The Hierarchical Construction of a Universal Qualitative Belief Space*

Satoshi Fukuda†

September 5, 2025

Abstract

This paper constructs a canonical representation of players' belief hierarchies—players' beliefs over some exogenously given values such as their action profiles or payoff functions, their beliefs about their beliefs about exogenously given values, and so on ad infinitum—in the context of non-probabilistic beliefs, including knowledge. This paper demonstrates that the idea that any "possible" belief hierarchy of a player can be captured as the player's type holds true regardless of whether players' beliefs are probabilistic or qualitative. Formally, the first main result is to construct a universal qualitative belief space as the set of players' belief hierarchies that can be induced by some qualitative belief space. The second is to show that the universal qualitative belief space coincides with the set of coherent belief hierarchies.

JEL Classification: C70; D83
Keywords: Interactive Belief; Qualitative Belief; Universal Belief Space; Belief Hierarchies

---

*This paper is based on part of the first chapter of my Ph.D. thesis submitted to the University of California at Berkeley. I would especially like to thank David Ahn, William Fuchs, and Chris Shannon for their encouragement, support, and guidance. I am also grateful to the editor, associate editor, and two anonymous referees for their excellent suggestions, which have significantly improved the manuscript. All remaining errors are mine.

†Department of Economics, Leavey School of Business, Santa Clara University, Santa Clara, CA 95053, USA.

1 Introduction

In strategic situations, game theoretic analyses often call for considerations of belief hierarchies about players' action profiles or payoff functions. For example, the underlying idea behind iterated elimination of strictly dominated actions is that players are rational and commonly believe their rationality: every player is rational, and every player believes that every player is rational, every player believes that every player believes that every player is rational, and so on. In games with incomplete information where players face payoff uncertainty, they hold beliefs about their payoff functions, they have beliefs about beliefs about their payoff functions, and so on.

Call a space of exogenous uncertainty such as players' action profiles or payoff functions the set of states of nature (or nature states). When it comes to probabilistic beliefs, a belief hierarchy is naturally formulated as a sequence of probability measures on the set of states of nature: the first-order belief (i.e., the first element of the sequence) is a probability measure over the nature states, the second-order belief is a probability measure over the nature states and the opponents' first-order beliefs, and so on.

A type space (Harsanyi, 1967-68) provides a self-referential representation of players' belief hierarchies. Each player's type is a probability measure over the nature states and the opponents' types. Each type induces her belief about the nature states (i.e., the first-order belief), her belief about the nature states and the other players' beliefs about the nature states (i.e., the second-order belief), and so on. Seminal papers such as Armbruster and Böge (1979), Böge and Eisele (1979), Brandenburger and Dekel (1993), Heifetz (1993), and Mertens and Zamir (1985) show that there exists a canonical representation—a universal type space. The universal type space is *the* largest type space in that it consists of all "possible" belief hierarchies.[1] It is characterized as the set of belief hierarchies that satisfies certain "coherency" conditions: any two levels of beliefs in a belief hierarchy do no contradict one another (a condition referred to as 'coherency' of the belief hierarchy), and the players are commonly certain that their belief hierarchies are coherent (a condition on the set of belief hierarchies referred to as 'common certainty of coherency'). The existence of the universal type space shows that, while a type induces a coherent belief hierarchy, a coherent belief hierarchy is itself a type.

The purpose of this paper is to show that this insight extends to non-probabilistic beliefs. To see this, consider a model of qualitative beliefs or knowledge. The first difficulty is how to model a belief hierarchy in such settings. This difficulty arises because the primitive of a model in the literature is typically not a type but an information set. Players face a set of states of the world Ω. Each player has her information set at each state ω, which consists of states ω' that she considers possible at ω.[2] A player believes an event E (i.e., a subset of states) at a state ω if the event

[1] Section 2.4 formally discusses the sense in which one can speak of *the* universal space.

[2] In Aumann (1976)'s partitional model of knowledge, the information sets form a partition: the

E includes her information set at ω (i.e., the event E contains each state that the player considers possible at ω). Since the set of states at which a player (say, Ann) believes an event E is an event, one can define the event that another player (say, Bob) believes that Ann believes the event E. In sum, the first difficulty of modeling belief hierarchies with qualitative beliefs lies in the fact that they are typically constructed indirectly, through information sets rather than directly from types.

This paper thus extends the notion of types to qualitative beliefs. In the context of qualitative beliefs, a type μ (of a player at a state) is a binary mapping on the collection of events. For any event E, the player believes the event E (at the state) if her type assigns 1 to the event: $\mu(E) = 1$. In contrast, she does not believe the event E if her type does not assign 1, i.e., assigns 0, to the event: $\mu(E) = 0$. This paper then formally defines a belief hierarchy for qualitative beliefs as a sequence of first- and higher-order beliefs on the underlying nature states. In doing so, I consider a general model of qualitative beliefs in a way such that one can add (resp., drop) various properties of qualitative beliefs one by one to (resp., from) the set of types.[3]

The second difficulty of modeling belief hierarchies for qualitative beliefs is to determine players' depth of reasoning. On the one hand, it appears to be natural to define players' belief hierarchies consisting of all finite levels of beliefs: for instance, Ann and Bob believe that they are rational (the first-order beliefs), they believe that they believe that they are rational (the second-order beliefs), and so on for all finite-level beliefs. For standard countably-additive probabilistic beliefs, this suffices because each belief hierarchy in the universal type space (Brandenburger and Dekel, 1993; Heifetz and Samet, 1998b; Mertens and Zamir, 1985) consists of finite-level beliefs that admit an extension to transfinite (but countable) levels owning to the countable additivity (i.e., continuity) of beliefs. However, since qualitative beliefs may not satisfy continuity, this paper considers belief hierarchies consisting of transfinite levels of beliefs.[4] For instance, in order to define the notion of common belief, one may need to consider an arbitrarily long sequence of mutual beliefs (i.e., everyone believes), which necessitates belief hierarchies consisting of transfinite levels; and for another instance, elimination of strictly dominated actions may call for an arbitrarily long elimination process.[5] With these in mind, this paper considers belief hierar-

information set at state ω consists of states ω' such that the player cannot distinguish between ω and ω'.

[3]As an analogy, in the context of quantitative beliefs, this corresponds to a framework that allows for studying, for instance, non-additive, finitely-additive, and countably-additive beliefs in a unified manner.

[4]Meier (2006) constructs the universal finitely-additive belief space, where each state in the universal space consists of logical formulas expressing players' interactive beliefs of possibly transfinite levels.

[5]For early papers that point out the role that transfinite levels of beliefs play in iterated elimination of strictly dominated actions, see, among others, Chen, Long, and Luo (2007), Dufwenberg and Stegeman (2002), and Lipman (1994). In the context of qualitative beliefs, see Fukuda (2020, 2024b).

chies consisting of transfinite levels, up to an arbitrary but predetermined ordinal level. This specification allows for the belief hierarchies consisting of all finite-level qualitative beliefs when the upper bound is the least infinite ordinal.

Formally, starting from an arbitrary number of players and without any topological assumptions, Theorem 1 constructs the universal qualitative belief space as the set of all "possible" belief hierarchies. Theorem 2 shows that the universal qualitative belief space is characterized as the set of coherent belief hierarchies closed under common certainty of coherency. Hence, this paper shows that Harsanyi (1967-68)'s type-space approach carries over also to non-probabilistic beliefs.

The paper is structured as follows. The rest of the Introduction discusses the related literature. Section 2 defines qualitative belief spaces. Section 3 establishes the main results: Theorem 1 in Section 3.1 and Theorem 2 in Section 3.2. Section 4 provides concluding remarks. The proofs are relegated to Appendix A.

Related Literature

This paper is related to strands of literature that study canonical representations of various forms of qualitative beliefs and knowledge.

First, this paper develops the "type-space" representation of a state-space model of knowledge and qualitative beliefs and constructs a canonical representation. The framework of this paper includes: (i) the partitional model of knowledge (e.g., Aumann, 1976), (ii) the non-partitional model of knowledge in which a player may lack the negative introspection property: when the player does not know an event, she may not know that she does not know the event (e.g., Bacharach, 1985; Brandenburger, Dekel, and Geanakoplos, 1992; Geanakoplos, 2021; Morris, 1996; Rubinstein and Wolinsky, 1990; Samet, 1990; Shin, 1993), and (iii) possibility correspondence models of qualitative beliefs (e.g., Bonanno, 2008, 2015; Bonanno and Tsakas, 2018; Hillas and Samet, 2020; Samet, 2013).[6]

The partitional model of knowledge is the oldest model in economics and game theory that enables one to analyze interactive knowledge about underlying nature states such as players' action profiles or payoff functions. Aumann (1999, Section 10 (c)) constructs his "canonical" knowledge space with his caveat that "[t]he hierarchy construction is so convoluted that we present it here with some diffidence. Specifically, we have not checked carefully that the three conditions ... really are the "right" conditions..."

The contributions of this paper regarding Aumann (1999, Section 10 (c)) are as follows. Firstly, unlike Aumann (1999), this paper formally defines the notion of types that represents knowledge and qualitative beliefs. Thus, unlike Aumann (1999), this paper enables one to directly compare qualitative and probabilistic beliefs under a

[6]The interactive qualitative belief model is also related to the interactive preference model (e.g., Chen, 2010; Di Tillio, 2008; Epstein and Wang, 1996; Ganguli, Hiefetz, and Lee, 2016; Pivato, 2024) in that players' qualitative beliefs may be induced from their preferences.

unified framework.[7] Conceptually, the first contribution of this paper is to show that the type-space approach succeeds in obtaining a self-referential representation of players' infinite belief hierarchies regardless of whether players' beliefs are probabilistic or qualitative.

A companion paper, Fukuda (2024b), shows the existence of the universal qualitative belief space irrespective of properties of beliefs, where each state of the universal belief space consists of a collection of logical formulas expressing players' interactive beliefs.[8] With respect to Fukuda (2024b), the contribution of this paper is to demonstrate the versatility of the type-space approach irrespective of properties of beliefs: a type induces a belief hierarchy, and a belief hierarchy itself can be a type, irrespective of whether beliefs are probabilistic or qualitative.

Secondly, the construction of the universal qualitative belief space in this paper is not an application of Aumann (1999) because contributions from the previous literature such as Fagin (1994), Fagin, Geanakoplos, et al. (1999), Fagin, J. Y. Halpern, and Vardi (1991), Fukuda (2024b), Heifetz and Samet (1998a, 1999), and Meier (2005, 2008) have shown that, without an appropriate setup, the universal qualitative belief space does not exist.[9] In this regard, this paper provides a "right" setup in which the "canonical" representation of Aumann (1999)'s knowledge hierarchies turns out to be the universal knowledge space (which consists of players' knowledge hierarchies). In contrast, without such reformulations of the setup, as shown in the previous literature, the framework of Aumann (1999) does not admit the universal knowledge space.

Thirdly, this paper constructs the universal belief space irrespective of properties of beliefs and depth of reasoning. In game-theoretic applications, for instance, the truth axiom of knowledge may not necessarily be appropriate: the truth axiom states that if a player knows an event at a state then the event has to hold true at that state. Thus, if Ann knows that Bob is rational at a certain state, then it has to be the case that Bob is indeed rational at that state. However, in principle, whether Bob is rational or not has nothing to do with whether Ann knows that Bob is rational. This paper enables one to construct the universal belief space in a way that does not presuppose particular properties of qualitative beliefs. Moreover, Aumann (1999) considers knowledge hierarchies consisting only of finite-level interactive knowledge: Ann and Bob know that they are rational, they know that they know that they are rational, up to some arbitrarily-long finite levels. Thus, the framework of Aumann (1999) does not allow for reasoning about common knowledge (of rationality): Ann

[7] Fukuda (2025) uses the "type space" representation of a belief model to study the formal sense in which the players in the belief model are certain of the model itself.

[8] While Fukuda (2024b) and this paper provide different constructions, the resulting universal qualitative belief spaces are isomorphic. Roughly speaking, given the collection of logical formulas that express players' interactive beliefs at a state in Fukuda (2024b)'s universal space, there exists a unique belief hierarchy for each player in the universal space constructed in this paper, and vice versa.

[9] See also Heifetz and Samet (1998b, Section 6) on this point.

and Bob know that they are rational, they know that they know that they are rational, and so on *ad infinitum*.

Second, the framework of this paper is related to a possibility model first studied by Brandenburger (2003) and Brandenburger and Keisler (2006). The possibility model is another type-space representation of qualitative beliefs.[10] Each (possibility) type induces a subset of the nature states and the (possibility) types of the other players.[11] Mariotti, Meier, and Piccione (2005) and Salonen (2009a,b) establish the existence of the universal possibility model when players hold a compact set of beliefs. This paper establishes the existence of the universal qualitative belief space without such assumptions. Thus, the universal qualitative belief space constructed in this paper can be used for any set of states of nature.

2 Belief Spaces

Throughout the paper, let I be a non-empty set of players. Let S be a set of *states of nature* (or nature states). An element of S is regarded as a specification of the exogenous values such as action profiles or payoff functions that are relevant to the strategic interactions among the players. The set of states of nature S is endowed with a sub-collection $\mathcal{S}$ of $\mathcal{P}(S)$, i.e., $\mathcal{S} \subseteq \mathcal{P}(S)$, where $\mathcal{P}(\cdot)$ is the power-set operation. Each element E of $\mathcal{S}$, called an event of nature, is an object about which the players interactively reason.

2.1 Technical Preliminaries

In the literature on (probabilistic) type spaces, each player's type in a type space induces a belief (i.e., probability measure) over underlying states of nature S, a belief over states of nature S and the opponents' (first-order) beliefs over S, and so on. Thus, each player's type induces a belief hierarchy consisting of all finite levels of beliefs, i.e., the belief hierarchy consists of the first-order belief, the second-order belief, the third-order belief, and so on, along the set of natural numbers. When the underlying set S of states of nature is endowed with a topological structure and players' beliefs are countably additive, each belief hierarchy that consists of all finite levels of beliefs admits an extension to the belief hierarchy that consists of all transfinite but countable levels of beliefs.

[10] Guarino and Ziegler (2022), for instance, study rationalizability solution concepts when players have optimistic or pessimistic attitudes, using a possibility model.

[11] The assumption that a (possibility) type of a player induces a subset of the nature states and the (possibility) types of the *other* players corresponds to the idea that each type of the player is certain of her own type. In fact, in the context of probabilistic beliefs, papers such as Heifetz and Mongin (2001) and Meier (2012) axiomatize the class of type spaces (in which each type of a player has beliefs over the nature states and the types of the other players) in terms of players' introspection.

In contrast, since this paper studies non-probabilistic beliefs which may fail the continuity property (and this paper does not presuppose any topological structure on the underlying set S of states of nature), this paper explicitly considers transfinite levels of beliefs (in Definitions 7 and 8 in Section 3.1). Thus, this paper considers a belief hierarchy consisting of the first-order beliefs about S, the second-order beliefs, and so on, up to a pre-determined "ordinal level $\overline{\kappa}$."[12]

Hence, this subsection introduces a "logical" (precisely, a set-algebraic) structure on the collection $\mathcal{S}$ of events of nature (i.e., the objects of players' beliefs) that allows one to define interactive reasoning of transfinite levels. To that end, I introduce the following three technical definitions.

First, I denote by κ an infinite cardinal number. By the Axiom of Choice in set theory, the least ordinal number $\overline{\kappa}$ with its cardinality $\kappa = |\overline{\kappa}|$ exists. The Online Appendix provides the definitions of ordinal and cardinal numbers (ordinals and cardinals, for short).

Second, for an infinite cardinal κ, I introduce the notion of a κ-complete algebra (a κ-algebra, for short). For an underlying set X, a subset $\mathcal{X}$ of $\mathcal{P}(X)$ is a *κ-algebra* if the following four conditions are met.

1. The collection $\mathcal{X}$ contains $\emptyset$ and X: $\{\emptyset, X\} \subseteq \mathcal{X}$.
2. The collection $\mathcal{X}$ is closed under complementation: $E \in \mathcal{X}$ implies $E^c \in \mathcal{X}$. I also denote the complement of E by $\neg E$.
3. The collection $\mathcal{X}$ is closed under arbitrary union of any sub-collection with cardinality less than κ: if $E \in \mathcal{X}$ for all $E \in \mathcal{E}$ with $|\mathcal{E}| < \kappa$, then $\bigcup \mathcal{E} := \bigcup_{E \in \mathcal{E}} E \in \mathcal{X}$.
4. The collection $\mathcal{X}$ is closed under arbitrary intersection of any sub-collection with cardinality less than κ: if $E \in \mathcal{X}$ for all $E \in \mathcal{E}$ with $|\mathcal{E}| < \kappa$, then $\bigcap \mathcal{E} := \bigcap_{E \in \mathcal{E}} E \in \mathcal{X}$.

For example, an $\aleph_0$-algebra is an algebra of sets, because $\aleph_0$ is the least infinite cardinal. Similarly, an $\aleph_1$-algebra is a σ-algebra, because $\aleph_1$ is the least uncountable cardinal. When the context is clear, I also refer to $(S, \mathcal{S})$ as a κ-algebra.

Third, for any given infinite cardinal κ, I denote by $\mathcal{A}_\kappa(\cdot)$ the operation of generating the smallest κ-algebra containing a given collection. Since the intersection of a collection of κ-algebras is also a κ-algebra, for any given collection $\mathcal{X}$ of subsets of an underlying set X, the smallest κ-algebra $\mathcal{A}_\kappa(\mathcal{X})$ on X including $\mathcal{X}$ is given by:

$$\mathcal{A}_\kappa(\mathcal{X}) := \bigcap \{\mathcal{A} \in \mathcal{P}(\mathcal{P}(X)) \mid \mathcal{A} \text{ is a } \kappa\text{-algebra with } \mathcal{X} \subseteq \mathcal{A}\}.$$

[12] In fact, for any given limit ordinal number α, there exists a strategic game such that the unique prediction under iterated elimination of strictly dominated actions requites $\alpha + 1$ rounds of elimination (Fukuda, 2024b, Section 6.3.2). This implies that, when S is the set of action profiles of a strategic game, the analysts would need to examine belief hierarchies (over S) that consist of arbitrarily long transfinite levels of beliefs.

For instance, when $\kappa = \aleph_0$, $\mathcal{A}_\kappa(\cdot)$ is the operation of generating the smallest algebra including a given collection. Similarly, when $\kappa = \aleph_1$, $\mathcal{A}_\kappa(\cdot)$ is the operation of generating the smallest σ-algebra including a given collection.

With these definitions in mind, for a given set of states of nature $(S, \mathcal{S})$, I endow it with a κ-algebraic structure. Namely, I simply assume that $(S, \mathcal{S})$ is a κ-algebra for a given κ, because my arguments hold by replacing $\mathcal{S}$ with $\mathcal{A}_\kappa(\mathcal{S})$.[13]

This assumption means the following: (i) any tautology in the form of S (e.g., $E \cup E^c$) and any contradiction in the form of $\emptyset$ (e.g., $E \cap E^c$) are an event of nature (i.e., an object of players' beliefs regarding states of nature S); (ii) if E is an event of nature, then so is its complement E^c; and if E is an event of nature for each $E \in \mathcal{E}$ with $|\mathcal{E}| < \kappa$, then so are (iii) its union $\bigcup \mathcal{E}$ and (iv) its intersection $\bigcap \mathcal{E}$.

I have two further remarks regarding a κ-algebra $(S, \mathcal{S})$. First, one can accommodate the notion of mutual beliefs ("everybody believes") if one assumes $\kappa > |I|$.

Second, as mentioned in Meier (2006, Remark 1), it is without loss of generality to restrict attention to κ-algebras for infinite regular cardinals κ.[14] If an infinite cardinal κ is not regular, then any κ-algebra is indeed a κ^+-algebra, where κ^+ is the successor cardinal. Now, κ^+ is known to be regular (supposing the Axiom of Choice). Note also that $\aleph_0$ and $\aleph_1$ are regular. As it turns out in Remark 3 in Section 3.1, if one considers the set of belief hierarchies up to the ordinal level $\overline{\kappa}$, where κ is an infinite regular cardinal, then the set of belief hierarchies has a κ-algebraic structure.

With these in mind, henceforth, unless otherwise stated, fix a non-empty set of players I, an infinite regular cardinal κ, and a κ-algebra $(S, \mathcal{S})$ of states of nature.

2.2 Type Mappings

Throughout the subsection, fix a κ-algebra $(\Omega, \mathcal{D})$. Each *qualitative-type* (*type*, for short) is a mapping $\mu : \mathcal{D} \to \{0, 1\}$ (i.e., $\mu \in \{0, 1\}^{\mathcal{D}}$), where the belief of an event $E \in \mathcal{D}$ is captured by $\mu(E) = 1$. I represent each player's beliefs by a mapping, which I call a *qualitative-type mapping* (*type mapping*, for short), $t_i : \Omega \to \{0, 1\}^{\mathcal{D}}$ with the following interpretation: player i believes an event E at state ω if $t_i(\omega)(E) = 1$ with $t_i(\omega)$ being her type at ω. Note that player i does not believe an event E at state ω, i.e., $t_i(\omega)(E) < 1$, if and only if (henceforth, abbreviated as iff) $t_i(\omega)(E) = 0$.

[13] As an analogy, in the context of probabilistic beliefs, one often starts with a topological space $(S, \mathcal{S})$. One proceeds by endowing the measurable structure by generating the smallest σ-algebra $\mathcal{A}_{\aleph_1}(\mathcal{S})$ including $\mathcal{S}$. Then, $(S, \mathcal{A}_{\aleph_1}(S))$ is the measurable space of states of nature. In contrast, this paper considers an infinite cardinal κ so that, for any collection $\mathcal{S}$ of subsets of S, the space $(S, \mathcal{A}_\kappa(S))$ is a κ-algebra.

[14] Under the Axiom of Choice, an infinite cardinal κ is *regular* if, for any set A which is a union of less-than κ-many sets each of which has cardinality less than κ, the cardinality of A is less than κ: if $A = \bigcup_{j \in J} A_j$ satisfies $|J| < \kappa$ and $|A_j| < \kappa$ for all $j \in J$, then $|A| < \kappa$. For the expert reader familiar with the (equivalent) definition that an infinite cardinal κ is regular if its "cofinality" is κ, the Online Appendix briefly discusses this definition and properties of an (infinite) regular cardinal (see also the reference therein).

I define a type mapping t_i in the following three steps.

1. The first step defines the set $M(\Omega)$ of types (a subset of $\{0,1\}^{\mathcal{D}}$) that reflects given logical assumptions on beliefs. Put differently, just as $\Delta(\Omega)$ is the set of countably-additive probability measures over Ω, I formalize the set of legitimate binary "measures" $M(\Omega)$ which represents beliefs on Ω.

2. The second step defines a type mapping t_i as a mapping from Ω into $M(\Omega)$.

3. The third step introduces introspective properties of beliefs.

2.2.1 Logical Properties of Qualitative Beliefs

The first step defines $M(\Omega)$ as a subset of $\{0,1\}^{\mathcal{D}}$ based on given properties of beliefs, as well as a κ-algebra $\mathcal{M}(\mathcal{D})$ on $M(\Omega)$.

I start by defining the following five logical properties of beliefs in terms of types.

Definition 1 (Logical Properties of Qualitative Beliefs)**.** *Fix* $\mu \in \{0,1\}^{\mathcal{D}}$. *Then:*

1. *No-Contradiction:* $\mu(\emptyset) = 0$.

2. *Consistency:* $\mu(E) \leq 1 - \mu(E^c)$ *for all* $E \in \mathcal{D}$.

3. *Monotonicity:* $\mu(E) \leq \mu(F)$ *for all* $E, F \in \mathcal{D}$ *with* $E \subseteq F$.

4. λ*-Conjunction (where* λ *is an infinite (regular) cardinal with* $\lambda \leq \kappa$*): for all* $\mathcal{E} \subseteq \mathcal{D}$ *with* $0 < |\mathcal{E}| < \lambda$, $\min_{E \in \mathcal{E}} \mu(E) \leq \mu(\bigcap \mathcal{E})$.

5. *Necessitation:* $\mu(\Omega) = 1$.

The logical properties of μ are analogous to the corresponding logical properties of probabilistic beliefs. First, μ satisfies No-Contradiction if a contradiction of the form $\emptyset$ is never believed. This is similar to the corresponding axiom of probability.

Second, Consistency means that if an event E is believed (i.e., $\mu(E) = 1$) then its negation is not believed (i.e., $\mu(E^c) = 0$). Unlike the additivity of probability, however, the converse may not hold. That is, it is possible that a player does not believe an event E nor its negation E^c at a particular state: $\mu(E) + \mu(E^c) = 0$.

Third, Monotonicity is similar to the corresponding property of probability: if E is believed and E implies F (in the sense of $E \subseteq F$) then F is believed.

Fourth, letting λ be an infinite (regular) cardinal with $\lambda \leq \kappa$, λ-Conjunction states that, for a non-empty collection of events $\mathcal{E}$ with $|\mathcal{E}| < \lambda$, if each event $E \in \mathcal{E}$ is believed then its intersection (i.e., its conjunction) $\bigcap \mathcal{E}$ is believed.[15] When $\lambda = \aleph_0$,

[15] Note that λ-Conjunction does not require the converse (i.e., if $\bigcap \mathcal{E}$ is believed, then each event $E \in \mathcal{E}$ is believed). Indeed, the converse is equivalent to Monotonicity, which is a conceptually different property.

$\aleph_0$-Conjunction is equivalent to the property that if E and F are believed then its intersection $E \cap F$ is believed. Likewise, when $\lambda = \aleph_1$, $\aleph_1$-Conjunction is equivalent to the property that if E_n is believed for each $n \in \mathbb{N}$ then its intersection $\bigcap_{n \in \mathbb{N}} E_n$ is believed. For instance, the notion of probability-one belief satisfies $\aleph_1$-Conjunction.

Fifth, Necessitation states that a tautology of the form Ω is always believed, similarly to the corresponding axiom of probability.

2.2.2 A Player's Type Mapping

The second step defines a type mapping t_i on Ω in three sub-steps. To that end, recall that, in the case of probabilistic beliefs, a type mapping is a measurable mapping from Ω to $\Delta(\Omega)$. Thus, the first sub-step defines the set of types $M(\Omega)$, which corresponds to $\Delta(\Omega)$. The second sub-step defines a κ-algebra $\mathcal{M}(\mathcal{D})$ on $M(\Omega)$. This corresponds to a σ-algebra on $\Delta(\Omega)$ in the case of probabilistic beliefs. The third sub-step defines a type mapping as a measurable mapping from Ω to $M(\Omega)$.

The first sub-step thus defines $M(\Omega)$, which is the set of mappings $\mu \in \{0,1\}^{\mathcal{D}}$ which satisfy a given combination of properties in Definition 1.[16] For instance, if one assumes all the properties in Definition 1, then

$$M(\Omega) = \{\mu \in \{0,1\}^{\mathcal{D}} \mid \mu \text{ satisfies Definition 1}\}.$$

The second sub-step introduces a κ-algebra $\mathcal{M}(\mathcal{D})$ on $M(\Omega)$. In an analogous way to the standard type-space approach (e.g., Heifetz and Samet, 1998b), I define $\mathcal{M}(\mathcal{D})$ to be the κ-algebra generated by the sets of the form

$$\mathcal{M}_E := \{\mu \in M(\Omega) \mid \mu(E) = 1\} \text{ for some } E \in \mathcal{D}.$$

Thus, the κ-algebra $\mathcal{M}(\mathcal{D})$ on $M(\Omega)$ is defined as

$$\mathcal{M}(\mathcal{D}) := \mathcal{A}_\kappa(\{\mathcal{M}_E \in \mathcal{P}(M(\Omega)) \mid E \in \mathcal{D}\}).$$

Slightly abusing the notation, I sometimes write $\mathcal{M}^{\mathcal{D}}(E) := \mathcal{M}_E$ when the set E is notationally convoluted (e.g., when E has a super- or sub-script).

The third sub-step then defines a *type mapping* as a measurable mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{M}(\mathcal{D}))$. The measurablity condition of t_i means that the set $t_i^{-1}(\mathcal{M}_E)$ (i.e., the set of states at which player i believes an event E) is an event. Formally, $t_i : \Omega \to M(\Omega)$ is a type mapping if, for all $E \in \mathcal{D}$,

$$t_i^{-1}(\mathcal{M}_E) = t_i^{-1}(\{\mu \in M(\Omega) \mid \mu(E) = 1\}) \in \mathcal{D}.$$

I remark that the logical properties imposed on types are inherited on the type mapping t_i in the sense that each $t_i(\omega) \in M(\Omega)$ satisfies the the logical properties of

[16] In defining $M(\Omega)$, one can consider any possible combination of properties from Definition 1. Note, however, that some combination of properties implies other properties. For instance, Consistency and Necessitation imply No-Contradiction.

beliefs that are imposed on $M(\Omega)$. For each property in Definition 1, I say that the type mapping t_i satisfies the given property if $t_i(\omega)$ satisfies the property for all $\omega \in \Omega$. For example, t_i satisfies *No-Contradiction* if $t_i(\omega)$ satisfies it (i.e., $t_i(\omega)(\emptyset) = 0$) for all $\omega \in \Omega$.

2.2.3 Introspective Properties of Qualitative Beliefs

The third step defines introspective properties of beliefs. In the context of probabilistic beliefs, the idea that each player is certain of her own beliefs is formulated as the following two requirements: (i) if the player believes an event E with probability at least p, then she believes, with probability one, that she believes E with probability at least p; and (ii) if the player does not believe an event E with probability at least p, then she believes, with probability one, that she does not believe E with probability at least p (e.g., Heifetz and Mongin, 2001; Heifetz and Samet, 1998b; Meier, 2012; Mertens and Zamir, 1985).

In the context of qualitative beliefs, these reduce to (i) *Positive Introspection* (i.e., if a player believes an event E, then she believes that she believes E); and (ii) *Negative Introspection* (i.e., if a player does not believe an event E, then she believes that she does not believe E).

Since I study qualitative beliefs, I also consider the case in which qualitative beliefs satisfy the axiom of knowledge. That is, knowledge satisfies *Truth Axiom*: if a player knows an event E at a state, then E has to hold true at that state. Truth Axiom distinguishes knowledge from qualitative beliefs in the sense that the player can believe an event E at a state at which the event E does not hold true.

I also consider a condition—*Kripke Property*—under which qualitative beliefs are induced from a possibility correspondence. The possibility correspondence is a mapping $b_i : \Omega \to \mathcal{P}(\Omega)$ such that player i believes an event E at a state ω iff

$$b_i(\omega) \subseteq E$$

in the sense that E is implied by the set of states $b_i(\omega)$ that player i considers possible at ω. The set $b_i(\omega)$ corresponds to the intersection of the events E that player i believes at ω.

Only for ease of exposition, I simply refer to these four properties as introspective properties as opposed to logical properties. Below, I formalize the introspective properties of a type mapping.

Definition 2 (Introspective Properties of Qualitative Beliefs). *Let $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{M}(\mathcal{D}))$ be a type mapping. Then:*

1. *Truth Axiom: for any $(\omega, E) \in \Omega \times \mathcal{D}$, if $t_i(\omega)(E) = 1$ then $\omega \in E$.*
2. *Positive Introspection: for any $(\omega, E) \in \Omega \times \mathcal{D}$, if $t_i(\omega)(E) = 1$, then*

$$t_i(\omega)(\underbrace{\{\omega' \in \Omega \mid t_i(\omega')(E) = 1\}}_{=t_i^{-1}(\mathcal{M}_E)}) = 1.$$

3. *Negative Introspection: for any* $(\omega, E) \in \Omega \times \mathcal{D}$*, if* $t_i(\omega)(E) = 0$*, then*

$$t_i(\omega)(\underbrace{\{\omega' \in \Omega \mid t_i(\omega')(E) = 0\}}_{=\neg t_i^{-1}(\mathcal{M}_E)}) = 1.$$

4. *Kripke Property: for any* $(\omega, E) \in \Omega \times \mathcal{D}$*,* $t_i(\omega)(E) = 1$ *if (and only if)*

$$b_{t_i}(\omega) \subseteq E, \text{ where } b_{t_i}(\omega) := \bigcap\{F \in \mathcal{D} \mid t_i(\omega)(F) = 1\}.$$

Henceforth, fix any possible combination of logical and introspective properties of beliefs.[17] For any given combination of logical and introspective properties of beliefs, I construct the universal belief space. Section 2.3 defines belief spaces. Section 2.4 defines the universal belief space.

2.3 Belief Spaces

Section 2.3.1 defines a model of players' beliefs, a belief space. I represent players' interactive beliefs regarding $(S, \mathcal{S})$ on a "sample space" $(\Omega, \mathcal{D})$ by type mappings. Section 2.3.2 discusses the definition of a belief space. Section 2.3.3 provides examples.

2.3.1 The Definition of a Belief Space

Definition 3 (Belief Space)**.** *A* κ*-belief space of* I *on* $(S, \mathcal{S})$ *is a tuple* $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (t_i)_{i \in I}, \Theta\rangle$ *with the following three properties.*

1. $(\Omega, \mathcal{D})$ *is a* κ*-algebra.* Ω *is the set of states of the world. Each* $E \in \mathcal{D}$ *is an event (of the world).*

2. *For each* $i \in I$*,* $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{M}(\mathcal{D}))$ *is player* i*'s type mapping that satisfies the given logical and introspective properties of beliefs. It satisfies the measurability condition:*

$$t_i^{-1}(\mathcal{M}_E) = \{\omega \in \Omega \mid t_i(\omega)(E) = 1\} \in \mathcal{D} \text{ for each } E \in \mathcal{D}. \tag{1}$$

   Player $i \in I$ *believes an event* $E \in \mathcal{D}$ *at a state* $\omega \in \Omega$ *if* $t_i(\omega)(E) = 1$*.*

3. $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{S})$ *is a measurable mapping:* $\Theta^{-1}(E) \in \mathcal{D}$ *for any* $E \in \mathcal{S}$*.*

So far, I have defined belief spaces (Definition 3) in addition to the properties of beliefs (Definitions 1 and 2). The main result of this paper is to construct the universal belief space for any (non-empty) set I of players, any infinite (regular) cardinal κ,

[17] As discussed in Footnote 16, note that some properties of beliefs may imply others. For example, Truth Axiom implies No-Contradiction and Consistency. Also, Negative Introspection together with Truth Axiom imply Positive Introspection (e.g., Aumann, 1999, p. 270).

any κ-algebra $(S, \mathcal{S})$ of nature states, and any combination of properties of beliefs given in Definitions 1 and 2. Given such inputs (i.e., I, κ, $(S, \mathcal{S})$, and the properties of beliefs), one can define the class of κ-belief spaces $\overrightarrow{\Omega}$ of I on $(S, \mathcal{S})$ satisfying the given properties of beliefs. Since such inputs are fixed, I simply refer to $\overrightarrow{\Omega}$ as a *belief space*.

2.3.2 Remarks on the Definition

Five remarks on Definition 3 are in order. First, while any subset of underlying states of the world is deemed to be an event in standard partitional knowledge models (i.e., $\mathcal{D} = \mathcal{P}(\Omega)$), my framework is more general. As in Fukuda (2024b), this specification (of a κ-algebra) leads to the existence of the universal belief space.

Second, players' beliefs are represented through their type mappings instead of their belief operators.

Remark 1 (Belief Operator)**.** Player i's belief operator $B_i : \mathcal{D} \to \mathcal{D}$ associates, with each event $E \in \mathcal{D}$, the event that (i.e., the set of states at which) player i believes E. If player i's type mapping t_i is given, then one can define

$$B_{t_i}(E) := \{\omega \in \Omega \mid t_i(\omega)(E) = 1\} \text{ for each } E \in \mathcal{D},$$

where $B_{t_i}(E)$ is an event by Condition (1) of Definition 3. Conversely, if B_i is given, then one can define t_{B_i} as

$$t_{B_i}(\omega)(E) = 1 \text{ if } \omega \in B_i(E).$$

By definition, $t_i = t_{B_{t_i}}$ and $B_i = B_{t_{B_i}}$. Appendix A.1 shows that the properties of beliefs given in Definitions 1 and 2 are represented by well-known properties of the belief operator B_i. Thus, any belief operator on a κ-algebra can be equivalently represented by the corresponding type mapping.[18]

This means that notions derived from player i's belief operator B_i such as believing whether $J_{B_i}(E) := B_i(E) \cup B_i(E^c)$ (i.e., player i believes whether E holds if she believes E or she believes E^c) can also be expressed from her type mapping t_{B_i}: player i believes whether E holds at ω (i.e., $\omega \in J_{B_i}(E)$) iff $\max(t_{B_i}(\omega)(E), t_{B_i}(\omega)(E^c)) = 1$.

[18]As a remark, while this paper focuses on qualitative beliefs, the framework can also accommodate various forms of probabilistic beliefs defined on a κ-algebra as long as players' beliefs are represented through p-belief operators (Monderer and Samet, 1989): a player p-believes an event at a state if she believes the event at the state with probability at least p. The player possesses a collection of type mappings for each $p \in [0, 1]$, and the type (that corresponds to p) assigns the value of 1 when the player p-believes the event. Examples of various forms of probabilistic beliefs include, but are not limited to, countably-additive beliefs (Samet, 2000), finitely-additive beliefs (Meier, 2006), non-additive beliefs, and conditional beliefs (e.g., Di Tillio, J. Halpern, and Samet, 2014; Guarino, 2017, 2025; Tsakas, 2014a).

Third, within a belief space, the assumption that κ is an infinite cardinal implies that the players are able to interactively reason about their beliefs of any finite order without any bound. To see this, each player i's belief about an event $E \in \mathcal{S}$ of nature is represented as an event $B_{t_i}(\Theta^{-1}(E)) \in \mathcal{D}$ within the belief space $\overrightarrow{\Omega}$. Player i's belief about j's belief about the event E of nature is represented as an event $B_{t_i}B_{t_j}(\Theta^{-1}(E)) \in \mathcal{D}$. For any finite $k \in \mathbb{N}$, player i_1's belief about player i_2's belief ... about player i_k's belief about the event E of nature is represented as

$$B_{t_{i_1}}B_{t_{i_2}}\cdots B_{t_{i_k}}(\Theta^{-1}(E)) \in \mathcal{D}.$$

Thus, this paper differs from the literature that aims at capturing "bounded-depth reasoning" within a type space (e.g., Heifetz and Kets, 2018; Kets, 2017; Strzalecki, 2014), where player i has some limitations on reasoning above some level $k \in \mathbb{N}$.

Fourth, this framework can nest various classes of (qualitative) belief spaces in the previous literature.

Remark 2 (Various Belief Models in the Literature)**.**

1. The class of κ-belief spaces satisfying the properties in Definitions 1 and 2 is the class of partitional knowledge spaces (e.g., Aumann, 1976) defined on a κ-algebra.

2. If one drops Negative Introspection, then the class of κ-belief spaces satisfying possibly all but Negative Introspection is the class of non-partitional knowledge spaces (e.g., Bacharach, 1985; Brandenburger, Dekel, and Geanakoplos, 1992; Geanakoplos, 2021; Morris, 1996; Rubinstein and Wolinsky, 1990; Samet, 1990; Shin, 1993) defined on a κ-algebra.

3. If one drops Truth Axiom from Definitions 1 and 2, then the class of κ-belief spaces satisfying possibly all but Truth Axiom is the class of qualitative belief spaces (e.g., Bonanno, 2008, 2015; Bonanno and Tsakas, 2018; Hillas and Samet, 2020; Samet, 2013) defined on a κ-algebra.

Fifth, Tsakas (2014b,c) studies rational beliefs, which are Borel probability measures that assign a rational probability to every Borel event. The notions of rational beliefs and qualitative beliefs are not generalizations of one another. Consider the special case of "1-rational beliefs," i.e., consider the set $\Delta^1(\Omega)$ of probability measures on $(\Omega, \mathcal{D})$ which assign binary values (i.e., 0 or 1). Then, each element $\mu \in \Delta^1(\Omega)$ is assumed to satisfy the logical properties of beliefs in Definition 1. Especially, since μ is countably additive, it satisfies $\aleph_1$-Conjunction, which can be restated as follows: if $\mu(E_n) = 1$ for all $n \in \mathbb{N}$, then $\mu(\bigcap_{n\in\mathbb{N}} E_n) = 1$.[19] In contrast, in this paper, one can define the set $M(\Omega)$ of qualitative beliefs on $(\Omega, \mathcal{D})$ in a way such that some $\mu \in M(\Omega)$ may violate some properties in Definition 1 (e.g., one can consider "1-rational finitely-additive beliefs"). Also, Truth Axiom or Kripke Property

[19]In fact, given the lack of this continuity property, Section 3.1 considers belief hierarchies of transfinite levels. This also differs from the definition of belief hierarchies in Tsakas (2014b,c).

	s^1	s^2
s^1	$2, 2$	$0, 3$
s^2	$3, 0$	$1, 1$

Table 1: The Set of States of Nature $S := \{s^1, s^2\}^2$.

in Definition 2 may be incompatible with probabilistic beliefs (e.g., Fukuda, 2024a; Monderer and Samet, 1989).

2.3.3 Examples

I provide two simple examples of a belief space, where $S := \{s^1, s^2\}^2$ is the set of action profiles for Prisoners' Dilemma as depicted by Table 1. Thus, s^1 corresponds to cooperation, while s^2 defection. Two players, $I := \{1, 2\}$, are reasoning about any subsets of S: $\mathcal{S} := \mathcal{P}(S)$. I consider the class of κ-belief spaces of I on $(S, \mathcal{S})$, where κ is an infinite (regular) cardinal and beliefs are assumed to satisfy the properties in Definitions 1 and 2. To anticipate the definition of belief hierarchies in Section 3.1, I briefly outline how a given belief space induces belief hierarchies over S.

Example 1. First, let $\Omega := S$ and $\mathcal{D} := \mathcal{S}$, i.e., the set of states of the world is given by the set of action profiles, and any subset of S is an event (of the world) about which the players hold beliefs. The space $(\Omega, \mathcal{D})$ is a κ-algebra.

Second, before defining the players' type mappings, let $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{S})$ be the identity map. By definition, Θ is measurable. For any $\omega = (\omega_1, \omega_2) \in \Omega$, denote $\Theta(\omega) = (\Theta_1(\omega_1), \Theta_2(\omega_2))$. Thus, $(\Theta_1(\omega_1), \Theta_2(\omega_2)) = (\omega_1, \omega_2)$.

Third, suppose that, when each player i takes action $s_i \in \{s^1, s^2\}$ (i.e., at state $\omega = (\omega_1, \omega_2)$ at which $\Theta_i(\omega_i) = s_i$), she believes that her action is s_i. Formally, each player i's type mapping t_i is given as follows: for each $(\omega, F) \in \Omega \times \mathcal{D}$,

$$t_i(\omega)(F) := \begin{cases} 1 & \text{if } \{\tilde{\omega} \in \Omega \mid \tilde{\omega}_i = \omega_i\} \subseteq F \\ 0 & \text{otherwise} \end{cases}.$$

The belief space $\overrightarrow{\Omega}$ reduces to a standard partitional knowledge model, where each player's partition cell at $\omega \in \Omega$ is given by $\{\tilde{\omega} \in \Omega \mid \tilde{\omega}_i = \omega_i\}$: at ω, player i considers any $\tilde{\omega}$ with $\tilde{\omega}_i = \omega_i$ possible. Thus, it can be seen that t_i satisfies the properties in Definitions 1 and 2.

Since $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{S})$ is measurable, Θ^{-1} associates, with each event of nature $E \in \mathcal{S}$, the corresponding event $\Theta^{-1}(E) \in \mathcal{D}$. Thus, each player i's first-order beliefs about S at state $\omega \in \Omega$ are given by

$$h_i^1(\omega)(\cdot) := t_i(\omega)(\Theta^{-1}(\cdot)) \in H_i^1 := M(S).$$

By the definition of $t_i(\omega)$, when each player i takes action $s_i \in \{s^1, s^2\}$, she believes that she takes s_i (and vice versa). Using player i's belief operator, one can express it as:

$$\{\omega \in \Omega \mid \Theta_i(\omega_i) = s_i\} = B_{t_i}(\{\omega \in \Omega \mid \Theta_i(\omega_i) = s_i\}).$$

Next, I consider each player i's second-order beliefs about the nature states S and the first-order beliefs of the players $M(S)^I$ (i.e., each player i's second-order beliefs about the set of the players' belief hierarchies of order up to 1). Let

$$H^1 := S \times \prod_{i \in I} H_i^1, \text{ i.e., } H^1 = S \times M(S)^I$$

be the set of the players' belief hierarchies of order up to 1. Letting $h^1 := (\Theta, (h_i^1)_{i \in I}) : \Omega \to H^1$ be the mapping that associates, with each state, the players' belief hierarchies of order up to 1 at that state, player i's second-order beliefs about H^1 at state ω are given by

$$t_i(\omega)((h^1)^{-1}(\cdot)) \in M(H^1).$$

Using the players' belief operators, I represent player j's belief about player i's belief about i's own action (where $j \neq i$). Namely, player j does not believe that player i takes s_i because

$$\emptyset = B_{t_j} \underbrace{B_{t_i}(\{\omega \in \Omega \mid \Theta_i(\omega_i) = s_i\})}_{=\{\omega \in \Omega \mid \Theta_i(\omega_i) = s_i\}}.$$

In this way, the belief space $\overrightarrow{\Omega}$ induces each player i's (first-order) beliefs about the states of nature S, her (second-order) beliefs about the states of nature S and the (first-order) beliefs of the players about S, and so on. Section 3.1 defines the set of belief hierarchies, and constructs the universal belief space that contains all possible belief hierarchies (the formal definition of the universal belief space is given in Section 2.4).

Example 2. First, let $\Omega := \{(s^2, s^2)\}$ and $\mathcal{D} := \mathcal{P}(\Omega)$, i.e., the set of states of the world is given by the singleton action profile (s^2, s^2) of defection, and $\mathcal{D} = \{\emptyset, \Omega\}$ is a "degenerate" κ-algebra.

Second, before defining the players' type mappings, let $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{S})$ be the inclusion mapping: $\Theta(s^2, s^2) = (s^2, s^2)$. By definition, Θ is measurable: for each $E \in \mathcal{S}$,

$$\Theta^{-1}(E) = \begin{cases} \emptyset & \text{if } (s^2, s^2) \notin E \\ \Omega & \text{if } (s^2, s^2) \in E \end{cases}.$$

Third, each player i's type mapping t_i is given as follows: for each $(\omega, F) \in \Omega \times \mathcal{D}$,

$$t_i(\omega)(F) := \begin{cases} 1 & \text{if } F = \Omega \\ 0 & \text{if } F = \emptyset \end{cases}.$$

The belief space $\overrightarrow{\Omega}$ reduces to a standard partitional knowledge model, where each player's partition is the degenerate partition $\{\Omega\}$. Thus, it can be seen that t_i satisfies the properties in Definitions 1 and 2.

At the state $\omega = (s^2, s^2)$ at which both players take s^2, every player believes that both take s^2. This is because, for any $E \in \mathcal{S}$ with $(s^2, s^2) \in E$,

$$\underbrace{h_i^1(\omega)}_{\in M(S)}(E) := t_i(\omega)(\underbrace{\Theta^{-1}(E)}_{=\Omega}) = 1.$$

In terms of the belief operator,

$$B_{t_1}(\underbrace{\{\omega \in \Omega \mid \omega_1 = \omega_2 = s^2\}}_{=\Omega}) = B_{t_2}(\underbrace{\{\omega \in \Omega \mid \omega_1 = \omega_2 = s^2\}}_{=\Omega}) = \Omega.$$

Thus, at $\omega = (s^2, s^2)$, every player believes that both players take s^2, every player believes that every player believes that both players take s^2, and so forth ad infinitum. The belief spaces in Examples 1 and 2 induce different belief hierarchies over S. The universal belief space constructed in Section 3.1 contains belief hierarchies induced by these belief spaces, as it contains all possible belief hierarchies.

2.4 The Universal Belief Space

In this subsection, I define the universal belief space as a belief space to which every belief space is uniquely mapped in a belief-preserving manner. I start by formalizing the notion of a mapping that preserves states of nature and players' beliefs, i.e., the notion of a belief morphism between belief spaces.

Definition 4 (Belief Morphism)**.** *Let* $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (t_i)_{i\in I}, \Theta\rangle$ *and* $\overrightarrow{\Omega'} := \langle(\Omega', \mathcal{D}'), (t'_i)_{i\in I}, \Theta'\rangle$ *be belief spaces. A belief morphism* $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ *is a measurable mapping* $\varphi : (\Omega, \mathcal{D}) \to (\Omega', \mathcal{D}')$ *with the following two additional properties.*

1. *For all* $\omega \in \Omega$*,* $\Theta'(\varphi(\omega)) = \Theta(\omega)$*.*
2. *For each* $i \in I$*,* $\omega \in \Omega$*, and* $E' \in \mathcal{D}'$*,* $t'_i(\varphi(\omega))(E') = t_i(\omega)(\varphi^{-1}(E'))$*.*

Condition (1) requires that the same state of nature prevail for two associated belief spaces. Condition (2) requires that players' beliefs be preserved from one space to another in the following sense: for any event $E' \in \mathcal{D}'$, player i believes E' at $\varphi(\omega)$ iff she believes $\varphi^{-1}(E')$ at ω.

Two remarks are in order. First, for any given belief space $\overrightarrow{\Omega}$, the identity map $\mathrm{id}_\Omega : \Omega \to \Omega$ is a belief morphism from $\overrightarrow{\Omega}$ into itself.

Second, I call a belief morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ a *belief isomorphism*, if φ is bijective and its inverse φ^{-1} is a belief morphism. If this is the case, then the belief spaces $\overrightarrow{\Omega}$ and $\overrightarrow{\Omega'}$ are *isomorphic*.

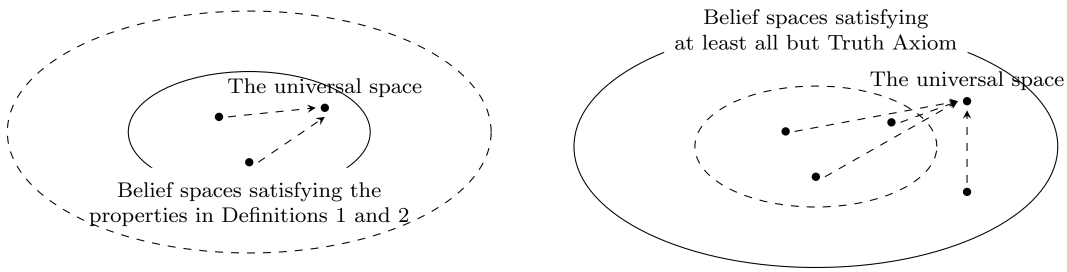

Figure 1: The Universal Belief Space within a Class of Belief Spaces.

Now, I define the universal belief space. I formalize below the idea that the universal belief space "contains" all belief spaces in the sense that any belief space can be mapped uniquely to the universal space by a belief morphism.

Definition 5 (Universal Belief Space)**.** *A belief space $\overrightarrow{\Omega^*}$ is universal if, for any belief space $\overrightarrow{\Omega}$, there is a unique belief morphism $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$.*

This definition follows that of Heifetz and Samet (1998b) and, more recently, that of Meier and Perea (2025). In the language of category theory, the universal belief space is a terminal object in the category of belief spaces. As it is well-known in category theory, a terminal object is unique up to isomorphism. Thus, the universal belief space is unique up to belief isomorphism. Thus, one can indeed speak of *the* universal belief space.

Figure 1 illustrates Definition 5. The left panel depicts the class of (κ-)belief spaces where beliefs satisfy the properties in Definition 1 and 2 (i.e., the class of partitional knowledge spaces defined on a κ-algebra) and its universal space to be constructed. The belief spaces defined in Examples 1 and 2 belong to this class of belief spaces. The right panel depicts the class of (κ-)belief spaces where beliefs do not necessarily satisfy Truth Axiom (otherwise all the other properties are still imposed) and the universal space within the class. Since the underlying assumptions on beliefs are relaxed, the class of belief spaces is enlarged. As the figure illustrates, fixing κ, I, and $(S, \mathcal{S})$, for any given combination of properties of beliefs, Theorem 1 in Section 3.1 constructs the universal belief space which may vary depending on the properties of beliefs.

3 Main Result

This section constructs the universal belief space as the set of all possible belief hierarchies. I follow Heifetz and Samet (1998b, Section 5)'s hierarchical approach to establishing the universal type space. Since I extensively use product κ-algebras, I formally define the product κ-algebra of a collection of κ-algebras.

Definition 6 (Product κ-algebra)**.** *Let $(\Omega_j, \mathcal{D}_j)$ be a κ-algebra for each $j \in J$, where J is a non-empty index set. Letting $\pi_j : \prod_{j\in J} \Omega_j \to \Omega_j$ be the projection for each $j \in J$, the product κ-algebra $\prod_{j\in J} \mathcal{D}_j$ on the product space $\prod_{j\in J} \Omega_j$ is defined as*

$$\prod_{j\in J} \mathcal{D}_j := \mathcal{A}_\kappa \left(\bigcup_{j\in J} \{\pi_j^{-1}(E) \mid E \in \mathcal{D}_j\} \right).$$

As usual, if $\mathcal{D} = \mathcal{D}_j$ for all $j \in J$, then I denote $\mathcal{D}^J := \prod_{j\in J} \mathcal{D}_j$. I also define finite products such as $\mathcal{D}_1 \times \mathcal{D}_2 = \prod_{j\in\{1,2\}} \mathcal{D}_j$ in the usual manner.

3.1 A Hierarchical Construction of the Universal Belief Space

I construct the universal belief space in four steps. The first step defines the hierarchies space, the space of the players' belief hierarchies over $(S, \mathcal{S})$ up to the ordinal level $\overline{\kappa}$.

Definition 7 (Hierarchies Space)**.** *I define the sequence of κ-algebras $(H^\alpha, \mathcal{H}^\alpha)$ for ordinals α with $0 \leq \alpha \leq \overline{\kappa}$ as follows.*

1. *For $\alpha = 0$, let $(H^0, \mathcal{H}^0) := (S, \mathcal{S})$.*
2. *For any ordinal α with $0 < \alpha < \overline{\kappa}$, let*

$$(H^\alpha, \mathcal{H}^\alpha) := \left(S \times \prod_{\beta<\alpha} M(H^\beta)^I, \mathcal{S} \times \prod_{\beta<\alpha} \mathcal{M}(\mathcal{H}^\beta)^I \right). \tag{2}$$

3. *For $\alpha = \overline{\kappa}$, define the hierarchies space $(H^{\overline{\kappa}}, \mathcal{H}^{\overline{\kappa}})$ as:*

$$H^{\overline{\kappa}} := S \times \prod_{\alpha<\overline{\kappa}} M(H^\alpha)^I \text{ and} \tag{3}$$

$$\mathcal{H}^{\overline{\kappa}} := \{(\pi^{\overline{\kappa},\alpha})^{-1}(E^\alpha) \mid E^\alpha \in \mathcal{H}^\alpha \text{ for some } \alpha < \overline{\kappa}\}, \tag{4}$$

where, for any ordinals (α, β) with $0 \leq \beta \leq \alpha \leq \overline{\kappa}$, denote by

$$\pi^{\alpha,\beta} : H^\alpha \to H^\beta$$

the projection.

For each ordinal $\alpha \leq \overline{\kappa}$, call a measurable space $(H^\alpha, \mathcal{H}^\alpha)$ the space of the players' belief hierarchies of order up to α. When $\alpha = \overline{\kappa}$, I omit the superscript $\overline{\kappa}$ from $H^{\overline{\kappa}}$, $\mathcal{H}^{\overline{\kappa}}$, and $\pi^{\overline{\kappa},\alpha}$. Thus, I denote

$$(H, \mathcal{H}) := (H^{\overline{\kappa}}, \mathcal{H}^{\overline{\kappa}}).$$

I remark on Expression (2) when α is a successor ordinal $\alpha = \beta + 1$. Then, the space $(H^\alpha, \mathcal{H}^\alpha)$ of the players' belief hierarchies of order up to α consists of the space $(H^\beta, \mathcal{H}^\beta)$ of the players' belief hierarchies of order up to β and their beliefs over $(H^\beta, \mathcal{H}^\beta)$:

$$\begin{aligned}(H^\alpha, \mathcal{H}^\alpha) &= (H^\beta \times M(H^\beta)^I, \mathcal{H}^\beta \times \mathcal{M}(\mathcal{H}^\beta)^I) \\ &= \left(S \times \prod_{\gamma<\alpha} M(H^\gamma)^I, \mathcal{S} \times \prod_{\gamma<\alpha} \mathcal{M}(\mathcal{H}^\gamma)^I \right),\end{aligned}$$

where the right-most side coincides with Expression (2).

With this in mind, I discuss Definition 7. Starting with $(H^0, \mathcal{H}^0) = (S, \mathcal{S})$, the space

$$(H^1, \mathcal{H}^1) = (S \times M(S)^I, \mathcal{S} \times \mathcal{M}(\mathcal{S})^I)$$

of the players' belief hierarchies of order up to 1 consists of their "0-th order beliefs" (i.e., states of nature) S and their first-order beliefs about S, i.e., $M(S)^I$, endowed with the corresponding κ-algebra. When $\alpha = 2$, the space

$$(H^2, \mathcal{H}^2) = (S \times M(\underbrace{S}_{=H^0})^I \times M(\underbrace{S \times M(S)^I}_{=H^1})^I, \mathcal{S} \times \mathcal{M}(\underbrace{\mathcal{S}}_{=\mathcal{H}^0})^I \times \mathcal{M}(\underbrace{\mathcal{S} \times \mathcal{M}(\mathcal{S})^I}_{=\mathcal{H}^1})^I)$$

of the players' belief hierarchies of order up to 2 consists of their "0-th order beliefs," their first-order beliefs about S, and their second-order beliefs about the nature states S and their own first-order beliefs $S \times M(S)^I$, endowed with the corresponding κ-algebra. In this way, $(H^\alpha, \mathcal{H}^\alpha)$ is defined for any finite ordinal $\alpha \in \mathbb{N} \cup \{0\}$.

If κ is the least infinite cardinal (i.e., $\overline{\kappa}$ corresponds to the least infinite ordinal, that is, the set of non-negative integers), then the hierarchy space $H^{\overline{\kappa}}$ defined by Expression (3) consists of the set S of states of nature (the "0-th order beliefs") and the set $M(H^\alpha)^I$ of the players' beliefs over H^α (i.e., "$(\alpha+1)$-th order beliefs") for all $\alpha \in \mathbb{N} \cup \{0\}$. Also, the hierarchy space $H^{\overline{\kappa}}$ is endowed with a κ-algebra (i.e., an algebra) $\mathcal{H}^{\overline{\kappa}}$ defined by Expression (4).[20]

More generally, when κ is an infinite regular cardinal, the hierarchy space $H^{\overline{\kappa}}$ defined by Expression (3) consists of the set S of states of nature (the "0-th order beliefs") and the set $M(H^\alpha)^I$ of the players' beliefs over H^α for all ordinals α less than $\overline{\kappa}$, endowed with $\mathcal{H}^{\overline{\kappa}}$ defined by Expression (4).

A key mathematical observation behind Definition 7 is that, when one defines the players' belief hierarchies up to the ordinal level $\overline{\kappa}$, the collection $\mathcal{H}^{\overline{\kappa}}$ on $H^{\overline{\kappa}}$ defined by Expression (4) forms a κ-algebra. Formally:

Remark 3 (Hierarchies Space)**.** The hierarchies space $(H, \mathcal{H})$ is a κ-algebra.

[20]Thus, the hierarchy space $(H^{\overline{\kappa}}, \mathcal{H}^{\overline{\kappa}})$ defined by Expressions (3) and (4) is consistent with Expression (2) in the following sense: if we let $\alpha = \overline{\kappa}$, then the hierarchy space $(H^{\overline{\kappa}}, \mathcal{H}^{\overline{\kappa}})$ coincides with the space $(H^\alpha, \mathcal{H}^\alpha)$ defined by Expression (2) (with $\alpha = \overline{\kappa}$).

To examine Remark 3, if we let $\alpha = \overline{\kappa}$ in Expression (2), then the collection $\mathcal{H}^{\overline{\kappa}}$ is defined as the product κ-algebra on $H^{\overline{\kappa}}$:

$$\mathcal{A}_\kappa(\{(\pi^\alpha)^{-1}(E^\alpha) \mid E^\alpha \in \mathcal{H}^\alpha \text{ for some } \alpha < \overline{\kappa}\}).$$

Thus, Remark 3 states that, if one defines the players' belief hierarchies up to the ordinal level $\overline{\kappa}$, the lower-order belief hierarchies $(H^\alpha)_{\alpha<\overline{\kappa}}$ determine the κ-algebra $\mathcal{H}^{\overline{\kappa}}$, i.e., the operation of $\mathcal{A}_\kappa$ is not needed.

Instead, consider the case of probabilistic beliefs. In the framework of this paper, this corresponds to the case where $\kappa = \aleph_1$, as the domain of each belief space is a σ-algebra. However, in the literature on (probabilistic) type spaces, one usually considers the sequence $(H^\alpha, \mathcal{H}^\alpha)_{\alpha \in \overline{\aleph_0}}$ of finite-order beliefs, where $\overline{\aleph_0} = \mathbb{N} \cup \{0\}$.[21] Then, one defines the σ-algebra (i.e., the $\aleph_1$-algebra) $\mathcal{A}_{\aleph_1}(\mathcal{H}^{\overline{\aleph_0}})$ from an $\aleph_0$-algebra (i.e., an algebra)

$$\mathcal{H}^{\overline{\aleph_0}} = \{(\pi^{\overline{\aleph_0},\alpha})^{-1}(E^\alpha) \mid E^\alpha \in \mathcal{H}^\alpha \text{ for some } \alpha < \overline{\aleph_0}\}.$$

Any probability measure on an algebra $\mathcal{H}^{\overline{\aleph_0}}$ then admits a unique extension to the σ-algebra $\mathcal{A}_{\aleph_1}(\mathcal{H}^{\overline{\aleph_0}})$ because the probability measure is countably additive (i.e., satisfies the continuity property).

Going back to the context of this paper, since qualitative beliefs may not satisfy such continuity property, I consider instead the transfinite levels of beliefs $(H^\alpha, \mathcal{H}^\alpha)_{\alpha<\overline{\aleph_1}}$ up to the ordinal level of $\kappa = \aleph_1$ to obtain a σ-algebra (i.e., an $\aleph_1$-algebra) $(H, \mathcal{H}) = (H^{\overline{\aleph_1}}, \mathcal{H}^{\overline{\aleph_1}})$. I will use this insight (in Lemma 2 below) to define each player's type mapping on the universal space.

I often identify the hierarchies space H with the product of the nature states S and the players' hierarchies spaces. For each $i \in I$ and α with $1 \leq \alpha \leq \overline{\kappa}$, let

$$H_i^\alpha := \prod_{\beta<\alpha} M(H^\beta).$$

Then, for any ordinal α with $1 \leq \alpha \leq \overline{\kappa}$, one can identify

$$H^\alpha = S \times \prod_{i \in I} H_i^\alpha. \tag{5}$$

For all ordinals (α, β) with $1 \leq \beta \leq \alpha \leq \overline{\kappa}$, denote by

$$\pi_i^{\alpha,\beta} : H_i^\alpha \to H_i^\beta$$

the projection. Again, I omit the superscript $\overline{\kappa}$ when $\alpha = \overline{\kappa}$.

[21] While each H^α (with $\alpha \geq 1$) is defined from the set $\Delta(S)$ of probability measures over S instead of $M(S)$, for ease of exposition, I use the same notation H^α.

The second step defines, for any given belief space $\overrightarrow{\Omega}$, a map $h : \Omega \to H$ that associates, with each state $\omega \in \Omega$, the players' belief hierarchies $h(\omega)$. Call $h(\omega)$ the (players') belief hierarchies at ω and h the hierarchy map. Since I establish results on belief hierarchies by induction on the formation of h, I inductively define h from h^α, depending on whether α is a successor ordinal or a limit ordinal.

Definition 8 (Belief Hierarchies)**.** *Given a belief space $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}), (t_i)_{i\in I}, \Theta\rangle$, I inductively define the hierarchy map $h : \Omega \to H$ as follows.*

1. *For $\alpha = 0$, let $h^0 : \Omega \to H^0$ by $h^0 := \Theta$.*
2. *For a successor ordinal $\alpha = \beta + 1$, define $h^\alpha : \Omega \to H^\alpha$ by*
$$\begin{aligned} h^\alpha(\omega) &:= (h^\beta(\omega), t(\omega) \circ (h^\beta)^{-1}) \\ &:= (h^\beta(\omega), (t_i(\omega) \circ (h^\beta)^{-1})_{i\in I}) \text{ for all } \omega \in \Omega. \end{aligned} \tag{6}$$
3. *For any (non-zero) limit ordinal α with $\alpha \leq \overline{\kappa}$, let $h^\alpha : \Omega \to H^\alpha$ be the unique mapping which satisfies*
$$h^\beta = \pi^{\alpha,\beta} \circ h^\alpha \text{ for all } \beta < \alpha. \tag{7}$$

I omit the superscript $\overline{\kappa}$ when $\alpha = \overline{\kappa}$, i.e., let $h = h^{\overline{\kappa}}$. Call each $h(\omega)$ the (players') belief hierarchies at ω.

In Definition 8, if $\alpha = 0$, then $h^0(\omega) = \Theta(\omega) \in S$ is the state of nature that corresponds to the state of the world ω. When $\alpha = 1$,
$$h^1(\omega) = (\underbrace{h^0(\omega)}_{\in S}, (\underbrace{t_i(\omega) \circ (h^0)^{-1}}_{\in M(S)})_{i\in I}) \in H^1$$
consists of the state of nature and each player's first-order beliefs about S (i.e., the players' belief hierarchies of order up to 1) at ω. When $\alpha = 2$, since one can identify $H^2 = H^1 \times M(H^1)^I$,
$$h^2(\omega) = (\underbrace{h^1(\omega)}_{\in H^1}, (\underbrace{t_i(\omega) \circ (h^1)^{-1}}_{\in M(H^1)})_{i\in I}) \in H^2.$$
If $\alpha = \beta + 1$ is a successor ordinal, then, since one can identify $H^\alpha = H^\beta \times M(H^\beta)^I$,
$$h^\alpha(\omega) = (\underbrace{h^\beta(\omega)}_{\in H^\beta}, (\underbrace{t_i(\omega) \circ (h^\beta)^{-1}}_{\in M(H^\beta)})_{i\in I}) \in H^\alpha$$
as in Expression (6). When α is a limit ordinal (including the case in which $\alpha = \overline{\kappa}$), Definition 8 defines $h^\alpha(\omega)$ as the "limit" of $(h^\beta(\omega))_{\beta<\alpha}$ in the sense of Expression (7).

In light of Equation (5), Remark 4 below shows that the players' belief hierarchies $h(\omega) \in H$ at ω can be decomposed into the state of nature and each player's belief hierarchy:
$$(h^0(\omega), (h_i(\omega))_{i\in I}) \in S \times \prod_{i\in I} H_i.$$

Remark 4 (Belief Hierarchy of a Player)**.** For each ordinal α with $1 \leq \alpha \leq \overline{\kappa}$, I can identify each $h^\alpha \in H^\alpha$ in Definition 8 with $h^\alpha = (h^0, (h_i^\alpha)_{i \in I})$ as follows.

1. Let $h_i^1(\omega) := t_i(\omega) \circ (h^0)^{-1}$ for all $\omega \in \Omega$.
2. For a successor ordinal $\alpha = \beta + 1$ with $\beta \geq 1$, define $h_i^\alpha : \Omega \to H_i^\alpha$ by
$$h_i^\alpha(\omega) := (h_i^\beta(\omega), t_i(\omega) \circ (h^\beta)^{-1}) \text{ for all } \omega \in \Omega.$$
3. For a limit ordinal α, let $h_i^\alpha : \Omega \to H_i^\alpha$ be the unique mapping which satisfies
$$h_i^\beta = \pi_i^{\alpha,\beta} \circ h_i^\alpha \text{ for all } \beta < \alpha.$$

Call $h_i(\omega)$ player i's belief hierarchy at ω.

Now, I define an underlying set Ω^* of a candidate universal belief space as the set of all possible belief hierarchies:

$$\Omega^* := \{\omega^* \in H \mid \omega^* = h(\omega) \text{ for some belief space } \overrightarrow{\Omega} \text{ and a state } \omega \in \Omega\}. \tag{8}$$

It can be seen that $\Omega^* \neq \emptyset$ because there is a belief space $\overrightarrow{\Omega}$ with $\Omega \neq \emptyset$.[22]

Since Ω^* is a subset of H, I can induce a κ-algebra $\mathcal{D}^*$ on Ω^* from $\mathcal{H}$ by

$$\mathcal{D}^* = \{(\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha) \in \mathcal{P}(\Omega^*) \mid E^\alpha \in \mathcal{H}^\alpha \text{ for some } \alpha < \kappa\}, \tag{9}$$

where $\pi^\alpha|_{\Omega^*} : \Omega^* \to H^\alpha$ is the restriction of $\pi^\alpha : H \to H^\alpha$ on Ω^* for each $\alpha < \overline{\kappa}$. Note that, since $(\pi^\alpha|_{\Omega^*})^{-1}(\cdot) = (\pi^\alpha)^{-1}(\cdot) \cap \Omega^*$, one has:

$$\mathcal{D}^* = \{E \cap \Omega^* \in \mathcal{P}(\Omega^*) \mid E \in \mathcal{H}\}.$$

From now on, for any belief space $\overrightarrow{\Omega}$, I identify the hierarchy map h as

$$h : \Omega \to \Omega^*.$$

I denote the hierarchy map by

$$h_{\overrightarrow{\Omega}} : \Omega \to \Omega^*$$

when I stress its domain. I establish that the hierarchy map $h : (\Omega, \mathcal{D}) \to (\Omega^*, \mathcal{D}^*)$ is measurable and that a belief morphism preserves belief hierarchies.

Lemma 1 (Belief Morphism Preserves Belief Hierarchies)**.** *1. For any belief space $\overrightarrow{\Omega}$, the hierarchy map $h_{\overrightarrow{\Omega}} : (\Omega, \mathcal{D}) \to (\Omega^*, \mathcal{D}^*)$ is measurable.*

[22] Since $S \neq \emptyset$, take $s \in S$, and one can define a belief space $\overrightarrow{\{s\}}$ which satisfies Definitions 1 and 2 as in Example 2.

2. *If $\varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega'}$ is a belief morphism, then $h_{\overrightarrow{\Omega}} = h_{\overrightarrow{\Omega'}} \circ \varphi$.*

A belief space $\overrightarrow{\Omega}$ is *non-redundant* if its hierarchy map h is injective. If a belief space $\overrightarrow{\Omega}$ is non-redundant, then $(\Theta, (t_i)_{i \in I}) : \Omega \to S \times M(\Omega)^I$ is injective. This follows because, if $(\Theta, (t_i)_{i \in I})(\omega) = (\Theta, (t_i)_{i \in I})(\omega')$ for some $\omega, \omega' \in \Omega$, then $h(\omega) = h(\omega')$ and thus $\omega = \omega'$.

The third step defines the mapping $\Theta^* : \Omega^* \to S$ and the players' type mappings $(t_i^*)_{i \in I}$. I define $\Theta^* : \Omega^* \to S$ by the projection

$$\Theta^* = \pi^0|_{\Omega^*}.$$

By construction, $\Theta^* : (\Omega^*, \mathcal{D}^*) \to (S, \mathcal{S})$ is measurable. Moreover, for any belief space $\overrightarrow{\Omega}$,

$$\Theta(\omega) = \pi^0(h(\omega)) = \Theta^*(h(\omega)) \text{ for all } \omega \in \Omega.$$

Hence, Θ^* preserves states of nature.

Next, I define the players' type mappings $(t_i^*)_{i \in I}$.

Lemma 2 (Players' Beliefs on Candidate Universal Space)**.** *For each $i \in I$ and $\omega^* \in \Omega^*$, let*

$$t_i^*(\omega^*)((\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha)) := (\omega^*)_i^{\alpha+1}(E^\alpha) \text{ for each } \alpha < \overline{\kappa} \text{ and } E^\alpha \in \mathcal{H}^\alpha, \tag{10}$$

where note that $(\omega^*)_i^{\alpha+1} \in M(H^\alpha)$.[23] *Then:*

1. *$t_i^* : (\Omega^*, \mathcal{D}^*) \to (M(\Omega^*), \mathcal{M}(\mathcal{D}^*))$ is a well-defined measurable mapping.*

2. *t_i^* inherits all the given properties of beliefs.*

3. *Fix a belief space $\overrightarrow{\Omega}$. Then, for any $\omega \in \Omega$ and $E^* \in \mathcal{D}^*$,*

$$t_i^*(h(\omega))(E^*) = t_i(\omega)(h^{-1}(E^*)). \tag{11}$$

For the left-hand side of Expression (10), Remark 3 allows one to define $t_i^*(\omega^*)$ on $\mathcal{D}^*$ as $\mathcal{H}$ itself is a κ-algebra on H (and similarly for the left-hand side of Expression (11)). The first part of the lemma shows that the right-hand side of Expression (10) is well-defined, i.e., $t_i^*(\omega^*)(E^*)$ does not depend on a particular choice of $\alpha < \overline{\kappa}$ with $E^* = (\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha)$.

So far, the three steps jointly imply the following: (i) the space $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (t_i^*)_{i \in I}, \Theta^* \rangle$ is a belief space with $\Omega^* \neq \emptyset$; and (ii) for any given belief space $\overrightarrow{\Omega} = \langle (\Omega, \mathcal{D}), (t_i)_{i \in I}, \Theta \rangle$, the hierarchy map $h : \Omega \to \Omega^*$ is a belief morphism.

The fourth step shows that the hierarchy map is a unique belief morphism from a given belief space into the belief space $\overrightarrow{\Omega^*}$. To that end, I show that the hierarchy map from $\overrightarrow{\Omega^*}$ into itself is the identity map.

[23] Denoting $\omega^* = (s, (\mu_i)_{i \in I}) \in \Omega^*$ and $\mu_i = (\mu_i^\alpha)_{0 \leq \alpha < \overline{\kappa}}$ with $\mu_i^\alpha \in M(H^\alpha)$, one has $(\omega^*)_i^{\alpha+1} = \mu_i^\alpha$.

Lemma 3 (Hierarchy Map of Candidate Universal Space)**.** *The hierarchy map* $h^* : \overrightarrow{\Omega^*} \to \overrightarrow{\Omega^*}$ *is the identity map on* Ω^*.

Now, I establish the main theorem:

Theorem 1 ($\overrightarrow{\Omega^*}$ is Universal)**.** *The belief space* $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (t_i^*)_{i \in I}, \Theta^* \rangle$ *is universal. The mapping* $(\Theta^*, (t_i^*)_{i \in I}) : \Omega^* \to \Omega^{**}$ *is bijective, where*

$$\Omega^{**} := \{(s, (\mu_i)_{i \in I}) \in S \times M(\Omega^*)^I \mid \textit{there are a belief space } \overrightarrow{\Omega} \textit{ and } \omega \in \Omega \\ \textit{such that } (s, (\mu_i)_{i \in I}) = (\Theta(\omega), (t_i(\omega) \circ h^{-1})_{i \in I})\}.$$

Proof of Theorem 1. First, the previous steps have shown that $\overrightarrow{\Omega^*} = \langle (\Omega^*, \mathcal{D}^*), (t_i^*)_{i \in I}, \Theta^* \rangle$ is a belief space. To show that it is universal, observe that if $h, \varphi : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ are a belief morphism, then

$$h = h_{\overrightarrow{\Omega^*}} \circ \varphi = \varphi,$$

where the first equality follows from Lemma 1 and the second from Lemma 3. Thus, the hierarchy map h is a unique belief morphism from the given belief space $\overrightarrow{\Omega}$ to $\overrightarrow{\Omega^*}$.

Second, since $\overrightarrow{\Omega^*}$ is non-redundant by Lemma 3, the mapping $(\Theta^*, (t_i^*)_{i \in I}) : \Omega^* \to S \times M(\Omega^*)^I$ is injective. The mapping is also surjective because $\overrightarrow{\Omega^*}$ is universal. Thus, the universal belief space $\overrightarrow{\Omega^*}$ is in a bijective relation to a subset of $S \times M(\Omega^*)^I$ that respects given introspective properties, establishing the second statement. □

As in the proof of Theorem 1, Lemma 3 implies that the universal belief space $\overrightarrow{\Omega^*}$ is non-redundant. Also, by definition, for any $\omega^* \in \Omega^*$, there exist a belief space $\overrightarrow{\Omega}$ and a state $\omega \in \Omega$ such that $\omega^* = h_{\overrightarrow{\Omega}}(\omega)$. Since $h_{\overrightarrow{\Omega}}$ associates, with each state $\omega \in \Omega$, the players' belief hierarchies $h(\omega) \in \Omega^*$ at the state ω, it follows that the space Ω^* consists of all possible belief hierarchies that can be attained by some state of some belief space.

For the rest of this subsection, technical remarks are in order. The proof of the second statement of the theorem suggests that, when no introspective properties (i.e., no properties in Definition 2) are assumed,

$$(\Theta^*, (t_i^*)_{i \in I}) : \Omega^* \to S \times M(\Omega^*)^I$$

is indeed a belief isomorphism.[24] Thus, the universal belief space Ω^* is (measurably) isomorphic to $S \times \prod_{i \in I} \Omega_i^*$ such that Ω_i^* is (measurably) isomorphic to $M(S \times \prod_{j \in I} \Omega_j^*)$.

[24]One can formalize this insight from the fact that, when a given combination of properties of beliefs is subsumed in the operation of $M(\cdot)$ (as, for instance, when no introspective properties are assumed), a belief space forms a coalgebra in category theory. See, for instance, Moss and Viglizzo (2004, 2006) and Viglizzo (2005) for an application of the theory of coalgebra to standard type spaces. For this particular context, see Fukuda (2017) for a proof.

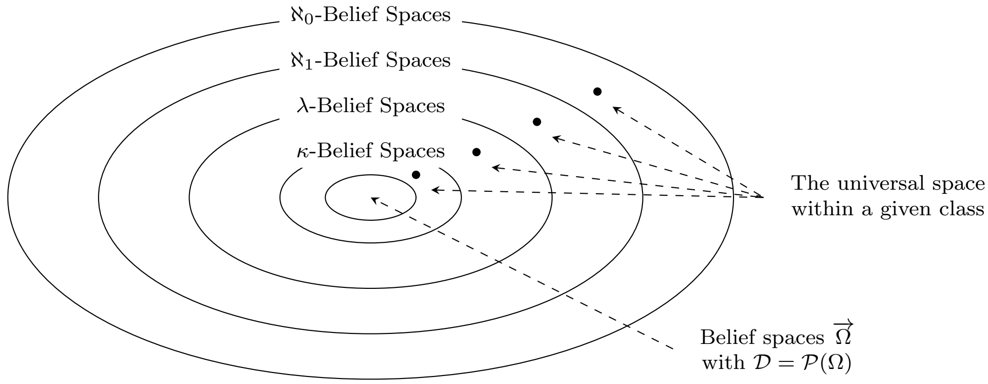

Figure 2: The Role of the Domain Specification: $\aleph_1 < \lambda < \kappa$.

When Truth Axiom is imposed, Ω^{**} is a strict (and non-product) subset of $S \times M(\Omega^*)^I$, because the players' knowledge is "correlated" in that they can only know, at each state, the events that hold true at that state.

In the special case in which Truth Axiom is not assumed while Positive Introspection and Negative Introspection are assumed, somewhat informally, since each player i is certain of her own beliefs, she would not need to reason about her own beliefs. Thus, consider the class of possibility models (e.g., Brandenburger, 2003; Brandenburger and Keisler, 2006) as follows: each state space has a product structure $\Omega = \prod_{i \in I_0} \Omega_i$ with $I_0 := I \cup \{0\}$ and $\Omega_0 = S$, and each player's type mapping is given by the marginal $t_i(\cdot) \circ \pi_{-i}^{-1} : \Omega \to M(\Omega_{-i})$ where (i) $\pi_{-i} : \Omega \to \Omega_{-i}$ is the projection with $\Omega_{-i} := \prod_{j \in I_0 \setminus \{i\}} \Omega_j$ and (ii) t_i only depends on Ω_i (more formally, $t_i((\omega_j)_{j \in I_0}) = t_i((\omega'_j)_{j \in I_0})$ if $\omega_i = \omega'_i$). While the formal and detailed comparison between the possibility models and the framework of this paper is beyond the scope of this paper, one can apply the analyses of this paper to this restricted class of belief spaces, and thus the universal belief space (i.e., the universal possibility model) would exist and satisfy

$$\Omega^* = S \times \prod_{i \in I} \Omega_i^*, \text{ where each } \Omega_i^* \text{ is isomorphic to } M\left(S \times \prod_{j \in I \setminus \{i\}} \Omega_j^*\right), \tag{12}$$

similarly to the case of the standard universal type space (replacing M with Δ).[25]

Finally, I briefly discuss the role of the domain specification. To make the exposition simplest, I impose no (logical or introspective) properties of beliefs. Then, I vary infinite regular cardinals: Figure 2 considers $\aleph_0$, $\aleph_1$, λ, and κ, where $\aleph_1 < \lambda < \kappa$. I

[25] In the context of probabilistic beliefs, Meier (2012) shows that, in the class of belief spaces in which players' beliefs satisfy introspection properties, its universal belief spaces admits the product structure in the sense of Expression (12) (where M is replaced with Δ).

also consider the class of belief spaces such that the domain of each belief space is the power set as depicted by the inner-most ellipse of Figure 2.

If the class of belief spaces where the domain of a belief space is always the power set has the universal belief space, then a bijection between Ω^* and $S \times \mathcal{P}(\mathcal{P}(\Omega^*))^I$ would exist, as $M(\Omega^*) = \{0,1\}^{\mathcal{P}(\Omega^*)}$ would be in a bijective relation to $\mathcal{P}(\mathcal{P}(\Omega^*))$. This is clearly impossible. This argument is closely related to the non-existence of a "complete" possibility structure by Brandenburger (2003). Figure 2 illustrates this point. On the one hand, each class of belief spaces admits the universal belief space under the domain specification ("•" in each ellipse denotes the universal belief space within the corresponding class).[26] On the other hand, the inner-most ellipse of Figure 2 does not admit the universal belief space. Another way to highlight the importance of domain specification is that, without fixing an infinite (regular) cardinal κ, one cannot define the space $(H, \mathcal{H}) = (H^{\overline{\kappa}}, \mathcal{H}^{\overline{\kappa}})$.

3.2 Coherent Belief Hierarchies

This subsection defines a coherent subset of the hierarchies space H and shows that the universal belief space $\overrightarrow{\Omega^*}$ (established in Theorem 1) is the largest coherent subset of H. I start with the definition of a coherent subset of H.

Definition 9 (Coherent Belief Hierarchies)**.** *A subset Ω of H is coherent if it satisfies the following. Take any $(s, \mu) \in \Omega$, where $\mu = (\mu_i)_{i \in I}$ and $\mu_i = (\mu_i^\alpha)_{0 \leq \alpha < \overline{\kappa}}$ with $\mu_i^\alpha \in M(H^\alpha)$.*

1. *For any ordinals (α, β) with $0 \leq \beta \leq \alpha < \overline{\kappa}$, if $(\pi^\alpha|_\Omega)^{-1}(E^\alpha) = (\pi^\beta|_\Omega)^{-1}(F^\beta)$ for some $E^\alpha \in \mathcal{H}^\alpha$ and $F^\beta \in \mathcal{H}^\beta$, then*

$$\mu_i^\alpha(E^\alpha) = \mu_i^\beta(F^\beta) \text{ for all } i \in I.$$

2. *If any of the introspective properties of beliefs (recall Definition 2) is imposed, then $\mu = (\mu_i)_{i \in I}$ satisfies the corresponding property.*
   (a) *Truth Axiom. If $\mu_i^\alpha(E^\alpha) = 1$, then $(s, \mu) \in (\pi^\alpha|_\Omega)^{-1}(E^\alpha)$.*
   (b) *Positive Introspection. If $\mu_i^\alpha(E^\alpha) = 1$, then*

$$\mu_i^\alpha(\{(s', (\mu'_j)_{j \in I}) \in H \mid (\mu'_i)^\alpha \in \mathcal{M}_{E^\alpha}\}) = 1.$$

   (c) *Negative Introspection. If $\mu_i^\alpha(E^\alpha) = 0$, then*

$$\mu_i^\alpha(\{(s', (\mu'_j)_{j \in I}) \in H \mid (\mu'_i)^\alpha \in \neg\mathcal{M}_{E^\alpha}\}) = 1.$$

[26] Fixing a set I of players, a set $(S, \mathcal{S})$ of states of nature, and some properties of beliefs, the universal κ-belief space of I on $(S, \mathcal{A}_\kappa(\mathcal{S}))$ differs from the universal λ-belief space of I on $(S, \mathcal{A}_\lambda(\mathcal{S}))$, whenever $\kappa \neq \lambda$ (see also Fukuda, 2024b, Section 7.4).

(d) *Kripke Property. If* $\bigcap\{E \in \mathcal{D} \mid E = (\pi^\alpha|_\Omega)^{-1}(E^\alpha)$ *and* $\mu_i^\alpha(E^\alpha) = 1$ *for some* $\alpha < \overline{\kappa}$ *and* $E^\alpha \in \mathcal{H}^\alpha\} \subseteq (\pi^\beta|_\Omega)^{-1}(F^\beta)$ *for some* $\beta < \overline{\kappa}$ *and* $F^\beta \in \mathcal{H}^\beta$*, then* $\mu_i^\beta(F^\beta) = 1$.

Condition (1) states that, for each player i's belief hierarchy $\mu_i = (\mu_i^\alpha)_{0 \leq \alpha < \overline{\kappa}}$, any two levels of beliefs (i.e., α and β) do not contradict one another. However, differently from the case of probabilistic beliefs (e.g., Brandenburger and Dekel, 1993), coherency requires that all different levels of beliefs, including transfinite levels, do not contradict one another. Condition (2) requires that each player i's belief hierarchy μ_i respects the given introspective properties.

The next definition enables one to define the belief space $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}), (t_i)_{i \in I}, \Theta\rangle$ if a subset Ω of H is coherent.

Definition 10. *If* Ω *is a coherent subset of* H*, then I define the induced belief space* $\overrightarrow{\Omega} = \langle(\Omega, \mathcal{D}), (t_i)_{i \in I}, \Theta\rangle$ *as follows.*

1. $\mathcal{D}$ *is a* κ*-algebra on* Ω *defined by*
$$\mathcal{D} := \{(\pi^\alpha|_\Omega)^{-1}(E^\alpha) \in \mathcal{P}(\Omega) \mid E^\alpha \in \mathcal{H}^\alpha \text{ for some } \alpha < \overline{\kappa}\}. \tag{13}$$
2. *Let* $\Theta : (\Omega, \mathcal{D}) \to (S, \mathcal{S})$ *be the projection* $\pi^0|_\Omega$.
3. *I define* $t_i : (\Omega, \mathcal{D}) \to (M(\Omega, \mathcal{D}), \mathcal{M}(\Omega, \mathcal{D}))$ *by*
$$t_i(s, (\mu_j)_{j \in I})((\pi^\alpha|_\Omega)^{-1}(E^\alpha)) := \mu_i^\alpha(E^\alpha) \text{ for each } E^\alpha \in \mathcal{H}^\alpha. \tag{14}$$

In Condition (1), the κ-algebra $\mathcal{D}$ is defined from $\mathcal{H}$ through Expression (13). The mapping Θ is naturally defined by the projection through Condition (2). Condition (3) defines each player's type mapping t_i. Condition (1) of Definition 9 ensures that t_i defined through Expression (14) is well-defined, i.e., the left-hand side of the expression does not depend on a particular choice of α. Also, Condition (2) of Definition 9 ensures that t_i defined through Expression (14) respects the given introspective properties.

Definitions 9 and 10 mean that a subset Ω of H is coherent (i) if each belief hierarchy of each player in Ω is coherent in that any two levels of beliefs do not contradict one another and (ii) if the set Ω is "belief-closed" in that Ω induces a belief structure on itself. Since Ω, consisting of coherent belief hierarchies, induces a belief structure on Ω itself, coherency of the belief hierarchies Ω is commonly certain.

Now, I show that $\overrightarrow{\Omega^*}$ is the largest coherent subset of H.

Theorem 2 (Ω^* is the Largest Coherent Space)**.** *The set* Ω^* *established in Theorem 1 is the largest coherent subset of* H*: (i)* Ω^* *is coherent; and (ii) for any coherent subset* Ω *of* H*, the hierarchy map* $h : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ *is an inclusion map (so that* $\Omega \subseteq \Omega^*$*).*

The theorem roughly states that each player's belief hierarchy in Ω^* is coherent and that any coherent belief hierarchies of the players belong to Ω^*. The key observation is the definition of coherency (in Definition 9), which requires that all different levels of beliefs, including transfinite levels, do not to contradict with each other. This enables one to define the underlying type mapping t_i on $(\Omega, \mathcal{D})$ through Expression (14) in Definition 10 as long as the given set Ω is coherent because the collection $\mathcal{D}$ forms a κ-algebra.[27]

4 Conclusion

Theorem 1 constructs the universal belief space as the set of belief hierarchies that can be induced by some state of some belief space when beliefs are qualitative. This paper shows that the following idea of Harsanyi (1967-68) extends to qualitative beliefs beyond probabilistic ones: while a type induces a belief hierarchy, a belief hierarchy that can be induced by some belief space is a type. Theorem 2 characterizes the universal qualitative belief space as the largest coherent set of belief hierarchies, extending the characterization of the universal type space for probabilistic beliefs (e.g., Brandenburger and Dekel, 1993; Mertens and Zamir, 1985) to qualitative beliefs.

A Appendix

A.1 Section 2.3 (Remark 1)

To see that type mappings and belief operators are equivalent, I provide the corresponding definitions of properties of beliefs in terms of belief operators.

Definition A.1 (Propeties of Beliefs)**.** *Fix $B_i : \mathcal{D} \to \mathcal{D}$.*

1. *The following properties of B_i are referred to as logical properties.*
   (a) *No-Contradiction: $B_i(\emptyset) = \emptyset$.*
   (b) *Consistency: $B_i(E) \subseteq (\neg B_i)(E^c)$ for any $E \in \mathcal{D}$.*
   (c) *Monotonicity: $B_i(E) \subseteq B_i(F)$ for any $E, F \in \mathcal{D}$ with $E \subseteq F$.*
   (d) *Necessitation: $B_i(\Omega) = \Omega$.*
   (e) *λ-Conjunction: $\bigcap_{E \in \mathcal{E}} B_i(E) \subseteq B_i(\bigcap \mathcal{E})$ for any $\mathcal{E} \subseteq \mathcal{D}$ with $0 < |\mathcal{E}| < \lambda$.*
2. *The following properties of B_i are referred to as introspective properties.*

[27] In the context of probabilistic beliefs, Fukuda (2024c) shows that, even when there is no topological assumption on an underlying set of states of nature, the universal type space is the largest set of coherent belief hierarchies that satisfies common certainty of coherency (when coherency requires that all different levels of beliefs, including transfinite but countable levels, do not contradict one another).

(a) *Truth Axiom:* $B_i(E) \subseteq E$ *for any* $E \in \mathcal{D}$.

(b) *Positive Introspection:* $B_i(\cdot) \subseteq B_i B_i(\cdot)$.

(c) *Negative Introspection:* $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$.

(d) *Kripke Property: for each* $(\omega, E) \in \Omega \times \mathcal{D}$, $\omega \in B_i(E)$ *if (and only if)*

$$b_{B_i}(\omega) \subseteq E, \text{ where } b_{B_i}(\omega) := \bigcap \{E \in \mathcal{D} \mid \omega \in B_i(E)\}.$$

See the main text for the interpretation of each property. It can be shown that B_i (resp., t_i) satisfies a given property iff t_{B_i} (resp., B_{t_i}) satisfies it.

A.2 Section 3.1

Proof of Remark 3. First, $H = (\pi^\alpha)^{-1}(H^\alpha) \in \mathcal{H}$. Second, if $(\pi^\alpha)^{-1}(E^\alpha) \in \mathcal{H}$ with $\alpha < \overline{\kappa}$, then $\neg(\pi^\alpha)^{-1}(E^\alpha) = (\pi^\alpha)^{-1}(\neg E^\alpha) \in \mathcal{H}$. Third, take any subset A of ordinals $\{\alpha \mid \alpha < \overline{\kappa}\}$ whose cardinality $|A|$ is less than κ. Consider $((\pi^\alpha)^{-1}(E^\alpha))_{\alpha \in A}$. Since κ is regular (recall Footnote 14 or see the Online Appendix for the definition), its supremum $\gamma := \sup A$ has cardinality less than κ. Since $(H^\gamma, \mathcal{H}^\gamma)$ is a κ-algebra,

$$\bigcap_{\alpha \in A} (\pi^\alpha)^{-1}(E^\alpha) = (\pi^\gamma)^{-1}\Bigg(\underbrace{\bigcap_{\alpha \in A} (\pi^{\gamma,\alpha})^{-1}(E^\alpha)}_{\in \mathcal{H}^\gamma}\Bigg) \in \mathcal{H}.$$

The proof is complete. □

Proof of Lemma 1. 1. Since $h^\alpha = \pi^\alpha|_{\Omega^*} \circ h$ for all $\alpha < \overline{\kappa}$, it suffices to show that

$$h^{-1}((\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha)) = (h^\alpha)^{-1}(E^\alpha) \in \mathcal{D} \text{ for all } \alpha < \overline{\kappa} \text{ and } E^\alpha \in \mathcal{H}^\alpha.$$

I prove the assertion by induction on $\alpha < \overline{\kappa}$.

Let $\alpha = 0$. For any $E^0 \in \mathcal{S}$,

$$(h^0)^{-1}(E^0) = \Theta^{-1}(E^0) \in \mathcal{D}.$$

For a successor ordinal $\alpha = \beta + 1$, it is sufficient to show that

$$(t_i \circ (h^\beta)^{-1})^{-1}(\mathcal{M}^{\mathcal{H}^\beta}(E^\beta)) \in \mathcal{D} \text{ for each } E^\beta \in \mathcal{H}^\beta \text{ and } i \in I.$$

Fix $E^\beta \in \mathcal{H}^\beta$ and $i \in I$. Then,

$$\begin{aligned}(t_i \circ (h^\beta)^{-1})^{-1}(\mathcal{M}^{\mathcal{H}^\beta}(E^\beta)) &= \{\omega \in \Omega \mid t_i(\omega)((h^\beta)^{-1}(E^\beta)) = 1\} \\ &= t_i^{-1}(\mathcal{M}^{\mathcal{D}}((h^\beta)^{-1}(E^\beta))) \in \mathcal{D}.\end{aligned}$$

For a limit ordinal α, if

$$h^{-1}((\pi^{\beta}|_{\Omega^*})^{-1}(E^{\beta})) = (h^{\beta})^{-1}(E^{\beta}) \in \mathcal{D} \text{ for all } \beta < \alpha,$$

then it is clear that

$$h^{-1}((\pi^{\alpha}|_{\Omega^*})^{-1}(E^{\alpha})) = (h^{\alpha})^{-1}(E^{\alpha}) \in \mathcal{D}.$$

The induction is complete.

2. For notational ease, I denote $h = h_{\overrightarrow{\Omega}}$ and $h' = h_{\overrightarrow{\Omega'}}$. It suffices to show by induction on ordinals $\alpha < \overline{\kappa}$ that

$$h^{\alpha}(\omega) = (h')^{\alpha}(\varphi(\omega)) \text{ for each } \omega \in \Omega.$$

Let $\alpha = 0$. Since φ is a belief morphism,

$$h^{0}(\omega) = \Theta(\omega) = \Theta'(\varphi(\omega)) = (h')^{0}(\varphi(\omega)) \text{ for each } \omega \in \Omega.$$

Let $\alpha = \beta + 1$ be a successor ordinal. Since I have

$$\begin{aligned} h^{\beta+1}(\omega) &= \left(h^{\beta}(\omega), t(\omega) \circ (h^{\beta})^{-1}\right) \text{ and} \\ (h')^{\beta+1}(\varphi(\omega)) &= \left(h'^{\beta}(\varphi(\omega)), t'(\varphi(\omega)) \circ (h'^{k})^{-1}\right), \end{aligned}$$

it suffices to show that

$$t(\omega) \circ (h^{\beta})^{-1} = t'(\varphi(\omega)) \circ (h'^{\beta})^{-1}.$$

Now, since φ is a belief morphism, it follows that, for each $i \in I$,

$$\begin{aligned} t'_i(\varphi(\omega)) \left((h'^{\beta})^{-1}(\cdot)\right) &= t_i(\omega) \left(\varphi^{-1}\left((h'^{\beta})^{-1}(\cdot)\right)\right) \\ &= t_i(\omega) \left(\left(h'^{\beta} \circ \varphi\right)^{-1}(\cdot)\right) = t_i(\omega) \left((h^{\beta})^{-1}(\cdot)\right). \end{aligned}$$

Let α be a limit ordinal. Fix $\omega \in \Omega$. By the definitions of h^{α} and $(h')^{\alpha}$, it is immediate that $h^{\alpha}(\omega) = (h')^{\alpha}(\varphi(\omega))$ if $h^{\beta}(\omega) = (h')^{\beta}(\varphi(\omega))$ for all $\beta < \alpha$. The induction is complete.

□

Proof of Lemma 2. Fix $i \in I$. I prove the results in the following five steps. The first step shows that t_i^* is a well-defined mapping on Ω^*. Fix $\omega^* \in \Omega^*$. I show that, for any ordinals (α, β) with $0 \leq \beta \leq \alpha < \overline{\kappa}$, if

$$(\pi^{\alpha}|_{\Omega^*})^{-1}(E^{\alpha}) = (\pi^{\beta}|_{\Omega^*})^{-1}(F^{\beta}) \text{ for some } E^{\alpha} \in \mathcal{H}^{\alpha} \text{ and } E^{\beta} \in \mathcal{H}^{\beta},$$

then

$$(\omega^*)_i^{\alpha+1}(E^\alpha) = (\omega^*)_i^{\beta+1}(F^\beta).$$

Observe first that, for any belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$,

$$\begin{aligned} t_i^*(\omega^*) \circ (\pi^\alpha|_{\Omega^*})^{-1} &= (h(\omega))_i^{\alpha+1} \\ &= t_i(\omega) \circ (h^\alpha)^{-1} \\ &= t_i(\omega) \circ (\pi^\alpha|_{\Omega^*} \circ h)^{-1}. \end{aligned} \tag{15}$$

Thus, I have

$$\begin{aligned} (\omega^*)_i^{\alpha+1}(E^\alpha) &= t_i(\omega)((\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha)) \\ &= t_i(\omega)((\pi^\beta|_{\Omega^*})^{-1}(F^\beta)) = (\omega^*)_i^{\beta+1}(F^\beta). \end{aligned}$$

This equation holds irrespective of a choice of belief spaces.

The second step establishes Equation (11). Indeed, it follows from Equation (15) that, for each $\alpha < \overline{\kappa}$ and $E^\alpha \in \mathcal{H}^\alpha$,

$$t_i^*(h(\omega))((\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha)) = t_i(\omega)(h^{-1}((\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha))).$$

The third step shows that t_i^* inherits logical properties of beliefs so that it is a mapping from Ω^* into $M(\Omega^*)$.

1. No-Contradiction. For any $\omega^* \in \Omega^*$, there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$, and thus

$$t_i^*(\omega^*)(\emptyset) = t_i(\omega)(\underbrace{h^{-1}(\emptyset)}_{=\emptyset}) = 0,$$

where the last equality follows because t_i satisfies No-Contradiction.

2. Consistency. Suppose that there are $\omega^* \in \Omega^*$ and $E^* \in \mathcal{D}^*$ such that $t_i^*(\omega^*)(E^*) = 1$. Then, there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$ and $t_i(\omega)(h^{-1}(E^*)) = 1$. Consistency of t_i implies

$$t_i^*(\omega^*)(\neg E^*) = t_i(\omega)(h^{-1}(\neg E^*)) = t_i(\omega)(\neg h^{-1}(E^*)) = 0.$$

3. Monotonicity. Take any $\omega^* \in \Omega^*$ and $E^*, F^* \in \mathcal{D}^*$ with $E^* \subseteq F^*$. Now, there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$ and

$$t_i^*(\omega^*)(E^*) = t_i(\omega)(h^{-1}(E^*)) \leq t_i(\omega)(h^{-1}(F^*)) = t_i^*(\omega^*)(F^*),$$

where the inequality follows from $h^{-1}(E^*) \subseteq h^{-1}(F^*)$ and Monotonicity of t_i.

4. λ-Conjunction. Take any non-empty subset $\mathcal{F} \subseteq \mathcal{D}^*$ with $|\mathcal{F}| < \lambda$. Suppose that $t_i^*(\omega^*)(F^*) = 1$ for all $F^* \in \mathcal{F}$. Now, there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$ and $t_i(\omega)(h^{-1}(F^*)) = 1$ for all $F^* \in \mathcal{F}$. Since t_i satisfies λ-Conjunction, it follows that

$$1 = t_i(\omega)(\bigcap_{F^* \in \mathcal{F}} h^{-1}(F^*)) = t_i(\omega)(h^{-1}(\bigcap \mathcal{F})),$$

establishing $t_i^*(\omega^*)(\bigcap \mathcal{F}) = 1$.

5. Necessitation. For any $\omega^* \in \Omega^*$, there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$ and

$$1 = t_i(\omega)(\Omega) = t_i(\omega)(h^{-1}(\Omega^*)) = t_i^*(\omega^*)(\Omega^*),$$

where the first equality follows because t_i satisfies Necessitation.

The fourth step shows that $t_i^* : (\Omega^*, \mathcal{D}^*) \to (M(\Omega^*), \mathcal{M}(\mathcal{D}^*))$ is measurable. Let $E^* = (\pi^\alpha|_{\Omega^*})^{-1}(E^\alpha)$ with $E^\alpha \in \mathcal{H}^\alpha$ and $\alpha < \overline{\kappa}$. Then,

$$\begin{aligned}(t_i^*)^{-1}(\mathcal{M}_{E^*}) &= \{\omega^* \in \Omega^* \mid (\omega^*)_i^{\alpha+1}(E^\alpha) = 1\} \\ &= \{\omega^* \in \Omega^* \mid (\omega^*)_i^{\alpha+1} \in \mathcal{M}_{E^\alpha}\} \in \mathcal{D}^*.\end{aligned}$$

The fifth step shows that t_i^* inherits the introspective properties of beliefs.

1. Truth Axiom. Fix $E^* \in \mathcal{D}^*$. If $t_i^*(\omega^*)(E^*) = 1$, then there exist a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$ and

$$1 = t_i^*(\omega^*)(E^*) = t_i(\omega)(h^{-1}(E^*)).$$

Now, Truth Axiom of t_i implies $\omega \in h^{-1}(E^*)$, and thus $\omega^* = h(\omega) \in E^*$.

2. Positive Introspection. Fix $E^* \in \mathcal{D}^*$ and $\omega^* \in \Omega^*$ such that $t_i^*(\omega^*)(E^*) = 1$. Then, there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$, $1 = t_i^*(\omega^*)(E^*) = t_i(\omega)(h^{-1}(E^*))$, and $t_i^*(\omega^*)((t_i^*)^{-1}(\mathcal{M}_{E^*})) = t_i(\omega)(h^{-1}((t_i^*)^{-1}(\mathcal{M}_{E^*})))$. Now, Positive Introspection of t_i implies that

$$1 = t_i(\omega)(t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)})).$$

Next, I show that $t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)}) = h^{-1}((t_i^*)^{-1}(\mathcal{M}_{E^*}))$:

$$\begin{aligned}\omega \in t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)}) &\text{ iff } t_i(\omega)(h^{-1}(E^*)) = 1 \text{ iff } t_i^*(\omega^*)(E^*) = 1 \\ &\text{ iff } h(\omega) = \omega^* \in (t_i^*)^{-1}(\mathcal{M}_{E^*}) \text{ iff } \omega \in h^{-1}((t_i^*)^{-1}(\mathcal{M}_{E^*})).\end{aligned}$$

Then, I obtain

$$1 = t_i(\omega)(t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)})) = t_i(\omega)(h^{-1}((t_i^*)^{-1}(\mathcal{M}_{E^*}))) = t_i^*(\omega^*)((t_i^*)^{-1}(\mathcal{M}_{E^*})).$$

3. Negative Introspection. Fix $E^* \in \mathcal{D}^*$ and $\omega^* \in \Omega^*$ with $t_i^*(\omega^*)(E^*) = 0$. Then, there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$ and $0 = t_i^*(\omega^*)(E^*) = t_i(\omega)(h^{-1}(E^*))$. Now, Negative Introspection of t_i implies that
$$1 = t_i(\omega)(\neg t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)})).$$
Then, it follows from the previous argument that $t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)}) = h^{-1}((t_i^*)^{-1}(\mathcal{M}_{E^*}))$, and hence I obtain
$$1 = t_i(\omega)(\neg t_i^{-1}(\mathcal{M}_{h^{-1}(E^*)})) = t_i(\omega)(h^{-1}(\neg(t_i^*)^{-1}(\mathcal{M}_{E^*}))) = t_i^*(\omega^*)(\neg(t_i^*)^{-1}(\mathcal{M}_{E^*})).$$

4. Kripke Property. Observe that $h(b_{t_i}(\omega)) \subseteq b^*_{t_i^*}(h(\omega))$ for each $\omega \in \Omega$. If $b^*_{t_i^*}(\omega^*) \subseteq E^*$, then there are a belief space $\overrightarrow{\Omega}$ and $\omega \in \Omega$ such that $\omega^* = h(\omega)$, and thus
$$b_{t_i}(\omega) \subseteq h^{-1}(h(b_{t_i}(\omega))) \subseteq h^{-1}(b^*_{t_i^*}(\omega^*)) \subseteq h^{-1}(E^*).$$
By Kripke Property of $\overrightarrow{\Omega}$, it follows that
$$1 = t_i(\omega)(h^{-1}(E^*)) = t_i^*(\omega^*)(E^*),$$
as desired.

□

Proof of Lemma 3. I show by induction on ordinals $\alpha < \overline{\kappa}$ that $(h^*)^\alpha : \Omega^* \to H^\alpha$ is the projection $\pi^\alpha|_{\Omega^*}$, where note that $(h^*)^\alpha = \pi^\alpha \circ h^*$. For $\alpha = 0$, $(h^*)^0 = \pi^0|_{\Omega^*}$.

Let $\alpha = \beta + 1$ be a successor ordinal. Then, for each ω^*,
$$\begin{aligned}(h^*)^{\beta+1}(\omega^*) &= ((h^*)^\beta(\omega^*), t^*(\omega^*) \circ ((h^*)^\beta)^{-1}) \\ &= (\pi^\beta|_{\Omega^*}(\omega^*), t^*(\omega^*) \circ (\pi^\beta|_{\Omega^*})^{-1}) \\ &= (\pi^\beta|_{\Omega^*}(\omega^*), (\omega^*)^{\beta+1}) = \pi^{\beta+1}|_{\Omega^*}(\omega^*),\end{aligned}$$
where $t^*(\omega^*) \circ (\pi^\beta|_{\Omega^*})^{-1} = (\omega^*)^{\beta+1}$ follows from Equation (15).

For a limit ordinal α, the statement holds by construction. The induction is complete. □

A.3 Section 3.2

Proof of Theorem 2. Part (i). Consider the universal belief space $\overrightarrow{\Omega^*}$ established in Theorem 1. By definition, Ω^* is a subset of H. The entire proof of Theorem 1 implies that Ω^* is coherent and that the κ-algebra $\mathcal{D}^*$ and the mappings Θ^* and $(t_i^*)_{i\in I}$ are defined as in Definition 10.

Part (ii). Let Ω be a coherent subset of H which respects the required introspective properties. First, I show that $\langle(\Omega, \mathcal{D}), (t_i)_{i\in I}, \Theta\rangle$ is a belief space. To do so, it is enough to show that t_i satisfies the required logical properties of beliefs.

1. No-Contradiction. I have $t_i(s,\mu)(\emptyset) = \mu_i^\alpha(\emptyset) = 0$.

2. Consistency. If $t_i(s,\mu)((\pi^\alpha|_\Omega)^{-1}(E^\alpha)) = 1$ and $t_i(s,\mu)(\neg(\pi^\alpha|_\Omega)^{-1}) = 1$, then $\mu_i^\alpha(E^\alpha) = 1$ and $\mu_i^\alpha(\neg E^\alpha) = 1$, which contradicts the assumption that μ_i^α is a type satisfying Consistency (i.e., $\mu_i^\alpha \in M(H^\alpha)$).

3. Monotonicity. Suppose that $(\pi^\alpha|_\Omega)^{-1}(E^\alpha) \subseteq (\pi^\beta|_\Omega)^{-1}(F^\beta)$. Without loss of generality, I can assume as if $\alpha = \beta$. Since $E^\alpha \subseteq F^\beta$, I have
$$t_i(s,\mu)((\pi^\alpha|_\Omega)^{-1}(E^\alpha)) = \mu_i^\alpha(E^\alpha) \leq \mu_i^\beta(F^\beta) = t_i(s,\mu)((\pi^\beta|_\Omega)^{-1}(F^\beta)).$$

4. λ-Conjunction. Without loss of generality, suppose that $t_i(s,\mu)((\pi^\alpha|_\Omega)^{-1}(E^\alpha)) = 1$ for all $E^\alpha \in \mathcal{E}$, where $\mathcal{E}$ is a non-empty subset of $\mathcal{H}^\alpha$ with $|\mathcal{E}| < \lambda$. Then,
$$t_i(s,\mu)(\bigcap_{E^\alpha \in \mathcal{E}} (\pi^\alpha|_\Omega)^{-1}(E^\alpha)) = t_i(s,\mu)((\pi^\alpha|_\Omega)^{-1}(\bigcap \mathcal{E})) = \mu_i^\alpha(\bigcap \mathcal{E}) = 1.$$

5. Necessitation. I have $t_i(s,\mu)(\Omega) = \mu_i^\alpha(H^\alpha) = 1$.

Second, I show that the hierarchy map $h : \overrightarrow{\Omega} \to \overrightarrow{\Omega^*}$ is an inclusion map. If $\alpha = 0$, then $h^0 = \pi^0|_\Omega$. For a successor ordinal $\alpha = \beta + 1$, since $t(s,\mu) \circ (\pi^\beta|_\Omega)^{-1} = \mu^\beta$, it follows that $h^\alpha = \pi^\alpha|_\Omega$. For a limit ordinal α, by construction, if $h^\beta = \pi^\beta|_\Omega$ for all $\beta < \alpha$, then $h^\alpha = \pi^\alpha|_\Omega$.

□

References

[1] W. Armbruster and W. Böge. "Bayesian Game Theory". Game Theory and Related Topics. Ed. by O. Moeschlin and D. Pallaschke. North-Holland, 1979, 17–28.

[2] R. J. Aumann. "Agreeing to Disagree". Ann. Statist. 4 (1976), 1236–1239.

[3] R. J. Aumann. "Interactive Epistemology I, II". Int. J. Game Theory 28 (1999), 263–300, 301–314.

[4] M. Bacharach. "Some Extensions of a Claim of Aumann in an Axiomatic Model of Knowledge". J. Econ. Theory 37 (1985), 167–190.

[5] W. Böge and T. Eisele. "On Solutions of Bayesian Games". Int. J. Game Theory 8 (1979), 193–215.

[6] G. Bonanno. "A Syntactic Approach to Rationality in Games with Ordinal Payoffs". Logic and the Foundations of Game and Decision Theory (LOFT 7). Ed. by G. Bonanno, W. van der Hoek, and M. Wooldridge. Amsterdam University Press, 2008, 59–86.

[7] G. Bonanno. "Epistemic Foundations of Game Theory". Handbook of Epistemic Logic. Ed. by H. van Ditmarsch, J. Y. Halpern, W. van der Hoek, and B. P. Kooi. College Publications, 2015, 443–487.

[8] G. Bonanno and E. Tsakas. "Common Belief of Weak-dominance Rationality in Strategic-form Games: A Qualitative Analysis". Games Econ. Behav. 112 (2018), 231–241.

[9] A. Brandenburger. "On the Existence of a "Complete" Possibility Structure". Cognitive Processes and Economic Behavior. Ed. by N. Dimitri, M. Basili, and I. Gilboa. Routledge, 2003, 30–34.

[10] A. Brandenburger and E. Dekel. "Hierarchies of Beliefs and Common Knowledge". J. Econ. Theory 59 (1993), 189–198.

[11] A. Brandenburger, E. Dekel, and J. Geanakoplos. "Correlated Equilibrium with Generalized Information Structures". Games Econ. Behav. 4 (1992), 182–201.

[12] A. Brandenburger and H. J. Keisler. "An Impossibility Theorem on Beliefs in Games". Studia Logica 84 (2006), 211–240.

[13] Y.-C. Chen. "Universality of the Epstein-Wang Type Structure". Games Econ. Behav. 68 (2010), 389–402.

[14] Y.-C. Chen, N. V. Long, and X. Luo. "Iterated Strict Dominance in General Games". Games Econ. Behav. 61 (2007), 299–315.

[15] A. Di Tillio. "Subjective Expected Utility in Games". Theor. Econ. 3 (2008), 287–323.

[16] A. Di Tillio, J. Halpern, and D. Samet. "Conditional Belief Types". Games Econ. Behav. 87 (2014), 253–268.

[17] M. Dufwenberg and M. Stegeman. "Existence and Uniqueness of Maximal Reductions under Iterated Strict Dominance". Econometrica 70 (2002), 2007–2023.

[18] L. G. Epstein and T. Wang. ""Beliefs about Beliefs" without Probabilities". Econometrica 64 (1996), 1343–1373.

[19] R. Fagin. "A Quantitative Analysis of Modal Logic". J. Symb. Log. 59 (1994), 209–252.

[20] R. Fagin, J. Geanakoplos, J. Y. Halpern, and M. Y. Vardi. "The Hierarchical Approach to Modeling Knowledge and Common Knowledge". Int. J. Game Theory 28 (1999), 331–365.

[21] R. Fagin, J. Y. Halpern, and M. Y. Vardi. "A Model-theoretic Analysis of Knowledge". J. ACM 38 (1991), 382–428.

[22] S. Fukuda. "The Existence of Universal Knowledge Spaces". Essays in the Economics of Information and Epistemology. Ph.D. Dissertation, the University of California at Berkeley, 2017, 1–113.

[23] S. Fukuda. "Formalizing Common Belief with No Underlying Assumption on Individual Beliefs". Games Econ. Behav. 121 (2020), 169–189.

[24] S. Fukuda. "On the Consistency among Prior, Posteriors, and Information Sets". Econ. Theory 78 (2024), 521–565.

[25] S. Fukuda. "The Existence of Universal Qualitative Belief Spaces". J. Econ. Theory 216 (2024), 105784.

[26] S. Fukuda. Topology-free Constructions of a Universal Type Space as Coherent Belief Hierarchies. Working Paper. 2024.

[27] S. Fukuda. "Are the Players in an Interactive Belief Model Meta-certain of the Model itself?" Econ. Theory (2025). Forthcoming.

[28] J. Ganguli, A. Hiefetz, and B. S. Lee. "Universal Interactive Preferences". J. Econ. Theory 162 (2016), 237–260.

[29] J. Geanakoplos. "Game Theory without Partitions, and Applications to Speculation and Consensus". B. E. J. Theor. Econ. 21 (2021), 361–394.

[30] P. Guarino. "The Topology-Free Construction of the Universal Type Structure for Conditional Probability Systems". Proceedings of the 16th Conference on Theoretical Aspects of Rationality and Knowledge. Ed. by J. Lang. 2017.

[31] P. Guarino. "Topology-Free Type Structures with Conditioning Events". Econ. Theory 79 (2025), 1107–1166.

[32] P. Guarino and G. Ziegler. "Optimism and Pessimism in Strategic Interactions under Ignorance". Games Econ. Behav. 136 (2022), 559–585.

[33] J. C. Harsanyi. "Games with Incomplete Information Played by "Bayesian" Players, I-III". Manag. Sci. 14 (1967-68), 159–182, 320–334, 486–502.

[34] A. Heifetz. "The Bayesian Formulation of Incomplete Information—the Non-Compact Case". Int. J. Game Theory 21 (1993), 329–338.

[35] A. Heifetz and W. Kets. "Robust Multiplicity with a Grain of Naiveté". Theor. Econ. 13 (2018), 415–465.

[36] A. Heifetz and P. Mongin. "Probability Logic for Type Spaces". Games Econ. Behav. 35 (2001), 31–53.

[37] A. Heifetz and D. Samet. "Knowledge Spaces with Arbitrarily High Rank". Games Econ. Behav. 22 (1998), 260–273.

[38] A. Heifetz and D. Samet. "Topology-Free Typology of Beliefs". J. Econ. Theory 82 (1998), 324–341.

[39] A. Heifetz and D. Samet. "Hierarchies of Knowledge: An Unbounded Stairway". Math. Soc. Sci. 38 (1999), 157–170.

[40] J. Hillas and D. Samet. "Dominance Rationality: A Unified Approach". Games Econ. Behav. 119 (2020), 189–196.

[41] W. Kets. Bounded Reasoning and Higher-Order Uncertainty. Working Paper. 2017.

[42] B. L. Lipman. "A Note on the Implications of Common Knowledge of Rationality". Games Econ. Behav. 6 (1994), 114–129.

[43] T. Mariotti, M. Meier, and M. Piccione. "Hierarchies of Beliefs for Compact Possibility Models". J. Math. Econ. 41 (2005), 303–324.

[44] M. Meier. "On the Nonexistence of Universal Information Structures". J. Econ. Theory 122 (2005), 132–139.

[45] M. Meier. "Finitely Additive Beliefs and Universal Type Spaces". Ann. Probab. 34 (2006), 386–422.

[46] M. Meier. "Universal Knowledge-Belief Structures". Games Econ. Behav. 62 (2008), 53–66.

[47] M. Meier. "An Infinitary Probability Logic for Type Spaces". Isr. J. Math. 192 (2012), 1–58.

[48] M. Meier and A. Perea. Forward Induction in a Backward Inductive Manner. Working Paper. 2025.

[49] J.-F. Mertens and S. Zamir. "Formulation of Bayesian Analysis for Games with Incomplete Information". Int. J. Game Theory 14 (1985), 1–29.

[50] D. Monderer and D. Samet. "Approximating Common Knowledge with Common Beliefs". Games Econ. Behav. 1 (1989), 170–190.

[51] S. Morris. "The Logic of Belief and Belief Change: A Decision Theoretic Approach". J. Econ. Theory 69 (1996), 1–23.

[52] L. S. Moss and I. D. Viglizzo. "Harsanyi Type Spaces and Final Coalgebras Constructed from Satisfied Theories". Electron. Notes Theor. Comput. Sci. 106 (2004), 279–295.

[53] L. S. Moss and I. D. Viglizzo. "Final Coalgebras for Functors on Measurable Spaces". Inf. Comput. 204 (2006), 610–636.

[54] M. Pivato. Universal Recursive Preference Structures. Working Paper. 2024.

[55] A. Rubinstein and A. Wolinsky. "On the Logic of "Agreeing to Disagree" Type Results". J. Econ. Theory 51 (1990), 184–193.

[56] H. Salonen. "Common Theories". Math. Soc. Sci. 58 (2009), 279–289.

[57] H. Salonen. "On Completeness of Knowledge Models". 2009.

[58] D. Samet. "Ignoring Ignorance and Agreeing to Disagree". J. Econ. Theory 52 (1990), 190–207.

[59] D. Samet. "Quantified Beliefs and Believed Quantities". J. Econ. Theory 95 (2000), 169–185.

[60] D. Samet. "Common Belief of Rationality in Games with Perfect Information". Games Econ. Behav. 79 (2013), 192–200.

[61] H. S. Shin. "Logical Structure of Common Knowledge". J. Econ. Theory 60 (1993), 1–13.

[62] T. Strzalecki. "Depth of Reasoning and Higher Order Beliefs". J. Econ. Behav. Organ. 108 (2014), 108–122.

[63] E. Tsakas. "Epistemic Equivalence of Extended Belief Hierarchies". Games Econ. Behav. 86 (2014), 126–144.

[64] E. Tsakas. "Rational Belief Hierarchies". J. Math. Econ. 51 (2014), 121–127.

[65] E. Tsakas. "Universally Rational Belief Hierarchies". Int. Game Theory Rev. 16 (2014), 1440003.

[66] I. D. Viglizzo. "Final Sequences and Final Coalgebras for Measurable Spaces". Algebra and Coalgebra in Computer Science: First International Conference, CALCO 2005, Swansea, UK, September 3-6, 2005. Proceedings. Ed. by J. L. Fiadeiro, N. Harman, M. Roggenbach, and J. Rutten. Springer Berlin Heidelberg, 2005, 395–407.