Constructing the Universal Space for Expectation Hierarchies

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September 14, 2025

Abstract

This paper provides a framework—an expectation space—that allows for directly analyzing players' higher-order expectations as the primary object. In the expectation space, players interactively reason about their expectations of a random variable: they hold first-order expectations of the random variable, their second-order expectations about their first-order expectations of the random variable, and so forth. This paper imposes weak assumptions on expectations: (i) the law of iterated expectations (i.e., one's expectation about her own expectation coincides with her expectation) and (ii) continuity (for a sequence of random variables, the limit of expectations coincides with the expectation of the limit). The main result is to construct a universal expectation space: for any given expectation space, there exists a unique structure-preserving map into the universal expectation space. The universal expectation space consists of all possible expectation hierarchies.

JEL Classification: C70, D83
Keywords: Higher-Order Expectations; Expectation Hierarchies; Law of Iterated Expectations; Universal Space

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1 Introduction

In many economic and game-theoretic settings, a player's expectation about another player's expectation can significantly influence behavior. Traditionally, such higher-order expectations are modeled indirectly through underlying beliefs. However, in strategic environments where expectations over others' expectations are central, computing these hierarchies can be analytically intractable—except in special cases where players' expectations exhibit linear structure. Although closely related to beliefs, expectations reflect a distinct modeling perspective and warrant being treated as primitive objects in their own right.

This paper formulates the notion of an expectation space, in which players' expectation operators are a primitive, and constructs the universal expectation space that contains all possible forms of expectation hierarchies (Theorem 1 in Section 4).

Technical Challenge. To see the technical challenge behind deriving players' expectations from beliefs, let Ω be a set of states of the world. When a given state of the world ω realizes, each player is assumed to have a (conditional) belief $m_i(\omega)$ over the set Ω of states of the world, i.e., $m_i(\omega)$ is a probability measure over Ω. Her (conditional) belief $m_i(\omega)$ at ω may depend on the state realization ω because she may receive additional information at the state ω. At any given state ω, for any (integrable) random variable f on the set Ω of states of the world, her (conditional) expectation at the state ω is derived from her underlying belief $m_i(\omega)$ over Ω:

$$\mathbb{E}_i(\omega)[f] := \int_\Omega f(\tilde{\omega}) m_i(\omega)(d\tilde{\omega}).$$

While the computation of $\mathbb{E}_i(\omega)[f]$ itself may be a challenge, the additional and substantial difficulty is that, in a situation in which players reason about their expectations, the computation of a higher-order expectation is more challenging. Even player i's expectation about her own expectation about f at ω, which I denote by $(\mathbb{E}_i\mathbb{E}_i)(\omega)[f]$, involves multiple integrals:

$$(\mathbb{E}_i\mathbb{E}_i)(\omega)[f] := \int_\Omega \left(\int_\Omega f(\tilde{\omega}) m_i(\omega')(d\tilde{\omega}) \right) m_i(\omega)(d\omega').$$

Expectation Spaces. This paper, therefore, formulates the notion of an *expectation space*, where players' (conditional) expectation operators are a primitive. Fix a set I of players who reason about random variables defined over a set S of states of nature. For instance, random variables over the parameter space S can be players' payoff functions, their action profiles, and/or a price function.

The expectation space consists of three primitives. The first primitive is a set Ω of states of the world. The space Ω encodes the given parameter space S, the players' expectations over (random variables on) S, their expectations over their expectations over S, and so forth. Also, the space Ω is endowed with a measurable structure.

The second primitive is a measurable function Θ that associates, with each state ω of the world, the corresponding state of nature $s = \Theta(\omega)$. Through the function Θ, the space Ω represents the underlying set S of states of nature.

The third primitive is a profile of the players' expectation operators $(\mathbb{E}_i)_{i \in I}$. Each player's expectation operator $\mathbb{E}_i$ is a function that associates, with each state ω of the world, her conditional expectation $\mathbb{E}_i(\omega)[\cdot]$ over the set of random variables on Ω held at that state: that is, for any (integrable) random variable f on Ω, $\mathbb{E}_i(\omega)[f]$ is player i's expectation of f at ω.

How do the analysts represent players' higher-order expectations over random variables on S? Letting g be a random variable on S, one can identify g as a random variable $g \circ \Theta$ on Ω. Then, player i's expectation of g at a state ω is represented by $\mathbb{E}_i(\omega)[g \circ \Theta]$.

Next, one can identify $\mathbb{E}_i[g \circ \Theta] := \mathbb{E}_i(\cdot)[g \circ \Theta]$ as a function, which associates, with each ω, her expectation $\mathbb{E}_i(\omega)[g \circ \Theta]$. This structure allows one to iterate players' expectation operators. Since player j holds the expectation $\mathbb{E}_j(\omega)\left[\mathbb{E}_i[g \circ \Theta]\right]$ at each ω, one can express $(\mathbb{E}_j\mathbb{E}_i)(\omega)[g \circ \Theta]$, j's expectation over i's expectation over the random variable g on S, by

$$(\mathbb{E}_j\mathbb{E}_i)(\omega)[g \circ \Theta] = \mathbb{E}_j(\omega)[\mathbb{E}_i[g \circ \Theta]].$$

In the traditional type space approach, each player is supposed to be certain of her own beliefs. Here, I assume that each player is supposed to be certain of her own expectations. Thus, each player i's expectation operator is assumed to satisfy the *law of iterated expectations*:

$$\mathbb{E}_i\mathbb{E}_i = \mathbb{E}_i.$$

Nature of Expectations. Since players' expectation operators $(\mathbb{E}_i)_{i \in I}$ have not been studied as a primitive in the previous literature to analyze interactive expectations such as i's expectation over j's expectation over a random variable f, this paper provides a general framework with which to study interactive expectations. Instead of committing to expectations that are derived from countably-additive beliefs, I study a general notion of expectations the special case of which is the one that is derived from countably-additive beliefs. This is also an advantage of formulating players' expectation operators $(\mathbb{E}_i)_{i \in I}$ directly as a primitive.

Formally, this paper assumes the following three properties for expectations. The first is monotonicity. If two random variables satisfy $f \geq g$, then one's expectation of f is always as high as that of g. The second is constancy: one's expectation of a constant function c is always c. In the context of beliefs, this corresponds to the property that one's belief in the entire space (or a tautology) is always (probability) 1. The third is continuity (from below and above): if $(f_n)_{n \in \mathbb{N}}$ is a non-decreasing (resp., non-increasing) sequence of random variables with $f_n \uparrow f$ (resp., $f_n \downarrow f$), then player i's expectations always satisfy $\mathbb{E}_i[f_n] \uparrow \mathbb{E}_i[f]$ (resp., $\mathbb{E}_i[f_n] \downarrow \mathbb{E}_i[f]$).

Under these three assumptions, I construct a universal expectation space—an expectation space into which any other expectation space embeds (the universal expectation space, if it exists, exists uniquely up to isomorphism, and thus one can speak about *the* universal expectation space). The universal space consists of infinite hierarchies of expectations of the form, the players' expectations over (random variables on) S, their expectations over their expectations over S, their expectations over their expectations over their expectations over S, and so on *ad infinitum*.

Since this requires the notion of expectations over S, that of the expectations over the expectations over S, and so forth, I introduce the set of expectations over random variables on a set X, where an expectation is a mapping, which associates, with each random variable f on X, a corresponding real number (the expectation of f). Denote it by $F(X)$. In the special case in which one considers only indicator functions (instead of all possible random variables) on X and expectations are also assumed to be linear, the operation F is (measurably) isomorphic to that of Δ, where $\Delta(X)$ is the set of probability measures over X.[1] Indeed, the framework of this paper generalizes various notions of beliefs (i.e., countably-additive, finitely-additive, and non-additive beliefs) as a special case.

Expectation Hierarchies. With these in mind, I discuss the formalization of expectation hierarchies. For each player i, the set H_i^1 of her first-order expectations is the set of expectations over S: $H_i^1 := F(H^0)$ with $H^0 := S$. The set of second-order expectations of player i consists of expectations over the set S and the first-order expectations of the players: $F(H^1)$ with $H^1 := S \times \prod_{j \in I} H_j^1$. Denote by $H_i^2 := H_i^1 \times F(H^1)$ the set of player i's expectation hierarchies of order up to 2. Denoting by H_j^k the set of player j's expectation hierarchies of order up to k, the set H_i^{k+1} of player i's expectation hierarchies of order up to $k+1$ is the set $H_i^k \times F(H^k)$ with $H^k := S \times \prod_{j \in I} H_j^k$. The set $F(H^k) = F(S \times \prod_{j \in I} H_j^k)$ of the $(k+1)$-th order expectations is the set of expectations over the exogenously given parameters S and the players' expectation hierarchies order up to k. The paper shows that there exists a subset of expectation hierarchies $H = S \times \prod_{i \in I} H_i$ with $H_i = \prod_{k \in \mathbb{N} \cup \{0\}} F(H^k)$ which contains any possible expectation hierarchies.[2]

Universal Expectation Space. I discuss the universal expectation space that contains any possible expectation hierarchies. More formally, this paper constructs an expectation space $\overrightarrow{\Omega^*}$ such that, for any given expectation space, there exists a unique

[1] Technically, F is a functor in the language of category theory.

[2] For the reader who is familiar with the previous literature on the universal belief/type space, in the definitions of H_i^2 and more generally H_i^{k+1}, the previous literature on beliefs usually assumes that player i is certain of her own beliefs in the sense that each H_i^{k+1} is given by the set $H_i^k \times \Delta(S \times \prod_{j \in I \setminus \{i\}} H_j^k)$, where $\Delta(S \times \prod_{j \in I \setminus \{i\}} H_j^k)$ is the set of beliefs over the parameters S and the opponents' belief hierarchies of order up to k. Since this paper introduces an introspection condition through the law of iterated expectations, which imposes a condition on the entire set H, I included, in the description of player i's expectation hierarchies of order up to $k+1$, her own expectation hierarchies of order up to k.

structure-preserving map from the given space to the space $\overrightarrow{\Omega^*}$.[3]

In the universal expectation space, each state corresponds to a particular expectation hierarchy. Thus, the unique structure-preserving map has the following property: for any given state ω of any given expectation space $\overrightarrow{\Omega}$, denoting by h the unique structure-preserving map from $\overrightarrow{\Omega}$ to $\overrightarrow{\Omega^*}$, $h(\omega)$ corresponds to the expectation hierarchy associated with ω. Thus, the universal expectation space $\overrightarrow{\Omega^*}$ contains all possible expectation hierarchies $h(\omega)$ attained by some state ω of some expectation space $\overrightarrow{\Omega}$.[4]

Moreover, the universal expectation space $\overrightarrow{\Omega^*}$ to be constructed has the following three additional properties. First, the expectation space $\overrightarrow{\Omega^*}$ is *non-redundant* (Mertens and Zamir, 1985): two different states in $\overrightarrow{\Omega^*}$ correspond to two different expectation hierarchies. Second, the expectation space $\overrightarrow{\Omega^*}$ is *complete* (e.g., Brandenburger, 2003; Brandenburger and Keisler, 2006) in the following sense: for any expectation structure on the space Ω^* (i.e., suppose that there exists an expectation space on the state space Ω^*), the set of expectation hierarchies induced by the new expectation space is subsumed in the original universal expectation space. Third, the expectation space $\overrightarrow{\Omega^*}$ is *minimal* (Di Tillio, 2008; Friedenberg and Meier, 2011): the informational content of the space $\overrightarrow{\Omega^*}$ can be expressed by players' expectation hierarchies alone.[5]

The rest of the paper is organized as follows. The rest of the Introduction discusses related literature. Section 2 introduces a notion of expectations studied in this paper. Section 3 defines expectation spaces and the universal expectation space. Section 4 constructs the universal expectation space in terms of expectation hierarchies. Section 5 provides discussions on the main result followed by concluding remarks. Proofs are relegated to Appendix A.

Related Literature

First, I discuss the literature that studies interactive expectations. Second, I discuss the methodological contributions within the literature on universal spaces.

Interactive Expectations. Higher-order expectations have been studied in various strands of literature in economics and game theory. First, papers such as Golub and Morris (2017), Hellman (2011), and Samet (1998, 2000) study iterated expectations in various contexts including the characterization of the existence of a common prior. The framework of this paper allows for capturing iterated expectations as a primary object of study.

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[3]In the language of category theory, $\overrightarrow{\Omega^*}$ is a terminal object in the category of expectation spaces. In the category theory, it is well-known that the terminal object exists uniquely up to isomorphism.

[4]For the reader who is familiar with the literature on the universal belief/type space, this construction relies on the one pioneered by Heifetz and Samet (1998), who constructed the universal type space as the set of belief hierarchies attained by some type profile of some type space.

[5]For those properties, more precisely, see Sections 4 and 5.

Second, Jagau and Perea (2018, 2022) introduce interactive expectations into psychological game theory (Battigalli and Dufwenberg, 2009; Geanakoplos, Pearce, and Stacchetti, 1989). This paper contributes to providing the framework of an expectation space instead of deriving players' higher-order expectations from a type space (where the primitives are players' beliefs).

Third, while this paper focuses on the foundations of modeling higher-order expectations, reasoning about expectations plays a role at a broader level in economics, because, for instance, (future) inflation expectations may influence the current inflation rate. When firms may posses private information, firms' expectations about their expectations (about their current and future pricing behaviors and consequently future inflation rates) may affect the current inflation rate: see, for instance, Phelps (1970, 1983) for the pioneering idea. In the imperfect-common-knowledge New-Keynesian Phillips curve (e.g., Nimark, 2008), indeed, the current inflation depends on an entire hierarchy of firms' expectations about the next-period inflation rate.

Methodological Contributions. This paper belongs to the literature that studies the existence of a universal structure. For beliefs, i.e., Harsanyi (1967-1968) type spaces, pioneering works include, among others, Armbruster and Böge (1979), Böge and Eisele (1979), Brandenburger and Dekel (1993), Heifetz (1993), and Mertens and Zamir (1985). I discuss the methodological contributions of this paper in the literature on universal structures.

In constructing the universal expectation space, this paper utilizes the notion of a functor from category theory (recall footnote 1). The use of a functor establishes the existence of the universal expectation space regardless of some particular properties of expectations. More specifically, this paper establishes the existence of the universal expectation space when players' expectations are derived from countably-additive, finitely-additive, or non-additive beliefs. This paper also constructs the universal expectation space that consists of all finite levels of interactive expectations by imposing (and identifying) the continuity condition on the functor that represents expectations.

Thus, first, this paper is related to the literature on universal structures that utilizes category theory. The pioneering papers are Moss and Viglizzo (2004, 2006) and Viglizzo (2005).[6] More recently, papers such as Guarino (2025) and Pivato (2024a,b) apply category theory to constructing a universal structure in various contexts (i.e., conditional beliefs and preferences). While the main focus of this paper is on expectations, Section 5.5 discusses possible applications to various other modes of reasoning.

Second, this paper identifies a "continuity" property on a functor under which the universal expectation space consists of all finite levels of interactive expectations (see Lemma 2 in Section 3.4). In the literature, some authors call a canonical space to be

[6]The formulation of expectations in this paper is different from the "co-algebra" approach by Moss and Viglizzo (2004, 2006) and Viglizzo (2005) to the extent that the introspection property in the co-algebraic approach (i.e., each type of each is certain of her own type) is different from the one considered in this paper (i.e., the law of iterated expectations). Section 5.1 discusses further the co-algebra approach by Moss and Viglizzo (2004, 2006) and Viglizzo (2005).

a universal space if, in the canonical space, all finite-level reasoning (e.g., beliefs or expectations) extend to all subsequent countable-level reasoning (e.g., Brandenburger, 2003; Friedenberg, 2010). The canonical space constructed in this paper also satisfies this definition of universality. In the previous literature, such paper as Heifetz and Samet (1998) and Ganguli, Hiefetz, and Lee (2016) construct a canonical space with this universality property without the aid of any topological structure on an underlying space about which players interactively reason. This paper constructs the canonical space following their methodology. In this regard, one contribution of this paper is to formalize conditions on underlying properties of reasoning, i.e., at the level of a functor, under which a canonical space is universal—the question posed by Brandenburger (2003). This paper formulates properties on a functor under which reasoning about all finite-level reasoning is extended to subsequent countable levels.

2 Formulation of Expectations

The aim of this section is to formulate a notion of expectations through a functor in category theory: the functor F associates, with each measurable space X, another measurable space $F(X)$ that represents the set of expectations over X. To that end, this section starts with technical preliminaries on measurable spaces and a functor. Section 2.1 provides definitions on measurable spaces and functions. Since the paper formulates expectation hierarchies, the subsection also provides formal definitions on product measurable spaces. Section 2.2 defines the category-theoretic notion of a functor. Section 2.3 defines the notion of expectations through a functor.

2.1 Measurable Spaces and Functions

This subsection provides technical preliminaries on measurable spaces, measurable functions, and product measurable spaces.[7]

Measurable Spaces. Let X be a set. A sub-collection $\mathcal{X}$ of the power set $\mathcal{P}(X)$, i.e., $\mathcal{X} \subseteq \mathcal{P}(X)$, is an *algebra* if the following three conditions are met.

1. The collection $\mathcal{X}$ contains the entire set X and the empty set $\emptyset$.
2. The collection $\mathcal{X}$ contains the complement E^c whenever it contains E.
3. The collection $\mathcal{X}$ contains the union $E \cup F$ and the intersection $E \cap F$ whenever it contains E and F.

The collection $\mathcal{X}$ is a *σ-algebra* or a tuple $(X, \mathcal{X})$ is a *measurable space* if the collection $\mathcal{X}$, in addition, satisfies the following property (instead of Condition (3)).

[7] The materials in this subsection can be found, for instance, in Ash and Doléans-Dade (2000).

4. The collection $\mathcal{X}$ contains the countable union $\bigcup_{n\in\mathbb{N}} E_n$ and the countable intersection $\bigcap_{n\in\mathbb{N}} E_n$ whenever it contains E_n for all $n \in \mathbb{N}$.

If $(X, \mathcal{X})$ is a measurable space, then each $E \in \mathcal{X}$ is referred to as a *measurable set*. For any sub-collection of the power set $\mathcal{P}(X)$ of a set X, denote by $\sigma(\cdot)$ the operation of generating the (smallest) σ-algebra (on X) that includes the given collection. For a non-decreasing sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$ of σ-algebras on X (i.e., $m \leq n$ implies $\mathcal{X}_m \subseteq \mathcal{X}_n$), the collection $\bigcup_{n\in\mathbb{N}} \mathcal{X}_n$ of subsets of X forms an algebra, not necessarily a σ-algebra.

If $(X, \mathcal{X})$ is a measurable space and if Y is any subset of X, then $(Y, \mathcal{X} \cap Y)$ is a measurable space, where

$$\mathcal{X} \cap Y := \{E \cap Y \in \mathcal{P}(Y) \mid E \in \mathcal{X}\}.$$

For any subset A of $\mathbb{R}$, denote by $\mathcal{B}_A$ the Borel σ-algebra on A.

Measurable Functions. For any functions $\varphi, \psi : X \to \mathbb{R}$, the notation $\varphi \geq \psi$ means that $\varphi(x) \geq \psi(x)$ for all $x \in X$. A sequence $(\varphi_n)_{n\in\mathbb{N}}$ of functions on X is *non-decreasing* with *limit* φ, denoted by $\varphi_n \uparrow \varphi$, if (i) $\varphi_m \leq \varphi_n$ for all $m, n \in \mathbb{N}$ with $m \leq n$ and if (ii) for each $x \in X$, $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$. Similarly, the sequence $(\varphi_n)_{n\in\mathbb{N}}$ of functions on X is *non-increasing* with *limit* φ, denoted by $\varphi_n \downarrow \varphi$, if (i) $\varphi_n \leq \varphi_m$ for all $m, n \in \mathbb{N}$ with $m \leq n$ and if (ii) for each $x \in X$, $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$. Also, a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of functions on X *converges (pointwise) to* a function φ, denoted by $\varphi_n \to \varphi$ or by $\varphi = \lim_{n\to\infty} \varphi_n$, if, for any $x \in X$, $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$.

A map $\varphi : (X, \mathcal{X}) \to (X', \mathcal{X}')$ between measurable spaces is *measurable* (or $\varphi : X \to X'$ is $(\mathcal{X}, \mathcal{X}')$*-measurable*) if

$$\varphi^{-1}(\mathcal{X}') \subseteq \mathcal{X}, \text{ i.e., } \varphi^{-1}(E') \in \mathcal{X} \text{ for all } E' \in \mathcal{X}'.$$

Sometimes, slightly abusing the terminology, I say that the map $\varphi : (X, \mathcal{A}) \to (X', \mathcal{X}')$ is measurable as long as the above condition is satisfied, even if $\mathcal{A}$ may not be a σ-algebra but an arbitrary collection of sets such as an algebra (in that case, the fact that $\mathcal{A}$ may not be a σ-algebra will be indicated).

A measurable function $\varphi : (X, \mathcal{X}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ is referred to as a Borel measurable function or a random variable. For ease of exposition, when I take the expectation of a Borel measurable function, I focus on that of a bounded Borel measurable function. A Borel measurable function φ is *bounded* if

$$\|\varphi\| := \sup_{\omega\in\Omega} |\varphi(\omega)| < \infty.$$

For any given measurable space $(X, \mathcal{X})$, denote by $B(X, \mathcal{X})$ (or $B(X)$ when it is clear from the context) the set of bounded Borel measurable functions $f : (X, \mathcal{X}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$.

Letting $(X, \mathcal{X})$ be a measurable space, denote the *indicator* of $E \in \mathcal{X}$ by $\mathbb{I}_E : (X, \mathcal{X}) \to (\{0, 1\}, \mathcal{P}(\{0, 1\}))$:

$$\mathbb{I}_E(x) = 1 \text{ if and only if } x \in E.$$

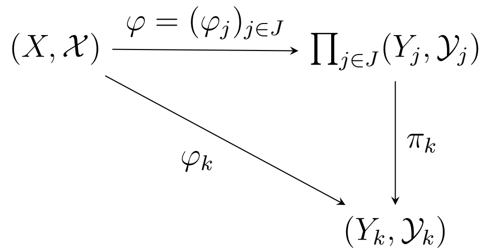

Figure 1: The Definition of $\varphi = (\varphi_j)_{j \in J}$

Product Measurable Spaces. Slightly abusing the notation, any product of measurable spaces $(X_j, \mathcal{X}_j)_{j \in J}$ (a shorthand for $((X_j, \mathcal{X}_j))_{j \in J}$), where J is a non-empty index set, is taken as the product measurable space:

$$\prod_{j \in J}(X_j, \mathcal{X}_j) := \left(\prod_{j \in J} X_j, \prod_{j \in J} \mathcal{X}_j \right).$$

That is, $\prod_{j \in J} X_j$ is the Cartesian product, and $\prod_{j \in J} \mathcal{X}_j$ is the product σ-algebra. Denoting by $\pi^X_{X_j}$ or simply π_j the projection from $\prod_{j \in J} X_j$ to X_j, the product σ-algebra $\prod_{j \in J} \mathcal{X}_j$ is defined as the smallest σ-algebra that makes every projection π_j measurable, i.e.,

$$\prod_{j \in J} \mathcal{X}_j := \sigma \left(\bigcup_{j \in J} \pi_j^{-1}(\mathcal{X}_j) \right).$$

As usual, if $(X, \mathcal{X}) = (X_j, \mathcal{X}_j)$ for all $j \in J$, then denote $(X^J, \mathcal{X}^J) := \prod_{j \in J}(X_j, \mathcal{X}_j)$. When $J = \{1, \ldots, n\}$, denote $\prod_{j=1}^n (X_j, \mathcal{X}_j) := \prod_{j \in J}(X_j, \mathcal{X}_j)$. In particular, denote $(X_1 \times X_2, \mathcal{D}_1 \times \mathcal{D}_2) := \prod_{j=1}^2 (X_j, \mathcal{D}_j)$.

Let $(X, \mathcal{X})$ be a measurable space, and let $(Y_j, \mathcal{Y}_j)_{j \in J}$ be a collection of measurable spaces. For a profile of measurable maps $\varphi_j : (X, \mathcal{X}) \to (Y_j, \mathcal{Y}_j)$, define $\varphi = (\varphi_j)_{j \in J} : (X, \mathcal{X}) \to \prod_{j \in J}(Y_j, \mathcal{Y}_j)$ as a measurable map satisfying

$$\varphi_k = \pi_k \circ \varphi \text{ for all } k \in J,$$

where π_k is the projection. Thus, $\varphi(x) = (\varphi_j(x))_{j \in J}$ for all $x \in X$. Figure 1 illustrates the definition.

2.2 A Functor

This paper formalizes the notion of expectations using the category-theoretic notion of a functor. When a player holds beliefs over a set X, the set of beliefs over X is represented by the set $\Delta(X)$ of probability measures over X. Category theoretically, this operation Δ is a functor. With this in mind, I represent the set of a player's

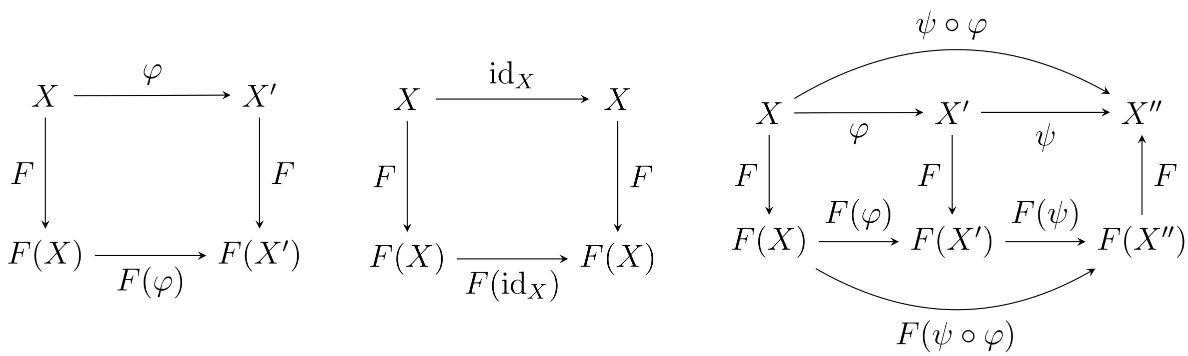

Figure 2: The Illustration of the properties of a Functor F

expectations over random variables on a set X using the set $F(X)$ where F is a functor (that represents expectations).

Before defining the functor that represents expectations, since the category-theoretic notion of a functor may not be a standard (or well-known) toolkit for economists or game theorists, I define a functor F. Namely:

Definition 1 (Functor)**.** The *functor* F (on the class of measurable spaces) is an operation that satisfies the following four properties.

1. F associates, with each measurable space $(X, \mathcal{X})$, a measurable space $F(X, \mathcal{X}) =: (F(X), \mathcal{X}_F)$. Namely, $F(X)$ is a set, and $\mathcal{X}_F$ is a σ-algebra on $F(X)$.

2. F associates, with each measurable map $\varphi : (X, \mathcal{X}) \to (X', \mathcal{X}')$ between measurable spaces, a measurable map $F(\varphi) : F(X, \mathcal{X}) \to F(X', \mathcal{X}')$.

3. For the identify map $\mathrm{id}_{(X,\mathcal{X})} : (X, \mathcal{X}) \to (X, \mathcal{X})$ (or id_X when $\mathcal{X}$ is clear from the context) on a measurable space $(X, \mathcal{X})$, F satisfies $F(\mathrm{id}_{(X,\mathcal{X})}) = \mathrm{id}_{F(X,\mathcal{X})}$.

4. For any measurable maps $\varphi : (X, \mathcal{X}) \to (X', \mathcal{X}')$ and $\psi : (X', \mathcal{X}') \to (X'', \mathcal{X}'')$, F satisfies $F(\psi) \circ F(\varphi) = F(\psi \circ \varphi)$.

The left panel of Figure 2 illustrates Condition (2). For a given measurable map φ from X to X', $F(\varphi)$ is a measurable map from $F(X)$ to $F(X')$. The central panel of Figure 2 depicts Condition (3). The condition requires the measurable map $F(\mathrm{id}_X)$ from $F(X)$ into itself (in the second "row" of the panel) to be equal to the identity map $\mathrm{id}_{F(X)}$. The right panel of Figure 2 illustrates Condition (4). The condition states that, for the composite $\psi \circ \varphi : X \to X''$ of two measurable functions $\varphi : X \to X'$ and $\psi : X' \to X''$, the measurable map $F(\psi \circ \varphi) : F(X) \to F(X'')$ satisfies $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$ as indicated in the lower part of the panel.

Remark 1 (Functor Δ)**.** To contrast expectations with beliefs (and to better understand the notion of a functor), consider Δ (i.e., beliefs). First, for any measurable space

$(X, \mathcal{X})$, let $\Delta(X)$ be the set of countably-additive probability measures over $(X, \mathcal{X})$. Let $\mathcal{X}_\Delta$ be the σ-algebra generated by

$$\{\{\mu \in \Delta(X) \mid \mu(E) \geq p\} \in \mathcal{P}(\Delta(X)) \mid (E, p) \in \mathcal{X} \times [0, 1]\}.$$

Second, for any measurable map $\varphi : (X, \mathcal{X}) \to (X', \mathcal{X}')$, define a mapping $\Delta(\varphi) : (\Delta(X), \mathcal{X}_\Delta) \to (\Delta(X'), \mathcal{X}'_\Delta)$ by the image measure $\Delta(\varphi)(\mu) := \mu \circ \varphi^{-1}$ for any $\mu \in \Delta(X)$: that is, for each $E' \in \mathcal{X}'$,

$$\Delta(\varphi)(\mu)(E') := \mu(\varphi^{-1}(E')).$$

The mapping $\Delta(\varphi)$ is measurable because, for any $(E', p) \in \mathcal{X}' \times [0, 1]$,

$$(\Delta(\varphi))^{-1}(\{\mu' \in \Delta(X') \mid \mu'(E') \geq p\}) = \{\mu \in \Delta(X) \mid \mu(\varphi^{-1}(E')) \geq p\} \in \mathcal{X}_\Delta.$$

Third, by definition, $\Delta(\mathrm{id}_{(X,\mathcal{X})}) = \mathrm{id}_{(\Delta(X),\mathcal{X}_\Delta)}$. Fourth, take any measurable maps $\varphi : (X, \mathcal{X}) \to (X', \mathcal{X}')$ and $\psi : (X', \mathcal{X}') \to (X'', \mathcal{X}'')$. For any $\mu \in \Delta(X)$,

$$\begin{aligned}(\Delta(\psi) \circ \Delta(\varphi))(\mu) &= \Delta(\psi)(\Delta(\varphi)(\mu)) = \Delta(\psi)(\mu \circ \varphi^{-1}) \\ &= (\mu \circ \varphi^{-1}) \circ \psi^{-1} = \mu \circ (\varphi^{-1} \circ \psi^{-1}) \\ &= \mu \circ (\psi \circ \varphi)^{-1} = \Delta(\psi \circ \varphi)(\mu),\end{aligned}$$

establishing that $\Delta(\psi) \circ \Delta(\varphi) = \Delta(\psi \circ \varphi)$.

2.3 An Expectation Functor

This subsection defines a functor that represents a notion of expectations.

Definition 2 (Expectation Functor)**.** An operation F is an *expectation functor* if the following three conditions are met.

1. For any measurable space $(X, \mathcal{X})$, the set $F(X)$ is a subset of the space $\mathbb{R}^{B(X,\mathcal{X})}$ of the mappings from $B(X, \mathcal{X})$ into $\mathbb{R}$ such that any $J \in F(X)$ satisfies the following four properties.

   (a) Monotonicity: $f \geq g$ implies $J[f] \geq J[g]$.

   (b) Constancy: $J[c \cdot \mathbb{I}_\Omega] = c$ for any $c \in \mathbb{R}$.

   (c) Continuity from Below: $f_n \uparrow f$ (in $B(X, \mathcal{X})$) implies $J[f_n] \uparrow J[f]$.

   (d) Continuity from Above: $f_n \downarrow f$ (in $B(X, \mathcal{X})$) implies $J[f_n] \downarrow J[f]$.

2. Let $\mathcal{X}_F$ be the σ-algebra on $F(X)$ generated by

$$\{\{J \in F(X) \mid J(f) \geq r\} \in \mathcal{P}(F(X)) \mid (f, r) \in B(X, \mathcal{X}) \times \mathbb{R}\}.$$

3. For any measurable map $\varphi : (X, \mathcal{X}) \to (X', \mathcal{X}')$, define $F(\varphi) : (F(X), \mathcal{X}_F) \to (F(X'), \mathcal{X}'_F)$ as follows: for any $J \in F(X)$ and $f' \in \mathbb{R}^{B(X', \mathcal{X}')}$,

$$F(\varphi)(J)(f') := J(f' \circ \varphi). \tag{1}$$

In Definition 2, Condition (1) defines the set $F(X)$ of expectation functionals on $(X, \mathcal{X})$.[8] Henceforth, I call $F(X)$ the *space of expectations over* X. The condition requires each expectation functional $J \in F(X)$ to satisfy Monotonicity, Constancy, and the continuity properties (i.e., (1c) and (1d)). Note that, since

$$J\left(\inf_{m \geq n} f_m\right) \leq J(f_n) \leq J\left(\sup_{m \geq n} f_m\right)$$

holds under Monotonicity, the continuity properties can be replaced with:

1. (e) Continuity: if $f = \lim_{n \to \infty} f_n$ (in $B(X, \mathcal{X})$), then $J[f] = \lim_{n \to \infty} J[f_n]$.

Remark 2 below considers the case in which the set $F(X)$ is defined as the set of expectation functionals on $(X, \mathcal{X})$ each of which is derived from some countably-additive probability measure on $(X, \mathcal{X})$.

Condition (2) defines a measurable structure on $F(X)$, i.e., the σ-algebra $\mathcal{X}_F$ on $F(X)$. The condition allows one to discuss whether one's expectation of a random variable f is at least as high as a real number r.

Condition (3) defines how the operation F preserves a measurable mapping $\varphi : X \to X'$ to $F(\varphi) : F(X) \to F(X')$. That is, for any expectation functional J on X, the expectation functional $F(\varphi)(J)$ on X' evaluates the random variable f' on X' as the expectation functional J evaluates the random variable $f' \circ \varphi$ on X.

The lemma below asserts that an expectation functor F is indeed a functor.

Lemma 1 (Expectation Functor)**.** *An expectation functor F is a functor.*

While Condition (1) accommodates various notions of (non-linear) expectation functionals, the following remark identifies the functor that represents the standard notion of expectations that are derived from countably-additive beliefs.

Remark 2 (Standard Expectations)**.** Define a (particular) expectation functor Γ as the one that satisfies the following properties in Condition (1) in Definition 2 in addition to Monotonicity, Constancy, Continuity from Below, and Continuity from Above. Namely, $\Gamma(X)$ is a subset of $\mathbb{R}^{B(X,\mathcal{X})}$ such that each $J \in \Gamma(X)$ satisfies the following three properties in addition to Monotonicity, Constancy, Continuity from Below, and Continuity from Above:

[8] That is, if $J \in F(X)$, then J associates, with any bounded Borel measurable function f, the expectation $J[f]$ of f. It is referred to as a functional because it maps a (bounded Borel measurable) function f to a real number.

(e) Sub-additivity: $J[f+g] \leq J[f] + J[g]$.

(f) Super-additivity: $J[f+g] \geq J[f] + J[g]$.

(g) Homogeneity: $J[cf] = cJ[f]$ for all $c \in \mathbb{R}$.

Note that, under Additivity (i.e., $J[f+g] = J[f] + J[g]$) and Homogeneity, Constancy and Monotonicity can be replaced by a weaker form of Constancy $J[\mathbb{I}_\Omega] = 1$ and Non-negativity: $f(\cdot) \geq 0$ implies $J[f] \geq 0$. Note also that, since $J[-f] = -J[f]$ under Homogeneity, Continuity from Below and Continuity from Above are equivalent.

In fact, as countably-additive beliefs and expectations (that are derived from countably-additive beliefs) are in a one-to-one correspondence, the next remark formally shows that the functors Γ and Δ are equivalent.

Remark 3 (Beliefs and Expectations)**.** To see the equivalence between the functors Γ and Δ, first, for any given $\mu \in \Delta(X)$, define $J_\mu \in \Gamma(X)$ as

$$J_\mu[f] := \int f(x)\mu(dx) \text{ for any } f \in B(X).$$

Second, for any $J \in \Gamma(X)$, define $\mu_J \in \Delta(X)$ as

$$\mu_J(E) := J[\mathbb{I}_E] \text{ for all } E \in \mathcal{X}.$$

Then, it can be seen that

$$\mu_{J_\mu} = \mu \text{ and } J_{\mu_J} = J.$$[9]

Another way to identify Γ and Δ is to restrict attention to the set of indicator functions $I(X, \mathcal{X})$ instead of the set of bounded Borel measurable functions $B(X, \mathcal{X})$. Then, the functor Γ reduces to Δ. This way, it can be seen that the methodology of this paper also subsumes the case in which players' beliefs are finitely-additive or non-additive.

3 Expectation Spaces

This section defines expectation spaces. An expectation space encodes players' interactive reasoning about their expectations. Section 3.1 defines an expectation space. Section 3.2 defines the universal expectation space, an expectation type space to which every expectation space is mapped in a structure-preserving manner. Section 3.3 analyzes economic examples in which players' strategies depend on their iterated expectations. Finally, Section 3.4 provides a measure-theoretic result on the expectation spaces which turns out to be useful for constructing the universal expectation space in Section 4.

[9]Thus, one can establish the category-theoretic equivalence between functors Δ and Γ. See also Fukuda (2024b).

Let I denote a non-empty set of *players*. By adding nature (without loss, denote it by 0) to the set I of players, define $I_0 := I \cup \{0\}$. Also, let $(S, \mathcal{S})$ a measurable space of *states of nature*. The players hold their expectations about (bounded) random variables $f : (S, \mathcal{S}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. An element of S is regarded as a specification of the exogenous values that are relevant to the strategic interactions among the players. For instance, $(S, \mathcal{S})$ is the set of action profiles and/or payoff functions endowed with a measurable structure (see Section 3.3 for examples).

3.1 Expectation Spaces

This subsection defines an expectation space, in which the players' expectation operators are a primitive. The expectation space allows one to study interactive expectations over random variables on a given set S of states of nature. Formally:

Definition 3 (Expectation Space)**.** An *expectation space* (of I on $(S, \mathcal{S})$) is a tuple $\overrightarrow{\Omega} := \langle (\Omega, \mathcal{D}), \Theta, (\mathbb{E}_i)_{i \in I} \rangle$ with the following properties.

1. $(\Omega, \mathcal{D})$ is a measurable space.
2. $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{S})$ is a measurable map.
3. Each $\mathbb{E}_i : (\Omega, \mathcal{D}) \to (F(\Omega), \mathcal{D}_F)$ is player i's expectation operator, where F is an expectation functor: it is a measurable mapping such that the law of iterated expectations holds:
$$\mathbb{E}_i = \mathbb{E}_i \mathbb{E}_i. \tag{2}$$

In Condition (1), the state space Ω is a sample space on which the set S of states of nature and the players' expectations about random variables on S are represented. In Condition (2), the mapping Θ associates, with each state of the world $\omega \in \Omega$, the corresponding state of nature $\Theta(\omega) \in S$.

In Condition (3), at each state $\omega \in \Omega$, the functional $\mathbb{E}_i(\omega) : B(\Omega, \mathcal{D}) \to \mathbb{R}$ is player i's expectation functional over random variables on $(\Omega, \mathcal{D})$. Thus, for any bounded measurable function f, $\mathbb{E}_i(\omega)[f]$ is player i's expectation of f at ω. For the law of iterated expectations, Expression (2) states that, for any $\omega \in \Omega$ and for any bounded Borel measurable function f on Ω,

$$\mathbb{E}_i(\omega)[f] = \mathbb{E}_i(\omega)[\mathbb{E}_i(\cdot)[f]]. \tag{3}$$

The left-hand side of Expression (3) is player i's expectation of f at ω. For the right-hand side of Expression (3), $\mathbb{E}_i(\cdot)[f]$ is a function that associates, with each state, player i's expectation of f at that state. Thus, the right-hand side is player i's expectation of her expectation of f.

One can also analyze players' interactive expectations. For instance, similarly to the right-hand side of Expression (3), $\mathbb{E}_i(\omega)[\mathbb{E}_j(\cdot)[f]]$ is player i's expectation at ω of player j's expectation of f, because $\mathbb{E}_j(\cdot)[f]$ is a (bounded Borel measurable) function that associates, with each state, player j's expectation of f at that state.

Remark 4 (Two Interpretations of $\mathbb{E}_i$)**.** To better understand iterations of expectations, I discuss two ways to interpret the expectation operator $\mathbb{E}_i$. The first is, as in Definition 3, to identify $\mathbb{E}_i$ with

$$\mathbb{E}_i : (\Omega, \mathcal{D}) \ni \omega \mapsto \mathbb{E}_i(\omega)[\cdot] \in (F(\Omega), \mathcal{D}_F).$$

That is, player i's expectation operator $\mathbb{E}_i$ associates, with each state $\omega \in \Omega$, her expectations of bounded Borel measurable functions on Ω.

The second is to identify $\mathbb{E}_i$ with

$$\mathbb{E}_i : B(\Omega, \mathcal{D}) \ni f \mapsto \mathbb{E}_i(\cdot)[f] \in B(\Omega, \mathcal{D}).$$

That is, player i's expectation operator associates, with each bounded Borel measurable function f on Ω, the bounded Borel measurable function $\mathbb{E}_i(\cdot)[f]$ on Ω that indicates her expectations of f. Put differently, a random variable f is mapped to another random variable $\mathbb{E}_i(\cdot)[f]$.

I show that, for any bounded Borel measurable function f, the function $\mathbb{E}_i(\cdot)[f]$ is indeed a bounded Borel measurable function. For measurability, for each $r \in \mathbb{R}$,

$$(\mathbb{E}_i)^{-1}(\{J \in F(\Omega) \mid J[f] \geq r\}) = \{\omega \in \Omega \mid \mathbb{E}_i(\omega)[f] \geq r\} \in \mathcal{D}.$$

For boundedness, since $\mathbb{E}_i(\omega)$ satisfies Monotonicity and Constancy (recall Definition 2),

$$\sup_{\omega \in \Omega} |\mathbb{E}_i(\omega)[f]| \leq \sup_{\omega \in \Omega} |f(\omega)| < \infty.$$

To conclude this subsection, two remarks are in order. First, I introduce the notion of average expectations.

Remark 5 (Average Expectations)**.** For ease of exposition, let $I = (0, 1]$ be the set of players.[10] Consider the class of expectation spaces $\overrightarrow{\Omega}$ of I on $(S, \mathcal{S})$ such that there exists a measurable mapping $\overline{\mathbb{E}} : (\Omega, \mathcal{D}) \to (F(\Omega), \mathcal{D}_F)$ such that

$$\overline{\mathbb{E}}(\omega)[f] = \int_I \mathbb{E}_i(\omega)[f] di \text{ for each } (\omega, f) \in \Omega \times B(\Omega).$$

Then, each player can reason about the average expectations, because $\overline{\mathbb{E}}$ maps $f \in B(\Omega, \mathcal{D})$ to $\overline{\mathbb{E}}[f] \in B(\Omega, \mathcal{D})$.

[10] Two remarks are in order. First, I consider $I = (0, 1]$ instead of $I = [0, 1]$ only so as to avoid the clash of notation with nature 0. Second, when $I = \{1, \ldots, n\}$, one can introduce the average expectation operator $\overline{\mathbb{E}} : (\Omega, \mathcal{D}) \to (F(\Omega), \mathcal{D}_F)$ by

$$\overline{\mathbb{E}}(\omega)[f] := \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_i(\omega)[f] \text{ for each } (\omega, f) \in \Omega \times B(\Omega, \mathcal{D}).$$

Second, I briefly discuss the case in which a player's expectation operator is derived from other primitives such as information sets or a σ-algebra.

Remark 6 (Other Primitives that Induce Expectation Operators)**.** I briefly discuss the case in which each player's expectation operator is derived from any other primitive such as a collection of information sets or a σ-algebra. Let $(\Omega, \mathcal{D})$ be a measurable space of states of the world, and let μ be a probability measure μ on $(\Omega, \mathcal{D})$ which plays a role of a common prior. Now, suppose that each player i's expectation operator is derived from her conditional expectation operator

$$\mathbb{E}_i[\cdot] := \mathbb{E}_\mu[\cdot \mid \mathcal{J}_i],$$

where $\mathbb{E}_\mu$ is the expectation functional derived from the common prior μ and $\mathcal{J}_i$ is a σ-algebra that represents player i's information.[11]

3.2 Universal Expectation Space

This subsection defines the universal expectation space. An expectation space $\overrightarrow{\Omega^*}$ is universal if, for any expectation space $\overrightarrow{\Omega}$, there is a unique structure-preserving map from $\overrightarrow{\Omega}$ to $\overrightarrow{\Omega^*}$. Definition 4 defines a structure-preserving map, i.e., an (expectation) morphism. Then, Definition 5 defines the universal expectation space.

Definition 4 (Morphism)**.** Let $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}), \Theta, (\mathbb{E}_i)_{i\in I}\rangle$ and $\overrightarrow{\Omega'} = \langle(\Omega', \mathcal{D}'), \Theta', (\mathbb{E}'_i)_{i\in I}\rangle$ be expectation spaces (of I on $(S, \mathcal{S})$). An *(expectation) morphism* $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a measurable map $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ satisfying the following two conditions:

1. $\Theta = \Theta' \circ \varphi$.
2. $\mathbb{E}'_i \circ \varphi = F(\varphi) \circ \mathbb{E}_i$ for each $i \in I$.

As depicted in the left panel of Figure 3, Condition (1) requires φ to preserve states. As depicted in the right panel of Figure 3, Condition (2) requires φ to preserve each player's expectations.

For any expectation space $\overrightarrow{\Omega}$, the identity map id_Ω forms a morphism from $\overrightarrow{\Omega}$ into itself. Denote by $\mathrm{id}_{\overrightarrow{\Omega}} : \overrightarrow{\Omega} \to \overrightarrow{\Omega}$ the identity (expectation) morphism. Next, a morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is an *(expectation) isomorphism*, if there is a morphism $\psi : \overrightarrow{\Omega'} \to \overrightarrow{\Omega}$ with $\psi \circ \varphi = \mathrm{id}_{\overrightarrow{\Omega}}$ and $\varphi \circ \psi = \mathrm{id}_{\overrightarrow{\Omega'}}$ (that is, φ is bijective and its inverse φ^{-1} is a morphism). If φ is an isomorphism then its inverse φ^{-1} is unique. Expectation spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ are *isomorphic*, if there is an isomorphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$.

I define the universal expectation space. It "includes" all expectation spaces in that any expectation space can be mapped to the universal space under a unique morphism.

[11] Fukuda (2024a) considers a belief model in which each player's posterior beliefs are induced from conditional probabilities (i.e., a prior conditional on the player's information sets) and studies conditions on a prior and information sets under which her posterior beliefs are Bayes conditional probabilities (and consequently the law of iterated expectations hold).

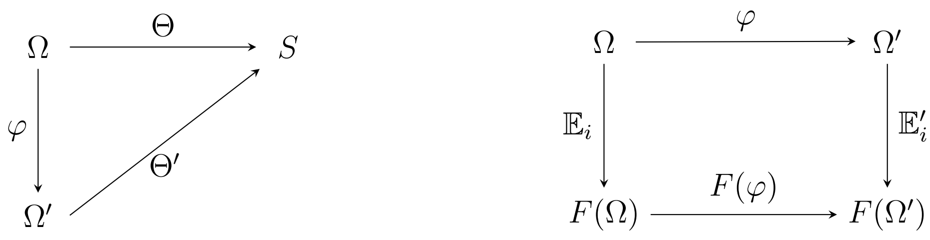

Figure 3: Illustration of Definition 4

Definition 5. Consider the class of expectation spaces of I on $(S, \mathcal{S})$. An expectation space $\overrightarrow{\Omega^*}$ (in the class) is *universal* if, for any expectation space $\overrightarrow{\Omega}$ (in the class), there is a unique morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$.

The terminology follows from the usage of Heifetz and Samet (1998) and more recently that of Meier and Perea (2025). Since the non-empty set of players I, the expectation functor F, and the measurable space $(S, \mathcal{S})$ of states of nature are all fixed, the class of expectation spaces of I on $(S, \mathcal{S})$ forms a *category*, where an expectation space $\overrightarrow{\Omega}$ is an *object* and an expectation morphism is a *morphism*. In fact, any composite of two morphisms is a morphism; composites of morphisms are associative; and an identity morphism satisfies the identity law. In the language of category theory, the universal expectation space in the class is a terminal (final) object in the given category of expectation spaces. The universal expectation space (in the category of expectation spaces) is unique up to type isomorphism (and thus one can speak about *the* universal space).

3.3 Economic Examples

To sum up the previous discussions on the framework of the paper, this subsection provides two examples of expectation spaces in applied contexts, in which players' equilibrium strategies depend on their iterated expectations.

3.3.1 Cournot Competition from Weinstein and Yildiz (2007)

I consider the Cournot duopoly model from Weinstein and Yildiz (2007), where each firm has its linear best-response function. Denote by P the price of a good, and denote by $Q = q_1 + q_2$ the total quantity supplied by the duopoly firms, where $q_i \in S_i := [0, \infty)$ is the supply by firm $i \in I := \{1, 2\}$. Suppose that the inverse demand function, which I also denote by P, is given by

$$P(Q) := s_0 - Q,$$

where $s_0 \in S_0$ is an unknown demand parameter. Assume that S_0 is a bounded subset of $(0, \infty)$ endowed with the Borel σ-algebra $\mathcal{S}_0 := \mathcal{B}_{S_0}$. For ease of analysis, assume that the production costs are zero. Letting $S := S_0 \times S_1 \times S_2$ be the profile of a demand parameter and the firms' actions, firm i's payoff function $u_i : S \to \mathbb{R}$ is written by

$$u_i(s_0, q_1, q_2) := q_i(s_0 - q_1 - q_2).$$ [12]

A model of the (Cournot duopoly) game $\langle (S_i)_{i\in I}, (u_i)_{i\in I}\rangle$ is an expectation space $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), \Theta, (\mathbb{E}_i)_{i\in I}\rangle$. First, $(\Omega, \mathcal{D})$ is a measurable space of states of the world. Second, $\Theta = (\Theta_i)_{i\in I_0}$ is a profile of measurable functions with the following properties: (i) $\Theta_0 : (\Omega, \mathcal{D}) \to (S_0, \mathcal{S}_0)$ associates, with each state of the world $\omega \in \Omega$, the corresponding demand parameter $\Theta_0(\omega) \in S_0$; and (ii) for each player $i \in I$, the mapping $\Theta_i : (\Omega, \mathcal{D}) \to (S_i, \mathcal{S}_i)$, where $\mathcal{S}_i := \mathcal{B}_{[0,\infty)}$ is the Borel σ-algebra on $S_i = [0, \infty)$, associates, with each state of the world $\omega \in \Omega$, the action taken by firm i at that state: $\Theta_i(\omega) \in S_i$ (i.e., Θ_i is firm i's strategy).[13] Third, each $\mathbb{E}_i$ is player i's expectation operator. For ease of analysis, assume that each $\mathbb{E}_i(\omega)[\cdot]$ is linear (i.e., satisfies Sub-additivity, Super-additivity, and Homogeneity as in Remark 2).

In the expectation space, player i's expected payoff at $\omega \in \Omega$ is given by

$$\mathbb{E}_i(\omega)[u_i \circ \Theta],$$

where, for any $\tilde{\omega} \in \Omega$,

$$(u_i \circ \Theta)(\tilde{\omega}) = u_i(\Theta_0(\tilde{\omega}), \Theta_1(\tilde{\omega}), \Theta_2(\tilde{\omega})).$$

To see the role of higher-order expectations, here I consider a particular Θ such that each firm best-responds to the other (i.e., (Θ_1, Θ_2) constitutes a Bayes Nash equilibrium). Thus, for each $i \in I = \{1, 2\}$ and the opponent firm $j = 3 - i$, Θ_i maximizes, at each $\omega \in \Omega$, the firm's expected profit at that state given Θ_{-i}:

$$\Theta_i(\omega) \in \operatorname*{argmax}_{q_i} \mathbb{E}_i(\omega)[q_i(\Theta_0(\tilde{\omega}) - q_i - \Theta_j(\tilde{\omega}))].$$

Thus, Θ_i satisfies

$$\Theta_i(\cdot) = \frac{\mathbb{E}_i(\cdot)[\Theta_0] - \mathbb{E}_i(\cdot)[\Theta_j]}{2}.$$

Substituting the corresponding equation for firm j into the above one yields:

$$\Theta_i = \frac{\mathbb{E}_i[\Theta_0]}{2} - \frac{\mathbb{E}_i\mathbb{E}_j[\Theta_0]}{2^2} + \frac{\mathbb{E}_i\mathbb{E}_j[\Theta_i]}{2^2}.$$

By repeated substitutions and by slightly abusing the notation, Θ_i can be written as a convergent sum of higher-order expectations about $s_0 = \Theta_0(\omega)$:

$$\Theta_i = \frac{\mathbb{E}_i[s_0]}{2} - \frac{\mathbb{E}_i\mathbb{E}_j[s_0]}{2^2} + \frac{\mathbb{E}_i\mathbb{E}_j\mathbb{E}_i[s_0]}{2^3} - \frac{\mathbb{E}_i\mathbb{E}_j\mathbb{E}_i\mathbb{E}_j[s_0]}{2^4} + \cdots\cdots.$$

[12]Two remarks are in order. First, although the payoff function u_i is not bounded, one can fit it to the framework of this paper by considering a bounded Borel measurable function $\max(q_i(s_0 - q_1 - q_2), 0)$. For ease of exposition, I adopt the simpler notation. Second, this example considers expectation hierarchies over the entire set S, i.e., the parameter space S_0 and the firms' actions $S_1 \times S_2$. This is different from the analyses of Bayesian games in the literature in which players interactively reason about their payoff uncertainty S_0 alone (e.g., Friedenberg and Meier, 2017).

[13]One can restrict attention to the case in which Ω is always given by the product space $\Omega = S_0 \times \Omega_{-0}$ and accordingly Θ_0 is the projection from $\Omega = S_0 \times \Omega_{-0}$ into S_0.

3.3.2 Keynesian Beauty Contest from Morris and Shin (2002)

Let $I := (0, 1]$. Player $i \in I$ chooses her action $a_i \in S_i := \mathbb{R}$. Letting $s_0 \in S_0 := \mathbb{R}$ be a realization of an underlying fundamental and letting $a = (a_j)_{j \in I} \in S_{-0} := \mathbb{R}^I$ be the action profile of the players, player i receives a payoff of

$$u_i(s_0, a) := -(1-r)(a_i - s_0)^2 - r(L_i - \overline{L}), \tag{4}$$

where $r \in (0, 1)$ is a constant and $L_i - \overline{L}$ is player i's loss function given as follows:

$$L_i := \int_I (a_j - a_i)^2 dj \text{ and } \overline{L} := \int_I L_j dj.$$

The first term of player i's utility function is a standard quadratic loss in the distance between player i's own action a_i and the underlying fundamental s_0. Denote by $(s_0, a) \in S := S_0 \times S_{-0}$. The second term of player i's utility function is the "beauty contest" term. The loss L_i is increasing in the average distance between player i's action a_i and the actions of the other players. The loss L_i is normalized by its average $\overline{L}$. Each player i second-guesses the actions of the other players. A higher r corresponds to a higher incentive to second guess the actions of the other players.

For each player i, who maximizes the expected payoff where the payoff is given by Expression (4), it is optimal to take her own estimate of the weighted average of the fundamental and the average action of the other players. Denoting by $\mathbb{E}_i$ the expectation operator of player i (to be discussed below in more detail), her best-response is to take

$$a_i = \mathbb{E}_i[(1-r)s_0 + r\overline{a}], \text{ where } \overline{a} := \int_I a_j dj. \tag{5}$$

Below, I consider two cases. First, in the public information benchmark where the players have access to public information, suppose that s_0 is drawn from an improper uniform prior over $S_0 = \mathbb{R}$ but the players observe a public signal

$$y = s_0 + \eta,$$

where η is normally distributed, independent of s_0, with mean 0 and variance α^{-1}. The players choose their actions after observing $y \in \Omega := \mathbb{R}$. Thus, $\Theta_0 : \Omega \to S_0$ associates, with each $y \in \Omega = \mathbb{R}$, $s_0 = y - \eta \in S_0 = \mathbb{R}$. For each player $i \in I$, her expectation operator associates, with each $y \in \Omega$, her conditional expectation conditional on the observation of y:

$$\mathbb{E}_i(y)[\cdot] := \mathbb{E}[\cdot \mid y].$$

Moreover, a measurable function Θ_i associates, with each $y \in \Omega$, her action $\Theta_i(y) \in S_i = \mathbb{R}$. Since $\mathbb{E}_i(y)[s_0] = y$ and $\mathbb{E}_i(y)[\Theta_j] = \Theta_j(y)$, it follows from Expression (5) that, in the unique equilibrium,

$$\Theta_i(y) = y \text{ for all } i \in I.$$

In sum, $\langle(\Omega,\mathcal{D}),\Theta,(\mathbb{E}_i)_{i\in I}\rangle$ is an expectation space, where $\mathcal{D}=\mathcal{B}_{\mathbb{R}}$ and $\Theta=(\Theta_i)_{i\in I_0}$.

Second, consider the situation in which, in addition to the public signal, player i observes the realization of a private signal

$$x_i = s_0 + \varepsilon_i,$$

where noise terms ε_i of the continuum of population are normally distributed with mean 0 and variance β^{-1}, independent of s_0 and η, so that $\mathbb{E}[\varepsilon_i\varepsilon_j]=0$ for all $i,j\in I$ with $i\neq j$. Denoting by $x=(x_i)_{i\in I}$ the profile of (the realizations of) private signals, a state of the world consists of $(y,x)\in\Omega=\mathbb{R}^{[0,1]}$. The state space Ω is endowed with the Borel σ-algebra $\mathcal{D}$. For Θ_0, it depends only on y and maps y to s_0. For each player i, the measurable map Θ_i depends only on (y,x_i) and maps (y,x) to player i's action. Similarly, for each player i, her expectation operator $\mathbb{E}_i$ associates, with each state $(y,x)\in\Omega$, the conditional expectation

$$\mathbb{E}_i(y,x)[\cdot] := \mathbb{E}[\cdot \mid y, x_i].$$

Note that

$$\mathbb{E}_i(y,x)[s_0] = \frac{\alpha y + \beta x_i}{\alpha+\beta} = (1-\mu)y + \mu x_i, \text{ where } \mu = \frac{\beta}{\alpha+\beta}.$$

To solve for an equilibrium, denoting by $\overline{\mathbb{E}}$ the average expectation, it follows from Expression (5) that

$$\begin{aligned} a_i &= (1-r)\mathbb{E}_i[s_0] + r\mathbb{E}_i[\overline{a}] \\ &= (1-r)\mathbb{E}_i[s_0] + (1-r)r\mathbb{E}_i\overline{\mathbb{E}}[s_0] + r^2\mathbb{E}_i\overline{\mathbb{E}}[\overline{a}] \\ &= (1-r)\mathbb{E}_i[s_0] + (1-r)r\mathbb{E}_i\overline{\mathbb{E}}[s_0] + (1-r)r^2\mathbb{E}_i\overline{\mathbb{E}}^2[s_0] + r^3\mathbb{E}_i\overline{\mathbb{E}}^2[\overline{a}] \\ &= \cdots\cdots \\ &= (1-r)\sum_{k=0}^{\infty} r^k\mathbb{E}_i\overline{\mathbb{E}}^k[s_0], \end{aligned} \tag{6}$$

provided that the last infinite sum is bounded. Morris and Shin (2002, Lemma 1) show that, for any $k\in\mathbb{N}$,

$$\mathbb{E}_i(y,x)\overline{\mathbb{E}}^k[s_0] = (1-\mu^{k+1})y + \mu^{k+1}x_i.$$

Substituting the above expression into Expression (6) and rearranging yields:

$$\Theta_i(y,x) = \frac{\alpha y + \beta(1-r)x_i}{\alpha+\beta(1-r)}.$$

In sum, $\langle(\Omega,\mathcal{D}),\Theta,(\mathbb{E}_i)_{i\in I}\rangle$ is an expectation space, where $\Theta=(\Theta_i)_{i\in I_0}$.

3.4 A Key Measure-Theoretic Result on an Expectation Functor

To conclude the section on expectation spaces, this subsection provides a technical but key measure-theoretic result on an expectation functor. The result, which hinges on the continuity of expectations, turns out to play a key role in establishing the main result (Theorem 1 in Section 4).

I present the key measure-theoretic result at an abstract level. Throughout this subsection, let $(\Omega_\ell, \mathcal{D}_\ell)_{\ell \in \mathbb{N}}$ be a collection of measurable spaces. For each $k \in \mathbb{N}$, define product measurable spaces

$$(\Omega^k, \mathcal{D}^k) := \prod_{\ell=1}^{k} (\Omega_\ell, \mathcal{D}_\ell). \tag{7}$$

Also, define the product measurable space

$$(\Omega, \mathcal{D}) := \prod_{\ell \in \mathbb{N}} (\Omega_\ell, \mathcal{D}_\ell). \tag{8}$$

For each $k \in \mathbb{N}$, denote by $\pi^k : (\Omega, \mathcal{D}) \to (\Omega^k, \mathcal{D}^k)$ the projection. Note that $\left((\pi^k)^{-1}(\mathcal{D}^k)\right)_{k \in \mathbb{N}}$ is a non-decreasing sequence of σ-algebra on Ω and

$$\mathcal{D} = \sigma \left(\bigcup_{k \in \mathbb{N}} (\pi^k)^{-1}(\mathcal{D}^k) \right).$$

In Section 4, the space of a player's expectation hierarchies of order up to k turns out to be of the form of $(\Omega^k, \mathcal{D}^k)$, and the space of a player's expectation hierarchies turns out to be of the form of $(\Omega, \mathcal{D})$. However, in this subsection, each $(\Omega_\ell, \mathcal{D}_\ell)$ is an arbitrary measurable space.

With these definitions in mind, the first part of the following lemma roughly states that if two expectation functionals J and J' on $B(\Omega, \mathcal{D})$ are identical if (and only if) they are identical on $B(\Omega^k, \mathcal{D}^k)$ for all $k \in \mathbb{N}$. In words, an expectation functional on $B(\Omega, \mathcal{D})$ admits a unique extension from $(B(\Omega^k, \mathcal{D}^k))_{k \in \mathbb{N}}$. The second part of the following lemma roughly states that $\mathbb{E}_i : \Omega \to F(\Omega)$ is $(\mathcal{D}, \mathcal{D}_F)$-measurable if (and only if) $\mathbb{E}_i$ is $(\mathcal{D}, (F(\pi^k))^{-1}(\mathcal{D}_F^k))$-measurable for all $k \in \mathbb{N}$. As I will illustrate it below, in words, the second part states that F is continuous in the sense that the order of the operations σ and F can be exchanged. Formally:

Lemma 2 (The Key Measure-Theoretic Lemma)**.**

1. *For any $J, J' \in F(\Omega)$, $J = J'$ if (and only if) $F(\pi^k)(J) = F(\pi^k)(J')$ for all $k \in \mathbb{N}$.*
2. *A mapping $\mathbb{E}_i : (\Omega, \mathcal{D}) \to (F(\Omega), \mathcal{D}_F)$ is measurable if (and only if) $F(\pi^k) \circ \mathbb{E}_i : (\Omega, \mathcal{D}) \to (F(\Omega^k), \mathcal{D}_F^k)$ is measurable for all $k \in \mathbb{N}$.*

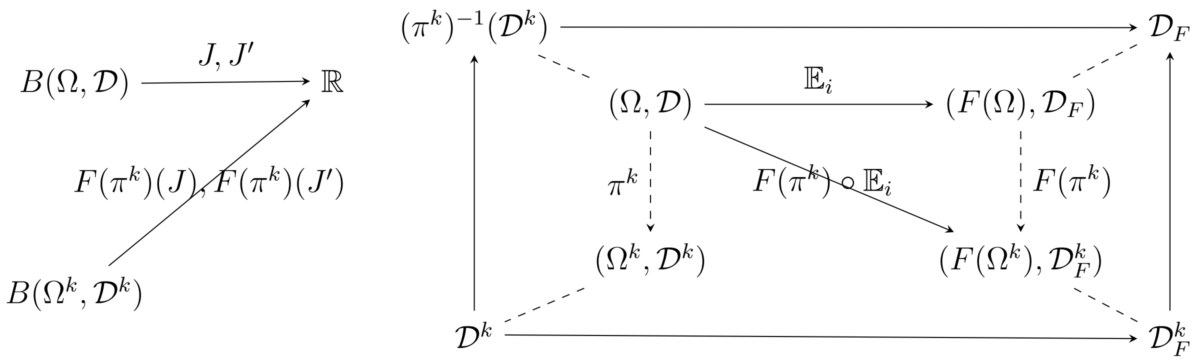

Figure 4: Illustration of Lemma 2: Part (1) (Left) and Part (2) (Right).

The left panel of Figure 4 illustrates Part (1) of Lemma 2. The upper arrow depicts two expectation functionals J and J' on $B(\Omega, \mathcal{D})$, and the lower arrow two expectation functionals $F(\pi^k)(J)$ and $F(\pi^k)(J')$ on $B(\Omega^k, \mathcal{D}^k)$ for a given $k \in \mathbb{N}$.

The right panel of Figure 4 illustrates Part (2) of Lemma 2. The solid arrow from Ω into $F(\Omega)$ depicts $\mathbb{E}_i$, and the solid arrow from Ω into $F(\Omega^k)$ depicts $F(\pi^k)\circ\mathbb{E}_i$. I discuss the sense in which the second part states that F is continuous, i.e., the operations of σ and F can be exchanged. On the one hand, starting from $\mathcal{D}^k$, the solid arrows from $\mathcal{D}^k$ to $\mathcal{D}_F$ through $(\pi^k)^{-1}(\mathcal{D}^k)$ illustrates the fact that $\mathbb{E}_i$ is

$$\left(\mathcal{D}, \underbrace{\left(\sigma\left(\bigcup_{k\in\mathbb{N}}(\pi^k)^{-1}(\mathcal{D}^k)\right)\right)_F}_{=\mathcal{D}_F}\right)\text{-measurable.}$$

I start with the operation σ of generating the smallest σ-algebra on Ω from an algebra $\bigcup_{k\in\mathbb{N}}(\pi^k)^{-1}(\mathcal{D}^k)$. Then, I apply the operation F to generate the σ-algebra on $F(\Omega)$. On the other hand, starting from $\mathcal{D}^k$, the solid arrows from $\mathcal{D}^k$ to $\mathcal{D}_F$ through $\mathcal{D}^k_F$ illustrates the fact that $\mathbb{E}_i$ is

$$\left(\mathcal{D}, \sigma\left(\bigcup_{k\in\mathbb{N}}(F(\pi^k))^{-1}(\mathcal{D}^k_F)\right)\right)\text{-measurable.}$$

I start with the operation F of generating the σ-algebra $\mathcal{D}^k_F$ on $F(\Omega^k)$, which defines an algebra $\bigcup_{k\in\mathbb{N}}(F(\pi^k))^{-1}(\mathcal{D}^k_F)$ on $F(\Omega)$. Then, I apply the operation σ to generate the smallest σ-algebra on $F(\Omega)$.

Remark 7 (Lemma 2 when $F = \Delta$)**.** I remark on Lemma 2 when $F = \Delta$.

1. Lemma 2 (1) states that if two probability measures (i.e., beliefs) μ and μ' on Ω (i.e., $\mu, \mu' \in \Delta(\Omega)$) satisfy

$$\mu\left((\pi^k)^{-1}(\cdot)\right) = \mu'\left((\pi^k)^{-1}(\cdot)\right) \text{ for all } k \in \mathbb{N},$$

then $\mu = \mu'$. That is, if $\mu = \mu'$ on an algebra $\mathcal{A} := \bigcup_{k \in \mathbb{N}} (\pi^k)^{-1}(\mathcal{D}^k)$, then $\mu = \mu'$ on $\mathcal{D} = \sigma(\mathcal{A})$. This is, however, a well-known unique extension result: a probability measure defined on an algebra admits a unique extension on the generated σ-algebra.

2. Denoting a belief mapping m_i (instead of an expectation operator $\mathbb{E}_i$) that associates, with each state $\omega \in \Omega$, player i's belief over Ω (i.e., $m_i(\omega) \in \Delta(\Omega)$), Lemma 2 (2) states that $m_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ is measurable if (and only if) $\Delta(\pi^k) \circ m_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega^k), \mathcal{D}^k_\Delta)$ is measurable for all $k \in \mathbb{N}$. Assume that $m_i^{-1}\left((\Delta(\pi^k))^{-1}(\mathcal{D}^k_\Delta)\right) \subseteq \mathcal{D}$ for all $k \in \mathbb{N}$. Thus,

$$m_i^{-1}\left(\sigma\left(\bigcup_{k \in \mathbb{N}} (\Delta(\pi^k))^{-1}(\mathcal{D}^k_\Delta)\right)\right) \subseteq \mathcal{D}.$$

Lemma 2 (2) then allows one to exchange the order of operating σ and Δ:

$$m_i^{-1}\left(\underbrace{\left(\sigma\left(\bigcup_{k \in \mathbb{N}} (\pi^k)^{-1}(\mathcal{D}^k)\right)\right)_\Delta}_{=\mathcal{D}_\Delta}\right) \subseteq \mathcal{D},$$

establishing the measurability of m_i.[14]

4 The Construction of a Terminal Expectation Space

This section constructs the universal expectation space as the set of all possible expectation hierarchies. Specifically, I extend Heifetz and Samet (1998)'s hierarchical approach to constructing the universal type space.[15] Specifically, I prove:

Theorem 1. *Fix an expectation functor F, a non-empty set I of players, and a measurable space $(S, \mathcal{S})$ of states of nature. Then, the universal expectation space $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}), (\mathbb{E}^*_i)_{i \in I} \rangle$ exists in the category of expectation spaces of I on $(S, \mathcal{S})$.*

The construction consists of eight steps.

[14] In the context of $F = \Delta$, Lemma 2 (2) is closely related to Heifetz and Samet (1998, Lemma 4.5), which allows one to exchange the order of σ and Δ in the following way: if $\mathcal{A}$ is an algebra on a set X then $(\sigma(\mathcal{A}))_\Delta = \sigma(\mathcal{A}_\Delta)$, where $\mathcal{A}_\Delta := \{\{\mu \in \Delta(\Omega) \mid \mu(E) \geq p\} \in \mathcal{P}(\Delta(\Omega)) \mid (E, p) \in \mathcal{A} \times [0,1]\}$. Lemma 2 (2) is also closely related to Ganguli, Hiefetz, and Lee (2016, Lemma 1), which generalize Heifetz and Samet (1998, Lemma 4.5).

[15] That is, in the context of this paper, one can interpret the construction by Heifetz and Samet (1998) as the special case of $F = \Delta$ (aside from the fact that an expectation space is defined over a non-product space Ω instead of the product type space).

First Step. The fist step defines, for each $i \in I_0$, the measurable space $(H_i, \mathcal{H}_i)$ of expectation hierarchies. The set H_0 represents the set of states of nature, and for each player $i \in I$, the set H_i encodes her expectation hierarchies (i.e., her first-order expectations over (random variables on) S, her second-order expectations over S and the players' first-order expectations, and so on). The universal space $(\Omega^*, \mathcal{D}^*)$ is carved out from the product measurable space $(H, \mathcal{H}) := \prod_{i \in I_0} (H_i, \mathcal{H}_i)$.

For $i = 0$, let

$$(H_0, \mathcal{H}_0) := (S, \mathcal{S}).$$

Next, I inductively define the spaces $(H_i, \mathcal{H}_i)_{i \in I}$ of expectation hierarchies from $\{(H_j^k, \mathcal{H}_j^k) \mid j \in I_0 \text{ and } k \in \mathbb{N}\}$. For $j = 0$ and any $k \in \mathbb{N}$, let

$$(H_0^k, \mathcal{H}_0^k) := (S, \mathcal{S}).$$

For each player $j \in I$ and any $k \in \mathbb{N}$, I define the space of player j's expectation hierarchies of order up to k by

$$(H_j^k, \mathcal{H}_j^k) := \begin{cases} (F(S), \mathcal{S}_F) & \text{if } k = 1 \\ (H_j^{k-1} \times F(H^{k-1}), \mathcal{H}_j^{k-1} \times (\mathcal{H}^{k-1})_F) & \text{if } k \geq 2 \end{cases},$$

where

$$(H^{k-1}, \mathcal{H}^{k-1}) := \prod_{i \in I_0} (H_i^{k-1}, \mathcal{H}_i^{k-1}).$$

That is, $H_j^1 = F(S)$ is the space of player j's first-order expectations over (random variables on) S. For $k \geq 2$, the space H_j^k of player j's expectation hierarchies of order up to k consists of the space of her expectation hierarchies of order up to $k-1$ and the expectations over the spaces of the players' (including nature) expectation hierarchies of order up to $k-1$. For instance, when $k = 2$, the space $H_j^2 = F(S) \times F(S \times (F(S))^I)$ of player j's expectation hierarchies of order up to 2 consists of the space of her first-order expectation hierarchies and the expectations over the spaces of the first-order expectation hierarchies of the players. Note that, for any $k \geq 2$, the space $(H_j^k, \mathcal{H}_j^k)$ satisfies:

$$(H_j^k, \mathcal{H}_j^k) = \left(F(S) \times \prod_{\ell=1}^{k-1} F(H^\ell), \mathcal{S}_F \times \prod_{\ell=1}^{k-1} \mathcal{H}_F^\ell \right),$$

where

$$(H^\ell, \mathcal{H}^\ell) := \prod_{i \in I_0} (H_i^\ell, \mathcal{H}_i^\ell).$$

That is, the space H_j^k of player j's expectation hierarchies of order up to k consists of the space of expectations over S (i.e., the first-order expectations) and the space of expectations over the players' expectation hierarchies of order up to $\ell \in \{1, \ldots, k-1\}$ (i.e., the $(\ell+1)$-th order expectations).

With these definition in mind, I define the space $(H_i, \mathcal{H}_i)$ of expectation hierarchies of player i:

$$(H_i, \mathcal{H}_i) := \left(F(S) \times \prod_{k \in \mathbb{N}} F(H^k), \mathcal{S}_F \times \prod_{k \in \mathbb{N}} \mathcal{H}_F^k \right).$$

Thus, the space H_i of expectation hierarchies of player i consists of the k-th order expectations of player i for all $k \in \mathbb{N}$.

Finally, call the product space $(H, \mathcal{H})$, which can also be written as

$$(H, \mathcal{H}) = \left(S \times \prod_{i \in I} H_i, \mathcal{S} \times \prod_{i \in I} \mathcal{H}_i \right),$$

the *(expectation) hierarchy space*. That is, the (expectation) hierarchy space H consists of the set of states of nature and the spaces of expectation hierarchies of the players I.

To conclude the first step, I remark on the generation of the product σ-algebra $\mathcal{H}$ on H:

Remark 8 (The product σ-algebra $\mathcal{H}$)**.** Since $(H, \mathcal{H})$ is a product measurable space, letting $\pi^k := \pi^H_{H^k}$ be the projection from H onto H^k for all $k \in \mathbb{N}$, the hierarchy space H can be written as $H = \prod_{k \in \mathbb{N}} H^k$ and its product σ-algebra $\mathcal{H}$ is given by

$$\mathcal{H} = \sigma \left(\bigcup_{k \in \mathbb{N}} (\pi^k)^{-1}(\mathcal{H}^k) \right).$$

Second Step. The second step defines, for each given expectation space $\overrightarrow{\Omega}$, a mapping $h : \Omega \to H$ by defining

$$h_i^k : (\Omega, \mathcal{D}) \to (H_i^k, \mathcal{H}_i^k) \text{ for every } (i, k) \in I_0 \times \mathbb{N}.$$

The interpretation of each h_i^k is the following. For $i = 0$, the mapping $h_0^k : (\Omega, \mathcal{D}) \to (H_0^k, \mathcal{H}_0^k)$, associates, with each state $\omega \in \Omega$, the corresponding state of nature $h_0^k(\omega) \in H_0^k (= S)$. For each player $i \in I$, the mapping $h_i^k : (\Omega, \mathcal{D}) \to (H_i^k, \mathcal{H}_i^k)$, associates, with each state $\omega \in \Omega$, player i's expectation hierarchy $h_i^k(\omega) \in H_i^k$ of order up to k.

Thus, the mapping h associates, with each state $\omega \in \Omega$ in each given expectation space, the expectation hierarchy $h(\omega) \in H$ induced by the state. When I stress the dependence of h on the underlying expectation space $\overrightarrow{\Omega}$, I denote $h_{\overrightarrow{\Omega}}$. As a preview, in the third step, the universal expectation space $(\Omega^*, \mathcal{D}^*)$ is carved out from the hierarchy space $(H, \mathcal{H})$ as the set of expectation hierarchies that can attain at some state of some expectation space. Also, h turns out to be the unique morphism from the given expectation space to the universal space.

Formally, take an expectation space $\overrightarrow{\Omega}$. I inductively define $h_i = (h_i^k)_{k \in \mathbb{N}}$ for each $(i, k) \in I_0 \times \mathbb{N}$. Let $i = 0$. For every $k \in \mathbb{N}$, define a measurable function $h_0^k : T_0 \to H_0^k$ by

$$h_0^k := \Theta.$$

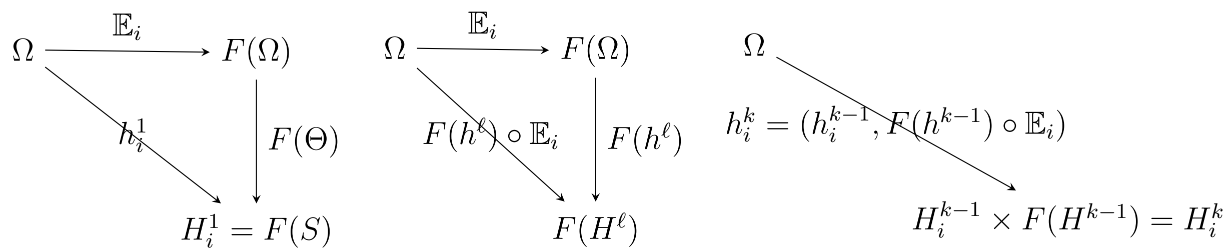

Figure 5: Illustration of h_i^k: h_i^1 (Left), $F(h^\ell) \circ \mathbb{E}_i$ (Center), and h_i^k (Right).

Define a measurable function $h_0 : T_0 \to H_0$ by

$$h_0 := \Theta.$$

Next, I define a profile $(h_i)_{i \in I}$ of measurable functions $h_i : (\Omega, \mathcal{D}) \to (H_i, \mathcal{H}_i)$ by inductively defining a profile $(h_i^k)_{i \in I}$ of measurable functions $h_i^k : (\Omega, \mathcal{D}) \to (H_i^k, \mathcal{H}_i^k)$ for each $k \in \mathbb{N}$. Let $k = 1$. As illustrated in the left panel of Figure 5, for each $i \in I$, let $h_i^1 : (\Omega, \mathcal{D}) \to (F(S), \mathcal{S}_F)$ be the (composite) measurable function defined by

$$h_i^1 := F(\Theta) \circ \mathbb{E}_i.$$

That is, recalling Expression (1), h_i^1 satisfies

$$h_i^1(\omega)[f] = \mathbb{E}_i(\omega)[f \circ \Theta] \text{ for any } \omega \in \Omega \text{ and } f \in B(S, \mathcal{S}).$$

For each $k \geq 2$, suppose that the profile $(h_i^\ell)_{i \in I}$ of measurable functions is defined for each $\ell \in \{1, \ldots, k-1\}$. Then, I define a measurable function $h_i^k : (\Omega, \mathcal{D}) \to (H_i^k, \mathcal{H}_i^k)$ as

$$h_i^k := \left(h_i^{k-1}, F(h^{k-1}) \circ \mathbb{E}_i\right),$$

where $h^{k-1} : (\Omega, \mathcal{D}) \to (H^{k-1}, \mathcal{H}^{k-1})$ is a measurable function defined by $h^{k-1} := (h_i^{k-1})_{i \in I_0}$. The function h_i^k is measurable because h_i^{k-1} and $F(h^{k-1}) \circ \mathbb{E}_i$ are measurable. Note that h_i^k satisfies

$$h_i^k = (h_i^1, F(h^1) \circ \mathbb{E}_i, \ldots, F(h^{k-1}) \circ \mathbb{E}_i),$$

where, for each $\ell \in \{1, \ldots, k-1\}$, $h^\ell : (\Omega, \mathcal{D}) \to (H^\ell, \mathcal{H}^\ell)$ is a measurable function defined by $h^\ell := (h_i^\ell)_{i \in I_0}$. The central panel of Figure 5 illustrates $F(h^\ell) \circ \mathbb{E}_i : \Omega \to F(H^\ell)$, where $\ell \in \{1, \ldots, k-1\}$. The right panel of Figure 5 illustrates $h_i^k = \left(h_i^{k-1}, F(h^{k-1}) \circ \mathbb{E}_i\right)$. For instance, one can see that h_i^2 (the right panel with $k = 2$) is defined from h_i^1 (the left panel) and $F(h^1) \circ \mathbb{E}_i$ (the central panel with $\ell = 1$).

Given that $(h_i^k)_{k \in \mathbb{N}}$ is defined, let

$$h_i : (\Omega, \mathcal{D}) \to (H_i, \mathcal{H}_i)$$

be such that

$$h_i^k = \pi_i^k \circ h_i \text{ for every } (i, k) \in I_0 \times \mathbb{N},$$

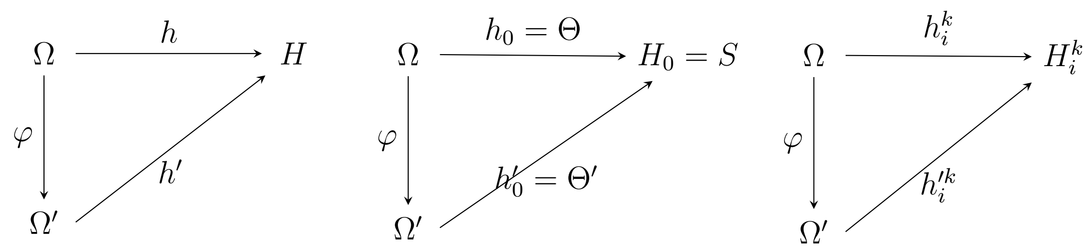

Figure 6: Illustrations of Lemma 3 (Left) and its Proof (Center and Right).

where $\pi_i^k := \pi^{H_i}_{H_i^k}$ is the projection from $(H_i, \mathcal{H}_i)$ into $(H_i^k, \mathcal{H}_i^k)$. Call h_i the (expectation) hierarchy map of player i. Since each $h_i^k : (\Omega, \mathcal{D}) \to (H_i^k, \mathcal{H}_i^k)$ is measurable, by construction, $h_i : (\Omega, \mathcal{D}) \to (H_i, \mathcal{H}_i)$ is measurable. Also, call $h_i(\omega)$ the (expectation) hierarchy of player i at ω. Note that h_i satisfies

$$h_i = (h_i^1, (F(h^k) \circ \mathbb{E}_i)_{k \in \mathbb{N}}).$$

Finally, since $(h_i)_{i \in I_0}$ is defined, I define the *(expectation) hierarchy map* $h : (\Omega, \mathcal{D}) \to (H, \mathcal{H})$ by

$$h^k = \pi^k \circ h \text{ for every } (i, k) \in I_0 \times \mathbb{N},$$

where each $\pi^k := \pi^H_{H^k}$ is the projection from $(H, \mathcal{H})$ into $(H^k, \mathcal{H}^k)$. Since each $h^k : (\Omega, \mathcal{D}) \to (H^k, \mathcal{H}^k)$ is measurable, by construction, $h : (\Omega, \mathcal{D}) \to (H, \mathcal{H})$ is measurable. Call $h(\omega)$ the *expectation hierarchy* at $\omega \in \Omega$. Precisely, for $h(\omega) = (h_i(\omega))_{i \in I_0}$, $h_0(\omega) \in S$ is the state of nature that corresponds to the given state of the world ω; and for each player $i \in I$, $h_i(\omega) = (h_i^k(\omega))_{k \in \mathbb{N}} \in H_i = \prod_{k \in \mathbb{N}} H_i^k$ is the expectation hierarchy of player i.

As depicted in the left panel of Figure 6, I show that a morphism preserves expectation hierarchies.

Lemma 3. *If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a morphism, then $h' \circ \varphi = h$.*

Lemma 3 generalizes Heifetz and Samet (1998, Proposition 5.1) in that the proof hinges only on the fact that F is a functor (in addition to the definition of a morphism in Definition 4 and those of hierarchy maps h and h'). More formally, the proof in Appendix A.3 shows that $h_i = h'_i \circ \varphi$ for each $i \in I_0$. For $i = 0$, as illustrated in the central panel of Figure 6, the statement follows from the fact that φ is a morphism (see the left panel of Figure 3). For each player $i \in I$, as illustrated in the right panel of Figure 6, the proof inductively shows that $h_i^k = h_i'^k \circ \varphi$ for all $k \in \mathbb{N}$.

Third Step. The third step defines the measurable state space $(\Omega^*, \mathcal{D}^*)$ from $(H, \mathcal{H})$: define Ω^* from H as follows:

$$\Omega^* := \{\omega^* \in H \mid \omega^* = h_{\overrightarrow{\Omega}}(\omega) \text{ for some expectation space } \overrightarrow{\Omega} \text{ and } \omega \in \Omega\}.$$

The underlying σ-algebra $\mathcal{D}^*$ is inherited from the one on H:

$$\mathcal{D}^* := \mathcal{H} \cap \Omega^* \left(= \{E \cap \Omega^* \in \mathcal{P}(\Omega^*) \mid E \in \mathcal{H}\}\right).$$

To see that Ω^* is not empty, it is enough to show that there exists an expectation space. Since S is not empty, choose $s \in S$ and let $(\Omega, \mathcal{D}) = (\{s\}, \mathcal{P}(\{s\}))$. Then, the function $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{S})$ defined by $\Theta(s) = s$ is measurable. One can also introduce $\mathbb{E}_i$ such that $\mathbb{E}_i(s)[f] = f(s)$ for any $f \in B(\Omega, \mathcal{D})$. Each player i's expectation operator thus is a measurable mapping from $(\Omega, \mathcal{D})$ to $(F(\Omega), \mathcal{D}_F)$. Hence:

Remark 9. The set Ω^* is not empty.

To conclude the third step, two remarks are in order. First, I briefly discuss the "coherency" of expectation hierarchies. In the literature on the universal belief space, papers such as Brandenburger and Dekel (1993) and Mertens and Zamir (1985) construct the universal space as the (largest) space of coherent belief hierarchies, where a belief hierarchy is coherent if all its (finite) levels of beliefs do not contradict one another. In the construction of the universal expectation space in this paper, each expectation hierarchy $h_i \in H_i$ of player i satisfies the same sense of coherency.[16]

Second, from now on, identify h as a mapping $h : (\Omega, \mathcal{D}) \to (\Omega^*, \mathcal{D}^*)$. Since the original function $h : (\Omega, \mathcal{D}) \to (H, \mathcal{H})$ is measurable, by construction, $h : (\Omega, \mathcal{D}) \to (\Omega^*, \mathcal{D}^*)$ is measurable.

Fourth Step. The fourth step defines a measurable mapping $\Theta^* : (\Omega^*, \mathcal{D}^*) \to (S, \mathcal{S})$ that associates, with each state of the world $\omega^* \in \Omega^*$, the corresponding state of nature $\Theta^*(\omega^*) \in S$. Observing that Ω^* is a subset of $H = S \times \prod_{i \in I} H_i$, let $\Theta^* : (\Omega^*, \mathcal{D}^*) \to (S, \mathcal{S})$ be (the restriction of) the projection, i.e.,

$$\Theta^* := \pi_S^H|_{\Omega^*}.$$

By construction, $\Theta^* : (\Omega^*, \mathcal{D}^*) \to (S, \mathcal{S})$ is measurable.

Also, $\Theta^* \circ h = h_0 = \Theta$ for any expectation space $\overrightarrow{\Omega}$: the first equality follows because Θ^* is the projection, and the second equality follows from the definition of h_0. The equation $\Theta^* \circ h = \Theta$ turns out to correspond to Condition (1) in Definition 4 once the expectation space $\overrightarrow{\Omega^*}$ is defined (i.e., $(\varphi, \Omega', \Theta') = (h, \Omega^*, \Theta^*)$ in the left panel of Figure 3).

[16]For the expert reader, each expectation hierarchy $h_i \in H_i$ of player i satisfies a stronger notion of coherency: namely, it admits an expectation hierarchy $\overline{h}_i$ consisting of all possible countable levels of expectations such that all its (countable) levels of expectations do not contradict one another. In the context of beliefs, Fukuda (2024c) considers the stronger notion of coherency, and shows that the construction of the universal space by coherency (e.g., Brandenburger and Dekel, 1993; Mertens and Zamir, 1985) and the construction of the universal space by the set of all possible belief hierarchies attained by some state of some belief space (i.e., Heifetz and Samet, 1998) coincide with each other, when the underlying space of states of nature is a measurable space (without any topological assumption).

Fifth Step. The fifth step defines each player's expectation operator. For each $i \in I$, define $\mathbb{E}_i^* : (\Omega^*, \mathcal{D}^*) \to (F(\Omega^*), \mathcal{D}_F^*)$ as follows. For each $\omega^* \in \Omega^*$, there are an expectation space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$. Then, as illustrated in Figure 8, define

$$\mathbb{E}_i^*(\omega^*) := F(h) \circ \mathbb{E}_i(\omega). \tag{9}$$

In order to show that $\overrightarrow{\Omega^*} := \langle (\Omega^*, \mathcal{D}^*), \Theta^*, (\mathbb{E}_i^*)_{i \in I} \rangle$ is an expectation space, one needs to show: (i) $\mathbb{E}_i^*$ defined by Expression (9) is well-defined, i.e., $\mathbb{E}_i^*(\omega^*)$ does not depend on a specific choice of an expectation space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ such that $\omega^* = h(\omega)$; (ii) the mapping $\mathbb{E}_i^* : (\Omega^*, \mathcal{D}^*) \to (F(\Omega^*), \mathcal{D}_F^*)$ is measurable; and (iii) $\mathbb{E}_i^*$ satisfies the law of iterated expectations.

For the law of iterated expectations for $\mathbb{E}_i^*$, for any state $\omega^* \in \Omega^*$, there exist an expectation space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ such that $\omega^* = h(\omega)$. Then, the Appendix shows that $\mathbb{E}_i^*$ inherits the law of iterated expectations from $\mathbb{E}_i$ in the expectation space $\overrightarrow{\Omega}$.

To show that each $\mathbb{E}_i^*(\omega^*)$ is well-defined and that each $\mathbb{E}_i^*$ is measurable, I start with:

Lemma 4. *Take any $\omega^* \in \Omega^*$ and $i \in I$. First,*

$$(F(\Theta^*) \circ \mathbb{E}_i^*)(\omega^*) = (\omega_i^*)^1, \tag{10}$$

where $(\omega_i^*)^1 \in F(S)$ is the projection of $\omega^* \in \Omega^*$ on the space $F(S)$ of the first-order expectations over S (of player i). Second, for any $k \in \mathbb{N}$,

$$(F(\pi^k) \circ \mathbb{E}_i^*)(\omega^*) = (\omega_i^*)^{k+1}, \tag{11}$$

where $\pi^k : \Omega^* \to H^k$ is the projection and $(\omega_i^*)^{k+1} \in F(H^k)$ is the projection of $\omega^* \in \Omega^*$ on the space $F(H^k)$ of the expectations over the $(k+1)$-th order expectation hierarchies of player i (i.e., the last component of H_i^{k+1}).

The left panel of Figure 7 illustrates Expression (10). For any $\omega^* \in \Omega^*$, take an expectation space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $\omega^* = h(\omega)$. On the one hand, the outer solid arrows from Ω to $F(S)$ through Ω^* and $F(\Omega^*)$ depict the left-hand of Expression (10):

$$(F(\Theta^*) \circ \mathbb{E}_i^*)(h(\omega)).$$

One the other hand, the direct solid arrow from Ω to $F(S)$ indicates

$$h_i^1(\omega) = (\omega_i^*)^1,$$

which is the right-hand side of Expression (10).

The right panel of Figure 7 illustrates Expression (11). For any $\omega^* \in \Omega^*$, take an expectation space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $\omega^* = h(\omega)$. On the one hand, the

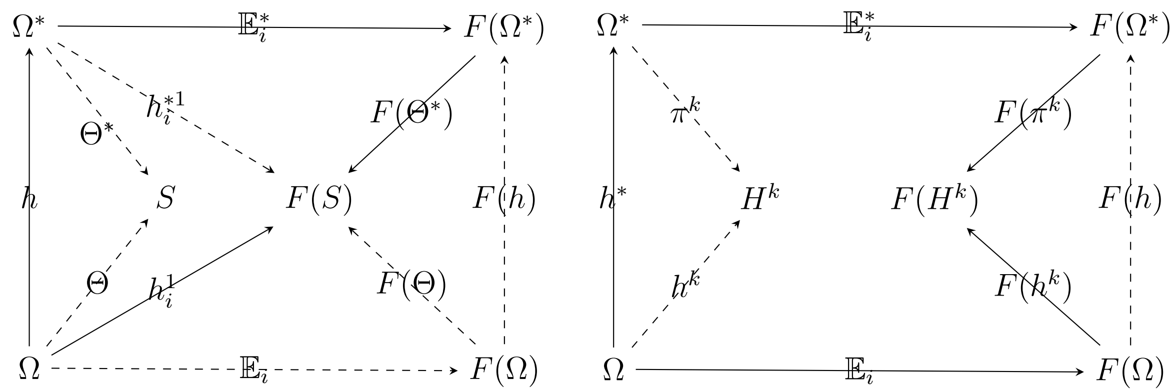

Figure 7: Illustration of Lemma 4: Expression (10) (Left) and Expression (11) (Right)

upper solid arrows from Ω to $F(H^k)$ through Ω^* and $F(\Omega^*)$ depict the left-hand side of Expression (11):

$$(F(\pi^k) \circ \mathbb{E}_i^*)(h(\omega)).$$

On the other hand, the lower solid arrows from Ω to $F(H^k)$ through $F(\Omega)$ indicate the right-hand side of Expression (11), as

$$(\omega_i^*)^{k+1} = (h(\omega))_i^{k+1} = (F(h^k) \circ \mathbb{E}_i)(\omega).$$

Lemma 4 implies that $(F(\pi^k) \circ \mathbb{E}^*)(\omega^*)$ is well-defined (i.e., it does not depend on any particular choice of an expectation space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ with $\omega^* = h(\omega)$) for all $k \in \mathbb{N}$. Then, Lemma 2 (1) in Section 3.4 implies that $\mathbb{E}_i^*(\omega^*)$ is well-defined.

Lemma 4 also implies that $F(\pi^k) \circ \mathbb{E}_i^* : (\Omega^*, \mathcal{D}^*) \to (F(H^k), \mathcal{H}_F^k)$ is measurable. Then, Lemma 2 (2) in Section 3.4 implies that $\mathbb{E}_i^* : (\Omega^*, \mathcal{D}^*) \to (F(\Omega^*), \mathcal{D}_F^*)$ is measurable.

In sum:

Lemma 5 (Expectation Operator $\mathbb{E}_i^*$)**.** *For each $i \in I$, the mapping $\mathbb{E}_i^* : (\Omega^*, \mathcal{D}^*) \to (F(\Omega^*), \mathcal{D}_F^*)$ is player i's expectation operator: it is a well-defined measurable mapping that satisfies the law of iterated expectations.*

Thus far, I have established that $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), \Theta^*, (\mathbb{E}_i^*)_{i \in I} \rangle$ is an expectation space. Before moving on to the sixth step, I remark that, within the class of expectation spaces in which the average expectation is well-defined, $\overrightarrow{\Omega^*}$ belongs to the class (i.e., the average expectation is well-defined in $\overrightarrow{\Omega^*}$) and reasoning about average expectations in a given expectation space is preserved in $\overrightarrow{\Omega^*}$.

Remark 10 (Average Expectations in $\overrightarrow{\Omega^*}$)**.** Let $I = (0, 1]$ as in Remark 5. Fix $\omega^* \in \Omega^*$ and $f^* \in B(\Omega^*, \mathcal{D}^*)$. Then, there exist an expectation space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$

$$
\begin{array}{ccc}
\Omega \ni \omega & \xrightarrow{\quad h \quad} & \omega^* \in \Omega^* \\
\Big\downarrow \mathbb{E}_i & & \Big\downarrow \mathbb{E}_i^* \\
F(\Omega) \ni \mathbb{E}_i(\omega) & \xrightarrow{\quad F(h) \quad} & \mathbb{E}_i^*(\omega^*) \in F(\Omega^*)
\end{array}
$$

Figure 8: Definition of $\mathbb{E}_i^*$

such that $\omega^* = h(\omega)$ and $\mathbb{E}_i^*(\omega^*)[f^*] = \mathbb{E}_i(\omega)[f^* \circ h]$ for all $i \in I$. Thus, one can define $\overline{\mathbb{E}}^*(\omega^*)$ such that

$$
\begin{aligned}
\overline{E}^*(\omega^*)[f^*] &= \int_I \mathbb{E}_i^*(\omega^*)[f^*]di \\
&= \int_I \mathbb{E}_i(\omega)[f^* \circ h]di = \overline{\mathbb{E}}(\omega)[f^* \circ h].
\end{aligned}
$$

Identifying the average expectation operator with the expectation operator of a hypothetical player, Lemma 5 implies that $\overline{\mathbb{E}}^* : (\Omega^*, \mathcal{D}^*) \to (F(\Omega^*), \mathcal{D}_F^*)$, which satisfies the above expression, is a well-defined measurable map.

Sixth Step. The sixth step shows that, for any expectation space $\overrightarrow{\Omega}$, the hierarchy map $h_{\overrightarrow{\Omega}}$ is a morphism. Take an expectation space $\overrightarrow{\Omega}$. As discussed at the end of the third step, the hierarchy map $h_{\overrightarrow{\Omega}}$ is measurable. Firstly,

$$\Theta^* \circ h = h_0 = \Theta.$$

The first equality follows because Θ^* is the projection. The second equality follows from the definition of h_0. Recall also the discussion at the end of the fourth step.

Secondly, for every $i \in I$, by definition (see Figure 8),

$$(\mathbb{E}_i^* \circ h)(\omega) = F(h) \circ \mathbb{E}_i(\omega) \text{ for all } \omega \in \Omega.$$

Since the two conditions in Definition 4 are satisfied, it follows that $h_{\overrightarrow{\Omega}}$ is a morphism.

Seventh Step. In order to prove that $\overrightarrow{\Omega^*}$ is universal, what remains is to show that the morphism $h_{\overrightarrow{\Omega}}$ is unique. To that end, the seventh step shows that $h^* := h_{\overrightarrow{\Omega^*}}$ is the identity.

Lemma 6. *The hierarchy map $h^* : \Omega^* \to \Omega^*$ is the identity.*

The proof in the Appendix shows that, given $h^* = (h_i^*)_{i \in I_0}$, each h_i^* is a projection from Ω^* into H_i. For $i = 0$, the proof in the Appendix shows that the assertion follows from the definition of h_0. For each $i \in I$, the proof in the Appendix utilizes Lemma 4 which implies that

$$(h_i^*(\omega^*))^k = (\omega_i^*)^k \text{ for all } k \in \mathbb{N}.$$

The rest of the seventh step discusses the following two implications of Lemma 6. Firstly, the lemma implies that the expectation space $\overrightarrow{\Omega^*}$ is non-redundant (Mertens and Zamir, 1985). Formally, an expectation space $\overrightarrow{\Omega}$ is *non-redundant* if two different states ω and ω' induce different expectation hierarchies $h(\omega)$ and $h(\omega')$, i.e., the hierarchy map h is injective.

Secondly, the lemma implies that the expectation space $\overrightarrow{\Omega^*}$ is minimal (Di Tillio, 2008; Friedenberg and Meier, 2011). To formalize the notion of minimality, recall that the hierarchy map h on a given expectation space $\overrightarrow{\Omega}$ associates, with each state ω, its expectation hierarchy $h(\omega)$. Since $\mathcal{D}^*$ is also defined through expectation hierarchies, the σ-algebra $\sigma(h) = h^{-1}(\mathcal{D}^*)$ corresponds to the collection of events in the given expectation space $\overrightarrow{\Omega}$ that can be expressed in terms of expectation hierarchies. The following lemma asserts that one can define an expectation space on $(\Omega, \sigma(h))$.

Lemma 7 (Expectation Space on $(\Omega, \sigma(h))$)**.** *Let $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}), \Theta, (\mathbb{E}_i)_{i\in I}\rangle$ be an expectation space. Denoting by h the hierarchy map and by $\sigma(h) := h^{-1}(\mathcal{D}^*)$ the σ-algebra generated by h, define $\overrightarrow{\Omega'} = \langle(\Omega', \mathcal{D}'), \Theta', (\mathbb{E}'_i)_{i\in I}\rangle$ as follows: (i) $(\Omega', \mathcal{D}') = (\Omega, \sigma(h))$; (ii) $\Theta' = \Theta$; and (iii) $\mathbb{E}'_i = \mathbb{E}_i$ (i.e., $\mathbb{E}'_i(\omega)[f] = \mathbb{E}_i(\omega)[f]$ for each $\omega \in \Omega$ and $f \in B(\Omega, \sigma(h))$) for each $i \in I$. Then, $\overrightarrow{\Omega'}$ is an expectation space.*

With this in mind, an expectation space $\overrightarrow{\Omega}$ is *minimal* if $\mathcal{D} = h^{-1}(\mathcal{D}^*)$, i.e., the collection of events $\mathcal{D}$ consists only of those events that are expressed in terms of expectation hierarchies. Lemma 6 implies that $\overrightarrow{\Omega^*}$ is minimal.

Eighth Step. Finally, the eighth step shows that $\overrightarrow{\Omega^*}$ is universal. Given the previous lemmas (whose proofs are in the Appendix), one can show the main theorem as follows:

Proof of Theorem 1. For any expectation space $\overrightarrow{\Omega}$, the above arguments show that the hierarchy map h is a morphism from $\overrightarrow{\Omega}$ to the expectation space $\overrightarrow{\Omega^*}$. Thus, it suffices to show that a morphism is unique. Let $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ be a morphism. Then, $\varphi = h^* \circ \varphi = h$, where the first equality follows from Lemma 6 and the second from Lemma 3. Thus, the morphism is unique, and the proof is complete. □

5 Discussions

This section discusses the main result and provides concluding remarks. Specifically, Section 5.1 discusses the alternative formulation of an expectation space in which each "type" of each player induces expectations over (random variables on) the types of the other players. Under this alternative specification, the universal "expectation-type" space has the property that the expectation map is an isomorphism between the set of types of a given player and the expectations over the types of the other players' types. Section 5.2 discusses the possibility to axiomatize (in the sense of providing a syntactical system of) interactive expectations. Section 5.3 discusses the possibility of incorporating expectation dynamics. Section 5.4 discusses the possibility to further generalize the properties of expectations. Section 5.5 discusses the possibility to apply the functor approach to other modes of reasoning such as interactive preferences. Section 5.6 briefly discusses the introduction of a topological structure to study the continuity of a strategic outcome in expectation hierarchies. Section 5.7 provides concluding remarks.

5.1 Completeness

This paper formulates an expectation space slightly differently from the type-space literature, in which each type of each player induces a belief over the types of the *other* players.[17] This is because (i) the expectation-space approach taken in this paper allows for directly iterating players' expectation operators $(\mathbb{E}_i)_{i \in I}$ and (ii) the underlying introspection property is formulated as the law of iterated expectations.

Specifically, under the "type-space" formulation, an expectation-type space is a tuple $\overrightarrow{\Omega} = \langle (T_i, \mathcal{T}_i)_{i \in I_0}, (\mathbb{E}_i)_{i \in I} \rangle$ with the following properties: (i) $(T_0, \mathcal{T}_0) = (S, \mathcal{S})$; (ii) each $(T_i, \mathcal{T}_i)$ is a measurable space; and (iii) each $\mathbb{E}_i : (T_i, \mathcal{T}_i) \to (F(T_{-i}), (\mathcal{T}_{-i})_F)$ is a measurable map that associates, with each type $t_i \in T_i$, the expectation $\mathbb{E}_i(t_i)$ over the set of bounded Borel measurable functions on $(T_{-i}, \mathcal{T}_{-i}) := \prod_{j \in I_0 \setminus \{i\}} (T_j, \mathcal{T}_j)$. The analysis in this paper suggests that one can construct the universal expectation-type space $\overrightarrow{\Omega^*} = \langle (T_i^*, \mathcal{T}_i^*)_{i \in I_0}, (\mathbb{E}_i^*)_{i \in I} \rangle$.[18]

One advantage of this alternative formulation is that the universal expectation-type space is also complete (e.g., Brandenburger, 2003): each $\mathbb{E}_i^* : T_i^* \to F(T_{-i}^*)$ is surjective (in fact, a measurable isomorphism). This would follow because each expectation-type space is regarded as a coalgebra in the language of category theory (e.g., Moss and Viglizzo, 2004, 2006), and then Lambek (1968)'s Lemma in category theory asserts that the mapping $(\mathrm{id}_S, (\mathbb{E}_i^*)_{i \in I}) : S \times \prod_{i \in I} T_i^* \to S \times \prod_{i \in I} F(T_{-i}^*)$ associated with a terminal object is a measurable isomorphism.

Instead, for the universal expectation space $\overrightarrow{\Omega^*}$, the profile of measurable mappings

$$(\Theta^*, (\mathbb{E}_i^*)_{i \in I})$$

[17] In this respect, the modeling approach is closer to a belief space a la Mertens and Zamir (1985) in which, at each state, each player holds a belief over the state space.

[18] In the context of beliefs, Fukuda (2025a) shows that, in a class of belief spaces (where each player at each state holds a belief over the entire set of states of the world) in which the players are fully introspective about their beliefs, the universal belief space has a type-space structure (indeed, the universal belief space is the universal type space).

defines a bijection between the underlying state space Ω^* and the space of all possible players' expectations over Ω^*, i.e.,

$$\{(s, (J_i)_{i \in I}) \in S \times F(\Omega^*)^I \mid \text{there are an expectation space } \overrightarrow{\Omega} \text{ and } \omega \in \Omega \\ \text{such that } (s, (J_i)_{i \in I}) = (\Theta(\omega), (\mathbb{E}_i(\omega) \circ h^{-1})_{i \in I})\}.$$

The mapping $(\Theta^*, (\mathbb{E}_i^*)_{i \in I})$ is injective because $\overrightarrow{\Omega^*}$ is non-redundant, and it is surjective because $\overrightarrow{\Omega^*}$ is universal.[19]

5.2 Syntactical and Hierarchical Constructions

This paper formulates a functor (i.e., an expectation functor) to construct the terminal expectation space as the space consisting of expectation hierarchies. This paper takes the "hierarchical" approach (which may be useful for other applications to be briefly discussed in Section 5.5). In the literature, however, there is an alternative approach to constructing the universal space, namely, the "syntactical approach," in which players' reasoning is expressed syntactically as formulas. In the context of beliefs, for example, a syntactical formula saying "player i beliefs a formula e with probability at least p" is represented as another formula (say, $\beta_i^p(e)$), and each state in the universal belief space consists of a set of formulas that are satisfied at that state (e.g., see Aumann, 1999; Fukuda, 2024b; Heifetz and Samet, 1998; Meier, 2006, 2012).

It would be possible (and interesting) to axiomatize players' interactive reasoning about their expectations. To that end, one would define a collection of random variables (on a given set S of states of nature) syntactically: that is, for each random variable f (on S) and each real number r, one can define a formula $(f \geq r)$ that can be read as "the value of the random variable f is at least as large as r" (this formula may be true at some states and false at the other states). One can define whether the formula holds (or not) at a state (in a particular expectation space). One can also define each player i's syntactic expectation operator E_i in order to define such formula as $(E_i[f] \geq r)$ saying that "the value of player i's expectation $E_i(f)$ of f is at least as large as r."[20]

5.3 Allowing for Expectation Dynamics

Some applications may call for incorporating one's expectations over time.[21] In the framework of this paper, one can introduce time indices on players' expectations. For

[19] For the sense in which the universal belief space is complete, papers such as Brandenburger and Keisler (2006), Fukuda (2024b), and Meier (2012) reformulate the notion of completeness in terms of syntactic languages. Fukuda (2025a) also shows that the universal belief space admits a type-space representation, which is universal (among the class of type spaces) and consequently complete.

[20] For a syntactical approach to expectations, see, for instance, Halpern and Pucella (2002, 2007).

[21] In fact, in some strands of literature in macroeconomics in which reasoning about expectations would matter, players (e.g., firms) reason about dynamic expectations of future economic variables such as future inflation rates or future policy interest rates (recall the related literature).

instance, suppose that player i's expectation operator at time $t \in \mathbb{N}$ is given by $\mathbb{E}_{i,t}$. The corresponding introspection property, i.e., the law of iterated expectations, is

$$\mathbb{E}_{i,t} = \mathbb{E}_{i,t}\mathbb{E}_{i,t'} \text{ for any } t, t' \in \mathbb{N} \text{ with } t' \geq t.$$

The construction of the terminal expectation space would go through by identifying the set of players as $I \times \mathbb{N}$, with the law of iterated expectations of the above form. As will be discussed in Section 5.5, one could alternatively analyze players' expectations induced from conditional probability systems (CPSs) as in Battigalli and Siniscalchi (1999) and Guarino (2017, 2025).

5.4 Allowing for Discontinuous Expectations

In this paper, Definition 2 in Section 2.3 assumes the continuity properties for expectations. Such continuity properties make it possible to define the expectation operator of the terminal expectation space as in Lemma 5 through the properties of an expectation functor established in Section 3.4.

An alternative approach that dispenses with the continuity properties is to consider the hierarchy space $(H, \mathcal{H})$ that consists of transfinite (more precisely, all possible countable) levels of expectation hierarchies. Once the space $(H^\alpha, \mathcal{H}^\alpha)$ of expectation hierarchies of order up to α is defined for each countable ordinal α (instead of each natural number k), the union of $(\pi^\alpha)^{-1}(\mathcal{H}^\alpha)$ itself generates a σ-algebra on H, as opposed to Remark 8, which uses the operation of $\sigma(\cdot)$.

In the context of qualitative beliefs (which do not satisfy the continuity properties or monotonicity), Fukuda (Forthcoming) constructs the universal belief space consisting of transfinite levels of qualitative belief hierarchies. An extension of the framework of expectation spaces that allows for the violation of the properties of expectations in Definition 2 in Section 2.3 would be an interesting avenue for future research.

Lastly, while it is a different context from the violation of the continuity properties of expectations (i.e., consider an expectation functor that satisfies Definition 2 in Section 2.3), as discussed in footnote 16, the use of transfinite levels of expectations may play a role in defining the notion of coherency under which the universal expectation space is characterized by the set of coherent expectation hierarchies (when the underlying space of states of nature is a measurable space without any topological assumptions). Namely, as in footnote 16, an expectation hierarchy $h_i \in H_i$ of player i is coherent if it admits an expectation hierarchy $\overline{h}_i = (\overline{h}_i^\alpha)_\alpha$ consisting of all possible countable levels of expectations such that all its (countable) levels of expectations do not contradict one another. It would be an interesting future research avenue to generalize the universal-type-space construction by Brandenburger and Dekel (1993) and Mertens and Zamir (1985) to the universal expectation space in the context of this paper in which the space $(S, \mathcal{S})$ of states of nature is a measurable space without any topological assumptions.

5.5 Other Forms of Interactive Reasoning

This paper uses the notion of a functor as a representation of expectations. The use of a functor may be useful to study other modes of reasoning. First, the applications of a functor F to countably-additive, finitely-additive, and non-additive beliefs are immediate.[22] Second, the functor approach may also be useful for interactive preferences (e.g., Di Tillio, 2008; Epstein and Wang, 1996; Ganguli, Hiefetz, and Lee, 2016; Pivato, 2024b): players hold preferences (instead expectations) over the set of bounded measurable functions. Third, the approach of this paper may be useful for ambiguous beliefs (e.g., Ahn, 2007; Alon and Heifetz, 2014), conditional beliefs (e.g., Battigalli and Siniscalchi, 1999; Guarino, 2017, 2025), and lexicographic beliefs (e.g., Brandenburger, Friedenberg, and Keisler, 2008; Catonini and Nicodemo, 2024; Halpern, 2010; Tsakas, 2014).[23]

5.6 Additional Topological Structures

As an avenue for future research, it would also be interesting to introduce some topological structure on an expectation space to study the impact of higher-order expectations on strategic outcomes. On the one hand, it would be conceptually useful to construct the universal expectation space without using any underlying topological structure in the sense that players' reasoning about their higher-order expectations has nothing to do with the underlying topological structure. In the context of the literature on type spaces, Heifetz and Samet (1998) construct such "topology-free" universal type space. In this spirit, this paper constructs the universal expectation space without the aid of a topology.

On the other hand, for applications, it would be interesting to study the approximation of the effect of an entire expectation hierarchy by a finite one. A topological structure on an underlying space may be useful for such analyses.[24] For instance,

[22] That is, as long as one considers a situation in which one's belief (at each state) is represented by a monotone and continuous set function, the methodology of this paper guarantees the existence of the universal space that consists of all possible finite levels of interactive beliefs. Note that, without the continuity assumption, the universal belief space may consist of transfinite levels of interactive beliefs (e.g., see Meier (2006) for finitely-additive beliefs). For an alternative construction of a canonical non-additive belief space, see also Pintér (2012). Fukuda (2025b) characterizes various forms of non-additive beliefs using an alternative approach using players' "p-belief" operators.

[23] Also, the functor approach may be useful for qualitative beliefs and knowledge: see Brandenburger (2003), Brandenburger and Keisler (2006), Mariotti, Meier, and Piccione (2005), and Salonen (2009a,b) for "possibility structures;" and see Aumann (1999), Fagin (1994), Fagin et al. (1999), Fagin, Halpern, and Vardi (1991), Fukuda (Forthcoming), and Heifetz and Samet (1999) for state space models of knowledge or qualitative beliefs. However, due to the lack of continuity, the universal space may consist of transfinite levels of interactive beliefs (see, for instance, Fukuda, Forthcoming).

[24] Also, it would be interesting to study whether the iteration of the average expectation operator $\overline{E}^k (= \overline{\mathbb{E}}\overline{\mathbb{E}}^{k-1})$ converges to some expectation operator (e.g., the expectation operator derived from a common prior in a setting in which players' expectation operators are derived from the common prior conditional on their information): see, for instance, Golub and Morris (2017), Hellman (2011), and

Nyarko (1997) studies a model in which the unique Nash equilibrium is continuous in the product topology on the universal type space. Also, Weinstein and Yildiz (2007) approximate equilibrium actions (of normal-form games) by finitely many orders of beliefs (see also Weinstein and Yildiz, 2011).

5.7 Conclusion

Theorem 1 of this paper constructs the universal expectation space: it is an expectation space to which any expectation space is mapped to the universal space in a structure-preserving manner. The universal expectation space consists of players' expectation hierarchies. To formulate the space of expectations and to construct the universal expectation space for various notions of expectations such as those derived from standard countably-additive, finite-additive, or non-additive beliefs, the paper takes the category-theoretic approach to formulate the space of expectations as a functor.

The formulation of expectation spaces is slightly different from that of type spaces on beliefs in order to impose the law of iterated expectations as a primary introspection property. Section 3.3 discusses economic examples that fit into the framework of this paper. As discussed in Section 5.1, the universal expectation would also exist under the alternative formulation. Section 5.2 suggests that a syntactical approach to establishing the universal expectation space would also be possible. As discussed in Section 5.5, it is an interesting avenue for future research to explore the category-theoretic approach to other forms of beliefs, preferences, and expectations. As briefly discussed in Section 5.6, it would also be interesting to introduce an additional topological structure to explore the impact of higher-order expectations on strategic outcomes.

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A Appendix

A.1 Section 2

Proof of Lemma 1. Given the definition of F in Definition 2, it suffices to show that F satisfies Conditions (2)-(4) in Definition 1.

First, for Condition (2), the mapping $F(\varphi)$ is measurable because, for any $(f', r) \in B(X,' \mathcal{X}') \times \mathbb{R}$,

$$(F(\varphi))^{-1}(\{J' \in F(X') \mid J'(f') \geq r\}) = \{J \in F(X) \mid J(f' \circ \varphi) \geq r\} \in \mathcal{X}_F.$$

Second, for Condition (3), it follows from Expression (1) that, for any measurable space $(X, \mathcal{X})$,

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}.$$

Third, for Condition (4), take any measurable mappings $\varphi : (X, \mathcal{X}) \to (X', \mathcal{X}')$ and $\psi : (X', \mathcal{X}') \to (X'', \mathcal{X}'')$. The mapping $F(\psi) \circ F(\varphi) : F(X) \to F(X'')$ satisfies, for any $J \in F(X)$ and $f'' \in B(X'', \mathcal{X}'')$,

$$\begin{aligned}(F(\psi) \circ F(\varphi))(J)(f'') &= F(\psi)\left(F(\varphi)(J)\right)(f'') \\ &= (F(\varphi)(J))[f'' \circ \psi] \\ &= J[(f'' \circ \psi) \circ \varphi] \\ &= J[f'' \circ (\psi \circ \varphi)] \\ &= F(\psi \circ \varphi)(J)(f''),\end{aligned}$$

implying that $F(\psi) \circ F(\varphi) = F(\psi \circ \varphi)$ as desired. □

A.2 Section 3

To prove Lemma 2, Appendix A.2.1 provides further technical preliminaries on measure theory.[25] Appendix A.2.2 then proves Lemma 2.

A.2.1 Further Technical Preliminaries on Measure Theory

A sequence $(E_n)_{n\in\mathbb{N}}$ of subsets of an underlying set X is a *non-decreasing sequence* with *limit* E, denoted by $E_n \uparrow E$, if (i) $E_m \subseteq E_n$ for any $m, n \in \mathbb{N}$ with $m \leq n$ and if (ii) $E = \bigcup_{n\in\mathbb{N}} E_n$. Similarly, a sequence $(E_n)_{n\in\mathbb{N}}$ of subsets of X is a *non-increasing sequence* with *limit* E, denoted by $E_n \downarrow E$, if (i) $E_n \subseteq E_m$ for any $m, n \in \mathbb{N}$ with $m \leq n$ and if (ii) $E = \bigcap_{n\in\mathbb{N}} E_n$.

If a sequence $(E_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{X}$ satisfies $E_n \uparrow E$, then $\mathbb{I}_{E_n} \uparrow \mathbb{I}_E$. Similarly, if the sequence $(E_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{X}$ satisfies $E_n \downarrow E$, then $\mathbb{I}_{E_n} \downarrow \mathbb{I}_E$.

A collection $\mathcal{M}$ of subsets of an underlying set X is a *monotone class* if the following holds: if $(E_n)_{n\in\mathbb{N}}$ is either a non-decreasing sequence or a non-increasing sequence of elements of $\mathcal{M}$ with limit E, then $E \in \mathcal{M}$. The monotone class theorem states that if $\mathcal{M}$ is a monotone class including an algebra $\mathcal{A}$ then $\mathcal{M}$ includes the smallest σ-algebra $\sigma(\mathcal{A})$. In particular, if the algebra $\mathcal{A}$ is a monotone class, then it is a σ-algebra.

A function $\varphi : (X, \mathcal{X}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is *simple* if there are a finite disjoint collection of measurable sets $(E_i)_{i=1}^n$ and real numbers $(a_i)_{i=1}^n$ such that $\varphi = \sum_{i=1}^n a_i \mathbb{I}_{E_i}$. It is well-known in measure theory that a non-negative Borel measurable function φ on a measurable space $(X, \mathcal{X})$ is the limit of a non-decreasing sequence of non-negative finite-valued simple functions $(\varphi_n)_{n\in\mathbb{N}}$. This result implies that any bounded Borel measurable function $\psi \in B(X, \mathcal{X})$ is the limit of a non-decreasing sequence of finite-valued simple functions $(\psi_n)_{n\in\mathbb{N}}$.

[25] As in footnote 7, the materials in Appendix A.2.1 can be found, for instance, in Ash and Doléans-Dade (2000).

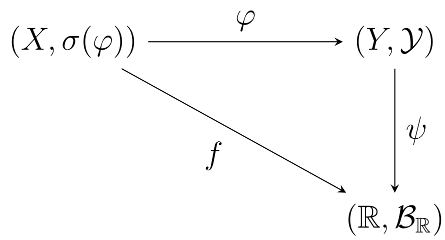

Figure 9: Doob-Dynkin Lemma

The following result is known as Doob-Dynkin Lemma. Let $\varphi : X \to Y$ be a mapping. Let $\mathcal{Y}$ be a σ-algebra on Y, and let $\sigma(\varphi) := \varphi^{-1}(\mathcal{Y})$ be the σ-algebra induced by φ. Then, $f : (X, \sigma(\varphi)) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is measurable if and only if there exists a measurable function $\psi : (Y, \mathcal{Y}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ with $f = \psi \circ \varphi$. See Figure 9.

A.2.2 Proof of Lemma 2

As in the beginning of Section 3.4, let $(\Omega_\ell, \mathcal{D}_\ell)_{\ell \in \mathbb{N}}$ be a collection of measurable spaces. For each $k \in \mathbb{N}$, let $(\Omega^k, \mathcal{D}^k)$ be defined as in Expression (7). Also, let $(\Omega, \mathcal{D})$ be defined as in Expression (8). To prove Lemma 2, I provide the following auxiliary lemmas.

Lemma 8. *For any $A \in \mathcal{D}$, there exists a non-decreasing sequence of sets $(A^k)_{k \in \mathbb{N}}$ with limit A such that, for each $k \in \mathbb{N}$, there exists $B^k \in \mathcal{D}^k$ with $A^k = (\pi^k)^{-1}(B^k)$.*

Proof of Lemma 8. To show the statement, let $\mathcal{M}$ be a sub-collection of $\mathcal{D}$ such that $A \in \mathcal{M}$ if and only if there exists a non-decreasing sequence of sets $(A^k)_{k \in \mathbb{N}}$ with limit A such that, for each $k \in \mathbb{N}$, there exists $B^k \in \mathcal{D}^k$ with $A^k = (\pi^k)^{-1}(B^k)$.

First, I show $\bigcup_{k \in \mathbb{N}} (\pi^k)^{-1}(\mathcal{D}^k) \subseteq \mathcal{M}$. Take $A \in \bigcup_{k \in \mathbb{N}} (\pi^k)^{-1}(\mathcal{D}^k)$. Then, there exists $k_0 \in \mathbb{N}$ such that $A = (\pi^{k_0})^{-1}(B)$ for some $B \in \mathcal{D}^{k_0}$. For any $k \in \mathbb{N}$ with $k < k_0$, let $B^k = \emptyset \in \mathcal{D}^k$. For any $k \in \mathbb{N}$ with $k \geq k_0$, choose $B^k \in \mathcal{D}^k$ with $(\pi^k)^{-1}(B^k) = (\pi^{k_0})^{-1}(B^{k_0})$. Then, $A \in \mathcal{M}$.

Second, observe that $\bigcup_{k \in \mathbb{N}} (\pi^k)^{-1}(\mathcal{D}^k)$ is an algebra. Since $\bigcup_{k \in \mathbb{N}} (\pi^k)^{-1}(\mathcal{D}^k) \subseteq \mathcal{M} \subseteq \mathcal{D}$, if $\bigcup_{k \in \mathbb{N}} (\pi^k)^{-1}(\mathcal{D}^k)$ is a monotone class, then it follows from the monotone class lemma (recall Appendix A.2.1) that $\mathcal{M} = \mathcal{D}$, which will complete the proof of the statement.

Third, therefore, it suffices to show that $\mathcal{M}$ is a monotone class. Firstly, let $(A_n)_{n \in \mathbb{N}}$ be a non-decreasing sequence from $\mathcal{M}$ with $A := \bigcup_{n \in \mathbb{N}} A_n$. For each $A_n \in \mathcal{M}$, there exists a sequence $(A_n^k)_{k \in \mathbb{N}}$ with $A_n^k \uparrow A_n$. To show that $A \in \mathcal{M}$, define, for each $k \in \mathbb{N}$,

$$A^k = \bigcup_{n \in \mathbb{N}} A_n^k.$$

For each $k \in \mathbb{N}$, since $A_n^k \in (\pi^k)^{-1}(\mathcal{D}^k)$ for each $n \in \mathbb{N}$ and $(\pi^k)^{-1}(\mathcal{D}^k)$ is a σ-algebra, it follows that $A^k \in (\pi^k)^{-1}(\mathcal{D}^k)$. If $k \leq \ell$ then it follows from $A_n^k \subseteq A_n^\ell$ for all $n \in \mathbb{N}$

that, taking the union over $n \in \mathbb{N}$, $A^k \subseteq A^\ell$. Then,

$$\bigcup_{k\in\mathbb{N}} A^k = \bigcup_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} A_n^k = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} A_n^k = \bigcup_{n\in\mathbb{N}} A_n = A.$$

Secondly, if $(A_n)_{n\in\mathbb{N}}$ is a non-increasing sequence from $\mathcal{M}$ with $A := \bigcap_{n\in\mathbb{N}} A_n$, then a similar proof replacing a union with an intersection proves this case. More precisely, for each $A_n \in \mathcal{M}$, there exists a non-increasing sequence $(A_n^k)_{k\in\mathbb{N}}$ with limit A_n. To show that $A \in \mathcal{M}$, define, for each $k \in \mathbb{N}$,

$$A^k = \bigcap_{n\in\mathbb{N}} A_n^k.$$

For each $k \in \mathbb{N}$, since $A_n^k \in (\pi^k)^{-1}(\mathcal{D}^k)$ for each $n \in \mathbb{N}$ and $(\pi^k)^{-1}(\mathcal{D}^k)$ is a σ-algebra, it follows that $A^k \in (\pi^k)^{-1}(\mathcal{D}^k)$. If $k \leq \ell$ then it follows from $A_n^k \subseteq A_n^\ell$ for all $n \in \mathbb{N}$ that, taking the intersection over $n \in \mathbb{N}$, $A^k \subseteq A^\ell$. Then,

$$\bigcap_{k\in\mathbb{N}} A^k = \bigcap_{k\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} A_n^k = \bigcap_{n\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} A_n^k = \bigcap_{n\in\mathbb{N}} A_n = A.$$

The proof is complete. □

Lemma 9. *For any bounded Borel measurable function $f \in B(\Omega, \mathcal{D})$, there exists a sequence $(\varphi^k)_{k\in\mathbb{N}}$ of bounded Borel measurable functions $\varphi^k \in B(\Omega, (\pi^k)^{-1}(\mathcal{D}^k))$ such that $\varphi^k \uparrow f$.*

Proof of Lemma 9. The proof consists of four steps. In the first step, recalling Appendix A.2.1, for any $f \in B(\Omega, \mathcal{D})$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions on $(\Omega, \mathcal{D})$ such that $f_n \uparrow f$. Denote each f_n by

$$f_n = \sum_{j=1}^{m_n} x_{n,j} \mathbb{I}_{A_{n,j}},$$

where $A_{n,j} \in \mathcal{D}$ and $x_{n,j} \in \mathbb{R}$ for each $n \in \mathbb{N}$ and $j \in \{1, \dots, m_n\}$.

The second step invokes Lemma 8 to approximate each f_n. Fix $n \in \mathbb{N}$. For each $A_{n,j} \in \mathcal{D}$, it follows from Lemma 8 that there exists a non-decreasing sequence $(A_{n,j}^k)_{n\in\mathbb{N}}$ of sets with limit $A_{n,j}$ such that, for each $k \in \mathbb{N}$, there exists $B_{n,j}^k \in \mathcal{D}^k$ such that $A_{n,j}^k = (\pi^k)^{-1}(B_{n,j}^k)$. Define:

$$f_n^k := \sum_{j=1}^{m_n} x_{n,j} \mathbb{I}_{A_{n,j}^k}.$$

By construction, $f_n^k \in B(\Omega, (\pi^k)^{-1}(\mathcal{D}^k))$.

The third step defines a sequence $(\varphi^k)_{k\in\mathbb{N}}$. For each $k \in \mathbb{N}$, let

$$\varphi^k := \sup_{n\in\mathbb{N}} f_n^k.$$

For each $k \in \mathbb{N}$, the function φ^k is a well-defined bounded function because $\|f_n^k\| \leq \|f\|$ for all $n \in \mathbb{N}$. The function φ^k is also Borel measurable because it is the supremum of a sequence of Borel measurable functions. I show that the sequence $(\varphi^k)_{k\in\mathbb{N}}$ is non-decreasing. For each $k \in \mathbb{N}$, it follows from

$$f_n^k \leq f_n^{k+1} \leq \varphi^{k+1} \text{ for all } n \in \mathbb{N}$$

that

$$\varphi^k \leq \varphi^{k+1}.$$

The fourth step shows that, for each $\omega \in \Omega$, $\varphi^k(\omega) \uparrow f(\omega)$. Fix $\omega \in \Omega$, and take $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then

$$f(\omega) - \frac{\varepsilon}{2} < f_n(\omega).$$

For this $n_0 \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$ then

$$f(\omega) - \varepsilon < f_{n_0}(\omega) - \frac{\varepsilon}{2} < f_{n_0}^k(\omega) \leq \varphi^k(\omega) \leq f(\omega).$$

In sum, fixing $\omega \in \Omega$, for any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$ then

$$f(\omega) - \varepsilon < \varphi^k(\omega) < f(\omega) + \varepsilon.$$

Since $(\varphi^k(\omega))_{k\in\mathbb{N}}$ is a non-decreasing sequence, it follows that $\varphi^k(\omega) \uparrow f(\omega)$, as desired. □

Now, I prove Lemma 2.

Proof of Lemma 2. 1. It suffices to show the "if" part. Assuming the supposition, we show that $J[f] = J'[f]$ for all $f \in B(\Omega, \mathcal{D})$. Take $f \in B(\Omega, \mathcal{D})$. It follows from Lemma 9 that there is a sequence $(f^k)_{k\in\mathbb{N}}$ of Borel measurable maps $f^k : (\Omega, (\pi^k)^{-1}(\mathcal{D}^k)) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ such that $f^k \uparrow f$. For each $k \in \mathbb{N}$, it follows from Doob-Dynkin Lemma (recall Appendix A.2.1) that there is a Borel measurable map $\varphi^k : (\Omega^k, \mathcal{D}^k) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ such that

$$f^k = \varphi^k \circ \pi^k.$$

Then,

$$\begin{aligned} J[f^k] = J[\varphi^k \circ \pi^k] &= F(\pi^k)(J)[f^k] \\ &= F(\pi^k)(J')[f^k] = J'[\varphi^k \circ \pi^k] = J'[f^k], \end{aligned}$$

where the third equality follows from the supposition. By continuity,

$$J[f] = \lim_{k\to\infty} J[f^k] = \lim_{k\to\infty} J'[f^k] = J'[f],$$

as desired.

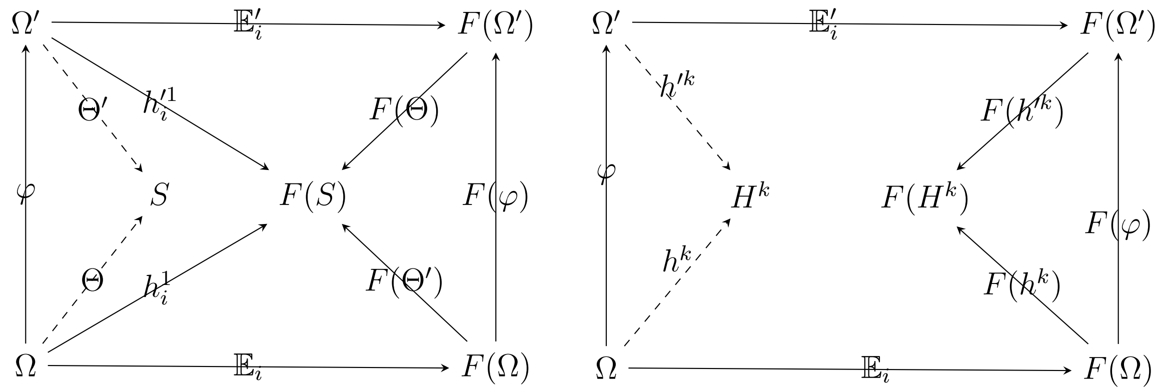

Figure 10: The Proof of Lemma 3

2. The "only if" part follows because the composite $F(\pi^k) \circ \mathbb{E}_i$ of two measurable functions is measurable. In order to show the "if" part, it is enough to show that, for any bounded Borel measurable function $f \in B(\Omega, \mathcal{D})$ and $r \in \mathbb{R}$, $(\mathbb{E}_i[f])^{-1}([r, \infty)) \in \mathcal{D}$. Take $f \in B(\Omega, \mathcal{D})$. It follows from Lemma 9 that there is a sequence $(f^k)_{k \in \mathbb{N}}$ of Borel measurable maps $f^k : (\Omega, (\pi^k)^{-1}(\mathcal{D}^k)) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $f^k \uparrow f$. For each $k \in \mathbb{N}$, it follows from Doob-Dynkin Lemma that there is a Borel measurable map $\varphi^k : (\Omega^k, \mathcal{D}^k) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $f^k = \varphi^k \circ \pi^k$. Take $r \in \mathbb{R}$, and I consider $r - \frac{1}{m}$ where $m \in \mathbb{N}$ (later I will take the limit as $m \to \infty$). Then, it follows from the supposition that

$$(\mathbb{E}_i[f^k])^{-1}([r - \frac{1}{m}, \infty)) = (\mathbb{E}_i[\varphi^k \circ \pi^k])^{-1}([r - \frac{1}{m}, \infty))$$
$$= ((F(\pi^k) \circ \mathbb{E}_i)[\varphi^k])^{-1}([r - \frac{1}{m}, \infty)) \in \mathcal{D}.$$

Then, by the continuity of $\mathbb{E}_i$, as in the standard proof of the fact that the limit of measurable functions is measurable, I have:

$$(\mathbb{E}_i[f])^{-1}([r, \infty)) = \bigcap_{m \in \mathbb{N}} \bigcup_{k_0 \in \mathbb{N}} \bigcap_{k \geq k_0} (\mathbb{E}_i[f^k])^{-1}([r - \frac{1}{m}, \infty)) \in \mathcal{D}.$$

The proof is complete.

□

A.3 Section 4

Proof of Lemma 3. As discussed in the main text, I show

$$h'_i \circ \varphi = h_i \text{ for all } i \in I_0.$$

First, let $i = 0$. I have

$$h'_0 \circ \varphi = \Theta' \circ \varphi = \Theta = h_0,$$

where the first equality follows from the definition of h'_0, the second from the definition of a morphism φ (i.e., Condition (1) in Definition 4), and the third from the definition of h_0. See also the central panel of Figure 6.

Second, I show that $h'_i \circ \varphi = h_i$ for all $i \in I$ by showing that

$$h'^k_i \circ \varphi = h^k_i \text{ for all } (k, i) \in \mathbb{N} \times I.$$

I prove the statement by induction on k. Let $k = 1$. For any $i \in I$,

$$\begin{aligned} h'^1_i \circ \varphi &= (F(\Theta') \circ \mathbb{E}'_i) \circ \varphi \\ &= F(\Theta') \circ (\mathbb{E}'_i \circ \varphi) \\ &= F(\Theta') \circ (F(\varphi) \circ \mathbb{E}_i) \\ &= (F(\Theta') \circ F(\varphi)) \circ \mathbb{E}_i \\ &= F(\Theta) \circ \mathbb{E}_i \\ &= h^1_i. \end{aligned}$$

The first equality follows from the definition of h'^1_i. Likewise, the sixth equality follows from the definition of h^1_i. The second and fourth equalities follow from the associativity of composite functions. The third equality follows from the definition of a morphism φ (i.e., Condition (2) in Definition 4). For the fifth equality, since φ is a morphism, it follows from Condition (1) in Definition 4 that $\Theta = \Theta' \circ \varphi$. Then, since F is a functor,

$$F(\Theta) = F(\Theta' \circ \varphi) = F(\Theta') \circ F(\varphi).$$

The left panel of Figure 10 illustrates this argument.

Suppose that the inductive hypothesis for k holds. Observe that

$$\begin{aligned} h'^{k+1}_i \circ \varphi &= \left(h'^k_i, F(h'^k) \circ \mathbb{E}'_i\right) \circ \varphi \\ &= \left(h'^k_i \circ \varphi, (F(h'^k) \circ \mathbb{E}'_i) \circ \varphi\right) \end{aligned}$$

and

$$h^{k+1}_i = \left(h^k_i, F(h^k) \circ \mathbb{E}_i\right).$$

Since $h'^k_i \circ \varphi = h^k_i$ follows from the induction hypothesis, it suffices to show

$$(F(h'^k) \circ \mathbb{E}'_i) \circ \varphi = F(h^k) \circ \mathbb{E}_i.$$

However, this latter equation follows because

$$\begin{aligned} \left(F(h'^k) \circ \mathbb{E}'_i\right) \circ \varphi &= F(h'^k) \circ (\mathbb{E}'_i \circ \varphi) \\ &= F(h'^k) \circ (F(\varphi) \circ \mathbb{E}_i) \\ &= \left(F(h'^k) \circ F(\varphi)\right) \circ \mathbb{E}_i \\ &= F(h'^k \circ \varphi) \circ \mathbb{E}_i \\ &= F(h^k) \circ \mathbb{E}_i. \end{aligned}$$

The first and third equalities follow because composite functions are associative. The second equality follows because φ is a morphism (i.e., Condition (2) in Definition 4). The fourth equality follows because F is a functor. The fifth equality follows from the induction hypothesis. The right panel of Figure 10 illustrates this argument.

Consequently, $h' \circ \varphi = h$. □

Proof of Lemma 4. For any $\omega^* \in \Omega^*$, take any expectation space $\overrightarrow{\Omega}$ and state $\omega \in \Omega$ such that $\omega^* = h(\omega)$. Let $i \in I$.

First, one has:

$$\begin{aligned}(F(\Theta^*) \circ \mathbb{E}_i^*)(\omega^*) &= (F(\Theta^*) \circ \mathbb{E}_i^*)(h(\omega)) = F(\Theta^*)\left((\mathbb{E}_i^* \circ h)(\omega)\right) \\ &= F(\Theta^*)\left((F(h) \circ \mathbb{E}_i)(\omega)\right) = \left((F(\Theta^*) \circ F(h)) \circ \mathbb{E}_i\right)(\omega) \\ &= \left(F(\Theta^* \circ h) \circ \mathbb{E}_i\right)(\omega) \\ &= \left(F(\Theta) \circ \mathbb{E}_i\right)(\omega) \\ &= h_i^1(\omega) = (\omega_i^*)^1.\end{aligned}$$

The first and last equalities follow because $\omega^* = h(\omega)$. The second and fourth equalities follow from the properties of composite functions. The third equality follows from Expression (9). The fifth equality follows because F is a functor. The sixth equality follows from the definition of h in the second step of the construction. The left panel of Figure 7 illustrates the argument.

Second, let $k \in \mathbb{N}$. Then,

$$\begin{aligned}F(\pi^k) \circ \mathbb{E}_i^*(\omega^*) &= F(\pi^k \circ h) \circ \mathbb{E}_i(\omega) \\ &= F(h^k) \circ \mathbb{E}_i(\omega) \\ &= (h_i(\omega))^{k+1} = (\omega_i^*)^{k+1}.\end{aligned}$$

The first equality follows from the definition of $\mathbb{E}_i^*$ in the fifth step of the construction. The second and third equalities follow from the definition of h. The fourth equality follows from $\omega^* = h(\omega)$. The right panel of Figure 7 illustrates the first and second equalities. □

Proof of Lemma 5. As discussed in the main text, the proof consists of three steps. First, I show that $\mathbb{E}_i^*(\omega^*)$ is well-defined irrespective of the choices of $\overrightarrow{\Omega}$ and $\omega \in \Omega$ with $\omega^* = h(\omega)$. Take $\omega^* \in \Omega^*$. Suppose that there are expectation spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ and states $\omega \in \Omega$ and $\omega' \in \Omega'$ such that $\omega^* = h(\omega)$ and $\omega^* = h'(\omega')$ (see also Figure 11).

Take any $f^* \in B(\Omega^*, \mathcal{D}^*)$. On the one hand,

$$\mathbb{E}_i(\omega)[f^* \circ h] = (F(h) \circ \mathbb{E}_i(\omega))[f^*].$$

On the other hand,

$$\mathbb{E}'_i(\omega')[f^* \circ h'] = (F(h') \circ \mathbb{E}'_i(\omega'))[f^*].$$

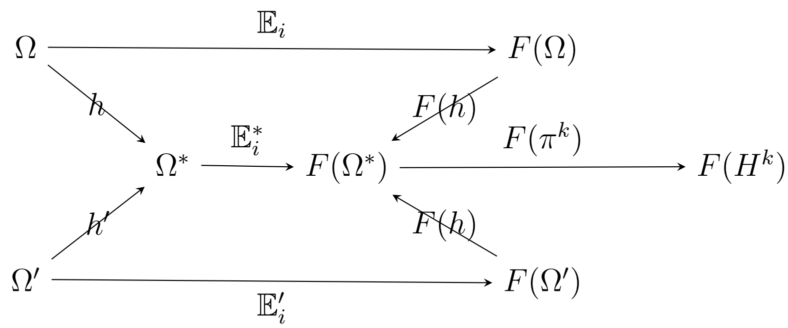

Figure 11: The Proof of Lemma 5: the Second Step.

Hence, if, as depicted in Figure 11,

$$(F(h) \circ \mathbb{E}_i(\omega))[\cdot] = (F(h') \circ \mathbb{E}_i'(\omega'))[\cdot],$$

then $\mathbb{E}_i^*(h(\cdot)) := F(h) \circ \mathbb{E}_i(\cdot)$ is well-defined. By Lemma 2 (1), it suffices to show that, for each $k \in \mathbb{N}$,

$$F(\pi^k) \circ (F(h) \circ \mathbb{E}_i(\omega)) = F(\pi^k) \circ (F(h') \circ \mathbb{E}_i'(\omega'))\,.$$

However, recalling that $h(\omega) = \omega^* = h'(\omega')$, it follows from the first step that

$$\begin{aligned} F(\pi^k) \circ (F(h) \circ \mathbb{E}_i(\omega)) &= F(h^k) \circ \mathbb{E}_i(\omega) \\ &= (\omega_i^*)^{k+1} \\ &= F(h'^k) \circ \mathbb{E}_i'(\omega') = F(\pi^k) \circ (F(h') \circ \mathbb{E}_i'(\omega'))\,. \end{aligned}$$

Thus, each $\mathbb{E}_i^*$ is well-defined.

Second, I show that $\mathbb{E}_i^* : (\Omega^*, \mathcal{D}^*) \to (F(\Omega^*), \mathcal{D}_F^*)$ is measurable. By Lemma 2 (2), it is enough to show that, for all $k \in \mathbb{N}$, $F(\pi^k) \circ \mathbb{E}_i^* : (\Omega^*, \mathcal{D}^*) \to (F(H^k), \mathcal{H}_F^k)$ is measurable, that is, for any $f^k \in B(H^k, \mathcal{H}^k)$ and $r \in \mathbb{R}$,

$$(F(\pi^k) \circ \mathbb{E}_i^*)^{-1}(\{J^k \in F(H^k) \mid J^k[f^k] \geq r\}) \in \mathcal{D}^*.$$

However, it follows from Expression (11) that

$$(F(\pi^k) \circ \mathbb{E}_i^*)^{-1}(\{J^k \in F(H^k) \mid J^k[f^k] \geq r\}) = \{\omega^* \in \Omega^* \mid (\omega_i^*)^{k+1}[f^k] \geq r\} \in \mathcal{D}^*.$$

Third, I show that the law of iterated expectations holds. Take any $f^* \in B(\Omega^*, \mathcal{D}^*)$. For any $\omega^* \in \Omega^*$, there exist an expectation space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$. Then,

$$\mathbb{E}_i^*(\omega^*)[f^*] = \mathbb{E}_i(\omega)[f^* \circ h].$$

By the law of iterated expectations in the expectation space $\overrightarrow{\Omega}$, the right-hand side of the above equation satisfies:

$$\mathbb{E}_i(\omega)[f^* \circ h] = \mathbb{E}_i(\omega)[\mathbb{E}_i(\cdot)[f^* \circ h]].$$

Then, by the definition of $\mathbb{E}_i^*$, the term inside the expectation functional $\mathbb{E}_i(\omega)$ in the right-hand side of the above equation satisfies

$$\begin{aligned}\mathbb{E}_i(\cdot)[f^* \circ h] &= \mathbb{E}_i^*(h(\cdot))[f^*] \\ &= (\mathbb{E}_i^*[f^*] \circ h)(\cdot),\end{aligned}$$

and operating $\mathbb{E}_i(\omega)$ yields

$$\mathbb{E}_i(\omega)[f^* \circ h] = \mathbb{E}_i(\omega)[\mathbb{E}_i^*[f^*] \circ h].$$

By the definition of $\mathbb{E}_i^*$, I have

$$\begin{aligned}\mathbb{E}_i(\omega)[\mathbb{E}_i^*[f^*] \circ h] &= \mathbb{E}_i^*(h(\omega))[\mathbb{E}_i^*[f^*]] \\ &= \mathbb{E}_i^*(\omega^*)[\mathbb{E}_i^*[f^*]].\end{aligned}$$

Hence, I obtain:

$$\mathbb{E}_i^*(\omega^*)[f^*] = \mathbb{E}_i^*\mathbb{E}_i^*(\omega^*)[f^*] \text{ for all } f^* \in B(\Omega^*, \mathcal{D}^*) \text{ and } \omega^* \in \Omega^*,$$

which establishes the law of iterated expectations: $\mathbb{E}_i^* = \mathbb{E}_i^*\mathbb{E}_i^*$. The proof is complete. □

Proof of Lemma 6. Given $h^* = (h_i^*)_{i \in I_0}$, I show that each h_i^* is a projection (precisely, $h_i^* = \pi_{H_i}^H|_{\Omega^*}$). First, for $i = 0$, the assertion follows because

$$h^{*0} = \Theta^* = \pi_S^H|_{\Omega^*}.$$

The first equality follows from the second step of the construction. The second equality follows from the fourth step of the construction.

Second, to prove that $h_i^* : \Omega^* \to H_i$ is a projection for each player $i \in I$, it suffices to show that, for each $\omega^* \in \Omega^*$ and each $i \in I$,

$$(h_i^*(\omega^*))^k = (\omega_i^*)^k \text{ for each } k \in \mathbb{N}.$$

Fix $\omega^* \in \Omega^*$ and $i \in I$. For $k = 1$, the assertion follows from Expression (10) in Lemma 4 because

$$(h_i^*(\omega^*))^1 = h_i^{*1}(\omega^*).$$

For $k \geq 2$,

$$(h_i^*(\omega^*))^k = F(\pi^{k-1}) \circ \mathbb{E}_i^*(\omega^*) = (\omega_i^*)^k,$$

where the first equality follows from the definition of h^* and the second from Expression (11) in Lemma 4. The proof is complete. □

Proof of Lemma 7. It suffices to show that each $\mathbb{E}'_i : (\Omega, \sigma(h)) \to (F(\Omega), (\sigma(h))_F)$ is measurable. To that end, it is enough to show that, for any $f \in B(\Omega, \sigma(h))$, $\mathbb{E}'_i[f] : (\Omega, \sigma(h)) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is measurable.

Fix $f \in B(\Omega, \sigma(h))$. It follows from Doob-Dynkin Lemma (recall Appendix A.2.1) that there exists $f^* \in B(\Omega^*, \mathcal{D}^*)$ such that

$$f = f^* \circ h.$$

For any $r \in \mathbb{R}$,

$$\begin{aligned}
(\mathbb{E}'_i[f])^{-1}([r, \infty)) &= (\mathbb{E}'_i[f^* \circ h])^{-1}([r, \infty)) \\
&= \{\omega \in \Omega \mid \mathbb{E}'_i(\omega)[f^* \circ h] \geq r\} \\
&= \{\omega \in \Omega \mid \mathbb{E}_i(\omega)[f^* \circ h] \geq r\} \\
&= \{\omega \in \Omega \mid (\mathbb{E}^*_i \circ h)(\omega) \geq r\} \\
&= (\mathbb{E}^*_i[f^*] \circ h)^{-1}([r, \infty)) \\
&= h^{-1}((\mathbb{E}^*_i[f^*])^{-1}([r, \infty))) \in h^{-1}(\mathcal{D}^*) = \sigma(h),
\end{aligned}$$

as desired. $\square$