The Hierarchical Construction of a Universal Qualitative Belief Space

Online Appendix

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Appendix B briefly introduces the concepts regarding ordinal and cardinal numbers that are used in the main text.[1]

B Ordinal and Cardinal Numbers

The main text of the paper (i.e., Definitions 7 and 8) explicitly considers transfinite levels of beliefs. Each state in the universal belief space induces the first-order beliefs about S, the second-order beliefs, and so on, up to a pre-determined "ordinal level $\overline{\kappa}$," beyond the least infinite ordinal number $\varpi = \{0, 1, 2, \dots\}$.[2] Moreover, the proof method of transfinite induction and the notion of cardinal numbers are based on ordinal numbers. Below, I formally define ordinal numbers, the principle of transfinite induction, and cardinal numbers.

Ordinal Numbers. Ordinal numbers are meant to generalize the non-negative integers, and the relation "$<$" (less than) on the non-negative integers is generalized to the set membership relation "$\in$."[3] Formally, an *ordinal number* (an ordinal, for short) α is a set with the following three properties:

1. The set-membership relation "$\in$" is a (strict) total order in α: for any two (distinct) elements $\beta \in \alpha$ and $\gamma \in \alpha$ with $\beta \neq \gamma$, either $\beta \in \gamma$ or $\gamma \in \beta$.
2. The relation "$\in$" is transitive in α: if $\beta \in \alpha$ and $\gamma \in \beta$ then $\gamma \in \alpha$.[4]
3. Any non-empty subset A of α has a least element with respect to "$\in$."[5]

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[1]Appendix B is not intended as a summary or overview of ordinal and cardinal numbers. For a textbook which covers the materials covered here, see, for instance, Hrbacek and Jech (1999).

[2]I use ϖ to denote the least infinite ordinal number instead of the standard notation ω, to avoid the possible confusion coming from the clash of notation (with a state of the world ω). Note also that ϖ appears only in the Online Appendix.

[3]I use the terminology "non-negative integers" because natural numbers are meant as positive integers in the main text.

[4]In other words, any element β of the set α is a subset of α (i.e., if $\beta \in \alpha$ then $\beta \subseteq \alpha$).

[5]In other words, if A is a non-empty subset of α, then there exists $\gamma \in A$ such that $\gamma \in \beta$ for all $\beta \in A \setminus \{\gamma\}$. Formally, such (strictly) totally-ordered set $\langle \alpha, \in \rangle$ is called a *well-ordered set*.

For any ordinal numbers α and β, denote by $\alpha < \beta$ if $\alpha \in \beta$. Also, denote by $\alpha \leq \beta$ if $\alpha < \beta$ or $\alpha = \beta$.

The empty set $\emptyset$ is an ordinal number and is identified as $0 := \emptyset$ (i.e., the non-negative integer 0, as an ordinal number, is set-theoretically identified as the empty set). The integer 1 is identified as an ordinal number $1 := 0 \cup \{0\} (= \{0\} = \{\emptyset\})$. The integer 2 is identified as an ordinal number $2 := 1 \cup \{1\} (= \{0, 1\} = \{\emptyset, \{\emptyset\}\})$. Given a non-negative integer n as an ordinal number, the non-negative integer $n + 1$ is identified as an ordinal number $n + 1 := n \cup \{n\} (= \{0, 1, \ldots, n\})$. Thus, we have finite ordinals $0, 1, 2, \ldots, n, n + 1, \ldots$. Then, one counts farther to define the least infinite ordinal as $\varpi = \{0, 1, 2, \ldots, \}$. The next ordinal is $\varpi + 1 := \varpi \cup \{\varpi\}$, and so forth indefinitely. Thus, one can enumerate ordinal numbers as:

$$
\begin{aligned}
&0, 1, 2, \ldots, n, n + 1, \ldots, \\
&\varpi, \varpi + 1, \varpi + 2, \ldots, \varpi + n, \varpi + (n + 1), \ldots, \\
&\varpi \cdot 2 (= \varpi + \varpi), \varpi \cdot 2 + 1, \ldots, \ldots\ldots\ldots, \\
&\varpi \cdot n, \varpi \cdot n + 1, \ldots, \ldots\ldots\ldots, \\
&\varpi^2 (= \varpi \cdot \varpi), \varpi^2 + 1, \ldots, \ldots\ldots\ldots, \\
&\varpi^n, \varpi^n + 1, \ldots, \ldots\ldots\ldots, \\
&\varpi^\varpi, \varpi^\varpi + 1, \ldots, \ldots\ldots\ldots.
\end{aligned}
$$

The enumeration lasts indefinitely. While these ordinals are all countable, once all the countable ordinals are enumerated, the next least ordinal is the least uncountable ordinal. The enumeration still continues.

Successor and Limit Ordinals. For any ordinal α, the *successor* of α is defined and denoted by $\alpha + 1 := \alpha \cup \{\alpha\}$. An ordinal α is a *successor ordinal* if $\alpha = \beta + 1$ for some ordinal β. An ordinal α is a *limit ordinal* if it is not a successor ordinal. In the above example, any positive integer n is a successor ordinal, while ϖ and $\varpi \cdot 2$ are a limit ordinal.

Transfinite Induction. Let $S(\alpha)$ be a statement for each ordinal α. If (i) $S(0)$ is true and if (ii) $S(\beta)$ is true for all $\beta < \alpha$ implies that $S(\alpha)$ is true, then $S(\alpha)$ is true for all ordinal α. In (ii), one can consider two cases for α, when α is a successor ordinal and when α is a limit ordinal.

Cardinal Numbers. Two sets A and B are defined to have the same cardinality if there is a bijection from A to B. Under the Axiom of Choice, cardinal numbers are sets with the following property: for any set A, there is a unique cardinal number having the same cardinality as A. With these in mind, a *cardinal number* (a cardinal, for short) is an ordinal number which does not have the same cardinality as any of its elements (recall that any of its elements itself is a smaller ordinal). By the Axiom of Choice, it is well-known that any set A has the same cardinality as some cardinal number. The least infinite cardinal is denoted by $\aleph_0$. The least uncountable cardinal is denoted by $\aleph_1$.

A Unique Identification of a Cardinal as an Ordinal. Although an (infinite) cardinal number κ is an ordinal number, the cardinal number κ may be in a bijective relation with multiple ordinal numbers. For instance, there exists a bijection between ordinal numbers $\varpi = \{0, 1, 2, \dots\}$ and $\varpi + 1 = \{0, 1, 2, \dots, \dots, \varpi\}$. To uniquely identify an (infinite) cardinal with the ordinal, call an ordinal α an *initial ordinal* if, for any $\beta \in \alpha$, there does not exist a bijection between α and β. Under the Axiom of Choice, for any (infinite) cardinal κ, there exists a unique initial ordinal $\overline{\kappa}$. Thus, we uniquely identify the (infinite) cardinal κ as an ordinal number $\overline{\kappa}$. For example, if $\kappa = \aleph_0$ then $\overline{\kappa} = \varpi$. Also, if $\kappa = \aleph_1$ then $\overline{\kappa}$ is the smallest uncountable ordinal.

Successor Cardinals. Under the Axiom of Choice, it is well-known that, for each cardinal κ, there is a unique least cardinal greater than κ. Denote by κ^+ this cardinal and call it the *successor cardinal* (to κ).

Regular Cardinals. Under the Axiom of Choice, an infinite cardinal κ is *regular* if, for any set A which is a union of less-than κ-many sets each of which has cardinality less than κ, the cardinality of A is less than κ: if $A = \bigcup_{i \in I} A_i$ satisfies $|I| < \kappa$ and $|A_i| < \kappa$ for all $i \in I$, then $|A| < \kappa$.[6] The proof of Remark 3 in the main text uses this definition. An infinite cardinal is *singular* if it is not regular. As mentioned in Section 2.1, it is well-known that any successor infinite cardinal κ^+ is regular (see also Hrbacek and Jech, 1999). Also, $\aleph_0$ and $\aleph_1$ are regular.[7]

References for the Online Appendix

[1] K. Hrbacek and T. Jech. Introduction to Set Theory. Third Edition. CRC Press, 1999.

[6] For the expert reader who knows the definition of the regularity of an infinite cardinal κ in terms of cofinality of κ (i.e., an infinite cardinal κ is regular if the cofinality of the infinite cardinal κ is κ), the aforementioned definition is equivalent under the Axiom of Choice. This is because the *cofinality* of κ is characterized as the least cardinal λ such that κ is the cardinality of the union of λ-many sets of cardinality less than κ (see, e.g., Hrbacek and Jech, 1999). Thus, if $|I| < \kappa$ then the union $A = \bigcup_{i \in I} A_i$ satisfies $|A| < \kappa$ as long as $|A_i| < \kappa$ for all $i \in I$.

[7] Technically, for $\aleph_0$, a union of finitely many finite sets is a finite set. For $\aleph_1$, a union of countably many countable sets is a countable set (or, $\aleph_1$ is indeed a successor cardinal).