well-defined and regular. In fact, by Limit Measurability and Dual Limit Measurability, one can admit another set of representations.

Corollary 1. *1. If $\overrightarrow{B^p}$ is regular, then $L^p_B(\cdot) = \bigcap_{n \in \mathbb{N}} (\neg B^{p+\frac{1}{n}})(\cdot) \in \mathcal{D}$.*

2. If $\overrightarrow{L^p}$ is regular, then $B^p_L(\cdot) = \bigcap_{n \in \mathbb{N}} (\neg L^{p-\frac{1}{n}})(\cdot) \in \mathcal{D}$.

Section 4 will characterize properties of the agent's beliefs with $\overrightarrow{B^p}$ being a primitive. By virtue of the equivalence established in Proposition 2, I sometimes use the dual p-belief operators $\overrightarrow{L^p_B}$. Also, I often drop the subscripts from the type mapping, the p-belief operator, and the dual p-belief operator (e.g., $\overrightarrow{L^p}$ instead of $\overrightarrow{L^p_B}$).

4 Representations of Properties of Beliefs

This section studies logical and introspective properties of beliefs. Section 4.1 studies logical properties of beliefs. The main result of this section, Theorem 1, allows for identifying the specific conditions on p-belief operators that lead to specific forms of non-additive beliefs such as Choquet and Dempster-Shafer beliefs. Section 4.2 studies introspective properties of beliefs. These properties also play a role in formalizing common p-beliefs.

4.1 Logical Properties of Beliefs

Proposition 2 has established the benchmark equivalence between a type mapping and a regular collection of p-belief operators. In order to extend the equivalence to a wide variety of beliefs, I define properties of beliefs in terms of a type mapping and p-belief operators.

I begin by defining standard properties of a set function.

Definition 1. A set function $\mu : \mathcal{D} \to [0,1]$ satisfies:

1. *No-Contradiction* if $\mu(\emptyset) = 0$;
2. *Normalization* if $\mu(\Omega) = 1$;
3. *Monotonicity* if $E \subseteq F$ implies $\mu(E) \subseteq \mu(F)$;
4. *Sub-additivity* if $\mu(E \cup F) \leq \mu(E) + \mu(E^c \cap F)$ for any $E, F \in \mathcal{D}$;
5. *Super-additivity* if $\mu(E \cup F) \geq \mu(E) + \mu(E^c \cap F)$ for any $E, F \in \mathcal{D}$;
6. *Finite Additivity* if μ satisfies Sub-additivity and Super-additivity.

A given type mapping t satisfies each property in Definition 1 if $t(\omega)$ satisfies it for every $\omega \in \Omega$. In Definition 1, Super-additivity implies Monotonicity. Also, Super-additivity and Normalization imply No-Contradiction.

A set function $\mu : \mathcal{D} \to [0,1]$ is a Choquet (1954) *capacity* (or a *non-additive probability measure*) if it satisfies No-Contradiction, Normalization, and Monotonicity (sometimes, the continuity properties (from above and from below) to be introduced in Definition 3 are assumed as well). A capacity is a *finitely-additive probability measure* if it additionally satisfies Finite Additivity.

To study Dempster-Shafer beliefs (Dempster, 1967; Shafer, 1976) or possibility beliefs (Dubois and Prade, 1988; Zadeh, 1978), I introduce four additional properties. To that end, let $\mathcal{P}_n := \mathcal{P}(\{1, \dots, n\}) \setminus \{\emptyset\}$ for each $n \in \mathbb{N}$.

Definition 2. A set function $\mu : \mathcal{D} \to [0,1]$ satisfies:

7. (a) *n-Monotonicity* (where $n \in \mathbb{N} \setminus \{1\}$) if

$$\mu\left(\bigcup_{j=1}^{n} E_j\right) \geq \sum_{J \in \mathcal{P}_n} (-1)^{|J|-1} \mu\left(\bigcap_{j \in J} E_j\right); \tag{11}$$

   (b) *∞-Monotonicity* if n-Monotonicity holds for all $n \in \mathbb{N} \setminus \{1\}$;

8. (a) *Alternating n-Monotonicity* (where $n \in \mathbb{N} \setminus \{1\}$) if

$$\mu\left(\bigcap_{j=1}^{n} E_j\right) \leq \sum_{J \in \mathcal{P}_n} (-1)^{|J|-1} \mu\left(\bigcup_{j \in J} E_j\right); \tag{12}$$

(b) *Alternating ∞-Monotonicity* if Alternating n-Monotonicity holds for all $n \in \mathbb{N} \setminus \{1\}$;

9. (a) *Maxitivity* if, for any $\{E_j\}_{j\in J} \subseteq \mathcal{D}$ with $\bigcup_{j\in J} E_j \in \mathcal{D}$,

$$\mu\left(\bigcup_{j\in J} E_j\right) \leq \sup_{j\in J} \mu(E_j);$$

   (b) *Finite Maxitivity* if $\mu(E \cup F) \leq \max(\mu(E), \mu(F))$;

10. (a) *Minitivity* if, for any $\{E_j\}_{j\in J} \subseteq \mathcal{D}$ with $\bigcap_{j\in J} E_j \in \mathcal{D}$,

$$\mu\left(\bigcap_{j\in J} E_j\right) \geq \inf_{j\in J} \mu(E_j);$$

   (b) *Finite Minitivity* if $\mu(E \cap F) \geq \min(\mu(E), \mu(F))$.

Again, a given type mapping t satisfies each property in Definition 2 if $t(\omega)$ satisfies it for every $\omega \in \Omega$. First, μ is a Dempster-Shafer belief function if it satisfies No-Contradiction, Normalization, and ∞-Monotonicity ("Continuity-from-above" property to be introduced in Definition 3 is sometimes assumed). The dual set function $\overline{\mu}$ defined by

$$\overline{\mu}(E) := 1 - \mu(E^c)$$

is referred to as a plausibility function. Groes et al. (1998) study mixed-strategy Nash equilibria when agents' beliefs are non-additive and satisfy n-Monotonicity.

Second, μ is *a possibility measure* if it satisfies No-Contradiction, Normalization, Monotonicity, and Maxitivity. Each set function $\pi_i(\omega)$ in Section 2 is an example of a possibility measure. The set function $\overline{\mu}$ is referred to as a necessity measure. It can be seen that a possibility measure (precisely, any set function satisfying Monotonicity and Finite Maxitivity) satisfies Alternating ∞-Monotonicity. Indeed, the possibility measure is a plausibility function (i.e., the necessity measure is a belief function). See, for example, Halpern (2017) and Wang and Klir (2010).

Also, qualitative belief or knowledge induced by a possibility correspondence (e.g., Aumann, 1976) satisfies Minitivity: the agent believes an event E (at a state) if the event E includes the information set at that state.[9]

Four remarks on Definition 2 are in order. First, the "converse" weak inequality in Finite Maxitivity, Finite Minitivity, Maxitivity, and Minitivity are all equivalent

[9] If the agent believes each event E_j (at a state), then her information set (at the state) is included in E_j for all $j \in J$. Whenever $\bigcap_{j\in J} E_j \in \mathcal{D}$ (i.e., it is an object of her beliefs), she believes the conjunction $\bigcap_{j\in J} E_j$ (at the state).

to Monotonicity. Thus, if μ is monotone then each of them is characterized by the equality as in the literature (e.g., Wang and Klir, 2010).

Second, Finite Maxitivity implies Sub-additivity. Third, consider n-Monotonicity. If $n = 2$ then

$$\mu(E_1) + \mu(E_2) \leq \mu(E_1 \cap E_2) + \mu(E_1 \cup E_2).$$

This is sometimes referred to as *Convexity*, and is associated with a notion of uncertainty aversion (Schmeidler, 1986, 1989).[10] If μ satisfies n-Monotonicity, then it satisfies k-Monotonicity for any $k \leq n$. Also, n-Monotonicity implies Monotonicity, given No-Contradiction.[11]

Fourth, consider Alternating n-Monotonicity. If $n = 2$ then

$$\mu(E_1 \cap E_2) + \mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2).$$

This is sometimes referred to as *Concavity*. If μ satisfies Alternating n-Monotonicity, then it satisfies Alternating k-Monotonicity for any $k \leq n$.

To sum up the properties of a set function, I introduce continuity properties.

Definition 3. For ease of exposition, suppose that $\mathcal{D}$ is a σ-algebra. The set function μ satisfies:

11. *Countable Sub-additivity* if $\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \mu\left(\left(\bigcap_{m=1}^{n-1} E_m^c\right) \cap E_n\right)$;

12. *Countable Super-additivity* if $\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) \geq \sum_{n\in\mathbb{N}} \mu\left(\left(\bigcap_{m=1}^{n-1} E_m^c\right) \cap E_n\right)$;

13. *Countable-additivity* if μ satisfies Countable Sub-additivity and Countable Super-additivity;

14. *Continuity from Above* if $E_n \downarrow E$ implies $\mu(E_n) \downarrow \mu(E)$;

15. *Continuity from Below* if $E_n \uparrow E$ implies $\mu(E_n) \uparrow \mu(E)$.

While I have introduce the continuity properties on a σ-algebra $\mathcal{D}$, it is possible to define these properties when $\mathcal{D}$ is an algebra by requiring the given collection $(E_n)_{n\in\mathbb{N}}$ to satisfy $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{D}$ or $E = \bigcap_{n\in\mathbb{N}} E_n \in \mathcal{D}$. Again, a given type mapping t satisfies

[10] For instance, Eichberger and Kelsey (2000) assume convexity throughout their analysis of games with non-additive beliefs.

[11] The proof goes as follows. Suppose $E \subseteq F$. Letting $(E_1, E_2) = (E, F \setminus E)$, it follows from 2-Monotonicity (i.e., Convexity) and No-Contradiction that

$$\mu(F) = \mu(E_1 \cup E_2) \geq \mu(E_1) + \mu(E_2) - \mu(E_1 \cap E_2) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

each property in Definition 3 if $t(\omega)$ satisfies it for every $\omega \in \Omega$. Monotonicity is implicit in Continuity from Above and Continuity from Below. As already discussed, a capacity is often assumed to satisfy Continuity from Above and Continuity from Below in addition to No-Contradiction, Normalization, and Monotonicity (when $\mathcal{D}$ is infinite). Also, a Dempster-Shafer belief function is also assumed to satisfy Continuity from Above in addition to No-Contradiction, Normalization, and ∞-Monotonicity (when $\mathcal{D}$ is infinite).

Now, I turn to defining the corresponding properties of beliefs in terms of p-belief operators. To that end, define M_J^p and N_J^p with $J \in \mathcal{P}_n = \mathcal{P}(\{1, \dots, n\}) \setminus \{\emptyset\}$ as

$$M_J^p := \begin{cases} B^p & \text{if } |J| \text{ is odd} \\ L^p & \text{if } |J| \text{ is even} \end{cases} \text{ and } N_J^p := \begin{cases} L^p & \text{if } |J| \text{ is odd} \\ B^p & \text{if } |J| \text{ is even} \end{cases}.$$

First, I introduce the standard properties of beliefs that correspond to Definition 1.

Definition 4. A regular collection $\overrightarrow{B^p}$ of p-belief operators satisfies:

1. *No-Contradiction* if $B^p(\emptyset) = \emptyset$ for any $p \in (0, 1]$;
2. *Normalization* if $B^1(\Omega) = \Omega$;
3. *Monotonicity* if $E \subseteq F$ implies $B^p(E) \subseteq B^p(F)$;
4. *Sub-additivity* if $L^p(E) \cap L^q(E^c \cap F) \subseteq L^{p+q}(E \cup F)$ for any $E, F \in \mathcal{D}$;
5. *Super-additivity* if $B^p(E) \cap B^q(E^c \cap F) \subseteq B^{p+q}(E \cup F)$ for any $E, F \in \mathcal{D}$;
6. *Finite Additivity* if $\overrightarrow{B^p}$ satisfies Sub-additivity and Super-additivity.

Two remarks on Definition 4 in relation to the literature are in order. First, as in Samet (2000, Section 4), Sub-additivity is equivalent to:

$$(\neg B^p)(\tilde{E} \cap \tilde{F}) \cap (\neg B^q)(\tilde{E} \cap \tilde{F}^c) \subseteq (\neg B^{p+q})(\tilde{E}) \text{ for any } \tilde{E}, \tilde{F} \in \mathcal{D}.$$

Second, as in Samet (2000, Section 4), Super-additivity is restated as:

$$B^p(\tilde{E} \cap \tilde{F}) \cap B^q(\tilde{E} \cap \tilde{F}^c) \subseteq B^{p+q}(\tilde{E}) \text{ for any } \tilde{E}, \tilde{F} \in \mathcal{D}.$$

Next, I introduce the properties of beliefs that correspond to Definition 2.

Definition 5. A regular collection $\overrightarrow{B^p}$ of p-belief operators satisfies:

7. (a) *n-Monotonicity* (where $n \in \mathbb{N} \setminus \{1\}$) if

$$\bigcap_{J \in \mathcal{P}_n} M_J^{p_J} \left(\bigcap_{j \in J} E_j \right) \subseteq B^{\sum_{J \in \mathcal{P}_n} (-1)^{|J|-1} p_J} \left(\bigcup_{j=1}^{n} E_j \right); \tag{13}$$

(b) ∞-*Monotonicity* if it satisfies n-Monotonicity for all $n \in \mathbb{N} \setminus \{1\}$;

8. (a) *Alternating n-Monotonicity* (where $n \in \mathbb{N} \setminus \{1\}$) if

$$\bigcap_{J \in \mathcal{P}_n} N_J^{p_J}\left(\bigcup_{j \in J} E_j\right) \subseteq B^{\sum_{J \in \mathcal{P}_n}(-1)^{|J|-1}p_J}\left(\bigcap_{j=1}^{n} E_j\right); \tag{14}$$

(b) *Alternating ∞-Monotonicity* if it satisfies Alternating n-Monotonicity for all $n \in \mathbb{N} \setminus \{1\}$;

9. (a) *Maxitivity* if $\bigcap_{j \in J} L^{p_j}(E_j) \subseteq L^{\sup_j p_j}\left(\bigcup_{j \in J} E_j\right)$, where $\bigcup_{j \in J} E_j \in \mathcal{D}$;

(b) *Finite Maxitivity* if $L^p(E) \cap L^q(F) \subseteq L^{\max(p,q)}(E \cup F)$;

10. (a) *Minitivity* if $\bigcap_{j \in J} B^{p_j}(E_j) \subseteq B^{\inf_j p_j}\left(\bigcap_{j \in J} E_j\right)$, where $\bigcap_{j \in J} E_j \in \mathcal{D}$;

(b) *Finite Minitivity* if $B^p(E) \cap B^q(F) \subseteq B^{\min(p,q)}(E \cap F)$.

In Definition 5, Finite Maxivity is restated as:

$$(\neg B^p)(E) \cap (\neg B^q)(F) \subseteq (\neg B^{\max(p,q)})(E \cup F).$$

Finally, I introduce the properties of beliefs that correspond to Definition 3.

Definition 6. As in Definition 3, for ease of exposition, suppose that $\mathcal{D}$ is a σ-algebra. A regular collection $\overrightarrow{B^p}$ of p-belief operators satisfies:

11. *Countable Sub-additivity* if

$$\bigcap_{n \in \mathbb{N}} L^{p_n}\left(\left(\bigcap_{m=1}^{n-1} E_m^c\right) \cap E_n\right) \subseteq L^{\sum_{n \in \mathbb{N}} p_n}\left(\bigcup_{n \in \mathbb{N}} E_n\right),$$

12. *Countable Super-additivity* if

$$\bigcap_{n \in \mathbb{N}} B^{p_n}\left(\left(\bigcap_{m=1}^{n-1} E_m^c\right) \cap E_n\right) \subseteq B^{\sum_{n \in \mathbb{N}} p_n}\left(\bigcup_{n \in \mathbb{N}} E_n\right);$$

13. *Countable-additivity* if $\overrightarrow{B^p}$ satisfies Countable Sub-additivity and Countable Super-additivity.

14. *Continuity from Above* if $E_n \downarrow E$ implies $B^p(E_n) \downarrow B^p(E)$;

15. *Continuity from Below* if $E_n \uparrow E$ implies $L^p(E_n) \downarrow L^p(E)$.

Now, I present the main result of this section, which extends Proposition 2 to any collection of axioms on set functions on $(\Omega, \mathcal{D})$ defined in Definitions 1 to 3.

Theorem 1. *Consider any possible combination of axiom(s) on set functions on $(\Omega, \mathcal{D})$ defined in Definitions 1 to 3, and let $M(\Omega)$ be the collection of such set functions.*

1. *If a measurable mapping $t : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ is given, then the induced collection $\overrightarrow{B_t^p}$ is well-defined, is regular, and satisfies the corresponding assumption(s) in Definitions 4 to 6.*

2. *Conversely, the mapping $t_B : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ induced by a regular collection $\overrightarrow{B^p}$ of p-belief operators respecting the given assumption(s) in Definitions 4 to 6 is a well-defined measurable mapping which satisfies the corresponding assumption(s) in Definitions 1 to 3.*

3. *Furthermore, $t = t_{B_t}$ and $\overrightarrow{B^p} = \overrightarrow{B^p_{t_B}}$.*

4.2 Introspective Properties of Beliefs

The previous subsection has characterized logical properties of beliefs. This subsection studies the agent's introspective properties on her own beliefs, and shows that the introspective properties can be formulated independently of the logical properties.

In the literature, an agent is *certain of her beliefs* if

$$t(\omega)(E) = 1 \text{ for any } (\omega, E) \in \Omega \times \mathcal{D} \text{ with } E \supseteq \{\tilde{\omega} \in \Omega \mid t(\omega) = t(\tilde{\omega})\}.$$

The idea is that, at each state ω, if the agent is certain of her beliefs, then she must be able to infer that the true state is in one of $\{\tilde{\omega} \in \Omega \mid t(\omega) = t(\tilde{\omega})\}$. For instance, in the literature on the construction of a terminal belief space, Fukuda (2024a, 2025b), Meier (2006), and Mertens and Zamir (1985) impose this property.[12]

As is well-known in the literature, if the agent is certain of her beliefs then the following two introspective properties hold (for all $p \in [0, 1]$).[13]

1. Positive Certainty: $B^p(\cdot) \subseteq B^1 B^p(\cdot)$.

2. Negative Certainty: $(\neg B^p)(\cdot) \subseteq B^1(\neg B^p)(\cdot)$.

[12] In different contexts, papers such as Fukuda (2025a) and Samet (1999, 2000) provide epistemic characterizations of this property.

[13] The converse holds, for example, under the following environment: $\mathcal{D}$ is a σ-algebra generated by a countable algebra and every $t_i(\omega)$ is a countably-additive probability measure (Samet, 1999).

While the literature assumes that each type $t(\omega)$ is at least a finitely-additive measure, it can be seen from Expressions (4) and (7) that one can formalize/rewrite these properties as follows.

1. Positive Certainty holds iff, for any $(E, p) \in \mathcal{D} \times [0, 1]$ with $t(\omega)(E) \geq p$,
$$1 = t(\omega)(\{\tilde{\omega} \in \Omega \mid p \leq t(\tilde{\omega})(E)\}). \tag{15}$$

2. Negative Certainty holds iff, for any $(E, p) \in \mathcal{D} \times [0, 1]$ with $t(\omega)(E) < p$,
$$1 = t(\omega)(\{\tilde{\omega} \in \Omega \mid t(\tilde{\omega})(E) < p\}). \tag{16}$$

Here, I study the following two weaker introspective properties, Positive Introspection and Negative Introspection, whenever each $t(\omega)$ is a set function and independently from the logical properties. Positive Introspection states that if the agent p-believes an event then she p-believes that she p-believes it. In contrast, Negative Introspection states that if the agent does not p-believe an event, then she p-believes that she does not p-believe it.

Lemma 1. *3. Positive Introspection: $B^p(\cdot) \subseteq B^p B^p(\cdot)$ iff*
$$t(\omega)(E) \leq t(\omega)(\{\tilde{\omega} \in \Omega \mid t(\omega)(E) \leq t(\tilde{\omega})(E)\}) \text{ for all } (\omega, E) \in \Omega \times \mathcal{D}.$$

4. Negative Introspection: $(\neg B^p)(\cdot) \subseteq B^p(\neg B^p)(\cdot)$ iff
$$t(\omega)(E) + \varepsilon \leq t(\omega)(\{\tilde{\omega} \in \Omega \mid t(\tilde{\omega})(E) < t(\omega)(E) + \varepsilon\})$$
for all $\omega \in \Omega$, $E \in \mathcal{D}$, and $\varepsilon \in (0, 1 - t(\omega)(E)]$.

Heifetz and Mongin (2001) characterize introspective properties of 1-belief (i.e., certainty) operators:
$$B^1(\cdot) \subseteq B^1 B^1(\cdot) \text{ and } (\neg B^1)(\cdot) \subseteq B^1(\neg B^1)(\cdot).$$
These properties can also be characterized by putting $p = 1$ in Expressions (15) and (16), respectively.

To conclude this subsection, I study Subpotency: if the agent p-believes that she p-believes an event then she p-believes it. Thus, Subpotency is the converse of Positive Introspection. Section 5.1 studies the implication of Subpotency on the iterative formulation of common belief.

Proposition 4. *1. Subpotency: $B^p B^p(\cdot) \subseteq B^p(\cdot)$ iff*
$$t(\omega)(E) \geq p$$
for any $(\omega, E, p) \in \Omega \times \mathcal{D} \times [0, 1]$ with $t(\omega)(\{\tilde{\omega} \in \Omega \mid t(\tilde{\omega})(E) \geq p\}) \geq p$.

2. *If a regular collection $\overrightarrow{B^p}$ of p-belief operators satisfies No-Contradiction, Normalization, 2-Monotonicity (i.e., Convexity), and Negative Certainty, then it satisfies Subpotency.*

The first part of the proposition states that one can formalize Subpotency solely in terms of the type mapping. The second part implies that if the agent's beliefs are represented by a type mapping t such that each $t(\omega)$ is a convex capacity and such that t satisfies Negative Certainty (i.e., the agent is introspective about the lack of her own beliefs) then her type mapping also satisfies Subpotency. The result in Section 5.1 shows that, in such a setting, the common p-belief operator reduces to the iteration of mutual p-belief operators.

The introspective properties in this subsection enable the analysts to study beliefs of agents who are introspective about their own beliefs by restricting attention to beliefs (either type mappings or p-belief operators) which satisfy the above introspective properties. Note that these characterizations hold irrespective of any underlying logical properties of beliefs.

5 Applications

This section discusses applications of the framework of this paper. Since the Introduction has discussed possible applications to epistemic analyses of game-theoretic solution concepts and the agreeing-to-disagree and no-trade theorems and Section 2 has analyzed Rubinstein (1989)'s e-mail game with non-additive beliefs, this section discusses other applications. Specifically, Section 5.1 defines the multi-agent framework including common p-beliefs. Section 5.2 demonstrates that the framework of this paper admits a terminal belief space. Section 5.3 considers the extension to conditional beliefs.

5.1 Common p-Belief

Throughout this subsection, let I be an at-most countable set of agents with $|I| \geq 2$. In this subsection, a *model* refers to a tuple $\langle(\Omega, \mathcal{D}), (\overrightarrow{B_i^p})_{i\in I}, \overrightarrow{C^p}\rangle$ with the following three ingredients. First, $(\Omega, \mathcal{D})$ is a measurable state space: I assume that $\mathcal{D}$ is a σ-algebra so as to ensure that the iterations of mutual p-beliefs are always well-defined.

Second, for each agent $i \in I$, $\overrightarrow{B_i^p} = (B_i^p)_{p\in[0,1]}$ is a regular collection of agent i's p-belief operators. Define the *mutual p-belief operators* $(B_I^p)_{p\in[0,1]}$ as

$$B_I^p(\cdot) := \bigcap_{i\in I} B_i^p(\cdot) \text{ for each } p \in [0, 1].$$

Call an event $F \in \mathcal{D}$ a *common p-basis* (Fukuda, 2020) if

$$F \subseteq E \text{ implies } F \subseteq B_I^p(E).$$

That is, F is a common p-basis if everybody p-believes any logical consequence of F. Note that, when the mutual p-belief operator B_I^p satisfies Monotonicity, an event $F \in \mathcal{D}$ is a common p-basis iff it is a p-evident event (e.g., Monderer and Samet, 1989): $F \subseteq B_I^p(F)$.

Third, $\overrightarrow{C^p} = (C^p)_{p \in [0,1]}$ is a collection of common p-belief operators $C^p : \mathcal{D} \to \mathcal{D}$ defined as follows (e.g., Fukuda, 2020; Monderer and Samet, 1989): for each $E \in \mathcal{D}$,

$$C^p(E) := \{\omega \in \Omega \mid \text{there is a common } p\text{-basis } F \in \mathcal{D} \text{ with } \omega \in F \subseteq B_I^p(E)\} \in \mathcal{D}.$$

By construction, the common p-belief implies any level of mutual p-beliefs:

$$C^p(\cdot) \subseteq (B_I^p)^n(\cdot) \text{ for any } n \in \mathbb{N}.$$

Denoting by

$$B_*^p(\cdot) := \bigcap_{n \in \mathbb{N}} (B_I^p)^n(\cdot)$$

the *iterative common p-belief operator*, it follows that

$$C^p(\cdot) \subseteq B_*^p(\cdot).$$

That is, if an event E is common p-belief, then everybody p-believes E, everybody p-believes that everybody p-believes E, and so on *ad infinitum*.

Now, if each agent's p-belief operators satisfy the logical and introspective properties of Monotonicity, Subpotency, and Continuity from Above, then the common p-belief operator C^p coincides with the iterative common p-belief operator B_*^p (Monderer and Samet, 1989, Proposition 4).

Remark 2. Suppose that, for every agent $i \in I$, $\overrightarrow{B_i^p}$ satisfies Monotonicity, Subpotency, and Continuity from Above. Then, $C^p = B_*^p$ (for all $p \in [0,1]$).

5.2 Terminal Non-Additive Belief Spaces

This subsection aims at showing the existence of a terminal non-additive belief space. To that end, let I be a set of agents with $|I| \geq 2$, and let S be a set of states of nature endowed with an algebra $\mathcal{S}$ on S. The states of nature are exogenously given parameter values such as strategies or payoffs about which agents interactively reason. Also, fix a possible combination of properties of (non-additive) beliefs specified in Definitions 4, 5, and 6.

With these definitions in mind, a belief space of I over $(S, \mathcal{S})$ is a tuple $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (B_i^p)_{(i,p) \in I \times [0,1]}, \Theta \rangle$ with the following three properties.

1. Ω is a set of states of the world, endowed with an algebra $\mathcal{D}$ on Ω.

2. For each agent i, $\overrightarrow{B_i^p}$, where $B_i^p : \mathcal{D} \to \mathcal{D}$, is agent i's regular collection of p-belief operators that satisfies the given assumptions on beliefs.

3. $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{S})$ is a measurable mapping that associates, with each state of the world, the corresponding state of nature.

One can define the class of belief spaces, depending on the choice of the assumptions on beliefs. Note that, by Theorem 1, even if one starts by defining a belief space through the agents' type mappings, one can equivalently define the belief space through p-belief operators. While, for ease of exposition, I omit introducing the common belief operator from a belief space, one can incorporate the common p-belief operators into the definition of a belief space.[14]

In the belief space $\overrightarrow{\Omega}$, the analysts can represent the agents' first- and higher-order beliefs about $(S, \mathcal{S})$ through Θ and the agents' belief operators $(B_i^p)_{(i,p)\in I\times[0,1]}$. For any $E \in \mathcal{S}$, the agents' first-order beliefs are represented by $B_i^p(\Theta^{-1}(E))$, their second-order beliefs are represented by $B_j^p B_i^p(\Theta^{-1}(E))$, and so on.

A terminal belief space is defined as a belief space to which every belief space in the given class is uniquely mapped in a belief-preserving manner. I start by formalizing the notion of a belief-preserving map, a belief morphism.

Namely, let $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}), (B_i^p)_{(i,p)\in I\times[0,1]}, \Theta\rangle$ and $\overrightarrow{\Omega'} = \langle(\Omega', \mathcal{D}'), (B_i'^p)_{(i,p)\in I\times[0,1]}, \Theta'\rangle$ be belief spaces (of the given class). A *belief morphism* $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ satisfying the following two conditions:

1. $\Theta = \Theta' \circ \varphi$;
2. $B_i^p(\varphi^{-1}(E')) = \varphi^{-1}(B_i'^p(E'))$ for each $(i, p, E') \in I \times [0, 1] \times \mathcal{D}'$.

The belief morphism φ associates, with each state $\omega \in \Omega$, the corresponding state $\varphi(\omega) \in \Omega'$ with the two conditions. The first condition requires that the same state of nature prevail between two states ω and $\varphi(\omega)$. The second condition states that the agents' p-beliefs are preserved from one space to another: agent i p-believes an event E' at $\varphi(\omega)$ iff she p-believes $\varphi^{-1}(E')$ at ω.

For any belief space $\overrightarrow{\Omega}$, the identity map id_Ω on Ω is a belief morphism from $\overrightarrow{\Omega}$ into itself. Next, call a belief morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ a *belief isomorphism*, if φ is bijective and its inverse φ^{-1} is a morphism. If φ is an isomorphism then its inverse φ^{-1} is unique. Belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ are *isomorphic*, if there is an isomorphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$.

Now, I define a terminal belief space. Namely a belief space $\overrightarrow{\Omega^*}$ is *terminal* if, for any belief space $\overrightarrow{\Omega}$ (in the class), there is a unique morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$.

Proposition 5. *Fix a combination of properties of beliefs specified in Definitions 4, 5, and 6. This defines the class of belief spaces respecting the specified properties. Then, there exists a terminal belief space $\overrightarrow{\Omega^*}$ in the given class.*

[14] Under certain assumptions (e.g., $\mathcal{D}$ is a σ-algebra and the agent's p-belief operators satisfies the properties in Remark 2), the common p-belief operator can be expressed in terms of the agents' p-belief operators.

In the context of this paper, the proposition follows from Fukuda (2024b, Section 5.1), which shows the existence of a terminal belief space irrespective of properties of beliefs once the properties of beliefs are represented through belief operators. Since Theorem 1 shows that various non-additive beliefs can be represented through p-belief operators, one can apply the arguments in Fukuda (2024b, Section 5.1) to the context of this paper to assert the existence of a terminal belief space.

In epistemic game theory, a terminal belief space is used to characterize certain solution concepts such as iterated elimination of strictly dominated actions (as an implication of rationality and common belief in rationality). It would be interesting to study an implication of rationality and common belief in rationality on a terminal belief space in the context of non-additive beliefs (see Dominiak and Schipper (2021) in this direction).

5.3 Conditional Non-Additive Beliefs

This subsection studies conditional non-additive beliefs represented by a conditional belief system. The conditional belief system specifies an agent's belief in an event conditional on each conditioning event. Rényi (1955) axiomatizes a conditional probability system that specifies countably-additive probabilistic beliefs on conditioning events.[15] Thus, a conditional type mapping is a mapping that associates, with each state of the world, a conditional belief system. Di Tillio, Halpern, and Samet (2014, Theorem 1) identify conditions on conditional p-belief operators under which they induce a conditional type mapping.

First, this subsection introduces conditional beliefs at an abstract level in line with the analyses in Section 3. Second, it replicates Di Tillio, Halpern, and Samet (2014, Theorem 1). Third, it formulates conditional belief systems for possibility measures.

5.3.1 Conditional Type Mappings and Conditional p-Belief Operators

Throughout the subsection, fix a tuple $(\Omega, \mathcal{D}, \mathcal{C})$: $\mathcal{D}$ is an algebra on a set Ω; and $\mathcal{C}$ is a non-empty sub-collection $\mathcal{D}$ with $\emptyset \notin \mathcal{C}$. While various set-algebraic assumptions on $\mathcal{C}$ are imposed in the literature, I simply consider the case with $\emptyset \notin \mathcal{C} \subseteq \mathcal{D}$. Let $M(\Omega)$ be a subset of $[0,1]^{\mathcal{D}}$ such that each element $\mu \in M(\Omega)$ satisfies some given properties of conditional beliefs irrespective of conditioning events (recall Section 3.2).[16]

Let $M^{\mathcal{C}}(\Omega) := (M(\Omega))^{\mathcal{C}}$ be the product of $M(\Omega)$ over $\mathcal{C}$, and denote by $\overrightarrow{\mu} := (\mu(\cdot|C))_{C \in \mathcal{C}}$ a profile of set functions in $M^{\mathcal{C}}(\Omega)$. Call $\overrightarrow{\mu}$ a *conditional set function*. For ease of notation, letting $M_C(\Omega) := M(\Omega)$, denote $M^{\mathcal{C}}(\Omega) = \prod_{C \in \mathcal{C}} M_C(\Omega)$.

[15] Battigalli and Siniscalchi (1999) and Guarino (2017, 2024) construct a terminal type space based on a conditional probability systems.

[16] I will define a conditional set function $\overrightarrow{\mu}$ as $\overrightarrow{\mu} := (\mu(\cdot \mid C))_{C \in \mathcal{C}}$, where $\mu(\cdot \mid C)$ is a set function defined on $\mathcal{D}$ conditional on a conditioning event $C \in \mathcal{C}$. I define $M(\Omega)$ as the collection of set functions on $\mathcal{D}$ which satisfy a given combination of properties (recall Section 3.2) irrespective of conditioning events $C \in \mathcal{C}$.

I introduce the product algebra $\mathcal{D}_M^{\mathcal{C}}$ on $M^{\mathcal{C}}(\Omega)$. Namely, denoting by $\mathrm{pr}_C : M^{\mathcal{C}}(\Omega) \to M_C(\Omega)$ the projection, $\mathcal{D}_M^{\mathcal{C}}$ is the smallest algebra including

$$\mathrm{pr}_C^{-1}(\{\mu \in M(\Omega) \mid \mu(E) \geq p\}) \text{ and } \mathrm{pr}_C^{-1}(\{\mu \in M(\Omega) \mid \mu(E) \leq p\})$$

for every $(C, E, p) \in \mathcal{C} \times \mathcal{D} \times [0, 1]$.

For a given property of a set function in Definitions 1 to 3, a conditional set function $\overrightarrow{\mu} := (\mu(\cdot|C))_{C \in \mathcal{C}} \in M^{\mathcal{C}}(\Omega)$ satisfies it if every $\mu(\cdot|C)$ satisfies it. For example, $\overrightarrow{\mu}$ satisfies No-Contradiction if

$$\mu(\emptyset|C) = 0 \text{ for all } C \in \mathcal{C}.$$

A *conditional type mapping* is a measurable mapping

$$\overrightarrow{t} : (\Omega, \mathcal{D}) \to (M^{\mathcal{C}}(\Omega), \mathcal{D}_M^{\mathcal{C}})$$

denoted by

$$\overrightarrow{t} = (t(\cdot)(\cdot|C))_{C \in \mathcal{C}}.$$

Namely, for each $(C, E, p) \in \mathcal{C} \times \mathcal{D} \times [0, 1]$,

$$B_t^p(E|C) := \{\omega \in \Omega \mid t(\omega)(E|C) \geq p\} \in \mathcal{D}.^{17}$$

A collection of *conditional p-belief operators* is

$$\overrightarrow{B^p} := (B^p(\cdot|C))_{(C,p) \in \mathcal{C} \times [0,1]}$$

such that $B^p(\cdot|C) : \mathcal{D} \to \mathcal{D}$ for every $(C, p) \in \mathcal{C} \times [0, 1]$. It is *regular* if

$$\overrightarrow{B^p(\cdot|C)} := (B^p(\cdot|C))_{p \in [0,1]}$$

is regular for every $C \in \mathcal{C}$. For each property in Definitions 4 to 6, a regular collection of conditional p-belief operators satisfies the property if $\overrightarrow{B^p(\cdot|C)}$ satisfies it for every $C \in \mathcal{C}$. Similarly, one can define the dual conditional p-belief operators $\overrightarrow{L^p}$.

The analysis in Section 3 implies: a conditional type mapping $\overrightarrow{t}$ induces the regular collection of conditional p-belief operators $\overrightarrow{B_t^p}$; and in turn, a regular collection of conditional p-belief operators $\overrightarrow{B_t^p}$ induces the conditional type mapping $\overrightarrow{t}$: for each $(\omega, C, E) \in \Omega \times \mathcal{C} \times \mathcal{D}$,

$$t_B(\omega)(E \mid C) := \max\{p \in [0, 1] \mid \omega \in B^p(E \mid C)\}.$$

[17] If each conditional p-belief $B_t^p(E|C)$ is required to be a conditional event, then a stronger "measurability" condition with respect to $\mathcal{C}$ has to be imposed: for all $(C, E, p) \in \mathcal{C} \times \mathcal{D} \times [0, 1]$,

$$B_t^p(E|C) \in \mathcal{C}.$$

This paper, however, does not consider this case.

Moreover, given $\overrightarrow{B^p}$, for all $(C, E) \in \mathcal{C} \times \mathcal{D}$,

$$B^p(E \mid C) = \{\omega \in \Omega \mid t_B(\omega)(E \mid C) \geq p\} = B^p_{t_B}(E \mid C);$$

and given $\overrightarrow{t}$, for all $(\omega, C, E) \in \Omega \times \mathcal{C} \times \mathcal{D}$,

$$t(\omega)(E \mid C) = \max\{p \in [0,1] \mid \omega \in B^p_t(E \mid C)\} = t_{B_t}(\omega)(E \mid C).$$

5.3.2 Conditional Probability Systems

I first revisit Di Tillio, Halpern, and Samet (2014, Theorem 1) by introducing two new axioms on conditional beliefs for both conditional set functions and conditional p-belief operators. The first axiom on a conditional set function $\overrightarrow{\mu}$ is *Normality*:

$$\mu(C|C) = 1 \text{ for all } C \in \mathcal{C}.$$

The second is the *Chain Rule*: for any $(C, D, E) \in \mathcal{C} \times \mathcal{C} \times \mathcal{D}$ with $E \subseteq D \subseteq C$,

$$\mu(E|C) = \mu(E|D)\mu(D|C). \tag{17}$$

In the literature, $\overrightarrow{\mu}$ is a (countably-additive) *conditional probability system* (Rényi, 1955) if it satisfies Normality, the Chain Rule, Normalization, and each $\mu(\cdot \mid C)$ is a countably-additive probability measure on $(\Omega, \mathcal{D})$.

Remark 3. If $\overrightarrow{\mu}$ satisfies Normality and Finite Additivity, then it satisfies the Chain Rule iff, for any $(C, D, E) \in \mathcal{C} \times \mathcal{C} \times \mathcal{D}$ with $E \subseteq D \subseteq C$,

$$\mu(E|C) \geq \mu(E|D)\mu(D|C).^{18} \tag{18}$$

Generally, if one does not impose Finite Additivity, then the Chain rule can be decomposed into the "$\leq$" and "$\geq$" parts.

As in Di Tillio, Halpern, and Samet (2014, Theorem 1), I now formulate the corresponding properties in terms of a collection of conditional p-belief operators. First, $\overrightarrow{B^p}$ satisfies *Normality* if

$$B^1(C|C) = \Omega \text{ for all } C \in \mathcal{C}.$$

Second, $\overrightarrow{B^p}$ satisfies the *Chain Rule* if, for any $(C, D, E) \in \mathcal{C} \times \mathcal{C} \times \mathcal{D}$ with $E \subseteq D \subseteq C$,

$$B^p(E|D) \cap B^q(D|C) \subseteq B^{pq}(E|C); \text{ and} \tag{19}$$

$$L^p(E|D) \cap L^q(D|C) \subseteq L^{pq}(E|C). \tag{20}$$

Here, the formalization of the Chain Rule by conditional p-belief operators is slightly different from that of Di Tillio, Halpern, and Samet (2014, Theorem 1) in the sense that they only require Expression (19). This is because, as implied by Remark 3, if $\overrightarrow{B^p}$ satisfies Normality and Finite Additivity, then it satisfies the Chain Rule, i.e., Expression (18), iff Expression (19) holds. The other part (i.e., the "$\leq$" part) of the Chain Rule (i.e., Expression (17)) holds iff Expression (20) holds. In sum,

[18] For completeness, the Appendix provides the proof.

Remark 4. 1. A conditional type mapping $\overrightarrow{t}$ satisfies Normality iff $\overrightarrow{B^p}$ satisfies it.

2. A conditional type mapping $\overrightarrow{t}$ satisfies the Chain Rule iff $\overrightarrow{B^p}$ satisfies Expressions (19) and (20).

I make additional remarks on the Chain Rule.

Remark 5. 1. (a) As in the literature (e.g., Halpern, 2017; Rényi, 1955), the Chain Rule can be restated as the following stronger form: for any $(C, D, E) \in \mathcal{C} \times \mathcal{D} \times \mathcal{D}$ with $D \cap C \in \mathcal{C}$,

$$\mu(E \cap D|C) = \mu(E|D \cap C)\mu(D|C).^{19} \tag{21}$$

(b) This stronger form of the Chain Rule can be expressed as follows: for any $(C, D, E) \in \mathcal{C} \times \mathcal{D} \times \mathcal{D}$ with $D \cap C \in \mathcal{C}$,

$$B^p(E|D \cap C) \cap B^q(D|C) \subseteq B^{pq}(E \cap D|C); \text{ and}$$
$$L^p(E|D \cap C) \cap L^q(D|C) \subseteq L^{pq}(E \cap D|C).$$

2. (a) Under Normality, $\overrightarrow{\mu}$ satisfies this stronger form of the Chain Rule iff it satisfies the Chain Rule and *Relativization*:

$$\mu(E|C) = \mu(E \cap C|C) \text{ for all } (E, C) \in \mathcal{D} \times \mathcal{C}. \tag{22}$$

(b) Relativization is expressed as:

$$B^p(E|C) = B^p(E \cap C|C) \text{ for all } (p, C, E) \in [0, 1] \times \mathcal{C} \times \mathcal{D}.^{20}$$

Di Tillio, Halpern, and Samet (2014) apply introspective properties of conditional beliefs to game-theoretic analyses of extensive-form games. As in Sections 4.2 and 5.1, one can study introspective properties of conditional (non-additive) beliefs and conditional common p-beliefs. As in Section 5.2, one can also consider a terminal conditional non-additive belief space.

5.3.3 Conditional Possibility Measures

Next, I consider conditional possibility measures. A conditional set function $\overrightarrow{\mu}$ is a *conditional possibility measure* (Dubois and Prade, 1998; Halpern, 2017; Hisdal, 1978) if it satisfies:

1. No-Contradiction;

[19] Note that if $E \subseteq D \subseteq C$ then $D \cap C = D$ and Expression (21) reduces to Expression (17).

[20] For completeness, the Appendix provides the proof of Part (2) of this remark.

2. Normality;
3. Maxitivity; and
4. The *Possibility Chain Rule*: for all $(E, D, C) \in \mathcal{D} \times \mathcal{D} \times \mathcal{C}$ with $D \cap C \in \mathcal{C}$,

$$\mu(E \cap D|C) = \min(\mu(E|D \cap C), \mu(D|C)). \tag{23}$$

Intuitively, in the Possibility Chain Rule, the minimum of $\mu(E|D \cap C)$ and $\mu(D|C)$ is taken instead of the product.

The Possibility Chain Rule can be expressed in terms of conditional p-belief operators as follows: for any $(C, D, E) \in \mathcal{C} \times \mathcal{C} \times \mathcal{D}$ with $E \subseteq D \subseteq C$,

$$B^p(D|C) \cap B^q(E|C \cap D) \subseteq B^{\min(p,q)}(E \cap D|C); \text{ and}$$
$$L^p(D|C) \cap L^q(E|C \cap D) \subseteq L^{\min(p,q)}(E \cap D|C).$$

The characterization for the Possibility Chain Rule can be obtained by replacing pq with $\min(p, q)$ in the characterization for the Chain Rule.

Remark 6. Under Relativization, the Possibility Chain Rule can be expressed as follows: for all $(C, D, E) \in \mathcal{C} \times \mathcal{C} \times \mathcal{D}$ with $E \subseteq D \subseteq C$,

$$\mu(E|C) = \min(\mu(E|D), \mu(D|C)).\text{[21]} \tag{24}$$

6 Concluding Remarks

In an interactive belief model, p-belief operators provide a convenient way to capture interactive higher-order beliefs. This paper studies representations of non-additive beliefs through p-belief operators. Section 2 studies the variant of Rubinstein (1989) e-mail game with non-additive beliefs in which the agents may succeed in coordinating when they receive a single message. Then, Section 3 provides the conditions on p-belief operators under which an agent's underlying type mapping is recovered (Proposition 2). Building on this benchmark result, Section 4 shows that one can axiomatize an interactive belief model that satisfies various logical and introspective properties of beliefs in terms of p-belief operators (Theorem 1). As a result, the paper demonstrates that one can analyze a wide variety of non-additive beliefs by p-belief operators. Examples include Choquet capacities, Dempster-Shafer beliefs, and possibility measures. The paper provides a foundation for studies of interactive beliefs such as implications of common belief when agents' beliefs are non-additive. As applications, Section 5 shows that one can incorporate common p-beliefs and that there exists a terminal belief space when agents' beliefs are non-additive. Section 5 discusses an extension to conditional beliefs. This paper leaves several avenues for future research such as the characterization of a common prior and/or the introduction of unawareness.

[21]For completeness, the Appeneix provides the proof.

A Proofs

A.1 Section 3.3

Proof of Proposition 2. 1. Let $t : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ be a type mapping. Since $t : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ is measurable, it follows that

$$B_t^p(E) := \{\omega \in \Omega \mid t(\omega)(E) \geq p\} \in \mathcal{D} \text{ for all } E \in \mathcal{D}.$$

That is, $B_t^p : \mathcal{D} \to \mathcal{D}$ is well-defined. Thus, I show that $\overrightarrow{B_t^p}$ is regular.

First, $t(\cdot)(E) \geq 0$ implies

$$B_t^0(E) = \{\omega \in \Omega \mid t(\omega)(E) \geq 0\} = \Omega.$$

Second, I start by observing p-Anti-Monotonicity: if $p \leq q$ then

$$B_t^q(E) = \{\omega \in \Omega \mid t(\omega)(E) \geq q\} \subseteq \{\omega \in \Omega \mid t(\omega)(E) \geq p\} = B_t^p(E).$$

Now, suppose $p_n \uparrow p$. Since $p \geq p_n$ for all $n \in \mathbb{N}$, it follows that

$$B_t^p(E) \subseteq \bigcap_{n \in \mathbb{N}} B_t^{p_n}(E).$$

If $\omega \in \bigcap_{n \in \mathbb{N}} B_t^{p_n}(E)$, then

$$t(\omega)(E) \geq p_n \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ yields $t(\omega)(E) \geq p$, i.e., $\omega \in B_t^p(E)$.

Third, I have:

$$\begin{aligned} \bigcap_{n \in \mathbb{N}} (\neg B^{p+\frac{1}{n}})(E) &= \bigcap_{n \in \mathbb{N}} \{\omega \in \Omega \mid t(\omega)(E) < p + \frac{1}{n}\} \\ &= \{\omega \in \Omega \mid t(\omega)(E) \leq p\} \in \mathcal{D}, \end{aligned}$$

where the last set containment follows because $t : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ is measurable.

2. Conversely, let $\overrightarrow{B^p}$ be regular. First, by Non-negativity,

$$0 \in \{p \in [0, 1] \mid \omega \in B^p(E)\} \neq \emptyset.$$

Second, I show that $\{p \in [0, 1] \mid \omega \in B^p(E)\} = [0, q]$, where, by the previous argument,

$$q := \sup\{p \in [0, 1] \mid \omega \in B^p(E)\} \in [0, 1].$$

Let $(q_n)_{n\in\mathbb{N}}$ be such that $q_n \uparrow q$ and $\omega \in B^{q_n}(E)$ for all $n \in \mathbb{N}$. By p-Continuity from Below,

$$\omega \in \bigcap_{n\in\mathbb{N}} B^{q_n}(E) = B^q(E).$$

Also, p-Continuity from Below implies p-Anti-Monotonicity, and thus

$$\{p \in [0,1] \mid \omega \in B^p(E)\} = [0,q].$$

Hence, $t_B(\omega)(E)$ is well-defined for all $(\omega, E) \in \Omega \times \mathcal{D}$.

Third, I show that $t_B : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ is measurable. For any $(p, E) \in [0,1] \times \mathcal{D}$,

$$(t_B)^{-1}(\{\mu \in M(\Omega) \mid \mu(E) \geq p\}) = \{\omega \in \Omega \mid t_B(\omega)(E) \geq p\} = B^p(E) \in \mathcal{D}.$$

Also, for any $(p, E) \in [0,1] \times \mathcal{D}$,

$$\begin{aligned}(t_B)^{-1}(\{\mu \in M(\Omega) \mid \mu(E) \leq p\}) &= \bigcap_{n\in\mathbb{N}} \{\omega \in \Omega \mid t_B(\omega)(E) < p + \frac{1}{n}\} \\ &= \bigcap_{n\in\mathbb{N}} (\neg B^{p+\frac{1}{n}})(E) \in \mathcal{D},\end{aligned}$$

where the last set containment follows from Limit Measurability.

3. Finally, $\omega \in B^p_{t_B}(E)$ iff $t_B(\omega)(E) \geq p$ iff $\omega \in B^p(E)$. Also,

$$\begin{aligned}t_{B_t}(\omega)(E) &= \max\{p \in [0,1] \mid \omega \in B^p_t(E)\} \\ &= \max\{p \in [0,1] \mid t(\omega)(E) \geq p\} = t(\omega)(E).\end{aligned}$$

□

Proof of Proposition 3. The proof of Part (1) is similar to that of Proposition 2. Somewhat roughly, it suffices to replace B^q_t and each axiom (Non-negativity, p-Continuity from Below, and Limit Measurability), respectively, with L^q_t and the corresponding axiom (Unit, p-Continuity from Above, and Dual Limit Measurability).

Hence, I only prove Part (2). First, suppose that $\overrightarrow{B^p}$ is regular. Since $L^p_B = L^p_{t_B}$, it follows from Propositions 2 and 3 (1) that $\overrightarrow{L^p_B}$ is well-defined and regular.

Second, suppose that $\overrightarrow{L^p}$ is regular. Since $B^p_L = B^p_{t_L}$, it follows from Propositions 2 and 3 (1) that $\overrightarrow{B^p_L}$ is well-defined and regular.

Third, I show $B^p_{L_B} = B^p$. Suppose $\omega \in B^p(E)$. Since one has

$$p \leq \max\{q \in [0,1] \mid \omega \in B^q(E)\},$$

it follows that

$$\begin{aligned} p &\le \min\{r \in [0,1] \mid \max\{q \in [0,1] \mid \omega \in B^q(E)\} \le r\} \\ &= \min\{r \in [0,1] \mid \omega \in L^r_B(E)\}. \end{aligned}$$

Thus, $\omega \in B^p_{L_B}(E)$. Conversely, suppose that $\omega \in B^p_{L_B}(E)$. Then,

$$\begin{aligned} p &\le \min\{q \in [0,1] \mid \omega \in L^q_B(E)\} \\ &= \min\{q \in [0,1] \mid \max\{r \in [0,1] \mid \omega \in B^r(E)\} \le q\} \\ &= \max\{r \in [0,1] \mid \omega \in B^r(E)\}. \end{aligned}$$

Thus, $\omega \in B^p(E)$.

Fourth, in a similar way, I show $L^p_{B_L} = L^p$. Suppose $\omega \in L^p(E)$. Since one has

$$p \ge \min\{q \in [0,1] \mid \omega \in L^q(E)\},$$

it follows that

$$\begin{aligned} p &\ge \max\{r \in [0,1] \mid \min\{q \in [0,1] \mid \omega \in L^q(E)\} \ge r\} \\ &= \max\{r \in [0,1] \mid \omega \in B^r_L(E)\}. \end{aligned}$$

Thus, $\omega \in L^p_{B_L}(E)$. Conversely, suppose that $\omega \in L^p_{B_L}(E)$. Then,

$$\begin{aligned} p &\ge \max\{q \in [0,1] \mid \omega \in B^q_L(E)\} \\ &= \max\{q \in [0,1] \mid \min\{r \in [0,1] \mid \omega \in L^r(E)\} \ge q\} \\ &= \min\{r \in [0,1] \mid \omega \in L^r(E)\}. \end{aligned}$$

Thus, $\omega \in L^p(E)$. □

Proof of Corollary 1.

1. Assume that $\overrightarrow{B^p}$ is regular. For each $(p, E) \in [0,1] \times \mathcal{D}$, one has:

$$\begin{aligned} L^p_B(E) &= \{\omega \in \Omega \mid t_B(\omega)(E) \le p\} \\ &= \bigcap_{n\in\mathbb{N}} \{\omega \in \Omega \mid t_B(\omega)(E) < p + \frac{1}{n}\} = \bigcap_{n\in\mathbb{N}} (\neg B^{p+\frac{1}{n}})(E) \in \mathcal{D}. \end{aligned}$$

2. Assume that $\overrightarrow{L^p}$ is regular. For each $(p, E) \in [0,1] \times \mathcal{D}$, one has:

$$\begin{aligned} B^p_L(E) &= \{\omega \in \Omega \mid t_B(\omega)(E) \ge p\} \\ &= \bigcap_{n\in\mathbb{N}} \{\omega \in \Omega \mid t_B(\omega)(E) > p - \frac{1}{n}\} = \bigcap_{n\in\mathbb{N}} (\neg L^{p-\frac{1}{n}})(E) \in \mathcal{D}. \end{aligned}$$

□

A.2 Section 4.1

Proof of Theorem 1. It suffices to show that, for each property of beliefs, the representations between a type mapping and p-belief operators are equivalent.

1. No-Contradiction. Since $t(\cdot)(\emptyset) = 0$, it follows that $B_t^p(\emptyset) = \emptyset$ for all $p \in (0,1]$. Conversely, if $t_B(\omega)(\emptyset) > 0$ for some $\omega \in \Omega$, then $t_B(\omega)(\emptyset) \geq p$ for some $p > 0$, i.e., $\omega \in B^p(\emptyset) = \emptyset$, a contradiction.

2. Normalization. If $t(\cdot)(\Omega) = 1$, then $B_t^1(\Omega) = \Omega$. Conversely, for all $\omega \in \Omega$, it follows from $\omega \in \Omega = B^1(\Omega)$ that $t_B(\omega)(\Omega) = 1$.

3. Monotonicity. Suppose $E \subseteq F$. If $\omega \in B_t^p(E)$ then $p \leq t(\omega)(E) \leq t(\omega)(F)$ and thus $\omega \in B_t^p(F)$. Conversely, $\omega \in B^{t_B(\omega)(E)}(E) \subseteq B^{t_B(\omega)(E)}(F)$ implies $t_B(\omega)(E) \leq t_B(\omega)(F)$.

4. Sub-additivity. Suppose that $\omega \in L_t^p(E) \cap L_t^q(E^c \cap F)$, i.e., $t(\omega)(E) \leq p$ and $t(\omega)(E^c \cap F) \leq q$. Since $t(\omega)$ is sub-additive,

$$t(\omega)(E \cup F) \leq t(\omega)(E) + t(\omega)(E^c \cap F) \leq p + q,$$

i.e., $\omega \in L_t^{p+q}(E \cup F)$. Conversely, since

$$\omega \in L_B^{t_B(\omega)(E)}(E) \cap L_B^{t_B(\omega)(E^c \cap F)}(E^c \cap F) \subseteq L_B^{t_B(\omega)(E)+t_B(\omega)(E^c \cap F)}(E \cup F),$$

it follows that $t_B(\omega)(E \cup F) \leq t_B(\omega)(E) + t_B(\omega)(E^c \cap F)$.

5. Super-additivity. The proof for Super-additivity is similar to that for Sub-additivity: roughly, replace L_t^r and $\leq$ with B_t^r and $\geq$, respectively.

6. Finite-additivity. The statement follows from the characterizations for Sub-additivity and Super-additivity.

7. n-Monotonicity and ∞-Monotonicity. Take $n \in \mathbb{N}$ with $n \geq 2$. If

$$\omega \in \bigcap_{J \in \mathcal{P}_n} M_{J,t}^{p_J}\left(\bigcap_{j \in J} E_j\right),$$

then (i) $t(\omega)\left(\bigcap_{j \in J} E_j\right) \geq p_J$ if $|J|$ is odd; and (ii) $t(\omega)\left(\bigcap_{j \in J} E_j\right) \leq p_J$ if $|J|$ is even. Since $t(\omega)$ satisfies n-Monotonicity,

$$t(\omega)\left(\bigcup_{j=1}^{n} E_j\right) \geq \sum_{J \in \mathcal{P}_n} (-1)^{|J|-1} t(\omega)\left(\bigcap_{j \in J} E_j\right) \geq \sum_{J \in \mathcal{P}_n} (-1)^{|J|-1} p_J.$$

Hence, $\omega \in B_t^{\sum_J (-1)^{|J|-1} p_J}\left(\bigcup_{j=1}^n E_j\right)$. Conversely, since

$$\omega \in \bigcap_{J \in \mathcal{P}_n} M_J^{t_B(\omega)\left(\bigcap_{j\in J} E_j\right)}\left(\bigcap_{j \in J} E_j\right) \subseteq B^{\sum_{J\in\mathcal{P}_n}(-1)^{|J|-1} t_B(\omega)\left(\bigcap_{j\in J} E_j\right)}\left(\bigcup_{j=1}^n E_j\right),$$

it follows that each $t_B(\omega)$ satisfies n-Monotonicity:

$$t_B(\omega)\left(\bigcup_{j=1}^n E_j\right) \geq \sum_{J\in\mathcal{P}_n} (-1)^{|J|-1} t_B(\omega)\left(\bigcap_{j\in J} E_j\right).$$

8. Alternating n-Monotonicity and Alternating ∞-Monotonicity. The proof of this part is omitted as it is similar to the proof of n-Monotonicity: roughly, replace $M_{J,t}^{p_J}$, $\bigcap_{j\in J} E_j$, $\bigcup_{j\in J} E_h$, B_t^r, $\geq$, and $\leq$, respectively, with $N_{J,t}^{p_J}$, $\bigcup_{j\in J} E_j$, $\bigcap_{j\in J} E_j$, L_t^r, $\leq$, and $\geq$.

9. Maxitivity and Finite Maxitivity. I only characterize Maxitivity. If $\omega \in \bigcap_{j\in J} L_t^{p_j}(E_j)$ then

$$t(\omega)\left(\bigcup_{j\in J} E_j\right) \leq \sup_{j\in J} t(\omega)(E_j) \leq \sup_{j\in J} p_j.$$

Thus, $\omega \in L_t^{\sup_j p_j}\left(\bigcup_{j\in J} E_j\right)$. Conversely, if

$$\omega \in \bigcap_{j\in J} L^{t_B(\omega)(E_j)}(E_j) \subseteq L^{\sup_j t_B(\omega)(E_j)}\left(\bigcup_{j\in J} E_j\right),$$

then

$$t_B(\omega)\left(\bigcup_{j\in J} E_j\right) \leq \sup_{j\in J} t_B(\omega)(E_j).$$

10. Minitivity and Finite Minitivity. The proof of this part is similar to that of Maxitivity and Finite Maxitivity: roughly, replace sup, $\bigcap_{j\in J} E_j$, L_t^r, and $\leq$, respectively, with inf, $\bigcup_{j\in J} E_j$, B_t^r, and $\geq$.

11. Countable Sub-additivity. Let $F_n := \left(\left(\bigcap_{m=1}^{n-1} E_m^c\right) \cap E_n\right)$ for each $n \in \mathbb{N}$. Note that $\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} F_n$. If $\omega \in \bigcap_n L_t^{p_n}(F_n)$, then

$$t(\omega)(F_n) \leq p_n \text{ for each } n \in \mathbb{N}.$$

Thus,

$$t(\omega)\left(\bigcup_n E_n\right) = t(\omega)\left(\bigcup_n F_n\right) \le \sum_n t(\omega)(F_n) \le \sum_n p_n,$$

i.e., $\omega \in L_t^{\sum_n p_n}\left(\bigcup_n E_n\right)$. Conversely, if

$$\omega \in \bigcap_n L_B^{t_B(\omega)(F_n)}(F_n) \subseteq L_B^{\sum_n t_B(\omega)(F_n)}\left(\bigcup_n F_n\right) = L_B^{\sum_n t_B(\omega)(F_n)}\left(\bigcup_n E_n\right),$$

then $t_B(\omega)\left(\bigcup_n E_n\right) \le \sum_n t_B(\omega)(F_n)$, as desired.

12. Countable Super-additivity. The proof for Countable Super-additivity is similar to that for Countable Sub-additivity: roughly, replace L_B^q and $\le$ with B^q and $\ge$, respectively.

13. Countable-additivity. The statement follows from the characterizations for Countable Sub-additivity and Countable Super-additivity.

14. Continuity from Above. Since each $t(\omega)$ is continuous from above, $t(\omega)$ and B_t^p are monotonic. Suppose that $E_n \downarrow E$. By Monotonicity,

$$B_t^p(E) \subseteq \bigcap_n B_t^p(E_n).$$

Let $\omega \in \bigcap_n B_t^p(E_n)$, i.e., $t(\omega)(E_n) \ge p$ for all $n \in \mathbb{N}$. Since $t(\omega)$ is continuous from above, it follows that $t(\omega)(E) \ge p$, i.e.,

$$\omega \in B_t^p(E).$$

Conversely, since $t_B(\omega)$ is monotone, $t_B(\omega)(E) \le \lim_n t_B(\omega)(E_n)$. Since

$$\omega \in B^{t_B(\omega)(E_n)}(E_n) \subseteq B^{\lim_n t_B(\omega)(E_n)}(E_n),$$

it follows from Continuity from Above that

$$\omega \in B^{\lim_n t_B(\omega)(E_n)}(E),$$

i.e., $t_B(\omega)(E) \ge \lim_n t_B(\omega)(E_n)$.

15. Continuity from Below. The proof of this part is similar to that of Continuity from Above: roughly, replace $\downarrow$, B_t^r, and $\ge$, respectively, with $\uparrow$, L_t^r, and $\le$. $\square$

A.3 Section 4.2

Proof of Lemma 1. 3. For the "if" part, $\omega \in B^p(E)$ implies $p \le t(\omega)(E)$. Then,

$$\begin{aligned} p \le t(\omega)(E) &\le t(\omega)(\{\tilde{\omega} \in \Omega \mid t(\omega)(E) \le t(\tilde{\omega})(E)\}) \\ &\le t(\omega)(\{\tilde{\omega} \in \Omega \mid p \le t(\tilde{\omega})(E)\}). \end{aligned}$$

Hence,

$$\omega \in B^p(\{\tilde{\omega} \in \Omega \mid p \le t(\tilde{\omega})(E)\}) = B^p B^p(E).$$

For the "only if" part, for any $(\omega, E) \in \Omega \times \mathcal{D}$, since

$$\omega \in B^{t(\omega)(E)}(E) \subseteq B^{t(\omega)(E)} B^{t(\omega)(E)}(E),$$

it follows that

$$t(\omega)(E) \le t(\omega)(B^{t(\omega)(E)}(E)) = t(\omega)(\{\tilde{\omega} \in \Omega \mid t(\omega)(E) \le t(\tilde{\omega})(E)\}).$$

4. For the "if" part, let $\varepsilon \in (0, 1 - t(\omega)(E)]$, and let $p := t(\omega)(E) + \varepsilon \le 1$. Since

$$\omega \in (\neg B^p)(E) \subseteq B^p(\neg B^p)(E),$$

it follows that

$$\underbrace{t(\omega)(E) + \varepsilon}_{=p} \le t(\omega)(\{\tilde{\omega} \in \Omega \mid t(\tilde{\omega})(E) < \underbrace{t(\omega)(E) + \varepsilon}_{=p}\}).$$

For the "only if" part, suppose that $\omega \in (\neg B^p)(E)$, i.e., $t(\omega)(E) < p$. Letting $\varepsilon := p - t(\omega)(E)$ yields

$$p = t(\omega)(E) + \varepsilon \le t(\omega)(\{\tilde{\omega} \in \Omega \mid t(\tilde{\omega})(E) < p\}),$$

i.e., $\omega \in B^p(\neg B^p)(E)$.

□

Proof of Proposition 4. 1. For the "if" part, $\omega \in B^p B^p(E)$ implies

$$t(\omega)(\{\tilde{\omega} \in \Omega \mid t(\tilde{\omega})(E) \ge p\}) = t(\omega)(B^p(E)) \ge p.$$

Thus, $t(\omega)(E) \ge p$, i.e., $\omega \in B^p(E)$. For the "only if" part, assume

$$t(\omega)(\{\tilde{\omega} \in \Omega \mid t(\tilde{\omega})(E) \ge p\}) \ge p.$$

Then, $\omega \in B^p B^p(E) \subseteq B^p(E)$, i.e., $t(\omega)(E) \ge p$.

2. The statement holds when $p = 0$ as $B^0(\cdot) = \Omega$ by Non-negativity. Thus, assume $p > 0$. Suppose to the contrary that $\omega \in B^p B^p(E) \cap (\neg B^p)(E)$. Then, it follows from Negative Certainty that

$$t(\omega)(B^p(E)) \geq p \text{ and } t(\omega)(\neg B^p(E)) = 1.$$

By No-Contradiction, Normalization, and 2-Monotonicity (i.e., Convexity),

$$1 = \underbrace{t(\omega)(\Omega)}_{=1} + \underbrace{t(\omega)(\emptyset)}_{=0} \geq \underbrace{t(\omega)(B^p(E))}_{\geq p} + \underbrace{t(\omega)(\neg B^p(E))}_{=1} \geq p + 1,$$

a contradiction.

□

A.4 Section 5.3

Proof of Remark 3. Suppose that a conditional set function μ satisfies Normality and Finite Additivity. I show that the Chain Rule in the form of Expression (18) implies that in the form of Expression (17). Take any $(C, D, E) \in \mathcal{C} \times \mathcal{C} \times \mathcal{D}$ with $E \subseteq D \subseteq C$. First, by assumption, Expression (18) holds:

$$\mu(E|C) \geq \mu(E|D)\mu(D|C). \tag{A.1}$$

Second, since $E^c \cap D \subseteq D \subseteq C$, it also follows that

$$\mu(E^c \cap D|C) \geq \mu(E^c \cap D|D)\mu(D|C). \tag{A.2}$$

Then, I obtain:

$$\begin{aligned} \mu(D|C) &= \mu(E|C) + \mu(E^c \cap D|C) \\ &\geq \mu(E|D)\mu(D|C) + \mu(E^c \cap D|D)\mu(D|C) \\ &= (\mu(E|D) + \mu(E^c \cap D|D))\mu(D|C) \\ &= \mu(D|D)\mu(D|C) = \mu(D|C). \end{aligned} \tag{A.3}$$

The first and third equalities follow from Finite Additivity. The weak inequality follows from Expressions (A.1) and (A.2). The last equality follows from Normality. Then, Expression (A.3) implies that Expressions (A.1) and (A.2) have to hold with equality. Thus, Expression (17) holds, as desired. □

Proof of Remark 5 (2). (a) Under Normality, substituting $D = C$ into Expression (21) yields Expression (22). Thus, it is enough to show that Expressions (17) and (22) imply Expression (21). For any $(C, D, E) \in \mathcal{C} \times \mathcal{D} \times \mathcal{D}$ with $D \cap C \in \mathcal{C}$,

$$\begin{aligned} \mu(E \cap D \mid C) &= \mu((E \cap D) \cap C \mid C) \\ &= \mu(E \cap D \cap C \mid D \cap C)\mu(D \cap C \mid C) \\ &= \mu(E \mid D \cap C)\mu(D \mid C), \end{aligned}$$

which establishes Expression (21). The first and third equalities follow from Relativization, and the second from Expression (17).

(b) First, suppose that t satisfies Relativization. If $\omega \in B^p(E|C)$, then

$$t(\omega)(E \cap C|C) = t(\omega)(E|C) \geq p,$$

i.e., $\omega \in B^p(E \cap C|C)$. If $\omega \in B^p(E \cap C|C)$, then

$$t(\omega)(E|C) = t(\omega)(E \cap C|C) \geq p,$$

i.e., $\omega \in B^p(E|C)$.

Conversely, for any $\omega \in \Omega$ and $(E, C) \in \mathcal{D} \times \mathcal{C}$,

$$\begin{aligned} t(\omega)(E|C) &= \max\{p \in [0,1] \mid \omega \in B^p(E|C)\} \\ &= \max\{p \in [0,1] \mid \omega \in B^p(E \cap C|C)\} = t(\omega)(E \cap C|C). \end{aligned}$$

□

Proof of Remark 6. Assume Expression (23). Let $(C, D, E) \in \mathcal{C} \times \mathcal{C} \times \mathcal{D}$ with $E \subseteq D \subseteq C$. Then,

$$\begin{aligned} \mu(E|C) &= \mu(E \cap D|C) \\ &= \min(\mu(E|D \cap C), \mu(D|C)) \\ &= \min(\mu(E|D), \mu(D|C)), \end{aligned}$$

which establishes Expression (24). The first equality follows because $E \subseteq D$. The second equality follows from Expression (23). The third equality follows because $D \subseteq C$.

Conversely, assume Expression (24). Take $(C, D, E) \in \mathcal{C} \times \mathcal{D} \times \mathcal{D}$ with $D \cap C \in \mathcal{C}$. Then,

$$\begin{aligned} \mu(E \cap D|C) &= \mu((E \cap D) \cap C|C) \\ &= \mu(D \cap C \cap E|C) \\ &= \min(\mu(D \cap C \cap E|D \cap C), \mu(D \cap C|C)) \\ &= \min(\mu(E|D \cap C), \mu(D|C)), \end{aligned}$$

which establishes Expression (23). The first and fourth equalities follow from Relativization. The second equality follows because $(E \cap D) \cap C = D \cap C \cap E$. The third equality follows from Expression (24), where observe that $D \cap C \cap E \subseteq D \cap C$. □

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