

Can the Crowd be Introspective? Modeling Distributed Knowledge from Collective Information through Inference*

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Abstract

This paper studies distributed knowledge among agents who may hold contradictory beliefs. Distributed knowledge is formalized as knowledge logically deduced from pooling agents' collective information, where an agent's information consists of events such that she believes any of their logical implications. Consequently, distributed knowledge is true, monotonic, and positively introspective, even if agents' beliefs are not. However, distributed knowledge can fail negative introspection even if agents' beliefs satisfy it. Agents may not have distributed knowledge of the lack of distributed knowledge. Thus, agents can be collectively unaware of events. If agents' beliefs are true, monotonic, positively introspective, and conjunctive, then distributed knowledge coincides with knowledge possessed by the least knowledgeable "wise man" who knows everything each agent knows. As a game-theoretic application, this paper shows that the informational content of a rational-expectations Nash equilibrium is the distributed knowledge formed from the informational content associated with agents' strategies.

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The peculiar character of the problem of a rational economic order is determined precisely by the fact that the knowledge of the circumstances of which we must make use never exists in concentrated or integrated form but solely as the dispersed bits of incomplete and frequently contradictory knowledge which all the separate individuals possess. The economic problem of society is thus not merely a problem of how to allocate “given” resources—if “given” is taken to mean given to a single mind which deliberately solves the problem set by these “data.” It is rather a problem of how to secure the best use of resources known to any of the members of society, for ends whose relative importance only these individuals know. Or, to put it briefly, it is a problem of the utilization of knowledge which is not given to anyone in its totality.

—Hayek (1945)

1 Introduction

Consider a group of agents reasoning about an underlying state of the world, say, the weight of an ox at a country fair as in the classic weight-judging experiment studied by Galton (1907a,b, 1908). Each agent holds qualitative beliefs about possible states—such as plausible weights or features of the ox—which may be inaccurate or even mutually contradictory. Galton (1907a,b, 1908) famously observed that the average of such estimates closely approximates the true weight. This paper focuses instead on the qualitative aspects of group knowledge: both about the state of the world (e.g., the ox’s weight) and about the group’s own knowledge. If agents share relevant information, what can the group collectively “know” about the world and about its own group knowledge? To address this, the paper studies group knowledge not merely as the aggregation of individual beliefs, but as a function of (i) what information agents pool and (ii) how they collectively reason from that shared information.

One cannot overstate the importance of group knowledge—knowledge dispersed among agents, often in inconsistent or incomplete forms. As Hayek (1945) emphasizes, a fundamental feature of economic and social activity is that knowledge is decentralized and individually held. A long-standing focus in economics has been on how such dispersed private information can be aggregated through public signals such as prices—see, for example, the pioneering work on rational-expectations equilibria by Radner (1979). In other contexts such as voting, the aggregation of dispersed information is likewise critical for collective decision-making.¹

More broadly, the aggregation of group knowledge—often dispersed across agents in inconsistent or contradictory forms—is a central topic in fields such as computer science and artificial intelligence, logic, and philosophy (e.g., Fagin et al., 2003; Meyer and van der Hoek, 1995). For instance, computer-based systems such as databases often contain

¹See, for instance, Austen-Smith and Banks (1996), Duggan and Martinelli (2001), Feddersen and Pesendorfer (1997), and Piccione and Rubinstein (2024). This paper does not address incentives to misreport or the costs of communication. However, the formal analysis of distributed knowledge and its relation to individual beliefs is independent of such incentive considerations.

contradictory pieces of information, raising the natural question of how to aggregate such inconsistent sources into a single, contradiction-free system (e.g., Konieczny and Pino Pérez, 2002).²

Specifically, this paper studies a notion of group knowledge termed *distributed knowledge*, which is first studied, independently, by Halpern and Moses (1990) and Hilpinen (1977) in computer science, logic, and philosophy.³ In the original context, a group of agents have distributed knowledge of a statement p if someone informed of everything that each agent knows would know p . For example, suppose that two agents, Ann and Bob, respectively know statements q and $(q \rightarrow p)$. Even if neither of them knows p , combining their knowledge together, Ann and Bob have distributed knowledge of p .

While the previous studies assume that individual agents have fully logical, fully introspective, and truthful knowledge (and thus so does group knowledge), this paper formalizes distributed knowledge among agents whose beliefs may fail to be truthful, consistent, or logically monotonic. The paper presents a formalization of distributed knowledge that turns out to be, in a minimal sense, logically monotonic, positively introspective, and truthful, independently of any particular assumptions on agents' beliefs.

Truthfulness of distributed knowledge is not a property imposed on the group but follows from what is taken to be an agent's *information*. More specifically, in order for an event E to be distributed knowledge among agents whose beliefs may be inconsistent with each other, this paper defines distributed knowledge in a way such that (i) the agents pool their information and that (ii) they make an inference to deduce E from their collective information. I define agents' collective information as those events such that some agent believes any of their implications whenever they are true. This idea follows the notion of self-evidence (e.g., Monderer and Samet, 1989) in modeling interactive knowledge in game theory. Thus, this paper studies distributed knowledge as opposed to "distributed belief," which would correspond to the aggregation of individual beliefs. False beliefs cannot become distributed knowledge. This paper systematically studies how the content of agents' collective information affects qualitative aspects of distributed knowledge and how distributed knowledge inherits properties of individual beliefs. This paper also presents some game-theoretic applications of distributed knowledge.

Distributed knowledge formalized in this paper is truthful, monotonic, and positively introspective. That is, if some event is distributed knowledge then it is true so that agents can only have truthful distributed knowledge. Distributed knowledge is thus consistent (i.e., if an event E is distributed knowledge, then the negation $\neg E$ is not distributed knowledge). When agents' beliefs are not true, their false beliefs do not lead to distributed knowledge. Even if agents are not logical reasoners, any log-

²In fact, information aggregation in a collective-decision setting has also been studied in computer science, logic, and philosophy: see, for example, Grossi and Pigozzi (2014).

³Two remarks are in order. First, Hilpinen (1977) calls it "impersonal knowledge." Second, see Halpern and Moses (1990, Footnote 1) and the references therein for their original terminology "implicit knowledge."

ical consequence of distributed knowledge turns out to be distributed knowledge. If agents have distributed knowledge of E , then they have distributed knowledge of the fact that E is distributed knowledge. These three properties embody the idea that a group of agents possess meaningful (e.g., contradiction-free and logically-closed) group knowledge.

In contrast, distributed knowledge is not necessarily negatively introspective for instance when individual beliefs are not truthful. If agents do not possess distributed knowledge of an event E , then it is not necessarily the case that they have distributed knowledge of the fact that E is not distributed knowledge. After aggregating collective information and making inferences about the world, agents may be collectively unaware of E .⁴ The result sheds new light on a limitation of group knowledge in that after pooling their information, agents may still not be able to resolve their own unawareness. Indeed, if agents' beliefs are truthful and logical in a certain sense as in the previous studies, then distributed knowledge inherits negative introspection.

If agents are positively introspective and logical reasoners as in the previous studies, distributed knowledge turns out to be knowledge of the least knowledgeable hypothetical individual ("wise man") who knows everything that each agent knows (e.g., Halpern and Moses, 1990). The formalization of distributed knowledge in this paper also accommodates Hilpinen (1977) if agents' knowledge is monotonic and positively introspective.

Returning to the economic relevance of group knowledge introduced earlier, this paper characterizes the informational content of a rational-expectations Nash equilibrium through the lens of distributed knowledge (Theorem 1). This application also provides a reason for studying distributed knowledge rather than distributed belief. A rational-expectations equilibrium is a fixed point of inference from behavior: each agent refines her information by what the other agents' strategies reveal, and such inference is sound only if it proceeds from true premises. Accordingly, the information each agent holds about the others' strategies corresponds to the distributed knowledge formed from the informational content of those strategies. In fact, if the informational content of a rational-expectations Nash equilibrium is formed as distributed belief instead, agents may make inferences from one another's actions that may be false, and the epistemic characterization of such equilibrium may fail.

Theorem 1 generalizes Minelli and Polemarchakis (2003, Proposition 1 and Corollary 2) to settings in which agents may hold beliefs rather than knowledge at the individual level. As discussed, knowledge must satisfy the truth axiom, whereas belief may be false. In interactive contexts, modeling agents as holding knowledge can raise conceptual issues: for instance, if Ann knows that Bob is rational, this implies that Bob truly is rational. For this reason, much of the game-theoretic literature tends to adopt belief rather than knowledge at the individual level as a modeling assumption. This paper contributes by bringing the concept of distributed knowledge—originally developed in computer science, logic, and philosophy as group knowledge—into economic and game-

⁴An agent is unaware of an event E if she does not know it and she does not know that she does not know it. Pioneering papers on unawareness include Fagin and Halpern (1987) and Modica and Rustichini (1994, 1999).

theoretic analysis.

The paper is structured as follows. The rest of this section discusses related literature. Section 2 presents an underlying framework. Section 3 formulates distributed knowledge. Section 4 studies properties of distributed knowledge. Section 5 provides game-theoretic applications. Section 6 provides concluding remarks. Proofs are relegated to Appendix A. The Online Appendix provides supplementary discussions.

Related Literature

This paper is related to two strands of literature: (i) modeling distributed knowledge in various disciplines; and (ii) applications of distributed knowledge in game theory and economics.

Modeling Distributed Knowledge. The formalization of distributed knowledge originates with Halpern and Moses (1990) and Hilpinen (1977), who assume that individual knowledge is truthful, fully logical, and fully introspective. This paper dispenses with those assumptions at the individual level. Pacuit (2017) and Van De Putte and Klein (2018) likewise impose no logical or introspective properties on individual beliefs, but study distributed *belief*, which may be inconsistent, so that a group may hold false group beliefs. While distributed belief is an interesting concept, this paper instead formalizes distributed knowledge, which is truthful and hence consistent (recall the contradiction-free database). The two notions coincide when agents' beliefs are truthful, monotonic, and positively introspective, and may differ otherwise.

Distributed knowledge has recently been brought into game theory and economics by Crescenzi (2023) and Tóbiás (2021), and this paper is complementary to both. Crescenzi (2023) shows how distributed knowledge arises as the outcome of an explicit communication process and establishes a rich set of properties for the resulting notion. This paper abstracts from any particular protocol, deriving distributed knowledge from (i) each agent's informational content and (ii) the pooling of that information, so that protocols of the kind he studies are among those accommodated. A further difference is that agents here may hold false beliefs rather than knowledge. Tóbiás (2021) develops the lattice-theoretic duality between distributed and common knowledge in the partitioned model, and gives it a striking application to criminal procedure. The duality he establishes reappears in the present framework: under suitable conditions, common knowledge is the knowledge of the most informative hypothetical agent who is no more informative than any member of the group, and distributed knowledge that of the least informative hypothetical agent who is at least as informative as every member. Here the application is instead to rational-expectations Nash equilibria.

Distributed knowledge has been studied extensively in computer science, logic, and philosophy, largely through syntactic characterizations and completeness results relating syntax to semantics: see Fagin, Halpern, and Vardi (1992), Fagin et al. (2003), Halpern and Moses (1992), Herzig et al. (2020), van der Hoek and Meyer (1992, 1997), and Meyer

and van der Hoek (1995).⁵ Since this is the first study of distributed knowledge that imposes no assumptions on individual beliefs, and since a semantic framework permits direct game-theoretic application, I work semantically. The two approaches are known to diverge in subtle ways (Ågotnes and Wáng, 2017; Gerbrandy, 1999; Roelofsen, 2007; van der Hoek and Meyer, 1992; van der Hoek, van Linder, and Meyer, 1999): with partitional knowledge, distributed knowledge is the coarsest common refinement of the agents’ partitions, but that equality may not be modally definable (Samet, 2010; van der Hoek and Meyer, 1992), and the “principle of full communication” may fail on an arbitrary state space (van der Hoek, van Linder, and Meyer, 1999). Since the purpose here is to study properties of distributed knowledge, I allow arbitrary unions and intersections of events, so that distributed knowledge is always well-defined: it reduces to the coarsest common refinement when beliefs are partitional, and the paper also characterizes it when beliefs are given by possibility correspondences (Aumann, 1976, 1999; Geanakoplos, 2021; Morris, 1996) or Kripke frames (Chellas, 1980; Fagin et al., 2003).

Game-Theoretic Applications. Distributed knowledge concerns what agents may come to know after communicating, explicitly or implicitly. Bacharach (1985), Cave (1983), Geanakoplos and Polemarchakis (1982), and McKelvey and Page (1986) study how communication among differentially informed agents leads to agreement, and Tallon, Vergnaud, and Zamir (2004) allow agents to hold erroneous beliefs. As in the literature on modeling distributed knowledge, this paper does not model an explicit communication process but asks how the content of agents’ beliefs shapes group knowledge. Following McKelvey and Page (1986) and Minelli and Polemarchakis (2003), this paper develops an epistemic characterization of rational-expectations Nash equilibria and relates it to distributed knowledge, allowing agents to hold beliefs rather than knowledge at the individual level.⁶

2 Framework

This section sets up the framework of the paper. Section 2.1 defines the basic framework which I call a belief space. Section 2.2 defines properties of individual agents’ beliefs. Since the previous literature on distributed knowledge has employed the possibility-correspondence models, Section 2.3 provides some definitions on possibility correspondences and shows that the framework of this paper nests possibility correspondences. Building on this framework, the next section formalizes distributed knowledge.

⁵See, for instance, Aumann (1999) and Fukuda (2024) for applications of the syntax approach to modeling agents’ knowledge and qualitative beliefs in game theory and economics.

⁶For other contexts including games with incomplete information, general equilibrium, and mechanism design in which information revelation occurs in equilibrium, see, for instance, Angeletos and Werning (2006), Dasgupta (2007), Forges and Minelli (1997, 1998), Grafenhofer and Kuhle (2016), and Huang (2021) and the papers cited therein.

2.1 Belief Spaces

The framework of this paper builds on a set-theoretic representation of agents' beliefs by their belief operators. Throughout the paper, let I denote a non-empty set of *agents*. A *belief space* (of I) is a tuple $\overline{\Omega} := \langle (\Omega, \mathcal{D}), (B_i)_{i \in I} \rangle$.

First, Ω is a non-empty set of *states* of the world. The state space Ω is a sample space on which agents' beliefs are represented. Second, $\mathcal{D}$ is a subset of the power set $\mathcal{P}(\Omega)$ called the *domain*. The domain $\mathcal{D}$ consists of *events* about which the agents reason. Each event $E \in \mathcal{D}$ is a subset of Ω which describes a certain aspect of the state space. For ease of analysis, I assume that $\mathcal{D}$ is a complete algebra on Ω : (i) $\mathcal{D}$ contains the empty set $\emptyset$ and the entire set Ω ; (ii) if E is an event, then its complement $\neg E$ (also denoted by E^c) is also an event; (iii) and if $\mathcal{E}$ is an arbitrary subset of $\mathcal{D}$, then its intersection and its union are events, i.e., $\bigcap \mathcal{E} := \bigcap_{E \in \mathcal{E}} E \in \mathcal{D}$ and $\bigcup \mathcal{E} := \bigcup_{E \in \mathcal{E}} E \in \mathcal{D}$.⁷ Third, each agent i 's *belief operator* $B_i : \mathcal{D} \rightarrow \mathcal{D}$ associates, with each event E , the event that (i.e., the set of states at which) agent i believes E .

The generality of the framework comes from the facts that the analysts can assign logical and introspective properties of agents' beliefs one by one through belief operators and that logical and introspective properties can be expressed by properties of the agents' belief operators in a transparent way. Indeed, as far as a semantic representation of agents' beliefs is concerned, this approach is the most general (or minimal) in that the only restriction is the closure of one's beliefs under logical equivalence: if two events E and F are set-theoretically equivalent (i.e., $E = F$) then the corresponding beliefs are equivalent (i.e., $B_i(E) = B_i(F)$).⁸

2.2 Properties of Beliefs

I define properties of beliefs. While I do not impose each property as an assumption a priori, I study how distributed knowledge inherits each property. Throughout this subsection, fix agent $i \in I$. I start with the following four logical properties of belief operators. First, B_i satisfies *Monotonicity* if

$$B_i(E) \subseteq B_i(F) \text{ for any } E, F \in \mathcal{D} \text{ with } E \subseteq F.$$

Second, B_i satisfies *Necessitation* if

$$B_i(\Omega) = \Omega.$$

Third, B_i satisfies *(Non-empty) Conjunction* if

$$\bigcap_{E \in \mathcal{E}} B_i(E) \subseteq B_i\left(\bigcap \mathcal{E}\right) \text{ for any } \mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}.$$

⁷Throughout the paper, I follow the convention that if $\mathcal{E} = \emptyset$ (i.e., $\mathcal{E}$ is the empty collection of subsets of Ω) then $\bigcap \mathcal{E} := \Omega$ and $\bigcup \mathcal{E} := \emptyset$.

⁸Previous contributions on distributed knowledge, such as Fagin et al. (2003) and Pacuit (2017), use the framework called neighborhood systems. As shown formally in Online Appendix D.1, when none of the properties on one's beliefs are assumed, the framework of neighborhood systems is equally general in the sense that the only restriction is the closure of one's beliefs under logical equivalence.

Fourth, B_i satisfies the *Kripke property* if, for each $(\omega, E) \in \Omega \times \mathcal{D}$,

$$\omega \in B_i(E) \text{ if and only if } E \supseteq b_{B_i}(\omega) := \bigcap \{F \in \mathcal{D} \mid \omega \in B_i(F)\}.$$

The mapping $b_{B_i} : \Omega \rightarrow \mathcal{D}$ is referred to as i 's possibility correspondence in economics and game theory (e.g., Aumann, 1976, 1999; Geanakoplos, 2021; Morris, 1996). Agent i considers ω' possible at ω if $\omega' \in b_{B_i}(\omega)$. Assigning i 's possibility correspondence is equivalent to assigning the accessibility relation in logic (e.g., Chellas, 1980; Fagin et al., 2003). Agent i believes E at ω if and only if (hereafter, often abbreviated as iff) E includes the set of states considered possible at ω . In the context of this paper, it can be shown that the Kripke property is jointly characterized as Monotonicity, Necessitation, and Conjunction.

Next, three introspective properties of belief operators are in order. First, B_i satisfies the axiom of Truth (*Truth Axiom* for short) if

$$B_i(E) \subseteq E \text{ for any } E \in \mathcal{D}.$$

Truth Axiom distinguishes between knowledge and belief: if agent i “knows” E at ω then E is true at ω .

Second, B_i satisfies *Positive Introspection* if

$$B_i(\cdot) \subseteq B_i B_i(\cdot).$$

Third, B_i satisfies *Negative Introspection* if

$$(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot).$$

Note that Truth Axiom and Negative Introspection yield Positive Introspection and Necessitation.

Finally, this paper also represents the informational content of one's beliefs as a collection of events in the sense that distributed knowledge is deduced from pooling the collections of events that represent the agents' beliefs. To that end, this subsection introduces one important definition.

Call an event E (*strongly*) *self-evident* to agent i if, for any $F \in \mathcal{D}$, $E \subseteq F$ implies $E \subseteq B_i(F)$, i.e., i believes any implication of E whenever E is true. Denote by $\mathcal{J}_{B_i}$ the collection of events self-evident to i :

$$\mathcal{J}_{B_i} := \{E \in \mathcal{D} \mid \text{for any } F \in \mathcal{D}, \text{ if } E \subseteq F \text{ then } E \subseteq B_i(F)\}^9 \quad (1)$$

In economics and game theory, the notion of (a weaker form of) self-evidence is used to define common knowledge (e.g., Monderer and Samet, 1989) if B_i is assumed to satisfy Monotonicity.¹⁰

⁹As a technical remark, $\mathcal{J}_{B_i}$ is not empty because $\emptyset \in \mathcal{J}_{B_i}$.

¹⁰The weaker form of self-evidence is defined as follows. An event E is self-evident to i if $E \subseteq B_i(E)$: agent i believes E whenever E is true. When agent i 's belief operator satisfies Monotonicity, this definition is equivalent to the one in the main text. Since this paper dispenses with Monotonicity, this paper, following Fukuda (2020), adopts the stronger form of self-evidence.

2.3 Possibility Correspondences

In the literature on modeling distributed knowledge, distributed knowledge has been studied under the framework of possibility correspondences/accessibility relations. This subsection shows that the framework of this paper nests it.

Call a mapping $b : \Omega \rightarrow \mathcal{D}$ a *possibility correspondence* if it satisfies

$$B_b(E) := \{\omega \in \Omega \mid b(\omega) \subseteq E\} \in \mathcal{D} \text{ for any } E \in \mathcal{D}.$$

If B_i satisfies the Kripke property, then b_{B_i} is a possibility correspondence because the belief operator $B_{b_{B_i}}$ induced by b_{B_i} coincides with the original belief operator B_i (i.e., $B_i = B_{b_{B_i}}$).

Fix a possibility correspondence $b : \Omega \rightarrow \mathcal{D}$. It is well-known that the introspective properties of the belief operator B_b can be stated in terms of the corresponding properties of the possibility correspondence b (e.g., Chellas, 1980). First, b is *reflexive* (i.e., $\omega \in b(\omega)$ for all $\omega \in \Omega$) iff B_b satisfies Truth Axiom. Second, b is *transitive* (i.e., $\omega' \in b(\omega)$ implies $b(\omega') \subseteq b(\omega)$) iff B_b satisfies Positive Introspection. Third, b is *Euclidean* (i.e., $\omega' \in b(\omega)$ implies $b(\omega) \subseteq b(\omega')$) iff B_b satisfies Negative Introspection.

Finally, I define the *reflexive transitive closure* of b as the smallest possibility correspondence $\bar{b} : \Omega \rightarrow \mathcal{D}$ which is reflexive and transitive and which includes b . That is, (i) $\bar{b}$ is reflexive and transitive; (ii) $b(\cdot) \subseteq \bar{b}(\cdot)$; and (iii) if $b' : \Omega \rightarrow \mathcal{D}$ is a reflexive and transitive possibility correspondence with $b(\cdot) \subseteq b'(\cdot)$, then $\bar{b}(\cdot) \subseteq b'(\cdot)$.¹¹

3 Formalization of Distributed Knowledge

This section defines distributed knowledge by knowledge logically induced from a pool of agents' collective information in two steps. The first step provides a general formalization of knowledge logically deduced from a collection of information. The second step defines a particular form of a pool of agents' collective information. Online Appendix B discusses the two-step formalization and provides an example.

The First Step. For any collection $\mathcal{I}$ of events (i.e., $\mathcal{I} \subseteq \mathcal{D}$), the first step defines an operator $K_{\mathcal{I}} : \mathcal{D} \rightarrow \mathcal{D}$ as follows. For each $E \in \mathcal{D}$, let

$$K_{\mathcal{I}}(E) := \{\omega \in \Omega \mid \text{there is } F \in \mathcal{I} \text{ with } \omega \in F \subseteq E\} = \bigcup \{F \in \mathcal{I} \mid F \subseteq E\}. \quad (2)$$

The operator $K_{\mathcal{I}}$ is a knowledge operator in the sense that an event E is known at ω (i.e., $\omega \in K_{\mathcal{I}}(E)$) if and only if E is deduced from an event $F \in \mathcal{I}$ that is true at ω (i.e., $\omega \in F \subseteq E$). Since $\mathcal{D}$ is a complete algebra, $K_{\mathcal{I}}(E)$ is itself an event, i.e., $K_{\mathcal{I}}$ is well-defined (see the right-most-hand side of Expression (2)). As Proposition 1 in Section 4.1 shows, $K_{\mathcal{I}}$ satisfies Truth Axiom, Monotonicity, and Positive Introspection.

¹¹Letting $\mathfrak{B}_b$ be the collection of reflexive and transitive possibility correspondences $b' : \Omega \rightarrow \mathcal{D}$ with $b(\cdot) \subseteq b'(\cdot)$, it can be seen that $\bar{b}(\omega) = \bigcap_{b' \in \mathfrak{B}_b} b'(\omega)$ for each $\omega \in \Omega$.

The Second Step. The second step specifies a particular form of a collection $\mathcal{I}$ of information as agents' collective information. To that end, fix a group of agents $G \in \mathcal{P}(I) \setminus \{\emptyset\}$. Recalling that $\mathcal{J}_{B_i}$ is the collection of events E such that agent i believes any logical consequence of E , I define $\mathcal{I} = \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$ as:

$$\mathcal{I}((\mathcal{J}_{B_i})_{i \in G}) := \left\{ \bigcap_{i \in H} E_i \in \mathcal{D} \mid \begin{array}{l} H \in \mathcal{P}(G) \setminus \{\emptyset\} \text{ and} \\ E_i \in \mathcal{J}_{B_i} \text{ for each } i \in H \end{array} \right\}. \quad (3)$$

Expression (3) states that the collective information $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$ consists of the intersection of a pool of agents' information among a non-empty sub-group H , where an agent's information refers to her self-evident event.

With Expression (3) in mind, I define the distributed knowledge operator $D_G : \mathcal{D} \rightarrow \mathcal{D}$ of G by

$$D_G := K_{\mathcal{I}((\mathcal{J}_{B_i})_{i \in G})}.$$

Thus, for each $E \in \mathcal{D}$, $\omega \in D_G(E)$ if there exist $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $E_i \in \mathcal{J}_{B_i}$ for each $i \in H$ such that $\omega \in \bigcap_{i \in H} E_i \subseteq E$.¹²

Suppose that events E_1 and E_2 are available to a hypothetical individual whose knowledge is associated with distributed knowledge among G in the sense that $E_i \in \mathcal{J}_{B_i}$ for each $i \in \{1, 2\} \subseteq G$. Expression (3) means that the hypothetical individual can combine E_1 and E_2 into one piece of information $E_1 \cap E_2$ to make inferences.¹³

Each element of the collective information $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$ is an event $\bigcap_{i \in H} E_i$ formed from self-evident collections $(\mathcal{J}_{B_i})_{i \in H}$. While each self-evident event E_i could be i 's belief $B_i(F_i)$ for some $F_i \in \mathcal{D}$ if $B_i(F_i) \in \mathcal{J}_{B_i}$, distributed knowledge is defined in a way that the distributed knowledge of an event is deduced from a conjunction of agents' collective information itself (i.e., E_i), not their beliefs (i.e., $B_i(E_i)$).¹⁴ Proposition 6 in Section 4.4 characterizes when distributed knowledge can be deduced from events of the form $\bigcap_{i \in H} B_i(E_i)$ where $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $E_i \in \mathcal{D}$ for each $i \in H$. Section 4.4 examines how the definition of distributed knowledge in this paper nests those in the previous literature when agents' beliefs satisfy additional properties.

¹²Technically, one can allow for $G = \emptyset$ because it leads to $\mathcal{I}((\mathcal{J}_{B_i})_{i \in \emptyset}) = \emptyset$ and $D_\emptyset(\cdot) = K_\emptyset(\cdot) = \emptyset$.

¹³While Section 4 compares individual beliefs and distributed knowledge in detail, note that $E_1 \cap E_2$ may constitute new information—even for identical agents—if their beliefs violate Conjunction. That is, although both E_1 and E_2 may be self-evident to agent i , the conjunction $E_1 \cap E_2$ may not be. To highlight the role of information aggregation, consider a group of identical agents who each possess positively introspective and monotonic knowledge. Even in such a case, distributed knowledge may differ from individual knowledge. For example, while each agent may not know statement “ q ” (event $E_1 \cap E_2$) despite knowing “ p ” and “ $p \rightarrow q$ ” (events E_1 and E_2), the group may possess distributed knowledge of “ q ” (event $E_1 \cap E_2$). Formally, it may occur when $\mathcal{J}_{B_1} = \mathcal{J}_{B_2} \subsetneq \mathcal{I}(\mathcal{J}_{B_1}, \mathcal{J}_{B_2})$. In this context, distributed knowledge captures what an agent could know if she were able to integrate the “dispersed knowledge within her own mind.” Remark 1 in Section 4.3 revisits this point.

¹⁴One can show that if B_i satisfies Truth Axiom, Positive Introspection, and Monotonicity, then $\mathcal{J}_{B_i} = \{B_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$.

4 Properties of Distributed Knowledge

This section examines properties of distributed knowledge.

4.1 General Properties of Distributed Knowledge

I start by studying general properties of a distributed knowledge operator.

Proposition 1. 1. If $G_1, G_2 \in \mathcal{P}(I) \setminus \{\emptyset\}$ satisfy $G_1 \subseteq G_2$, then $D_{G_1}(\cdot) \subseteq D_{G_2}(\cdot)$.

2. Let $G \in \mathcal{P}(I) \setminus \{\emptyset\}$, and partition G into $(G_j)_{j \in J}$. Denote by D_G^J the distributed knowledge operator among the group of hypothetical agents whose knowledge operators are $(D_{G_j})_{j \in J}$. Then, $D_G = D_G^J$.

3. For any $G \in \mathcal{P}(I) \setminus \{\emptyset\}$, the distributed knowledge operator D_G satisfies Truth Axiom, Monotonicity, and Positive Introspection.

The first part of Proposition 1 states that distributed knowledge is monotonic in group. This holds because the amount of collective information is monotonic in group and because distributed knowledge is logically derived from collective information.

The second part states that distributed knowledge is independent of the way in which knowledge is aggregated within a group when the group is partitioned. In order to find distributed knowledge among a group G , one can first find distributed knowledge among each subgroup G_j and then find distributed knowledge among hypothetical individuals who represent the knowledge possessed by subgroups $(G_j)_{j \in J}$. Roughly, the intuition follows from the facts that the amounts of collective information are identical irrespective of two different ways (i.e., $\bigcup_{i \in G} \mathcal{J}_{B_i} = \bigcup_{j \in J} \bigcup_{i \in G_j} \mathcal{J}_{B_i}$) and that distributed knowledge is logically deduced from collective information.

This second part implies that simultaneous pooling and sequential aggregation through disjoint subgroups yield the same distributed knowledge. Letting a group of agents be $G = \{1, \dots, n\}$, compare two ways in which distributed knowledge is formed. One is to extract agents' collective information simultaneously (i.e., in one step) to form directly $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$. In the other, one could find distributed knowledge among G by sequentially finding the distributed knowledge operator among two hypothetical agents whose belief operators are respectively $D_{\{1, \dots, k-1\}}$ and $D_{\{k\}}$ for $k \in \{2, \dots, n\}$.

The third part says that distributed knowledge is true, monotonic, and positively introspective irrespective of agents' individual beliefs. Informally, (i) by knowledge, distributed knowledge is truthful; (ii) if E is distributed knowledge then a (sub-)group of agents can “infer” E from a pool of their information. This inference step yields Monotonicity; and (iii) if E is distributed knowledge among a group, then the group as a whole can “recognize” that E is distributed knowledge. This third part of the proposition hinges on the way in which knowledge is defined through Expression (2) irrespective of a particular choice of $\mathcal{I}$.

As I formalize distributed knowledge as opposed to distributed “belief,” distributed knowledge is truthful and monotonic. In particular, even if individual agents' beliefs

are contradictory or non-monotonic, resulting distributed knowledge is consistent and monotonic. Consistency means that if E is distributed knowledge then $\neg E$ is not distributed knowledge.¹⁵ An agent's false belief does not lead to distributed knowledge because distributed knowledge is truthful.

Not only is distributed knowledge truthful and monotonic, but also it is positively introspective. Positive Introspection of distributed knowledge could be justified together with that of common belief or common knowledge (Aumann, 1976; Friedell, 1969). The idea behind axiomatizations (especially, the “fixed-point” axiomatization) of common belief is to ensure Positive Introspection because the intuitive definition of common belief by the iteration of mutual beliefs may not necessarily satisfy Positive Introspection (e.g., Lismont and Mongin, 1994).¹⁶ For example, in the possibility-correspondence model of knowledge, common knowledge is introduced by the transitive closure of (the union of) agents' possibility correspondences, where the transitivity of the possibility correspondence that represents common knowledge guarantees Positive Introspection. If agents' knowledge is truthful, monotonic, positively introspective, and conjunctive, then distributed knowledge turns out to be the “supremum” of individual agents' knowledge: distributed knowledge can be regarded as knowledge of the least knowledgeable “wise man” who knows everything that each agent knows (Halpern and Moses, 1990). In this case, common knowledge is considered to be the “infimum” of individual agents' knowledge as McCarthy et al. (1978) define the notion of what “any fool” knows.

The next proposition characterizes the informational content of distributed knowledge. To that end, let $\mathcal{J}_{D_G}$ be the collection of self-evident events according to D_G . Since D_G satisfies Monotonicity, recalling footnote 10, one has:

$$\mathcal{J}_{D_G} = \{E \in \mathcal{D} \mid E \subseteq D_G(E)\}.$$

Then, it can be seen that

$$D_G(E) = \bigcup \{F \in \mathcal{J}_{D_G} \mid F \subseteq E\} \text{ for all } E \in \mathcal{D}.$$

The next proposition characterizes this collection $\mathcal{J}_{D_G}$.

Proposition 2. *The collection $\mathcal{J}_{D_G}$ is the smallest collection (i) including $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$, (ii) containing the empty set $\emptyset$, and (iii) closed under any non-empty union.*

Proposition 2 characterizes the collection $\mathcal{J}_{D_G}$ which represents distributed knowledge. To the extent that the collective information $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$ plays an important role in formulating distributed knowledge, the informational content of distributed knowledge, $\mathcal{J}_{D_G}$, can be characterized as the smallest collection that includes the collective information $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$ and that is closed under arbitrary union.

¹⁵Consistency follows from Truth Axiom. Truth Axiom also implies that a contradiction in the form of $\emptyset$ can never be distributed knowledge.

¹⁶The iteration of mutual beliefs means that an event E is common belief if, everybody believes E , everybody believes that everybody believes E , and so forth *ad infinitum*.

I define $\delta((\mathcal{J}_{B_i})_{i \in G})$ as the smallest collection (i) that includes $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$, (ii) that contains the empty set $\emptyset$, and (iii) that is closed under any non-empty union. Formally, it can be seen that

$$\delta((\mathcal{J}_{B_i})_{i \in G}) = \left\{ \bigcup \mathcal{E} \in \mathcal{D} \mid \mathcal{E} \subseteq \mathcal{I}((\mathcal{J}_{B_i})_{i \in G}) \right\}.$$

The collection $\delta((\mathcal{J}_{B_i})_{i \in G})$ denotes the informational content of distributed knowledge as shown in Proposition 2. The operation $\delta(\cdot)$ will be used in Section 5 to formalize the informational content of agents' strategies.

4.2 Individual Beliefs and Distributed Knowledge

I study how distributed knowledge reflects logical properties of individual beliefs. I start with Necessitation, Conjunction, and the Kripke property.

Proposition 3. *Let $G \in \mathcal{P}(I) \setminus \{\emptyset\}$.*

1. (a) *The distributed knowledge operator D_G satisfies Necessitation if and only if, for each $\omega \in \Omega$, there is $F \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$ such that $\omega \in F$. In particular, if B_i satisfies Necessitation for some $i \in G$, then D_G satisfies Necessitation.*
- (b) *If B_i satisfies Necessitation for every $i \in G$, then $D_G = K_{\mathcal{I}_{\text{Nec}}^*}$, where*

$$\mathcal{I}_{\text{Nec}}^* = \left\{ \bigcap_{i \in G} E_i \in \mathcal{D} \mid E_i \in \mathcal{J}_{B_i} \text{ for each } i \in G \right\}.$$

2. *Assume that each B_i satisfies Monotonicity. If B_i satisfies Conjunction for every $i \in G$, then so does D_G . Moreover, $D_G = K_{\mathcal{I}_{\text{Con}}^*}$, where*

$$\mathcal{I}_{\text{Con}}^* = \left\{ \bigcap \mathcal{E} \in \mathcal{D} \mid \emptyset \neq \mathcal{E} \subseteq \bigcup_{i \in G} \mathcal{J}_{B_i} \right\}.$$

3. *Suppose that B_i satisfies the Kripke property for every $i \in G$.*

- (a) *The distributed knowledge operator D_G satisfies the Kripke property. Its possibility correspondence $b_{D_G} : \Omega \rightarrow \mathcal{D}$ satisfies*

$$b_{D_G}(\cdot) = \bigcap_{i \in G} \beta_{B_i}(\cdot),$$

where $\beta_{B_i}(\omega) := \bigcap \{F \in \mathcal{J}_{B_i} \mid \omega \in F\}$ is the reflexive transitive closure of b_{B_i} . The possibility correspondence b_{D_G} is reflexive and transitive.

- (b) Assume $\{\omega\} \in \mathcal{D}$ for all $\omega \in \Omega$. If each B_i satisfies Positive Introspection, then $\beta_{B_i}(\omega) = b_{B_i}(\omega) \cup \{\omega\}$ and $b_{D_G}(\omega) = (\bigcap_{i \in G} b_{B_i}(\omega)) \cup \{\omega\}$.¹⁷
- (c) If each B_i satisfies Truth Axiom and Positive Introspection, then $b_{B_i} = \beta_{B_i}$ and $b_{D_G}(\cdot) = \bigcap_{i \in G} b_{B_i}(\cdot)$.

Four remarks on Proposition 3 are in order. First, Proposition 3 (1) characterizes Necessitation. I remark on the tightness of the characterization: when no agent's belief operator satisfies Necessitation, distributed knowledge may or may not satisfy Necessitation. For example, suppose that each B_i satisfies Monotonicity but fails Necessitation. If $\omega \notin B_i(\Omega)$ for all $i \in G$, then $\omega \notin D_G(\cdot)$. In words, no event can be distributed knowledge at state ω . Consequently, D_G fails Necessitation.

For another example, consider two agents i and j with $E \in \mathcal{J}_{B_i}$ and $E^c \in \mathcal{J}_{B_j}$: while E is self-evident to agent $i \in G$, its complement E^c is self-evident to agent $j \in G$. If E is true at ω , then Ω is distributed knowledge at ω , as $\omega \in E \subseteq \Omega$. If E is false, then Ω is distributed knowledge at ω , as $\omega \in E^c \subseteq \Omega$. Generally, if there is $E \in \mathcal{D}$ with $E, E^c \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$, then D_G satisfies Necessitation.

Second, Proposition 3 (2) shows that if each B_i satisfies Monotonicity and Conjunction then distributed knowledge is also obtained through taking any non-empty conjunction of agents' information.¹⁸

Third, Proposition 3 (3) implies that if B_i satisfies the Kripke property, then so does $D_{\{i\}}$. Indeed, the possibility correspondence of $D_{\{i\}}$ amounts to β_{B_i} . Thus, Proposition 3 (3) states that distributed knowledge among a group G is generated from distributed knowledge among $\{i\}$ for all $i \in G$ through $b_{D_G}(\cdot) = \bigcap_{i \in G} \beta_{B_i}(\cdot)$.

Fourth, as illustrated in Figure 1, Proposition 3 (3) implies that $b_{D_G}(\cdot)$ is a reflexive and transitive possibility correspondence satisfying $\bigcap_{i \in G} b_{B_i}(\cdot) \subseteq b_{D_G}(\cdot)$. This sheds light on defining distributed knowledge from agents' possibility correspondences because such previous contributions as Halpern and Moses (1990) and Meyer and van der Hoek (1995) introduce the possibility correspondence that dictates D_G from the intersection of agents' possibility correspondences $\bigcap_{i \in G} b_{B_i}(\cdot)$ when agents' beliefs are fully introspective and satisfy the Kripke property.

For this point, two additional remarks are in order. Firstly, when the agents' belief operators do not satisfy Truth Axiom or Positive Introspection, $b_{D_G}(\cdot)$ may not necessarily be the intersection of agents' possibility correspondences $\bigcap_{i \in G} b_{B_i}(\cdot)$.

Secondly, it is not necessarily the case that $b_{D_G}(\cdot) = \bigcap_{i \in G} \beta_{B_i}(\cdot)$ is the reflexive transitive closure of $\bigcap_{i \in G} b_{B_i}(\cdot)$.¹⁹ To obtain the possibility correspondence that dictates D_G from the agents' beliefs which may violate either Truth Axiom or Positive Introspection, one needs to take the reflexive transitive closure of each b_{B_i} (which is β_{B_i}) instead of the reflexive transitive closure of $\bigcap_{i \in G} b_{B_i}(\cdot)$.

¹⁷More generally, without $\{\omega\} \in \mathcal{D}$, one has $\beta_{B_i}(\omega) = b_{B_i}(\omega) \cup \bigcap \{F \in \mathcal{D} \mid \omega \in F\}$ and $b_{D_G}(\omega) = (\bigcap_{i \in G} b_{B_i}(\omega)) \cup \bigcap \{F \in \mathcal{D} \mid \omega \in F\}$.

¹⁸Example OA.2 in Online Appendix C provides a counterexample when some agents' belief operators violate Conjunction.

¹⁹See Example OA.3 in Online Appendix C.

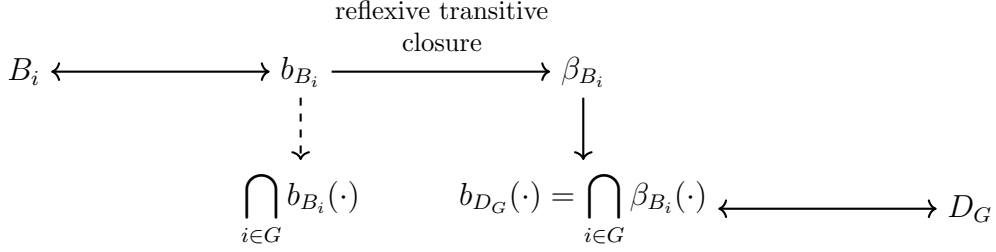


Figure 1: An Illustration of the Statement of Proposition 3 (3).

The next proposition studies relations between distributed knowledge and individual beliefs: when individual belief and distributed knowledge of a single agent coincide; and the sense in which distributed knowledge is the “supremum” of individual knowledge.

Proposition 4. *Let $G \in \mathcal{P}(I) \setminus \{\emptyset\}$, and let $i \in G$.*

1. (a) *The distributed knowledge operator $D_{\{i\}}$ is the maximum operator that simultaneously satisfies $D_{\{i\}}(\cdot) \subseteq B_i(\cdot)$, Truth Axiom, Positive Introspection, and Monotonicity. That is, if $K(\cdot)$ satisfies Truth Axiom, Positive Introspection, Monotonicity, and $K(\cdot) \subseteq B_i(\cdot)$, then $K(\cdot) \subseteq D_{\{i\}}(\cdot)$.*
- (b) *$B_i = D_{\{i\}}$ iff B_i satisfies Truth Axiom, Monotonicity, and Positive Introspection.*
2. *Let $K_G : \mathcal{D} \rightarrow \mathcal{D}$ satisfy $\bigcup_{i \in G} D_{\{i\}}(\cdot) \subseteq K_G(\cdot)$. If K_G additionally satisfies Monotonicity and Conjunction, then $D_G(\cdot) \subseteq K_G(\cdot)$.*

Proposition 4 (1) states that distributed knowledge of a single agent is a stronger notion than her belief and that they coincide with each other when her belief satisfies the three properties of Truth Axiom, Positive Introspection, and Monotonicity. By Proposition 1 (2), distributed knowledge among G coincides with distributed knowledge among the hypothetical agents whose belief (indeed, knowledge) operators are $(D_{\{i\}})_{i \in G}$.

On a related point, an agent i ’s false belief never enters the collective information on i ’s part in the following sense: if $\omega \in B_i(F) \setminus F$ then there is no $E \in \mathcal{J}_{B_i}$ with $\omega \in E \subseteq F$.²⁰ This means that if $\omega \in B_i(F) \setminus F$ then $\omega \notin D_{\{i\}}(F)$.

Proposition 4 (2) is related to Proposition 3 (3). To see this, assume that each B_i satisfies the Kripke property. Then, D_G satisfies the Kripke property. By Proposition 4 (2), D_G is the “minimal” belief operator satisfying $\bigcup_{i \in G} D_{\{i\}}(\cdot) \subseteq D_G(\cdot)$ (as well as Truth Axiom and Positive Introspection). Since $D_{\{i\}}$ is induced by β_{B_i} , $D_G(\cdot)$ is induced by $b_{D_G}(\cdot) = \bigcap_{i \in G} \beta_{B_i}(\cdot)$. I will discuss (in Corollary 1) the sense in which distributed knowledge is associated with the least knowledgeable “wise man” as knowledgeable as any individual agent (Halpern and Moses, 1990). Also, I remark that if $K_G : \mathcal{D} \rightarrow \mathcal{D}$ satisfies Monotonicity, Conjunction, and $\bigcup_{i \in G} B_i(\cdot) \subseteq K_G(\cdot)$, then $D_G(\cdot) \subseteq K_G(\cdot)$. This is because distributed knowledge among $\{i\}$ implies i ’s belief, i.e., $D_{\{i\}}(\cdot) \subseteq B_i(\cdot)$.

²⁰To see this, suppose to the contrary that there exists $E \in \mathcal{J}_{B_i}$ such that $\omega \in E \subseteq F$. Then, since $E \in \mathcal{J}_{B_i}$, it follows that $E \subseteq B_i(F)$. Thus, $\omega \in E \subseteq B_i(F) \cap F$, a contradiction.

4.3 Negative Introspection and Collective Unawareness

I study how distributed knowledge inherits or violates Negative Introspection. The proposition below shows that, even after aggregating agents' information, agents as a group may still be unable to resolve their unawareness.

Proposition 5. *Let $G \in \mathcal{P}(I) \setminus \{\emptyset\}$.*

1. *Suppose that each B_i satisfies the Kripke property, Truth Axiom, and Negative Introspection.*
 - (a) *The distributed knowledge operator D_G satisfies Negative Introspection.*
 - (b) *The distributed knowledge operator satisfies $D_G = K_{\mathcal{I}_{\text{NI}}^*}$, where $\mathcal{I}_{\text{NI}}^*$ is the complete algebra generated by $\bigcup_{i \in G} \mathcal{J}_{B_i}$. Indeed, $\mathcal{I}_{\text{NI}}^* = \mathcal{J}_{D_G}$.*
2. *Suppose that some B_i fails either the Kripke property or Truth Axiom. Then, D_G may fail Negative Introspection even if every B_i satisfies it.*

The proof of Proposition 5 (1a) is given as follows. Since each B_i satisfies Truth Axiom and Negative Introspection, it satisfies Positive Introspection. By Proposition 3, the distributed knowledge operator D_G satisfies the Kripke property with its possibility correspondence $\bigcap_{i \in G} b_{B_i}$. Since the Euclidean property for each b_{B_i} (which characterizes Negative Introspection of B_i) is preserved under the intersection, D_G satisfies Negative Introspection.

Proposition 5 (1b) states that if agents have fully introspective knowledge with the Kripke property, then distributed knowledge is associated with the knowledge deduced from the complete algebra $\mathcal{I}_{\text{NI}}^*$ generated by agents' self-evident events $\bigcup_{i \in G} \mathcal{J}_{B_i}$. Distributed knowledge does not have any more than the complete algebra $\mathcal{I}_{\text{NI}}^*$ in that $\mathcal{J}_{D_G} = \mathcal{I}_{\text{NI}}^*$.²¹

Proposition 5 (2) contrasts with Colombetti (1993), who provides a Kripke-frame example in which common belief fails to inherit Negative Introspection from agents' beliefs. Proposition 5 (2) establishes a related result for distributed knowledge.

I prove Proposition 5 (2) by providing (counter-)examples. The first example exhibits a violation of Truth Axiom while the second exhibits a violation of the Kripke property.

Example 1. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$, and let B_1 and B_2 be as in Table 1. Both B_1 and B_2 satisfy the Kripke property, Positive Introspection, and Negative Introspection. While B_2 satisfies Truth Axiom, B_1 fails it. The distributed knowledge operator $D_{\{1,2\}}$, depicted in Table 1, violates Negative Introspection: for instance,

$$(\neg D_{\{1,2\}})(\{\omega_2\}) = \{\omega_1, \omega_3\} \not\subseteq \{\omega_1\} = D_{\{1,2\}}(\neg D_{\{1,2\}})(\{\omega_2\}).$$

²¹In other words, if the agents' collective information is defined by the complete algebra $\mathcal{I}_{\text{NI}}^*$, then the resulting knowledge operator $K_{\mathcal{I}_{\text{NI}}^*}$ satisfies Negative Introspection.

E	$B_1(E)$	$B_2(E)$	$B_3(E) = B_4(E)$	$D_{\{1,2\}}(E)$	$D_{\{3\}}(E)$	$D_{\{3,4\}}(E)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\emptyset$	$\{\omega_1\}$	$\emptyset$	$\{\omega_1\}$
$\{\omega_2\}$	$\{\omega_2, \omega_3\}$	$\emptyset$	$\{\omega_2\}$	$\{\omega_2\}$	$\emptyset$	$\emptyset$
$\{\omega_3\}$	$\emptyset$	$\emptyset$	$\{\omega_3\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_2\}$	Ω	$\{\omega_1\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_1, \omega_3\}$	$\{\omega_1, \omega_3\}$
$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\emptyset$	$\{\omega_2, \omega_3\}$	$\emptyset$	$\emptyset$
Ω	Ω	Ω	Ω	Ω	Ω	Ω

Table 1: (Counter-)examples for Proposition 5 (2) (Examples 1 and 2).

To get an idea, suppose that agents 1 and 2 are reasoning about a company's performance. At state ω_1 , the company's performance is bad and the company is not window-dressing. At state ω_2 , the performance is good and the company is not window-dressing. At state ω_3 , the company is window-dressing so that the company's performance may appear good (even if the underlying performance is not good).

Agent 1 believes, at state $\omega \in \{\omega_1, \omega_2\}$, any event containing ω . At state ω_3 , she believes any event containing ω_2 . Thus, at ω_1 , agent 1 considers $\{\omega_1\}$ possible. At $\omega \in \{\omega_2, \omega_3\}$, she (naively) considers $\{\omega_2\}$ possible.

Agent 2 believes, at state ω_1 , any event containing ω_1 . At state $\omega \in \{\omega_2, \omega_3\}$, she believes $\{\omega_2, \omega_3\}$ and Ω . Thus, at ω_1 , agent 2 considers $\{\omega_1\}$ possible. At $\omega \in \{\omega_2, \omega_3\}$, she (cautiously) considers $\{\omega_2, \omega_3\}$ possible.

Consider the agents' distributed knowledge. While $D_{\{1,2\}}$ satisfies the Kripke property, $b_{D_{\{1,2\}}}$ does not partition Ω . While $b_{D_{\{1,2\}}}(\omega) = \{\omega\}$ for each $\omega \in \{\omega_1, \omega_2\}$, $b_{D_{\{1,2\}}}(\omega_3) = \{\omega_2, \omega_3\}$. That is, while the hypothetical individual whose knowledge is associated with distributed knowledge would consider $\{\omega_2\}$ possible at ω_2 (at which the company is not window-dressing), the hypothetical individual would consider $\{\omega_2, \omega_3\}$ (the company's real performance appears good, either because it is good or due to window-dressing) possible at ω_3 (at which the company is window-dressing). Now, let $E = \{\omega_2\}$ (i.e., the company is not window-dressing and is performing well). When the company is window-dressing (i.e., at ω_3), E is not distributed knowledge. At the same time, at ω_3 , it is not distributed knowledge that E is not distributed knowledge. Thus, to the extent that the failure of Negative Introspection of an agent's knowledge is associated with unawareness, the agents are "collectively unaware" of the event E at state ω_3 in the sense that E is not distributed knowledge at ω_3 and the event that E is not distributed knowledge is not distributed knowledge at ω_3 .²² Observe that agent 2's beliefs (knowledge) are fully introspective, and thus there is no possibility for agent 2 to be unaware of any event. $\square$

²²Following Modica and Rustichini (1994, 1999), an agent is unaware of an event E if she does not know E and she does not know that she does not know E .

Example 2. As in the previous example, let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Add agent 3 so that B_3 is defined in Table 1. While B_3 satisfies Truth Axiom, Negative Introspection, Positive Introspection, and Necessitation, it violates Conjunction and Monotonicity. For simplicity, consider the distributed knowledge operator $D_{\{3\}}$ depicted in Table 1. It violates Negative Introspection.

At state ω_1 (at which the company’s performance is “truly” bad), agent 3 believes $\{\omega_1, \omega_2\}$ (the company is not window-dressing), $\{\omega_1, \omega_3\}$ (the company’s underlying performance is bad), and a tautology Ω . Thus, agent 3 cannot identify the realization ω_1 at ω_1 , but otherwise she has correct beliefs. At state $\omega \in \{\omega_2, \omega_3\}$, agent 3 is able to identify the realization ω , i.e., she is able to discern whether the company is window-dressing or not. She, however, does not believe $\{\omega_2, \omega_3\}$. $\square$

Remark 1. Even if each agent does not exhibit any unawareness, group knowledge may exhibit unawareness. Consider agent 4 with $B_4 = B_3$. Then, $D_{\{3,4\}}$ exhibits unawareness while B_3 and B_4 do not. Incidentally, this is also an example in which a group of identical agents has distributed knowledge of an event which no member believes (knows) owing to the way in which distributed knowledge is generated from combining the agents’ collective information (recall footnote 13).

4.4 Comparisons

Finally, this subsection compares the distributed knowledge operator D_G of this paper with the previous literature: (i) the original formulations of distributed knowledge by Halpern and Moses (1990) and Hilpinen (1977) and (ii) the formulations of distributed beliefs by Pacuit (2017) and Van De Putte and Klein (2018).²³

4.4.1 The Original Formulations

The following proposition demonstrates that if agents’ beliefs (knowledge) satisfy Truth Axiom, Positive Introspection and Monotonicity then distributed knowledge can be seen as knowledge logically deduced from agents’ individual knowledge.

Proposition 6. *Let $G \in \mathcal{P}(I) \setminus \{\emptyset\}$, and let*

$$\mathcal{I}_B^* = \left\{ \bigcap_{i \in H} B_i(E_i) \in \mathcal{D} \mid H \in \mathcal{P}(G) \setminus \{\emptyset\} \text{ and } E_i \in \mathcal{D} \text{ for each } i \in H \right\}.$$

If each B_i satisfies Monotonicity and Positive Introspection, then $K_{\mathcal{I}_B^}(\cdot) \subseteq D_G(\cdot)$. If each B_i satisfies Truth Axiom, then $D_G(\cdot) \subseteq K_{\mathcal{I}_B^*}(\cdot)$.*

In Proposition 6, if each B_i satisfies Monotonicity and Positive Introspection, then any event deduced by putting agents’ beliefs together is distributed knowledge. If each

²³Online Appendix D.3 also discusses (iii) models of knowledge characterized by Truth Axiom, Positive Introspection, and Monotonicity.

B_i satisfies Truth Axiom, then distributed knowledge is deduced from pooling agents' beliefs.

The characterization of distributed knowledge in Proposition 6 is related to the original definition of distributed knowledge (impersonal knowledge) by Hilpinen (1977), which would be given as follows.

- An event E is distributed knowledge (impersonal knowledge) among $G \in \mathcal{P}(I) \setminus \{\emptyset\}$ at state ω , in the sense of Hilpinen (1977) (denoted by $\omega \in D_G^H(E)$), if there are $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $E_i \in \mathcal{D}$ for each $i \in H$ such that $\omega \in \bigcap_{i \in H} B_i(E_i)$ and $\bigcap_{i \in H} E_i \subseteq E$.

The definition of distributed knowledge in this paper can accommodate the original definition of distributed knowledge (impersonal knowledge) by Hilpinen (1977) if each agent has positively introspective, truthful, and monotonic knowledge. Formally:

Proposition 7. *Let $G \in \mathcal{P}(I) \setminus \{\emptyset\}$.*

1. $D_G(\cdot) \subseteq D_G^H(\cdot)$.
2. *If agents' beliefs satisfy Truth Axiom, Positive Introspection, and Monotonicity, then $D_G = D_G^H$.*

In words, the proposition states the following. First, if an event E is distributed knowledge at a state ω in the formulation of this paper then E is distributed knowledge at ω in the sense of Hilpinen (1977). Second, for the converse, suppose that agents' beliefs satisfy Truth Axiom, Positive Introspection, and Monotonicity. Then, the definition of distributed knowledge by Hilpinen (1977) coincides with that of this paper.

Next, to compare the distributed knowledge operator D_G with the formulation by Halpern and Moses (1990) in the context of a possibility-correspondence model, I present the following corollary of Propositions 3, 4, and 6.

Corollary 1. *Let $G \in \mathcal{P}(I) \setminus \{\emptyset\}$. Suppose that B_i satisfies Truth Axiom, Monotonicity, Positive Introspection, and Conjunction for each $i \in G$.*

1. D_G satisfies Truth Axiom, Monotonicity, Positive Introspection, Conjunction, and $\bigcup_{i \in G} B_i(\cdot) \subseteq D_G(\cdot)$.
2. *Let $K_G : \mathcal{D} \rightarrow \mathcal{D}$ satisfy Truth Axiom, Monotonicity, Positive Introspection, Conjunction, and $\bigcup_{i \in G} B_i(\cdot) \subseteq K_G(\cdot)$. Then, $D_G(\cdot) \subseteq K_G(\cdot)$.*

Corollary 1 sheds light on conditions under which distributed knowledge becomes the “supremum” of agents' knowledge (or the least knowledgeable “wise man” as in Halpern and Moses (1990)).²⁴

²⁴Under the assumptions of Corollary 1, common knowledge defined in Remark OA.1 in Online Appendix B.1 becomes the “infimum” of agents' knowledge so that distributed knowledge is the “dual” of common knowledge.

Remark 2. Assume that agents' beliefs (knowledge) satisfy Truth Axiom, Positive Introspection, Monotonicity, and Conjunction. Then, distributed knowledge also satisfies these axioms. Moreover, distributed knowledge is associated with knowledge of the least knowledgeable hypothetical individual who knows everything that each agent knows. Conjunction cannot be dropped because the hypothetical individual who dictates distributed knowledge can take intersections of agents' self-evident events.

Suppose that agents' knowledge additionally satisfies Necessitation. While each i 's knowledge is characterized by her (reflexive and transitive) possibility correspondence b_{B_i} , distributed knowledge is associated with $\bigcap_{i \in G} b_{B_i}$. This observation generalizes the introduction of distributed knowledge by $\bigcap_{i \in G} b_{B_i}$ in Halpern and Moses (1990), in which each b_{B_i} forms a partition of the state space Ω (i.e., each B_i also satisfies Negative Introspection). Thus, the definition of distributed knowledge coincides with that of Halpern and Moses (1990) if each agent's knowledge satisfies the Kripke property, Truth Axiom, Positive Introspection, and Negative Introspection. The previous argument shows that Negative Introspection could be dropped.

4.4.2 Distributed Beliefs

The definition of Hilpinen (1977), however, would reflect (monotonic) "collective belief" (or "distributed belief") when it comes to agents' beliefs on the following grounds. His notion of distributed knowledge may fail Truth Axiom if some agents' beliefs do not satisfy Truth Axiom (in contrast, if every agent's belief satisfies Truth Axiom, then his notion of distributed knowledge satisfies Truth Axiom). If each agent's belief satisfies Conjunction and Necessitation, then an event E is distributed knowledge (impersonal knowledge) at a state ω iff $\bigcap_{i \in G} b_{B_i}(\omega) \subseteq E$.²⁵ If each B_i satisfies Truth Axiom, Positive Introspection, and the Kripke property, then this characterization coincides with that in Proposition 3 (3).

Pacuit (2017) and Van De Putte and Klein (2018) study "distributed belief" which does not impose logical monotonicity.

- An event E is distributed belief among $G \in \mathcal{P}(I) \setminus \{\emptyset\}$ at state ω in the sense of Pacuit (2017) and Van De Putte and Klein (2018) (denoted by $\omega \in D_G^{\text{PPK}}(E)$), if there are $(E_i)_{i \in G} \in \mathcal{D}^G$ such that $\omega \in \bigcap_{i \in G} B_i(E_i)$ and $E = \bigcap_{i \in G} E_i$.

Then, I show: (i) if every B_i satisfies Monotonicity and Necessitation, then the definition of distributed belief by Pacuit (2017) and Van De Putte and Klein (2018) coincides with that by Hilpinen (1977); and (ii) if each B_i satisfies Truth Axiom, Positive Introspection, Monotonicity, and Necessitation, the definition of distributed belief by Pacuit (2017) and Van De Putte and Klein (2018) and the definition of distributed knowledge in this paper coincide with each other. Formally:

²⁵The proof goes as follows. For the "if" part, since each B_i satisfies Conjunction and Necessitation, it follows that $\omega \in \bigcap_{i \in G} B_i(b_{B_i}(\omega))$. Also, by supposition, $\bigcap_{i \in G} b_{B_i}(\omega) \subseteq E$. Then, E is distributed knowledge (impersonal knowledge) at ω in the sense of Hilpinen (1977). For the "only if" part, suppose that $\omega \in \bigcap_{i \in H} B_i(E_i)$ and $\bigcap_{i \in H} E_i \subseteq E$, where $H \in \mathcal{P}(G) \setminus \{\emptyset\}$. Then, $\bigcap_{i \in G} b_{B_i}(\omega) \subseteq \bigcap_{i \in H} b_{B_i}(\omega) \subseteq \bigcap_{i \in H} E_i \subseteq E$.

Proposition 8. *Let $G \in \mathcal{P}(I) \setminus \{\emptyset\}$.*

1. *If each B_i satisfies Monotonicity and Necessitation, then $D_G^{\text{PPK}} = D_G^{\text{H}}$.*
2. *If each B_i satisfies Truth Axiom, Positive Introspection, Monotonicity, and Necessitation, then $D_G^{\text{PPK}} = D_G^{\text{H}} = D_G$.*

If some agents fail Truth Axiom or Positive Introspection, however, the definition of distributed belief by Pacuit (2017) and Van De Putte and Klein (2018) and the definition of distributed knowledge in this paper may be different. To see this, suppose that each B_i satisfies the Kripke property. While their distributed belief (and indeed Hilpinen (1977)’s definition of distributed knowledge/belief) can be captured by $\bigcap_{i \in G} b_{B_i}(\cdot)$, distributed knowledge of this paper can be captured by $\bigcap_{i \in G} \beta_{B_i}(\cdot)$, which may properly include $\bigcap_{i \in G} b_{B_i}(\cdot)$. Thus, if an event E is distributed knowledge at a state then E is distributed belief at that state. However, it may not necessarily be the case that distributed knowledge is characterized by the reflexive transitive closure of distributed belief $\bigcap_{i \in G} b_{B_i}(\cdot)$.²⁶

Remark 3. I contrast distributed knowledge (formulated in this paper) and distributed belief in the following extreme example. Take any group $G \in \mathcal{P}(I) \setminus \{\emptyset\}$. For ease of exposition, assume that every agent’s belief operator satisfies the Kripke property. Then, denoting by B_G the distributed belief operator, it is induced from the agents’ possibility correspondences $\bigcap_{j \in G} b_{B_j}(\cdot)$.

Expression (1) implies that if there exist some agent $i \in G$ and event $E \in \mathcal{D}$ such that agent i believes E at any state (i.e., $B_i(E) = \Omega$), then group G is able to use event E to infer distributed knowledge at any state ω at which E occurs: $D_{\{i\}}(E) = D_G(E) = E$. Thus, in the extreme case in which B_i satisfies $B_i(\cdot) = \Omega$, the distributed knowledge operators $D_{\{i\}}$ and D_G satisfy $D_{\{i\}}(E) = D_G(E) = E$ for all $E \in \mathcal{D}$.

Two remarks are in order. First, in this case, Proposition 4 (1) states that $D_{\{i\}}$ is the *maximum* operator that satisfies $D_{\{i\}}(\cdot) \subseteq B_i(\cdot)$, Truth Axiom, Positive Introspection, and Monotonicity. The formulation of this paper provides a benchmark for distributed *knowledge* defined from individual *beliefs*.

Second, since $B_G(\cdot) = \Omega$, the maximum operator K_G that satisfies Truth Axiom, Positive Introspection, the Kripke property, and $K_G(\cdot) \subseteq B_G(\cdot)$ satisfies $K_G = D_G$.

5 Distributed Knowledge in Games with Rational Expectations

The solution concept of a rational-expectations Nash equilibrium captures the idea that agents simultaneously refine their information based on the informational content of others’ strategies. As a result, such an equilibrium is robust to any new knowledge each agent acquires from observing others’ strategies at equilibrium.

²⁶See Example OA.3 in Online Appendix C.

This section shows that the informational content of a rational-expectations Nash equilibrium corresponds to the distributed knowledge derived from pooling the informational content of agents' strategies. It also shows the sense in which, at such an equilibrium, agents' strategies are common knowledge.

5.1 Models

Let I be a (non-empty) set of agents.²⁷ For each $i \in I$, let A_i be a (non-empty) set of actions available to agent i , endowed with a (non-empty) collection $\mathcal{A}_i$ of subsets of A_i , i.e., $\mathcal{A}_i \subseteq \mathcal{P}(A_i)$.²⁸ A *model* on $((A_i, \mathcal{A}_i))_{i \in I}$ is a tuple $\mathcal{M} := \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, (\succsim_i)_{i \in I} \rangle$ with the following properties.

First, $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I} \rangle$ is a belief space. That is, Ω is a state space, and $\mathcal{D}$ is a domain, i.e., a complete algebra on Ω . For each $i \in I$, $B_i : \mathcal{D} \rightarrow \mathcal{D}$ is agent i 's belief operator. In this section, assume that each B_i satisfies Conjunction and Monotonicity. This reflects the idea that agent i is capable of aggregating her beliefs with the informational content provided by other agents.

Second, for each $i \in I$, the set S_i of agent i 's *behavioral strategies* consists of functions $s_i : \Omega \rightarrow A_i$ such that $s_i^{-1}(E_i) \in \mathcal{D}$ for each $E_i \in \mathcal{A}_i$. That is, $s_i : \Omega \rightarrow A_i$ is agent i 's behavioral strategy if it is “measurable” in the sense that, for any set of actions $E_i \in \mathcal{A}_i$, $s_i^{-1}(E_i)$ is an object of beliefs (i.e., an event). Succinctly written, S_i is the set of functions $s_i : \Omega \rightarrow A_i$ with $s_i^{-1}(\mathcal{A}_i) \subseteq \mathcal{D}$. Denote by $S := \times_{i \in I} S_i$ the set of behavioral strategy profiles. For each $i \in I$, then, $\succsim_i$ is agent i 's preference relation (i.e., a complete and transitive relation) on S . When each $\succsim_i$ admits a utility representation $u_i : S \rightarrow \mathbb{R}$, I denote by $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, (u_i)_{i \in I} \rangle$ the corresponding model.

Henceforth, I often omit the dependence of the model $\mathcal{M} = \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, (\succsim_i)_{i \in I} \rangle$ on $((A_i, \mathcal{A}_i))_{i \in I}$ as it is clear from the context. Also, for ease of notation, I denote $((A_i, \mathcal{A}_i))_{i \in I}$ simply by $(A_i, \mathcal{A}_i)_{i \in I}$.

5.2 Nash Equilibrium

I move on to defining Nash equilibria. Fix a model $\mathcal{M} := \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, (\succsim_i)_{i \in I} \rangle$. For each $i \in I$, let $\Phi_i(\mathcal{J}_{B_i})$ be the set of $s_i \in S_i$ such that $s_i^{-1}(E_i) \in \mathcal{J}_{B_i}$ for any $E_i \in \mathcal{A}_i$. Given that B_i is assumed to satisfy Monotonicity, this means that $s_i^{-1}(E_i) \subseteq B_i(s_i^{-1}(E_i))$ for any $E_i \in \mathcal{A}_i$. That is, whenever agent i takes an action in the set E_i , she believes that her action is in E_i . Hence, $\Phi_i(\mathcal{J}_{B_i})$ defines the set of agent i 's *strategies* (feasible with respect to her beliefs), while S_i is the set of agent i 's behavioral strategies.²⁹

²⁷To be consistent with the other parts of the paper, I refer to a player as an agent.

²⁸In applications, $\mathcal{A}_i$ is usually assumed to be a topology or a σ -algebra on A_i .

²⁹Consider a special case in which Ω is at most countable and B_i is induced by a partitional possibility correspondence b_{B_i} . As in, for instance, Aumann (1974, 1987) in the context of correlated equilibria, the self-evident collection $\mathcal{J}_{B_i}$ is the σ -algebra generated by the partition $\{b_{B_i}(\omega)\}_{\omega \in \Omega}$, and the set $\Phi_i(\mathcal{J}_{B_i})$ of strategies is the set of “measurable” functions $s_i : (\Omega, \mathcal{J}_{B_i}) \rightarrow (A_i, \sigma(\mathcal{A}_i))$.

A *Nash equilibrium* in the model $\mathcal{M}$ is a behavioral strategy profile $s^* \in S$ which satisfies the following two conditions for each $i \in I$:

1. $s_i^* \in \Phi_i(\mathcal{J}_{B_i})$;
2. $(s_i^*, s_{-i}^*) \succsim_i (s_i, s_{-i}^*)$ for all $s_i \in \Phi_i(\mathcal{J}_{B_i})$.

The first condition states that s_i^* is agent i 's strategy (feasible with respect to her beliefs). The second condition is the usual best-response condition. In other words, $s^* \in S$ is a Nash equilibrium in $\mathcal{M}$ if it is a Nash equilibrium of the corresponding strategic game in which each agent's strategy set is $\Phi_i(\mathcal{J}_{B_i})$, and each agent's preferences are given by the restriction of $\succsim_i$ to the set $\times_{i \in I} \Phi_i(\mathcal{J}_{B_i})$ of strategy profiles.

5.3 Rational-Expectations Nash Equilibrium

I now turn to the concept of rational-expectations Nash equilibria. In a rational-expectations Nash equilibrium, each agent infers the informational content of the strategies of the other agents. Thus, I consider a Nash equilibrium of a new game in which each agent i 's belief is augmented by the information generated from any other agent j 's strategy s_j . The collection $s_j^{-1}(\mathcal{A}_j)$ is the informational content of agent j 's strategy in the sense that it consists of events $s_j^{-1}(E_j)$ that agent j takes an action in E_j . Since each agent i may learn and make inferences from the informational content of the other agents, the set of agent i 's strategies may expand.

As discussed at the end of Section 4.1, the informational content generated from the agents' collections of events is summarized by the operation $\delta(\cdot)$. Since agent i 's beliefs are assumed to satisfy Monotonicity and Conjunction, I consider

$$\delta(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}}),$$

where $\mathcal{S}_j$ is the smallest collection that includes $s_j^{-1}(\mathcal{A}_j)$ and that is closed under arbitrary union and non-empty intersection (i.e., agent i can make inferences by taking any union or any non-empty intersection of any other agent j 's information $s_j^{-1}(\mathcal{A}_j)$).

This admits an interpretation in terms of distributed knowledge. For each $j \in I \setminus \{i\}$, consider a hypothetical agent whose knowledge satisfies Conjunction and is generated from the informational content $s_j^{-1}(\mathcal{A}_j)$. Then, $\mathcal{S}_j$ is the self-evident collection of such a hypothetical agent. Then, $\delta(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}})$ represents the informational content of distributed knowledge among i and those hypothetical agents.

Denote by $\Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i})$ the set of strategies of agent i feasible with respect to the information obtained by pooling her own beliefs $\mathcal{J}_{B_i}$ with the informational content $(\mathcal{S}_j)_{j \in I \setminus \{i\}}$:

$$\Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i}) := \Phi_i(\delta(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}})).$$

That is, $s_i \in \Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i})$ when

$$s_i^{-1}(E_i) \in \delta(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}}) \text{ for any } E_i \in \mathcal{A}_i.$$

In words, the behavioral strategy s_i is in $\Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i})$ if it is feasible with respect to the collection of information generated from pooling her own beliefs $\mathcal{J}_{B_i}$ and the informational content $(\mathcal{S}_j)_{j \in I \setminus \{i\}}$ derived from s_{-i} .

Now, a *rational-expectations Nash equilibrium* in the model $\mathcal{M}$ is a behavioral strategy profile $s^* \in S$ with the following two conditions for each $i \in I$:

1. $s_i^* \in \Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i}^*)$;
2. $(s_i^*, s_{-i}^*) \succsim_i (s_i, s_{-i}^*)$ for all $s_i \in \Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i}^*)$.

The first condition states that each s_i^* is agent i 's strategy feasible with respect to the informational content of s_{-i}^* in addition to her own beliefs. The second condition is the best-response condition in the expanded strategy space $\Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i}^*)$. The strategy space is expanded because each agent refines her information from the information revealed by the strategies of the other agents.

5.4 Common Knowledge of Strategies

Before proving the result that the informational content of a rational-expectations Nash equilibrium amounts to the distributed knowledge generated from the agents' strategies, I formalize the idea that, at a rational-expectations Nash equilibrium, the agents' strategies are common knowledge. To that end, I define what it means for the agents' strategies to be common knowledge.

In a model $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, (\succsim_i)_{i \in I} \rangle$, a behavioral strategy profile $s \in S$ is *common knowledge* if

$$s_i^{-1}(E_i) \in \bigcap_{j \in I} \mathcal{J}_{B_j} \text{ for any } i \in I \text{ and } E_i \in \mathcal{A}_i.$$

In words, $s \in S$ is common knowledge if any informational content $s_i^{-1}(E_i)$ of agent i is self-evident to any agent j (i.e., whenever agent i 's action is in E_i , agent j believes that agent i 's action is in E_i).³⁰

5.5 An Epistemic Characterization of Rational-Expectations Nash Equilibria

Section 5.5.1 presents the epistemic characterization of rational-expectations Nash equilibria through distributed knowledge. Section 5.5.2 presents an example that illustrates the epistemic characterization. Section 5.5.3 shows that distributed belief may not characterize rational-expectations Nash equilibria.

³⁰Using the common knowledge operator C_I introduced in Remark OA.1 in Online Appendix B.1, if $s \in S$ is common knowledge, then, at any state $\omega \in \Omega$ and for any set of actions $E_i \in \mathcal{A}_i$ of agent i with $s_i(\omega) \in E_i$, it is common knowledge that agent i takes an action in E_i at ω in that $\omega \in C_I(s_i^{-1}(E_i))$.

5.5.1 The Epistemic Characterization

In a rational-expectations Nash equilibrium, each agent incorporates information revealed by the strategies of the other agents. Since the informational content generated from collections of events is summarized by the operation $\delta(\cdot)$, the informational content derived from a behavioral strategy profile s is summarized by

$$\delta((\mathcal{S}_i)_{i \in I}).$$

This is equivalent to specifying distributed knowledge among agents i whose information (self-evident collection) is given by $\mathcal{S}_i$.

The following theorem shows that if a behavioral strategy profile s^* is a rational-expectations Nash equilibrium then it (i) is a Nash equilibrium and (ii) is common knowledge in a new model in which each agent's belief is given by the informational content $\delta((\mathcal{S}_i)_{i \in I})$ of the strategy profile s^* . The theorem also establishes the sense in which the converse also holds.

Theorem 1. *Let $\mathcal{M} := \langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, (\succsim_i)_{i \in I} \rangle$ be a model on $(A_i, \mathcal{A}_i)_{i \in I}$. Letting $s^* \in S$ be a behavioral strategy profile, let $\widetilde{\mathcal{M}} := \langle (\Omega, \mathcal{D}), (\widetilde{B}_i)_{i \in I}, (\succsim_i)_{i \in I} \rangle$ be a model in which each agent i 's belief is generated by the informational content $\delta((\mathcal{S}_j)_{j \in I})$ of s^* . Namely,*

$$\widetilde{B}_i(E) := \bigcup \{F \in \delta((\mathcal{S}_j)_{j \in I}) \mid F \subseteq E\} \text{ for each } E \in \mathcal{D}.$$

1. *If s^* is a rational-expectations Nash equilibrium in the model $\mathcal{M}$, then it is a Nash equilibrium and is common knowledge in $\widetilde{\mathcal{M}}$.*
2. *If s^* is a Nash equilibrium and is common knowledge in the model $\mathcal{M}$, then it is a rational-expectations Nash equilibrium in $\mathcal{M}$.*

Theorem 1 generalizes Minelli and Polemarchakis (2003, Proposition 1 and Corollary 2) in a way that agents are allowed to hold beliefs instead of knowledge on an arbitrary state space (provided that the domain is a complete algebra).³¹

In the first part, if s^* is a rational-expectations Nash equilibrium in $\mathcal{M}$, then each s_i^* is a strategy feasible with respect to the information generated from pooling agent i 's own information $\mathcal{J}_{B_i}$ and the informational content $(\mathcal{S}_j)_{j \in I \setminus \{i\}}$ of s_{-i}^* . In the new model $\widetilde{\mathcal{M}}$, s_i^* is, by construction, agent i 's strategy feasible with respect to $\delta(\mathcal{S}_i, (\mathcal{S}_j)_{j \in I \setminus \{i\}})$. The proof in Appendix A.5 shows that

$$\Phi_i(\delta(\mathcal{S}_i, (\mathcal{S}_j)_{j \in I \setminus \{i\}})) \subseteq \Phi_i(\delta(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}})) = \Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i}^*).$$

In the second part, if s^* is a Nash equilibrium and common knowledge in the original model $\mathcal{M}$, then for any agent i , the informational content of s_{-i}^* is already self-evident to i . The proof in Appendix A.5 shows that

$$\Phi_i(\mathcal{J}_{B_i}) = \Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i}^*).$$

³¹The theorem also generalizes McKelvey and Page (1986, Theorem 3), although the formalization of the theorem in this paper follows Minelli and Polemarchakis (2003).

5.5.2 Example

This section applies the formal results to the “opinion game” (e.g., Minelli and Polemarchakis, 2003).

Example 3. Let $I = \{1, \dots, n\}$. Let $(A_i, \mathcal{A}_i) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be agent i ’s action space, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ -algebra on $\mathbb{R}$. I start by defining a model $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, (u_i)_{i \in I} \rangle$. First, let Ω be a finite state space, endowed with its domain $\mathcal{D} = \mathcal{P}(\Omega)$. Let μ be a probability measure on Ω such that $\mu(\{\omega\}) > 0$ for all $\omega \in \Omega$.

Second, for each $i \in I$, let $\mathcal{J}_i$ be a sub- σ -algebra of $\mathcal{D}$ on Ω . Since Ω is finite, note that $\mathcal{J}_i$ is a complete algebra. For each $i \in I$, define agent i ’s belief operator $B_i : \mathcal{D} \rightarrow \mathcal{D}$ as follows:

$$B_i(E) := \bigcup \{F \in \mathcal{J}_i \mid F \subseteq E\} \text{ for each } E \in \mathcal{D}.$$

It can be shown that $\mathcal{J}_i = \mathcal{J}_{B_i}$.

Third, let $x : (\Omega, \mathcal{D}) \rightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be a given (measurable) function. For each $i \in I$, the set S_i of agent i ’s behavioral strategies is the set of (measurable) functions $s_i : (\Omega, \mathcal{D}) \rightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Let agent i ’s utility function $u_i : S \rightarrow \mathbb{R}$ be such that

$$u_i(s) := -\mathbb{E}_{\mu} [(x - s_i)^2] \text{ for all } s \in S, \quad (4)$$

where $\mathbb{E}_{\mu} [\cdot]$ is the expectation operator with respect to μ .

Given Expression (4), the opinion game has a unique Nash equilibrium s , where agent i ’s strategy is the conditional expectation of x given $\mathcal{J}_i$:

$$s_i(\cdot) = \mathbb{E}_{\mu}[x \mid \mathcal{J}_i](\cdot).$$

This is because s_i is the unique maximizer of u_i in $\Phi_i(\mathcal{J}_i)$ irrespective of the opponents’ strategies, where the uniqueness follows because $\mu(\{\cdot\}) > 0$.

Next, moving on to rational-expectations Nash equilibria, if $s^* \in S$ is a rational-expectations Nash equilibrium in $\mathcal{M}$, then it is a Nash equilibrium in the model $\widetilde{\mathcal{M}}$ as in the first part of Theorem 1. Since Ω is finite, each $(s_i^*)^{-1}(\mathcal{B}_{\mathbb{R}})$ is a complete algebra. Consequently, $\mathcal{S}_i = (s_i^*)^{-1}(\mathcal{B}_{\mathbb{R}})$ for each $i \in I$. Then, the informational content of s^* satisfies:

$$\delta((\mathcal{S}_i)_{i \in I}) = \sigma \left(\bigcup_{i \in I} \mathcal{S}_i \right),$$

which I denote by $\sigma(s^*)$ (consistently with the definition of the smallest σ -algebra generated by s^*). It follows that s^* satisfies:

$$s_i^*(\cdot) = \mathbb{E}_{\mu}[x \mid \sigma(s^*)](\cdot) \text{ for each } i \in I.$$

Thus, any rational-expectations Nash equilibrium s^* of the opinion game is symmetric: $s_i^* = s_j^*$ for all $i, j \in I$. Also, the strategy profile s^* is common knowledge in $\widetilde{\mathcal{M}}$. $\square$

Example 3 may capture some aspects of a parimutuel-style market (recall the experiment of Galton (1907a,b, 1908) in the Introduction) or a market under the strong form of the efficient-market hypothesis. There may also be situations in which an agent incorporates the informational content of the other agents' strategies.³²

5.5.3 Case with Distributed Belief

To sum up, I demonstrate a sense in which Theorem 1 may not hold if one considers distributed belief instead of distributed knowledge. For ease of analysis, I assume that each agent's belief operator B_i satisfies Necessitation in addition to Monotonicity and Conjunction (i.e., B_i satisfies the Kripke property) so that the definition of distributed belief does not depend on the choice among Hilpinen (1977), Pacuit (2017), and Van De Putte and Klein (2018).

Under distributed belief, agent i does not pool the event that agent j takes an action in $E_j \in \mathcal{A}_j$ (i.e., $F_j := s_j^{-1}(E_j)$) but pools the event that agent j believes that she takes an action in E_j (i.e., $B_j(F_j)$). Instead of $\tilde{B}_i$ in Theorem 1, in which each agent's belief operator $\tilde{B}_i$ is generated by distributed knowledge, consider a model $\widehat{\mathcal{M}} := \langle (\Omega, \mathcal{D}), (\hat{B}_i)_{i \in I}, (\succsim_i)_{i \in I} \rangle$ in which each agent i 's belief is generated by distributed belief: for each $E \in \mathcal{D}$, $\omega \in \hat{B}_i(E)$ if and only if there exist $E_j \in \mathcal{S}_j$ for each $j \in I$ such that $\omega \in \bigcap_{j \in I} B_j(E_j)$ and $\bigcap_{j \in I} E_j \subseteq E$.³³ Below, I provide a counterexample to Theorem 1 when one uses distributed belief (i.e., $\hat{B}_i$) instead of distributed knowledge (i.e., $\tilde{B}_i$).

Example 4. Consider the opinion game of Example 3 with $I = \{1, 2\}$, $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2\}, \mathcal{P}(\Omega))$, a uniform prior μ , and $x = \mathbb{I}_{\{\omega_2\}}$. Each agent i chooses $s_i : \Omega \rightarrow \mathbb{R}$ to maximize $u_i(s) = -\mathbb{E}_\mu[(x - s_i)^2]$. Let each B_i satisfy the Kripke property (i.e., Monotonicity, Conjunction, and Necessitation), with possibility correspondences

$$b_{B_1}(\omega_1) = \emptyset \text{ and } b_{B_1}(\omega_2) = \Omega; \text{ and } b_{B_2}(\omega_1) = \Omega \text{ and } b_{B_2}(\omega_2) = \emptyset.$$

³²For instance, in the context of Example 3, suppose that agent 1 represents the central bank and agent 2 the market. Let x be some given target (nominal) inflation rate (which depends on the state of the economy, which is uncertain). Both agents possess private information about the state of the economy (as both agents, in reality, spend resources to forecast the state of the economy). Assuming that the central bank can directly choose the (nominal) inflation rate policy s_1 , its objective is to minimize the loss between the chosen inflation rate s_1 and the target x . The objective of the market is to form its inflation expectation s_2 to match it with the target x . When both the central bank and the market have their own private information and consequently try to learn private information from their behaviors, it would be natural to consider a rational-expectations Nash equilibrium (instead of a Nash equilibrium). Conceptually, in such a setting, this paper assumes that each agent has beliefs (which may be false) instead of knowledge. As another illustration of similar reasoning, McKelvey and Page (1986) examine a Cournot competition in which firms produce a new chemical with unknown toxicity, which they attempt to infer from the market price.

³³It can be shown that if each B_j satisfies Truth Axiom, $\mathcal{S}_j \subseteq \mathcal{J}_{B_j}$ (as holds when $s_j \in \Phi_j(\mathcal{J}_{B_j})$), and $\Omega \in \mathcal{S}_j$ (as holds when $A_i \in \mathcal{A}_i$) then $\hat{B}_i = \tilde{B}_i$ for each i .

Neither operator satisfies Truth Axiom. The induced operators are $B_1(E) = \{\omega_1\}$ for every $E \neq \Omega$ with $B_1(\Omega) = \Omega$, and $B_2(E) = \{\omega_2\}$ for every $E \neq \Omega$ with $B_2(\Omega) = \Omega$. The agents' self-evident collections are:

$$\mathcal{J}_{B_1} = \{\emptyset, \{\omega_1\}, \Omega\} \text{ and } \mathcal{J}_{B_2} = \{\emptyset, \{\omega_2\}, \Omega\}.$$

First, the uninformative strategy profile $s^* = (\frac{1}{2}, \frac{1}{2})$ is a rational-expectations Nash equilibrium in $\mathcal{M}$. Since each s_j^* is constant, it follows that $\mathcal{S}_j = \{\emptyset, \Omega\}$. Thus, $\delta(\mathcal{J}_{B_1}, \mathcal{S}_2) = \mathcal{J}_{B_1}$ and $\delta(\mathcal{J}_{B_2}, \mathcal{S}_1) = \mathcal{J}_{B_2}$. Each agent is thus restricted to constants, and the unique best response is $\mathbb{E}_\mu[x] = \frac{1}{2}$.

Second, s^* is not a Nash equilibrium in $\widehat{\mathcal{M}}$. Since $\widehat{B}_i(E) = \Omega$ for every $E \in \mathcal{D}$, every behavioral strategy is feasible in $\widehat{\mathcal{M}}$. In particular agent 1 may play $s_1 = x$, which yields $u_1(x, s_2^*) = 0 > -\frac{1}{4} = u_1(s^*)$, a profitable deviation.

The two notions therefore disagree on the equilibrium set. At each state, one agent believes a contradiction. Pooling beliefs rather than actions transmits that contradiction to the other agent. Consequently, in $\widehat{\mathcal{M}}$, both agents forecast x exactly. This violates the idea that, in a rational-expectations Nash equilibrium, each agent *observes* the informational content of the other agents' strategies. In contrast, Theorem 1 (1) holds as stated for $\widetilde{\mathcal{M}}$: there $\delta((\mathcal{S}_j)_{j \in I}) = \{\emptyset, \Omega\}$, only constants are feasible, and s^* is both a Nash equilibrium and common knowledge. $\square$

6 Conclusion

This paper formalizes distributed knowledge as that which can be deduced from pooling agents' collective information, where each agent's information comprises self-evident events—events for which the agent believes all their logical implications. Irrespective of properties of agents' beliefs, distributed knowledge is true, monotonic, and positively introspective. The formalization of distributed knowledge in this paper nests the previous ones under particular assumptions on agents' beliefs.

Distributed knowledge inherits logical properties from individual agents. When agents' beliefs are induced by their possibility correspondences, the possibility correspondence that dictates distributed knowledge is the intersection of the reflexive transitive closures of agents' possibility correspondences. If individual beliefs satisfy Truth Axiom, Monotonicity, Positive Introspection, and Conjunction, then distributed knowledge coincides with knowledge of the least knowledgeable “wise man” who is at least as knowledgeable as any individual agent. Distributed knowledge of one agent coincides with her belief if and only if her belief is true, monotonic, and positively introspective.

Under the Truth Axiom and the Kripke property, distributed knowledge inherits Negative Introspection from individual agents. However, if agents' beliefs violate either of these conditions, distributed knowledge may fail to exhibit Negative Introspection. In such cases, it reflects “collective unawareness.” Agents, after aggregating their collective information, may not be able to resolve their unawareness.

Turning to game-theoretic applications, the informational content of a rational-expectations Nash equilibrium amounts to the distributed knowledge pooled from the informational content generated from the agents' strategies. This gives a sense in which, in a rational-expectations Nash equilibrium, the agents' strategies are common knowledge. This paper generalizes the epistemic characterization of rational-expectations Nash equilibria when agents possess beliefs instead of knowledge at the individual level.

Finally, two strands of future research seem promising. The first is to study the processes through which distributed knowledge is obtained—particularly by modeling explicit communication protocols or agents' strategic decisions to share information. The second is to explore alternative (possibly weaker) rational expectations equilibrium concepts in settings such as game theory, general equilibrium theory, or mechanism design, where agents may refine their strategic information only indirectly—e.g., through public signals like prices or allocations. Both directions aim to deepen our understanding of how distributed knowledge functions in decentralized systems. These questions are closely connected to the foundational insight emphasized by Hayek (1945) and discussed in the Introduction: one of the most fundamental facts in economic and social activities is to utilize the knowledge possessed by agents in a society that is dispersed among them in possibly contradictory forms.

A Proofs

A.1 Section 4.1

Proof of Proposition 1. 1. The statement follows because $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G_1}) \subseteq \mathcal{I}((\mathcal{J}_{B_i})_{i \in G_2})$.

2. Suppose $\omega \in D_G^J(E)$. There are $J' \in \mathcal{P}(J) \setminus \{\emptyset\}$ and $E_j \in \mathcal{J}_{D_{G_j}}$ for each $j \in J'$ such that $\omega \in \bigcap_{j \in J'} E_j \subseteq E$, where $\mathcal{J}_{D_{G_j}}$ is the self-evident collection according to D_{G_j} . Since $\omega \in E_j \subseteq D_{G_j}(E_j)$, there are $H_j \in \mathcal{P}(G_j) \setminus \{\emptyset\}$ and $F_i^j \in \mathcal{J}_{B_i}$ for all $i \in H_j$ such that $\omega \in \bigcap_{i \in H_j} F_i^j \subseteq E_j$. Let $H := \bigcup_{j \in J'} H_j \in \mathcal{P}(G) \setminus \{\emptyset\}$. For each $i \in H$, let $F_i := \bigcap_{j \in J': i \in H_j} F_i^j$. Since $(G_j)_{j \in J}$ partitions G (i.e., since it is mutually disjoint), $F_i \in \mathcal{J}_{B_i}$ is a single self-evident event. Then, $\omega \in \bigcap_{i \in H} F_i = \bigcap_{j \in J'} \bigcap_{i \in H_j} F_i^j \subseteq \bigcap_{j \in J'} E_j \subseteq E$. It follows that $\omega \in D_G(E)$.

Conversely, assume $\omega \in D_G(E)$. Then, there are $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $E_i \in \mathcal{J}_{B_i}$ for all $i \in H$ such that $\omega \in \bigcap_{i \in H} E_i \subseteq E$. Now, let $J_H := \{j \in J \mid G_j \cap H \neq \emptyset\}$. For each $j \in J_H$, $\omega \in E^j := \bigcap_{i \in G_j \cap H} E_i \in \mathcal{J}_{D_{G_j}}$. This is because, for any $\omega' \in E^j$, $\omega' \in \bigcap_{i \in G_j \cap H} E_i \subseteq E^j$ with $E_i \in \mathcal{J}_{B_i}$ for each $i \in G_j \cap H$, and thus $\omega' \in D_{G_j}(E^j)$. Hence, $\omega \in \bigcap_{j \in J_H} E^j = \bigcap_{j \in J_H} \bigcap_{i \in G_j \cap H} E_i = \bigcap_{i \in H} E_i \subseteq E$. Thus, $\omega \in D_G^J(E)$.

3. It can be seen from Expression (2) that Truth Axiom and Monotonicity hold. For Positive Introspection, if $\omega \in D_G(E)$ then there is $F \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$ such that $\omega \in F \subseteq E$. Since $\omega \in F \subseteq F$ and $F \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$, I have $\omega \in F \subseteq$

$D_G(F) \subseteq D_G(E)$, where the last set inclusion follows from Monotonicity of D_G . Thus, $\omega \in D_G D_G(E)$. $\square$

Proof of Proposition 2. First, I show $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G}) \subseteq \mathcal{J}_{D_G}$. Take $F \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$. For any $\omega \in F$, $\omega \in F \subseteq F$. Thus, $\omega \in D_G(F)$. Hence, $F \subseteq D_G(F)$, implying $F \in \mathcal{J}_{D_G}$. Thus, $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G}) \subseteq \mathcal{J}_{D_G}$.

Second, I show that $\mathcal{J}_{D_G}$ is closed under arbitrary union (indeed, the proof below implies that, for any operator on $\mathcal{D}$, its self-evident collection is closed under arbitrary union). Take $E_\lambda \in \mathcal{J}_{D_G}$ for each $\lambda \in \Lambda$. Suppose $\bigcup_{\ell \in \Lambda} E_\ell \subseteq E$. For each $\lambda \in \Lambda$, it follows from $E_\lambda \subseteq \bigcup_{\ell \in \Lambda} E_\ell \subseteq E$ that $E_\lambda \subseteq D_G(E)$. Thus, $\bigcup_{\lambda \in \Lambda} E_\lambda \subseteq D_G(E)$, establishing $\bigcup_{\lambda \in \Lambda} E_\lambda \in \mathcal{J}_{D_G}$. The collection $\mathcal{J}_{D_G}$ is also closed under the empty union: $\emptyset \in \mathcal{J}_{D_G}$ (which corresponds to the case in which $\Lambda = \emptyset$).

Third, suppose that $\mathcal{J}$ satisfies the three properties in the statement: it is closed under any non-empty union, contains $\emptyset$, and includes $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$. I show that $\mathcal{J}_{D_G} \subseteq \mathcal{J}$. For any $E \in \mathcal{J}_{D_G}$, if $E = \emptyset$ then $E \in \mathcal{J}$. If $E \neq \emptyset$, then $\omega \in E$ implies $\omega \in E \subseteq D_G(E)$. Thus, $\omega \in F_\omega \subseteq E$ for some $F_\omega \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G}) \subseteq \mathcal{J}$. Thus,

$$E = \bigcup_{\omega \in E} \{\omega\} \subseteq \bigcup_{\omega \in E} F_\omega \subseteq E.$$

Hence, $E \in \mathcal{J}$. Thus, $\mathcal{J}_{D_G} \subseteq \mathcal{J}$.

To conclude, $\mathcal{J}_{D_G}$ is the smallest collection that satisfies the three properties. $\square$

A.2 Section 4.2

To prove Proposition 3, I establish the following preliminary results.

Lemma A.1. *If B_i satisfies Monotonicity and Conjunction, then $\mathcal{J}_{B_i}$ is closed under any non-empty intersection.*

Proof of Lemma A.1. Take $\mathcal{E} \in \mathcal{P}(\mathcal{J}_{B_i}) \setminus \{\emptyset\}$. If $\bigcap \mathcal{E} \subseteq F$, then

$$\bigcap_{E \in \mathcal{E}} E \subseteq \bigcap_{E \in \mathcal{E}} B_i(E) \subseteq B_i(\bigcap \mathcal{E}) \subseteq B_i(F).$$

The first set inclusion follows because $E \in \mathcal{J}_{B_i}$ implies $E \subseteq B_i(E)$. The second and third, respectively, follow from Conjunction and Monotonicity. $\square$

Lemma A.2. *Recall $\beta_{B_i}(\omega) := \bigcap \{F \in \mathcal{J}_{B_i} \mid \omega \in F\}$ and $b_{B_i}(\omega) := \bigcap \{F \in \mathcal{D} \mid \omega \in B_i(F)\}$ for each $\omega \in \Omega$. Suppose that B_i satisfies the Kripke property. Then, β_{B_i} is the reflexive transitive closure of b_{B_i} .*

Proof of Lemma A.2. First, I show that β_{B_i} is reflexive and transitive. By construction, β_{B_i} is reflexive. Next, let $\omega' \in \beta_{B_i}(\omega)$ and $\omega'' \in \beta_{B_i}(\omega')$. Take any $F \in \mathcal{J}_{B_i}$ with $\omega \in F$. Then, $\omega' \in F$ and consequently $\omega'' \in F$. Thus, $\omega'' \in \beta_{B_i}(\omega)$. Hence, β_{B_i} is transitive.

Second, I show $b_{B_i}(\cdot) \subseteq \beta_{B_i}(\cdot)$. If $\omega' \in b_{B_i}(\omega) = \bigcap \{E \in \mathcal{D} \mid \omega \in B_i(E)\}$, then for any $F \in \mathcal{J}_{B_i}$ with $\omega \in F$, it follows from $\omega \in F \subseteq B_i(F)$ that $\omega' \in F$. Thus, $\omega' \in \beta_{B_i}(\omega)$.

Third, let b' be a reflexive and transitive possibility correspondence satisfying $b_{B_i}(\cdot) \subseteq b'(\cdot)$. Since B_i satisfies the Kripke property, it follows that $B_{b'}(\cdot) \subseteq B_i(\cdot)$. Take $\omega \in \Omega$, and let $\omega' \in \beta_{B_i}(\omega)$ (i.e., $\omega' \in F$ if $\omega \in F \subseteq B_i(F)$). For any $F \in \mathcal{D}$ with $\omega \in B_{b'}(F)$, I have $\omega \in B_{b'}(F) \subseteq B_{b'}B_{b'}(F) \subseteq B_iB_{b'}(F)$, and hence $\omega' \in B_{b'}(F) \subseteq F$. Thus, $\omega' \in b'(\omega) = \bigcap \{F \in \mathcal{D} \mid \omega \in B_{b'}(F)\}$. Hence, $\beta_{B_i}(\omega) \subseteq b'(\omega)$. $\square$

Proof of Proposition 3. 1. (a) Suppose that, for each $\omega \in \Omega$, there is $F \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$ such that $\omega \in F$. Since $F \subseteq \Omega$, it follows that $\omega \in D_G(\Omega)$. Conversely, for any $\omega \in \Omega = D_G(\Omega)$, there is $F \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$ such that $\omega \in F \subseteq \Omega$. The second assertion follows because if $B_i(\Omega) = \Omega$ for some $i \in G$, then $\Omega \in \mathcal{J}_{B_i} \subseteq \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$.

(b) Since $\mathcal{I}_{\text{Nec}}^* \subseteq \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$, it suffices to show $D_G(\cdot) \subseteq K_{\mathcal{I}_{\text{Nec}}^*}(\cdot)$. If $\omega \in D_G(E)$, then there are $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $E_i \in \mathcal{J}_{B_i}$ for each $i \in H$ such that $\omega \in \bigcap_{i \in H} E_i \subseteq E$. Letting $E_j = \Omega$ for each $j \in G \setminus H$, one has $\omega \in \bigcap_{i \in G} E_i \subseteq E$. Then, $\omega \in K_{\mathcal{I}_{\text{Nec}}^*}(E)$ as desired.

2. First, I show that D_G satisfies Conjunction. Take $\mathcal{E} \in \mathcal{P}(\mathcal{D}) \setminus \{\emptyset\}$, and let $\omega \in \bigcap_{E \in \mathcal{E}} D_G(E)$. For each $E \in \mathcal{E}$, there is $F_E \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$ such that $\omega \in F_E \subseteq E$. Thus, $\omega \in \bigcap_{E \in \mathcal{E}} F_E \subseteq \bigcap \mathcal{E}$.

To establish $\omega \in D_G(\bigcap \mathcal{E})$, it suffices to show that $\bigcap_{E \in \mathcal{E}} F_E \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$. For each $E \in \mathcal{E}$, there are $H_E \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $F_E^i \in \mathcal{J}_{B_i}$ with $i \in H_E$ such that $F_E = \bigcap_{i \in H_E} F_E^i$.

Letting $H = \bigcup_{E \in \mathcal{E}} H_E$, below I show $\bigcap_{E \in \mathcal{E}} F_E = \bigcap_{i \in H} \bigcap \mathcal{F}_i$, where

$$\mathcal{F}_i := \{F_E^i \in \mathcal{J}_{B_i} \mid E \in \mathcal{E} \text{ satisfies } i \in H_E\}.$$

By Lemma A.1, $\bigcap \mathcal{F}_i \in \mathcal{J}_{B_i}$, and thus $\bigcap_{E \in \mathcal{E}} F_E = \bigcap_{i \in H} \bigcap \mathcal{F}_i \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$.

Thus, what is left is to show $\bigcap_{E \in \mathcal{E}} F_E = \bigcap_{i \in H} \bigcap \mathcal{F}_i$. Suppose $\omega \in \bigcap_{i \in H} \bigcap \mathcal{F}_i$. For any $E \in \mathcal{E}$, I have $\omega \in \bigcap_{i \in H_E} F_E^i = F_E$. Conversely, suppose $\omega \in F_E = \bigcap_{i \in H_E} F_E^i$ for all $E \in \mathcal{E}$. Fix $i \in H$. For any $E \in \mathcal{E}$ with $i \in H_E$, I have $\omega \in \bigcap_{j \in H_E} F_E^j \subseteq F_E^i$. Thus, $\omega \in \bigcap \mathcal{F}_i$, and consequently $\omega \in \bigcap_{i \in H} \bigcap \mathcal{F}_i$.

Second, I prove $K_{\mathcal{I}_{\text{Con}}^*} = D_G$. Since $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G}) \subseteq \mathcal{I}_{\text{Con}}^*$, it is enough to show that $K_{\mathcal{I}_{\text{Con}}^*}(\cdot) \subseteq D_G(\cdot)$. If $\omega \in K_{\mathcal{I}_{\text{Con}}^*}(E)$, then there is $\emptyset \neq \mathcal{E} \subseteq \bigcup_{i \in G} \mathcal{J}_{B_i}$ such that $\omega \in \bigcap \mathcal{E} \subseteq E$. Since $\omega \in D_G(F)$ for each $F \in \mathcal{E}$ and since D_G satisfies Conjunction and Monotonicity, it follows $\omega \in \bigcap_{F \in \mathcal{E}} D_G(F) \subseteq D_G(\bigcap \mathcal{E}) \subseteq D_G(E)$.

3. (a) Since B_i satisfies Necessitation, Monotonicity, and Conjunction, $\mathcal{J}_{B_i}$ is closed under any non-empty intersection by Lemma A.1 and contains Ω .

First, I show that $\omega \in D_G(E)$ iff $\bigcap_{i \in G} \beta_{B_i}(\omega) \subseteq E$. Suppose that $\omega \in D_G(E)$. By Proposition 3 (1b), there are $F_i \in \mathcal{J}_{B_i}$ for all $i \in G$ such that $\omega \in \bigcap_{i \in G} F_i \subseteq E$. Then, $\omega \in \bigcap_{i \in G} \beta_{B_i}(\omega) \subseteq \bigcap_{i \in G} F_i \subseteq E$.

Conversely, suppose that $\bigcap_{i \in G} \beta_{B_i}(\omega) \subseteq E$. Firstly, $\omega \in \bigcap_{i \in G} \beta_{B_i}(\omega)$. Secondly, since $\mathcal{J}_{B_i}$ is closed under intersection by Lemma A.1 and Necessitation, it follows from $\beta_{B_i}(\omega) \in \mathcal{J}_{B_i}$ for each $i \in G$ that $\bigcap_{i \in G} \beta_{B_i}(\omega) \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$. Thus, $\omega \in D_G(E)$.

Second, I show that $b_{D_G}(\cdot) = \bigcap_{i \in G} \beta_{B_i}(\cdot)$ is reflexive and transitive. It suffices to show that each β_{B_i} is reflexive and transitive. However, the latter assertion follows from Lemma A.2.

- (b) If B_i satisfies Positive Introspection, then b_{B_i} is transitive and, since $\{\omega\} \in \mathcal{D}$, $b_{B_i}(\omega) \cup \{\omega\}$ is the reflexive transitive closure of b_{B_i} . Thus, $\beta_{B_i}(\omega) = b_{B_i}(\omega) \cup \{\omega\}$ for each $\omega \in \Omega$.
- (c) If B_i satisfies Truth Axiom and Positive Introspection, then $b_{B_i} = \beta_{B_i}$. □

Proof of Proposition 4. 1. (a) If $\omega \in D_{\{i\}}(E)$ then there is $F \in \mathcal{J}_{B_i}$ with $\omega \in F \subseteq E$. Then, $\omega \in F \subseteq B_i(E)$. Take any K satisfying the assumptions in the statement. Let $\omega \in K(E)$, and let $K(E) \subseteq F$. Then, $K(E) \subseteq KK(E) \subseteq K(F) \subseteq B_i(F)$. Thus, $K(E) \in \mathcal{J}_{B_i}$. Since $\omega \in K(E) \subseteq E$, it follows that $\omega \in D_{\{i\}}(E)$.

(b) This part follows from the previous part and Proposition 1.

- 2. If $\omega \in D_G(E)$, then there are $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $F_i \in \mathcal{J}_{B_i}$ for each $i \in H$ such that $\omega \in \bigcap_{i \in H} F_i \subseteq E$. Then,

$$\omega \in \bigcap_{i \in H} F_i \subseteq \bigcap_{i \in H} D_{\{i\}}(F_i) \subseteq \bigcap_{i \in H} K_G(F_i) \subseteq K_G \left(\bigcap_{i \in H} F_i \right) \subseteq K_G(E).$$

The first set inclusion follows because $F_i \in \mathcal{J}_{B_i}$ and thus $F_i \subseteq D_{\{i\}}(F_i)$, the second from the assumption $D_{\{i\}}(\cdot) \subseteq K_G(\cdot)$, the third from Conjunction of K_G , and the fourth from Monotonicity of K_G . □

A.3 Section 4.3

Proof of Proposition 5. It suffices to show Part (1b) as the proofs of the other parts are in the main text. I show the statement in two steps.

The first step shows $D_G = K_{\mathcal{I}_{\text{NI}}^*}$. Since $\mathcal{I}((\mathcal{J}_{B_i})_{i \in G}) \subseteq \mathcal{I}_{\text{NI}}^*$, it suffices to show $K_{\mathcal{I}_{\text{NI}}^*}(\cdot) \subseteq D_G(\cdot)$. If $\omega \in K_{\mathcal{I}_{\text{NI}}^*}(E)$, then $\omega \in F \subseteq E$ for some $F \in \mathcal{I}_{\text{NI}}^*$. Since $\mathcal{I}_{\text{NI}}^*$ is the smallest complete algebra including $\bigcup_{i \in G} \mathcal{J}_{B_i}$ and since each $\mathcal{J}_{B_i}$ is closed under

complementation, it can be seen that

$$F = \bigcap_{\lambda \in \Lambda} \bigcup_{\mu_\lambda \in M_\lambda} F_{\lambda, \mu_\lambda} \text{ with each } F_{\lambda, \mu_\lambda} \in \bigcup_{i \in G} \mathcal{J}_{B_i}.^{34}$$

Since D_G satisfies Necessitation, if $\Lambda = \emptyset$ then $F = \Omega = E$ and $\omega \in D_G(E)$. Thus, assume $\Lambda \neq \emptyset$. For each $\lambda \in \Lambda$, choose $\mu_\lambda \in M_\lambda$ and $i_\lambda \in G$ with $\omega \in F_{\lambda, \mu_\lambda} \in \mathcal{J}_{B_{i_\lambda}}$. Let $H := \{i_\lambda \in G \mid \lambda \in \Lambda\}$. For each $i \in H$, define $F_i := \bigcap_{\lambda \in \Lambda: i=i_\lambda} F_{\lambda, \mu_\lambda} \in \mathcal{J}_{B_i}$. Then, $\omega \in \bigcap_{i \in H} F_i \subseteq E$ and $\bigcap_{i \in H} F_i \in \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$. Thus, $\omega \in D_G(E)$.

The second step shows $\mathcal{I}_{\text{NI}}^* = \mathcal{J}_{D_G}$. For any $E \in \mathcal{I}_{\text{NI}}^*$, if $\omega \in E$ then $\omega \in E \subseteq E$ and thus $E \subseteq K_{\mathcal{I}_{\text{NI}}^*}(E) = D_G(E)$, where the last equality follows from the first step of the proof. Hence, $\mathcal{I}_{\text{NI}}^* \subseteq \mathcal{J}_{D_G}$. Conversely, take $E \in \mathcal{J}_{D_G}$. For any $\omega \in E$, $\omega \in D_G(E)$. Since every B_i satisfies Necessitation, there are $E_i^\omega \in \mathcal{J}_{B_i} \subseteq \mathcal{I}_{\text{NI}}^*$ for all $i \in G$ such that $\omega \in \bigcap_{i \in G} E_i^\omega \subseteq E$. Then, $E = \bigcup_{\omega \in E} \bigcap_{i \in G} E_i^\omega \in \mathcal{I}_{\text{NI}}^*$. $\square$

A.4 Section 4.4

Proof of Proposition 6. Suppose that each B_i satisfies Positive Introspection and Monotonicity. Since $\{B_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\} \subseteq \mathcal{J}_{B_i}$, it follows that $\mathcal{I}_{\text{B}}^* \subseteq \mathcal{I}((\mathcal{J}_{B_i})_{i \in G})$, and thus $K_{\mathcal{I}_{\text{B}}^*}(\cdot) \subseteq D_G(\cdot)$.

Next, suppose that each B_i satisfies Truth Axiom. Take $\omega \in D_G(E)$. There are $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $E_i \in \mathcal{J}_{B_i}$ for all $i \in H$ such that $\omega \in \bigcap_{i \in H} E_i \subseteq E$. Since $\bigcap_{i \in H} E_i = \bigcap_{i \in H} B_i(E_i)$, it follows that $\omega \in K_{\mathcal{I}_{\text{B}}^*}(E)$. $\square$

Proof of Proposition 7. First, suppose $\omega \in D_G(E)$: there are $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $E_i \in \mathcal{J}_{B_i}$ for each $i \in H$ such that $\omega \in \bigcap_{i \in H} E_i \subseteq E$. Since $E_i \subseteq B_i(E_i)$ for each $i \in H$, it follows that $\omega \in D_G^{\text{H}}(E)$.

Second, suppose that agents' beliefs satisfy Truth Axiom, Positive Introspection, and Monotonicity. It suffices to show $D_G^{\text{H}}(\cdot) \subseteq D_G(\cdot)$. If $\omega \in D_G^{\text{H}}(E)$, then there are $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $E_i \in \mathcal{D}$ for each $i \in H$ such that $\omega \in \bigcap_{i \in H} B_i(E_i)$ and $\bigcap_{i \in H} E_i \subseteq E$. Since B_i satisfies Positive Introspection and Monotonicity, $B_i(E_i) \in \mathcal{J}_{B_i}$. Since B_i satisfies Truth Axiom, $\omega \in \bigcap_{i \in H} B_i(E_i) \subseteq E$. Thus, $\omega \in D_G(E)$. Hence, $D_G^{\text{H}}(\cdot) \subseteq D_G(\cdot)$. $\square$

Proof of Proposition 8. 1. First, by inspection, $D_G^{\text{PPK}}(\cdot) \subseteq D_G^{\text{H}}(\cdot)$. Second, for the converse, assume that every B_i satisfies Monotonicity and Necessitation. Let $\omega \in D_G^{\text{H}}(E)$. Then, there are $H \in \mathcal{P}(G) \setminus \{\emptyset\}$ and $E_i \in \mathcal{D}$ with $\bigcap_{i \in H} E_i \subseteq E$ and $\omega \in \bigcap_{i \in H} B_i(E_i)$. It is without loss to assume that $H = G$ because one can let $E_j = \Omega$ for all $j \in G \setminus H$ as Necessitation of B_j implies $B_j(E_j) = \Omega$.

³⁴To see this, it suffices to show that the collection of such F is a complete algebra including $\bigcup_{i \in G} \mathcal{J}_{B_i}$. By construction, the collection includes $\bigcup_{i \in G} \mathcal{J}_{B_i}$ and is closed under arbitrary intersection. To see that it is closed under complementation, notice that, by the distributive law, any $F = \bigcap_{\lambda \in \Lambda} \bigcup_{\mu_\lambda \in M_\lambda} F_{\lambda, \mu_\lambda}$ can be written as $F = \bigcup_{(\mu_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} M_\lambda} \bigcap_{\lambda \in \Lambda} F_{\lambda, \mu_\lambda}$. Then, $F^c = \bigcap_{(\mu_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} M_\lambda} \bigcup_{\lambda \in \Lambda} F_{\lambda, \mu_\lambda}^c$ is of the desired form.

Now, replace each E_i with $E_i \cup E$. Then, since each B_i satisfies Monotonicity, $\omega \in \bigcap_{i \in G} B_i(E_i \cup E)$. Since

$$\bigcap_{i \in G} (E_i \cup E) = \left(\bigcap_{i \in G} E_i \right) \cup E = E,$$

it follows that $\omega \in D_G^{\text{PPK}}(E)$, as desired.

2. The assertion follows from the previous assertion and Proposition 7.

□

A.5 Section 5.5

Proof of Theorem 1. 1. Suppose that s^* is a rational-expectations Nash equilibrium in the model $\mathcal{M}$. Then, the following two conditions hold for each $i \in I$: (i) $s_i^* \in \Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i}^*)$; and (ii) $(s_i^*, s_{-i}^*) \succsim_i (s_i, s_{-i}^*)$ for all $s_i \in \Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i}^*)$.

To show that s^* is a Nash equilibrium in $\widetilde{\mathcal{M}}$, it suffices to show the following two conditions for each $i \in I$: (a) $s_i^* \in \Phi_i(\delta((\mathcal{S}_j)_{j \in I}))$; and (b) $\Phi_i(\delta((\mathcal{S}_j)_{j \in I})) \subseteq \Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i}^*)$. Fix $i \in I$.

Condition (a) follows because, by definition, $(s_i^*)^{-1}(\mathcal{A}_i) \subseteq \delta((\mathcal{S}_j)_{j \in I})$.

To show Condition (b), it suffices to show

$$\delta((\mathcal{S}_j)_{j \in I}) \subseteq \delta(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}}).$$

The right-hand side, $\delta(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}})$, is closed under any union by Proposition 2 and under non-empty intersection by Proposition 3 (2).³⁵ Also,

$$\mathcal{S}_j \subseteq \delta(\mathcal{J}_{B_i}, (\mathcal{S}_k)_{k \in I \setminus \{i\}}) \text{ for all } j \in I \setminus \{i\}.$$

Then, it is enough to show that

$$\mathcal{S}_i \subseteq \delta(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}}).$$

However, this follows from (i), i.e.,

$$(s_i^*)^{-1}(\mathcal{A}_i) \subseteq \delta(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}}).$$

Hence, the proof of the first assertion is complete.

Next, s^* is common knowledge in $\widetilde{\mathcal{M}}$ because, for each $i \in I$ and $E_i \in \mathcal{A}_i$,

$$(s_i^*)^{-1}(E_i) \in \delta((\mathcal{S}_j)_{j \in I}) = \bigcap_{j \in I} \mathcal{J}_{\widetilde{B}_j}.$$

³⁵To see this, by assumption, B_i satisfies Monotonicity and Conjunction. For each $j \in I \setminus \{i\}$, $\mathcal{S}_j$ can be identified with the self-evident collection associated with a knowledge operator $K_{\mathcal{S}_j}$ which satisfies Monotonicity and Conjunction. Then, $\delta(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}})$ can be identified with the self-evident collection associated with the resulting distributed knowledge operator, which inherits Conjunction by Proposition 3 (2) (in addition to satisfying Monotonicity). This implies that (by Lemma A.1) $\delta(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}})$ is closed under non-empty intersection.

2. It suffices to show that $\Phi_i(\mathcal{J}_{B_i}) = \Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i}^*)$ for each $i \in I$. Since s^* is common knowledge in $\mathcal{M}$, it follows that, for each $j \in I$,

$$(s_j^*)^{-1}(\mathcal{A}_j) \subseteq \bigcap_{i \in I} \mathcal{J}_{B_i}.$$

This implies that, for each $i \in I$,

$$\mathcal{J}_{B_i} = \mathcal{J}_{B_i} \cup \bigcup_{j \in I \setminus \{i\}} (s_j^*)^{-1}(\mathcal{A}_j).$$

Since B_i satisfies Monotonicity and Conjunction, Lemma A.1 implies that $\mathcal{J}_{B_i}$ is closed under non-empty intersection. It follows that

$$\mathcal{J}_{B_i} = \mathcal{I}(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}}).$$

Since it can be shown that $\mathcal{J}_{B_i}$ is closed under arbitrary union, it follows that

$$\mathcal{J}_{B_i} = \delta(\mathcal{J}_{B_i}, (\mathcal{S}_j)_{j \in I \setminus \{i\}}).$$

This implies $\Phi_i(\mathcal{J}_{B_i}) = \Phi_i^{\text{RE}}(\mathcal{J}_{B_i}, s_{-i}^*)$, as desired. □

References

- [1] T. Ågotnes and Y. N. Wáng. “Resolving Distributed Knowledge”. *Art. Intell.* 252 (2017), 1–21.
- [2] G.-M. Angeletos and I. Werning. “Crises and Prices: Information Aggregation, Multiplicity, and Volatility”. *Am. Econ. Rev.* 96 (2006), 1720–1736.
- [3] R. J. Aumann. “Subjectivity and Correlation in Randomized Strategies”. *J. Math. Econ.* 1 (1974), 67–96.
- [4] R. J. Aumann. “Agreeing to Disagree”. *Ann. Statist.* 4 (1976), 1236–1239.
- [5] R. J. Aumann. “Correlated Equilibrium as an Expression of Bayesian Rationality”. *Econometrica* 55 (1987), 1–18.
- [6] R. J. Aumann. “Interactive Epistemology I, II”. *Int. J. Game Theory* 28 (1999), 263–300, 301–314.
- [7] D. Austen-Smith and J. S. Banks. “Information Aggregation, Rationality and the Condorcet Jury Theorem”. *Amer. Polit. Sci. Rev.* 90 (1996), 34–45.
- [8] M. Bacharach. “Some Extensions of a Claim of Aumann in an Axiomatic Model of Knowledge”. *J. Econ. Theory* 37 (1985), 167–190.
- [9] J. A. K. Cave. “Learning to Agree”. *Econ. Lett.* 12 (1983), 147–152.
- [10] B. F. Chellas. *Modal Logic: An Introduction*. Cambridge University Press, 1980.
- [11] M. Colombetti. “Formal Semantics for Mutual Belief”. *Art. Intell.* 62 (1993), 341–353.
- [12] M. Crescenzi. *Group Knowledge and Individual Introspection*. Working Paper. 2023.
- [13] A. Dasgupta. “Coordination and Delay in Global Games”. *J. Econ. Theory* 134 (2007), 195–225.

- [14] J. Duggan and C. Martinelli. “A Bayesian Model of Voting in Juries”. *Games Econ. Behav.* 37 (2001), 259–294.
- [15] R. Fagin and J. Y. Halpern. “Belief, Awareness, and Limited Reasoning”. *Art. Intell.* 34 (1987), 39–76.
- [16] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. *Reasoning About Knowledge*. MIT Press, 2003.
- [17] R. Fagin, J. Y. Halpern, and M. Y. Vardi. “What Can Machines Know? On the Properties of Knowledge in Distributed Systems”. *J. ACM* 39 (1992), 328–376.
- [18] T. Feddersen and W. Pesendorfer. “Voting Behavior and Information Aggregation in Elections with Private Information”. *Econometrica* 65 (1997), 1029–1058.
- [19] F. Forges and E. Minelli. “Self-Fulfilling Mechanisms and Rational Expectations”. *J. Econ. Theory* 75 (1997), 388–406.
- [20] F. Forges and E. Minelli. “Self-Fulfilling Mechanisms in Bayesian Games”. *Games Econ. Behav.* 25 (1998), 292–310.
- [21] M. F. Friedell. “On the Structure of Shared Awareness”. *Behav. Sci.* 14 (1969), 28–39.
- [22] S. Fukuda. “Formalizing Common Belief with No Underlying Assumption on Individual Beliefs”. *Games Econ. Behav.* 121 (2020), 169–189.
- [23] S. Fukuda. “The Existence of Universal Qualitative Belief Spaces”. *J. Econ. Theory* 216 (2024), 105784.
- [24] F. Galton. “The Ballot-Box”. *Nature* 75 (1907), 509–510.
- [25] F. Galton. “Vox Populi”. *Nature* 75 (1907), 450–451.
- [26] F. Galton. *Memories of My Life*. Methuen, 1908.
- [27] J. Geanakoplos. “Game Theory without Partitions, and Applications to Speculation and Consensus”. *B. E. J. Theor. Econ.* 21 (2021), 361–394.
- [28] J. D. Geanakoplos and H. M. Polemarchakis. “We Can’t Disagree Forever”. *J. Econ. Theory* 28 (1982), 192–200.
- [29] J. Gerbrandy. “Bisimulations on Planet Kripke”. PhD thesis. University of Amsterdam, 1999.
- [30] D. Grafenhofer and W. Kuhle. “Observing Each Other’s Observations in a Bayesian Coordination Game”. *J. Math. Econ.* 67 (2016), 10–17.
- [31] D. Grossi and G. Pigozzi. *Judgement Aggregation: A Primer*. Morgan & Claypool, 2014.
- [32] J. Y. Halpern and Y. Moses. “Knowledge and Common Knowledge in a Distributed Environment”. *J. ACM* 37 (1990), 549–587.
- [33] J. Y. Halpern and Y. Moses. “A Guide to Completeness and Complexity for Modal Logics of Knowledge and Belief”. *Art. Intell.* 54 (1992), 319–379.
- [34] F. A. Hayek. “The Use of Knowledge in Society”. *Am. Econ. Rev.* XXXV (1945), 519–530.
- [35] A. Herzig, E. Lorini, E. Perrotin, F. Romero, and F. Schwarzentruher. “A Logic of Explicit and Implicit Distributed Belief”. *Proceedings of the 24th European Conference on Artificial Intelligence (ECAI 2020)*. 2020, 753–760.
- [36] R. Hilpinen. “Remarks on Personal and Impersonal Knowledge”. *Can. J. Philos* VII (1977), 1–9.
- [37] X. Huang. “Incentive Compatible Self-Fulfilling Mechanisms and Rational Expectations”. *Games Econ. Behav.* 126 (2021), 100–135.

- [38] S. Konieczny and R. Pino Pérez. “Merging Information Under Constraints: A Logical Framework”. *J. Log. Comput.* 12 (2002), 773–808.
- [39] L. Lismont and P. Mongin. “On the Logic of Common Belief and Common Knowledge”. *Theory Decis.* 37 (1994), 75–106.
- [40] J. McCarthy, M. Sato, T. Hayashi, and S. Igarashi. On the Model Theory of Knowledge. Stanford Artificial Intelligence Laboratory AIM-312, Computer Science Department STAN-CS-78-657. Apr. 1978.
- [41] R. D. McKelvey and T. Page. “Common Knowledge, Consensus, and Aggregate Information”. *Econometrica* 54 (1986), 109–127.
- [42] J.-J. C. Meyer and W. van der Hoek. *Epistemic Logic for AI and Computer Science*. Cambridge University Press, 1995.
- [43] E. Minelli and H. M. Polemarchakis. “Information at Equilibrium”. *Econ. Theory* 21 (2003), 573–584.
- [44] S. Modica and A. Rustichini. “Awareness and Partitional Information Structures”. *Theory Decis.* 37 (1994), 107–124.
- [45] S. Modica and A. Rustichini. “Unawareness and Partitional Information Structures”. *Games Econ. Behav.* 27 (1999), 265–298.
- [46] D. Monderer and D. Samet. “Approximating Common Knowledge with Common Beliefs”. *Games Econ. Behav.* 1 (1989), 170–190.
- [47] S. Morris. “The Logic of Belief and Belief Change: A Decision Theoretic Approach”. *J. Econ. Theory* 69 (1996), 1–23.
- [48] E. Pacuit. *Neighborhood Semantics for Modal Logic*. Springer, 2017.
- [49] M. Piccione and A. Rubinstein. Failing to Correctly Aggregate Signals. Working Paper. 2024.
- [50] R. Radner. “Rational Expectations Equilibrium: Generic Existence and the Information Revealed by Prices”. *Econometrica* 47 (1979), 655–678.
- [51] F. Roelofsen. “Distributed Knowledge”. *J. Appl. Non-Class. Log.* 17 (2007), 255–273.
- [52] D. Samet. “S5 Knowledge without Partitions”. *Synthese* 172 (2010), 145–155.
- [53] J.-M. Tallon, J.-C. Vergnaud, and S. Zamir. “Communication among Agents: A Way to Revise Beliefs in KD45 Kripke Structures”. *J. Appl. Non-Class. Log.* 14 (2004), 477–500.
- [54] Á. Tóbiás. “Meet Meets Join: The Interaction between Pooled and Common Knowledge”. *Int. J. Game Theory* 50 (2021), 989–1019.
- [55] F. Van De Putte and D. Klein. “Pointwise Intersection in Neighborhood Modal Logic”. *Advances in Modal Logic*. Ed. by G. Bezhanishvili, G. D’Agostino, G. Metcalfe, and T. Studer. Vol. 12. College Publications, 2018, 591–610.
- [56] W. van der Hoek and J.-J. Meyer. “Making Some Issues of Implicit Knowledge Explicit”. *Int. J. Found. Comput. Sci.* 3 (1992), 193–224.
- [57] W. van der Hoek and J.-J. Meyer. “A Complete Epistemic Logic for Multiple Agents”. *Epistemic Logic and the Theory of Games and Decisions*. Ed. by M. Bacharach, L.-A. Gérard-Varet, P. Mongin, and H. S. Shin. Kluwer, 1997, 35–68.
- [58] W. van der Hoek, B. van Linder, and J.-J. Meyer. “Group Knowledge is Not Always Distributed (Neither is It Always Implicit)”. *Math. Soc. Sci.* 38 (1999), 215–240.