

Information Correspondences: Bridging Knowledge-Belief Representations in Economics and Mathematical Psychology*

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Abstract

This paper studies an information correspondence, a model of beliefs that assigns multiple information sets to each state. This framework generalizes the possibility correspondence model in economics and game theory, which assigns a unique information set to each state. The information correspondence model allows for the analysis of agents who fail to believe the conjunction of their own beliefs or a tautology. It also provides a unified tractable representation for both qualitative and probabilistic beliefs. The model's only core assumption is logical monotonicity: an agent believes any logical consequence of her beliefs. The paper characterizes logical and introspective properties, as well as an agent's belief in those properties. Beliefs induced from expected utility or maxmin expected utility preferences on infinite state spaces are represented by an information correspondence but not, in general, by a possibility correspondence. The model also generalizes a knowledge representation in mathematical psychology known as a surmise function. The paper bridges seemingly different knowledge and belief representations in economics and mathematical psychology.

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1 Introduction

Representations of beliefs and knowledge have been studied across computer science, economics, game theory, logic, philosophy, and psychology. This paper studies a model of beliefs and knowledge, which I call an *information correspondence* model, with two aims. First, it generalizes the possibility correspondence model in economics and game theory in two respects: (i) it can relax some logical reasoning abilities of agents while keeping tractability; and (ii) it can represent agents' qualitative and probabilistic beliefs in the same framework. Second, it bridges, in a mathematical sense, seemingly different knowledge-belief representations in economics and mathematical psychology.

Information Correspondences. An information correspondence model has three ingredients. First, there is a set Ω of states of the world, each of which is a complete description of the world. Second, there is a collection of events, i.e., subsets E of Ω , each of which represents a property of the world. This collection determines the objects of agents' beliefs. Third, each agent is endowed with an information correspondence $\mathcal{I}$, which associates, with each state, possibly multiple information sets available at that state. The agent believes an event at a state if the event is implied by some information set at that state. If the information correspondence assigns a single information set to each state, then it reduces to a possibility correspondence (e.g., Aumann, 1976, 1999; Dekel and Gul, 1997; Morris, 1996): Proposition 1 in Section 2.1 characterizes when an information correspondence reduces to a possibility correspondence.

Unlike a possibility correspondence, an information correspondence can represent an agent who fails to believe the conjunction of events that she believes or a tautology (Example 1 in Section 2.2.1). It is thus a tractable generalization of the possibility correspondence model that dispenses with such logical sophistication. Its only requirement is logical monotonicity: an agent believes logical consequences of her own beliefs. Under this assumption, an information correspondence represents an agent's beliefs without specifying the entire collection of events that she believes at each state.¹

The relaxation of the conjunction property allows one to study both qualitative and probabilistic beliefs in the same framework (Section 2.2.2). For example, in a dynamic game, agents know their past observations but form probabilistic beliefs about their opponents' future behavior (see, for instance, Dekel and Gul, 1997). The arbitrary conjunction property rendered by a possibility correspondence (i.e., the agent believes an arbitrary conjunction of her beliefs) may be at odds with probabilistic reasoning: such a conjunction may not be an event (e.g., it may not be measurable when the events form a σ -algebra), and the agent may not assign probability one to the arbitrary conjunction of events to each of which she assigns probability one. Thus, a possibility correspondence cannot necessarily represent probabilistic beliefs. In contrast, an information correspondence can accommodate various forms of monotone probabilistic beliefs such as countably-additive, finitely-additive, and even non-additive beliefs. Section 5 provides

¹For the expert reader familiar with monotone neighborhood systems (which are to be discussed later), this feature distinguishes an information correspondence from a monotone neighborhood system.

economic applications in which an agent’s beliefs are induced from expected utility and maxmin expected utility preferences and cannot necessarily be captured by a possibility correspondence (Propositions 7 and 8).

The paper characterizes logical and introspective properties of beliefs in terms of an information correspondence (Propositions 2 and 3 in Sections 3.1 and 3.2). Since an information correspondence dispenses with Necessitation (i.e., an agent always believes a tautology), the paper also formulates the agent’s belief in a property (Proposition 4 in Section 3.3). For instance, Belief in Correct Belief states that the agent always believes that her belief is correct (i.e., whenever she believes an event at a state, the event holds true at that state), even if her belief may not always be correct.² Since logical monotonicity (if the agent believes E and E implies F , then she believes F) is the only assumption that any information correspondence imposes, the information correspondence approach is equivalent, for given logical and introspective properties, to the approach in which an agent’s belief operator maps each event E to the event that she believes E (Proposition 5 in Section 4.1).

Knowledge Representation in Mathematical Psychology. The second objective is to connect the information correspondence model to the knowledge representation known as a “surmise system” in the “knowledge space theory” of mathematical psychology developed by Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011).³ Knowledge is a special form of belief that satisfies the truth axiom: if an agent “knows” an event at a state then the event holds at that state. This paper links the knowledge representations in economics and mathematical psychology, which seem different given their respective aims, from a unified mathematical point of view.

To make the connection, I keep the same notation and, since each mathematical object has two interpretations, attach quotation marks to the terms from mathematical psychology.⁴ The literature studies an agent’s (e.g., a high school student’s) knowledge of subsets of Ω , whose elements are “questions” or “items.” Thus, Ω represents the entire body of knowledge in question (e.g., high-school algebra). A “surmise system” is a pair of the set Ω of questions and a “surmise function” $\mathcal{I}$, which associates, with each “question” ω , a collection of possibly multiple sets of “items,” each of which serves as a possible set of prerequisites for ω . Thus, if the agent has mastered ω , then she must have mastered all the “questions” in at least one member of $\mathcal{I}(\omega)$, and multiple members of $\mathcal{I}(\omega)$ represent multiple ways to master ω . The “surmise function” thus encodes all

²The previous literature has shown that an interpersonal version of Belief in Correct Belief (agent j believes that agent i ’s beliefs are correct) plays an important role in the existence of common prior and an epistemic characterization of backward induction as an implication of common knowledge of rationality in a possibility correspondence model (e.g., Bonanno and Nehring, 1998a,b; Samet, 2013).

³Doignon and Falmagne (1985) is the pioneering paper on the subject. Doignon and Falmagne (2016) is a more recent survey article. Doignon and Falmagne (1999) is the first survey book, while Falmagne and Doignon (2011) is an enriched edition of it. See also the references therein and in Hockemeyer (2026).

⁴Table A.1 in Appendix A.1 summarizes the correspondence between the terminology mainly used in this paper and that used in mathematical psychology.

possible (not necessarily unique) ways of making inferences from each question. This paper shows that an agent’s belief described by a “surmise function” is knowledge that exhibits positive introspection (if she knows a set of “questions” then she knows that she knows it) and necessitation (the agent knows a tautology).

On the technical side, the paper generalizes “surmise functions” to an arbitrary space Ω and provides a full characterization of the correspondence between “surmise functions” and another knowledge representation referred to as “knowledge structures” (Proposition 6 and Corollary 2 in Section 4.2).

More broadly, the two knowledge models in economics and mathematical psychology, which I demonstrate are firmly related, have evolved in quite different ways, which motivates connecting them in the same mathematical framework. On the one hand, interactive epistemology in economics and game theory has not developed empirical assessments of an agent’s beliefs and knowledge based on the formal model. The mathematical psychology literature (see Doignon and Falmagne (2016) and Falmagne and Doignon (2011) for surveys) has constructed and tested formal knowledge representations in practical contexts. In particular, “learning space theory,” a particular case of “knowledge space theory,” has developed assessments of students’ knowledge of their academic subjects. For example, the web-based system **ALEKS** (“**A**ssessment of **L**earning in **K**nowledge **S**paces”) has been used by “millions of students in schools and colleges, and by home schooled students” (Doignon and Falmagne, 2016).⁵

On the other hand, the economics literature has developed features that mathematical psychology has not. One is interactive higher-order beliefs, i.e., an agent’s beliefs about other agents’ beliefs. Unlike the possibility correspondence model, which only allows for knowledge and qualitative beliefs, the framework of this paper allows one to analyze agents’ qualitative and probabilistic beliefs (which, in contrast to knowledge, may be erroneous) and higher-order beliefs. Another is unawareness. In one formulation, an agent is unaware of an event if she does not know it and she does not know that she does not know it.⁶ Unlike the possibility correspondence model, the framework of this paper can capture some non-trivial forms of unawareness.

Other Related Literature. This paper is also closely related to knowledge and belief representations known as monotone neighborhood systems in computer science, logic, and philosophy (e.g., Chellas, 1980; Fagin et al., 2003; Pacuit, 2017) and to related models of limited reasoning in computer science. A neighborhood system (also called a Montague-Scott structure) associates, with each state of the world, the entire collection of events that an agent believes. It is monotone if the agent’s belief is logically monotone (i.e., she believes any logical consequence of an event that she believes). An information

⁵Also, see, for instance, Cosyn et al. (2021) and Sargin and Ünlü (2009) for empirical methodologies.

⁶Pioneering papers on unawareness include Fagin and Halpern (1987) and Modica and Rustichini (1994, 1999). In another formulation, some recent models such as Heifetz, Meier, and Schipper (2006, 2013) formalize unawareness as a lack of concepts. See Schipper (2015) for a survey. Whether unawareness is formalized as a lack of knowledge or a lack of concepts, it may offer novel insights into the assessments of one’s knowledge in mathematical psychology.

correspondence is a “generator” of a monotone neighborhood system, and thus describes an agent’s beliefs without specifying the entire collection of events that she believes at each state. A monotone neighborhood system is itself an information correspondence.

In economics and game theory, monotone neighborhood systems are used to represent common belief and common knowledge (e.g., Aumann, 1976; Friedell, 1969) by Heifetz (1996, 1999) and Lismont and Mongin (1994a,b). This paper instead formalizes logical and introspective properties of individual agents’ beliefs and knowledge, and briefly discusses how to introduce common belief and common knowledge into its framework. Salonen (2009a) studies a canonical syntactical interactive belief representation in which each agent’s beliefs are captured as a collection of propositions that she believes.

The closest paper in the literature on limited reasoning is the logic of local reasoning (or the “society-of-minds” approach) of Fagin and Halpern (1987).⁷ They study a “boundedly rational” agent who fails to believe (or know) the conjunction of her own beliefs (or knowledge): she is endowed with multiple information sets at each state and focuses on one of them at a time.

While an information correspondence can be regarded as a purely semantic counterpart of their model, this paper makes three additional contributions. First, relative to Fagin and Halpern (1987) and Thijsse (1993), it formulates properties of beliefs as conditions on the primitive (i.e., an information correspondence $\mathcal{I}$), and shows that each condition is necessary and sufficient for the corresponding property (Propositions 3 and 4 in Sections 3.2 and 3.3). Second, by defining the model on an appropriate set-algebraic structure, it captures probabilistic beliefs. Third, it connects the model to the mathematical psychology literature.

The paper is organized as follows. Section 2 defines information correspondences, shows that they generalize possibility correspondences, and provides examples that cannot be captured by possibility correspondences. Section 3 analyzes logical and introspective properties of information correspondences, including beliefs in those properties. Section 4 studies the equivalence between information correspondences and alternative knowledge-belief representations in economics and mathematical psychology. Section 5 studies economic applications in which an agent’s beliefs are induced from her preferences, providing an economic example of an information correspondence that cannot be captured by a possibility correspondence. Section 6 provides concluding remarks. Proofs are relegated to Appendix A, and Appendix B provides supplementary results.

2 An Information Correspondence

I represent agents’ beliefs (knowledge if they are truthful) on a *state space* $(\Omega, \mathcal{D})$, where Ω is a set of *states* of the world and $\mathcal{D}$ is a collection of *events*, i.e., subsets of states. To accommodate qualitative and probabilistic beliefs in the same framework and possibly

⁷Thijsse (1993) (see also Thijsse, 1992, Chapter 6.6) calls their model a cluster model. Also, Fagin et al. (2003, Chapter 9.6) and Meyer and van der Hoek (1995, Chapter 2.9) study this approach.

at the same time, I assume throughout that $(\Omega, \mathcal{D})$ is a measurable space, i.e., $\mathcal{D}$ is a σ -algebra on Ω .⁸ $\mathcal{D}$ contains $\emptyset$ and Ω and is closed under complementation and under countable union (and consequently countable intersection).⁹ Denote by $\mathcal{P}(\Omega)$ the power set of Ω , which is also a σ -algebra, and by E^c or $\neg E$ the complement of $E \in \mathcal{P}(\Omega)$.

2.1 Definition of an Information Correspondence

I represent an agent's belief (or knowledge) by an information correspondence. For ease of exposition, unless otherwise stated, I restrict attention to a single agent. As in a possibility correspondence, the agent's belief is logically entailed by her information, but the only requirement is logical monotonicity: the agent believes any logical consequence of her own belief. Thus, I dispense with the conjunction and necessitation properties endemic in possibility correspondences: the agent may believe events E and F without believing $E \cap F$, and she may fail to believe the tautology Ω .

The information correspondence $\mathcal{I}$ associates, with each state ω , a collection of events $\mathcal{I}(\omega) \in \mathcal{P}(\mathcal{D})$ that can be a source of beliefs at the state ω in the following sense: the agent believes an event E at the state ω if there is an event $F \in \mathcal{I}(\omega)$ which is included in E . Each element of $\mathcal{I}(\omega)$ can be understood as a piece of information available to the agent at state ω . Call E an *information set* at ω if $E \in \mathcal{I}(\omega)$.

Formally, an information correspondence is a mapping $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ satisfying a regularity condition. To define it, for any $\Gamma \in \mathcal{P}(\mathcal{D})$, let

$$\uparrow \Gamma := \{E \in \mathcal{D} \mid \text{there is } F \in \Gamma \text{ with } F \subseteq E\}.$$

I call $\Gamma \in \mathcal{P}(\mathcal{D})$ an *information collection*, as it is a collection of information sets. If $\Gamma = \mathcal{I}(\omega)$ for some $\omega \in \Omega$, then $E \in \uparrow \Gamma$ means that E is entailed by some information set $F \in \Gamma$. Note that $\uparrow \Gamma$ is closed under set inclusion: $\uparrow \uparrow \Gamma \subseteq \uparrow \Gamma$ (i.e., if $F \in \uparrow \Gamma$ and $F \subseteq E$ then $E \in \uparrow \Gamma$).

With these in mind, I formally define an information correspondence. An *information correspondence* on a state space $(\Omega, \mathcal{D})$ is a mapping $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ which satisfies the regularity condition that, for each $E \in \mathcal{D}$,

$$\{\omega \in \Omega \mid E \in \uparrow \mathcal{I}(\omega)\} \in \mathcal{D}. \quad (1)$$

Condition (1) states that the set of states at which the agent has information to support an event E itself is an event. I define the left-hand side of Expression (1) as:

$$B_{\mathcal{I}}(E) := \{\omega \in \Omega \mid E \in \uparrow \mathcal{I}(\omega)\}. \quad (2)$$

Thus, $B_{\mathcal{I}}(E)$ is the event that (i.e., the set of states at which) the agent *believes* E . Define the *belief operator* $B_{\mathcal{I}} : \mathcal{D} \rightarrow \mathcal{D}$ derived from $\mathcal{I}$ through Equation (2). Condition

⁸For example, Meier (2008) studies knowledge and σ -additive probabilistic beliefs on a σ -algebra.

⁹The analysis of this paper carries over when $\mathcal{D}$ is an algebra of sets (i.e., $\{\emptyset, \Omega\} \subseteq \mathcal{D}$ and $\mathcal{D}$ is closed under finite union and complementation (and consequently finite intersection)) or a complete algebra of sets (i.e., $\{\emptyset, \Omega\} \subseteq \mathcal{D}$ and $\mathcal{D}$ is closed under arbitrary union and complementation (and consequently arbitrary intersection)). See also Meier (2006) and Fukuda (2024b).

(1) guarantees that $B_{\mathcal{I}}$ is a well-defined operator. Higher-order beliefs are generated through iterating the belief operator.

I make five remarks on the belief operator $B_{\mathcal{I}}$. First, $\uparrow \mathcal{I}(\omega)$ is exactly the collection of events that the agent believes at ω :

$$\uparrow \mathcal{I}(\omega) = \{E \in \mathcal{D} \mid \omega \in B_{\mathcal{I}}(E)\},$$

that is, $E \in \uparrow \mathcal{I}(\omega)$ if and only if (hereafter, often abbreviated as iff) $\omega \in B_{\mathcal{I}}(E)$. Also, the mapping $\uparrow \mathcal{I}$ defined by $(\uparrow \mathcal{I})(\omega) := \uparrow \mathcal{I}(\omega)$ for each $\omega \in \Omega$ is itself an information correspondence.

Second, the belief operator $B_{\mathcal{I}}$ satisfies Monotonicity:

$$E \subseteq F \text{ implies } B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}(F).$$

That is, if the agent believes an event E at a state and if E implies F (in the sense of $E \subseteq F$), then she believes F at that state. Section 4.1 establishes the equivalence between an information correspondence and a monotone belief operator.

Third, $\uparrow \mathcal{I}$ is a monotone neighborhood system (recall Section 1): it assigns, to each state, the collection of events that the agent believes at that state, and the induced belief operator satisfies Monotonicity. Thus, $\mathcal{I}$ is itself a monotone neighborhood system iff $\mathcal{I} = \uparrow \mathcal{I}$.

Fourth, the agent considers an event E *possible* at a state ω if she does not believe E^c at ω . Denote by $L_{\mathcal{I}}$ the agent's possibility operator:

$$L_{\mathcal{I}}(E) := (\neg B_{\mathcal{I}})(E^c) \in \mathcal{D} \text{ for all } E \in \mathcal{D}.^{10}$$

The event that the agent considers E possible is

$$L_{\mathcal{I}}(E) = \{\omega \in \Omega \mid F \cap E \neq \emptyset \text{ for all } F \in \mathcal{I}(\omega)\}. \quad (3)$$

The agent considers E possible at ω when every information set at ω intersects E , i.e., none of her information at ω is inconsistent with E . Section 4.2 connects the possibility operator to a closure operator of Doignon and Falmagne (1985, 1999) and Falmagne and Doignon (2011) in mathematical psychology.

Fifth, if $\mathcal{I}$ is singleton-valued, i.e., $\mathcal{I}(\cdot) = \{P(\cdot)\}$, then it reduces to a possibility correspondence $P : \Omega \rightarrow \mathcal{D}$, where each possibility set $P(\omega)$ is an event and the regularity condition becomes

$$\{\omega \in \Omega \mid P(\omega) \subseteq E\} \in \mathcal{D} \text{ for each } E \in \mathcal{D}.^{11}$$

¹⁰Possibility (or compatibility) is often considered to be the dual of knowledge or belief (e.g., Fagin et al., 2003; Hintikka, 1962).

¹¹The regularity condition is needed for the belief operator induced by P to be well-defined. See Samet (2010) for examples of a state space $(\Omega, \mathcal{D})$ in which $\mathcal{D}$ is an algebra of sets and in which a partitional possibility correspondence $P : \Omega \rightarrow \mathcal{P}(\Omega)$ (i.e., a partition cell $P(\omega)$ may not be an event) does not induce a well-defined knowledge operator from $\mathcal{D}$ into itself.

More generally, $\mathcal{I}$ satisfies the *Kripke property* if each $\mathcal{I}(\omega)$ contains a minimum element, i.e., there is $P(\omega) \in \mathcal{I}(\omega)$ such that $P(\omega) \subseteq E$ for all $E \in \mathcal{I}(\omega)$. Then,

$$\uparrow \mathcal{I}(\omega) = \{E \in \mathcal{D} \mid P(\omega) \subseteq E\} = \uparrow \{P(\omega)\},$$

so that the agent’s beliefs are also represented by the singleton-valued information correspondence $\{P(\cdot)\}$. The following proposition formalizes this in terms of the belief operator.

Proposition 1. *Let $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ be an information correspondence. The following are equivalent.*

1. *The information correspondence $\mathcal{I}$ satisfies the Kripke property.*
2. *For each $\omega \in \Omega$, there is $P(\omega) \in \mathcal{I}(\omega)$ such that, for all $E \in \mathcal{D}$,*

$$\omega \in B_{\mathcal{I}}(E) \text{ iff } P(\omega) \subseteq E.$$

To conclude this subsection, I relate an information correspondence to a “surmise function” (see Table A.1 in Appendix A.1 for the correspondence of terminologies). Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011) introduce a “surmise system” $(\Omega, \mathcal{I})$ to model an agent’s knowledge, where a “surmise function” is a mapping $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{P}(\Omega))$ (i.e., $\mathcal{D} = \mathcal{P}(\Omega)$) satisfying certain logical and introspective properties of knowledge discussed in Section 3. If the agent is capable of solving the “question” ω , then there is $E \in \mathcal{I}(\omega)$ such that she is capable of solving all the “questions” in E . Such E is referred to as a “clause” (a “background” or a “foundation”) for ω .

2.2 Examples of Information Correspondences

This subsection provides two examples of information correspondences that cannot be reduced to possibility correspondences.

2.2.1 Dispensing with Conjunction or Necessitation

The first example shows that multiple information sets allow one to dispense with the agent’s conjunctive ability (i.e., she believes the conjunction of what she believes), and that an empty-valued information correspondence allows one to dispense with Necessitation (i.e., she believes a tautology).

Example 1. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Define $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ as follows:

$$\mathcal{I}(\omega) = \begin{cases} \{\{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}\} & \text{if } \omega = \omega_1 \\ \emptyset & \text{if } \omega \in \{\omega_2, \omega_3\} \end{cases}.$$

The agent believes $\{\omega_1, \omega_2\}$ and $\{\omega_1, \omega_3\}$ at state ω_1 but she does not believe $\{\omega_1, \omega_2\} \cap \{\omega_1, \omega_3\}$ at that state:

$$B_{\mathcal{I}}(\{\omega_1, \omega_2\}) \cap B_{\mathcal{I}}(\{\omega_1, \omega_3\}) \not\subseteq B_{\mathcal{I}}(\{\omega_1, \omega_2\} \cap \{\omega_1, \omega_3\}).$$

The failure of the conjunction property comes from the specification that an information collection $\mathcal{I}(\omega)$ may not be closed under intersection. At state ω_2 or ω_3 , she does not believe anything at all, and thus $B_{\mathcal{I}}(\Omega) = \{\omega_1\}$. Necessitation (i.e., $B_{\mathcal{I}}(\Omega) = \Omega$) fails because the information correspondence is empty-valued at some states. By inspection, one can show:

$$B_{\mathcal{I}}(E) = \begin{cases} \{\omega_1\} & \text{if } E \in \{\{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \Omega\} \\ \emptyset & \text{otherwise} \end{cases}. \quad (4)$$

The belief operator $B_{\mathcal{I}}$ cannot be induced by a possibility correspondence because any belief operator that is induced by a possibility correspondence has to satisfy the conjunction and necessitation properties. $\square$

This example also relates to the literature on unawareness, which uses two kinds of models: a standard-state-space model, in which the underlying state space consists of a single space, and a generalized-state-space model, in which it consists of multiple subspaces. Dekel, Lipman, and Rustichini (1998) show that a standard-state-space model cannot represent a non-trivial form of unawareness if the knowledge operator satisfies Monotonicity and Necessitation and if the knowledge and unawareness operators satisfy certain desirable properties. Thus, in any standard possibility correspondence model, whose knowledge operator satisfies Monotonicity and Necessitation, knowledge and unawareness cannot satisfy these properties. In contrast, an information correspondence can drop Necessitation.¹² In fact, Modica and Rustichini (1994, Section 4) provide a two-state model of unawareness in which the agent's knowledge fails Necessitation. Their knowledge operator is induced by a simplified version of this example in which $\omega_2 = \omega_3$.¹³

2.2.2 Modeling Qualitative and Probabilistic Beliefs in a Unified Manner

The second example shows that an information correspondence, unlike a possibility correspondence, can capture probabilistic beliefs, so that information correspondences capture qualitative and probabilistic beliefs in a unified manner.

Example 2. Consider a measurable space $(\Omega, \mathcal{D}) = ([0, 1], \mathcal{B}_{[0,1]})$, where $\mathcal{B}_{[0,1]}$ is the Borel σ -algebra on $[0, 1]$. For simplicity, suppose that the agent's beliefs are dictated by the Lebesgue measure μ on $([0, 1], \mathcal{B}_{[0,1]})$ at every state.

¹²Fukuda (2021) studies unawareness without the conjunction or necessitation properties, taking agents' knowledge and unawareness operators as primitives on standard and generalized state spaces. It would be interesting to extend information correspondences to generalized-state-space models of unawareness and knowledge such as Heifetz, Meier, and Schipper (2006, 2008) and of unawareness and qualitative belief such as Fukuda (2023).

¹³Holliday (2025), Rathke (2025), and Tada (2024) study, in different contexts, standard-state-space possibility-correspondence models of unawareness in which agents' knowledge may fail Necessitation.

First, her probability-one belief is not induced by a possibility correspondence. Suppose to the contrary that there is $P : \Omega \rightarrow \mathcal{P}(\Omega)$ such that she believes $E \in \mathcal{D}$ with probability one at ω iff

$$P(\omega) \subseteq E.^{14}$$

Since she assigns probability one to $E_r := \Omega \setminus \{r\} \in \mathcal{D}$ for each $r \in \Omega$ at each ω , one has $P(\omega) \subseteq E_r$ for all $r \in \Omega$, and hence $P(\omega) = \emptyset$. Then her probability-one belief operator $B : \mathcal{D} \rightarrow \mathcal{D}$ satisfies $B(E) = \Omega$ for all $E \in \mathcal{D}$; in particular, $B(\emptyset) = \Omega$, which contradicts $\mu(\emptyset) = 0$. Since one can generally only assert that she assigns probability one to any *countable* intersection of the events E_r , this contradiction comes from the arbitrary conjunction rendered by the possibility correspondence.

Next, an information correspondence captures her probability-one belief. Let

$$\mathcal{I}(\omega) := \{E \in \mathcal{D} \mid \mu(E) = 1\} \text{ for each } \omega \in \Omega.$$

Then, for any $E \in \mathcal{D}$,

$$B_{\mathcal{I}}(E) = \begin{cases} \emptyset & \text{if } \mu(E) \in [0, 1) \\ \Omega & \text{if } \mu(E) = 1 \end{cases},$$

so that $B_{\mathcal{I}}(E) \in \mathcal{D}$ and $\mathcal{I}$ is an information correspondence. Moreover, $\omega \in B_{\mathcal{I}}(E)$ iff there is $F \subseteq E$ with $\mu(F) = 1$, i.e., iff $\mu(E) = 1$. Thus, $B_{\mathcal{I}}$ is exactly her probability-one belief. $\square$

More generally, let $(\Omega, \mathcal{D})$ be an arbitrary measurable space. Call $t : \Omega \times \mathcal{D} \rightarrow [0, 1]$ a *type mapping* (of the agent) if:

1. For each $\omega \in \Omega$, the mapping $t(\omega, \cdot) : \mathcal{D} \rightarrow [0, 1]$ is a monotone set function with $t(\omega, \emptyset) = 0$ and $t(\omega, \Omega) = 1$ that dictates the agent's probabilistic beliefs at ω (e.g., a non-additive, finitely-additive, or countably-additive probability measure).
2. (Regularity) For all $(p, E) \in [0, 1] \times \mathcal{D}$, $\{\omega \in \Omega \mid t(\omega, E) \geq p\} \in \mathcal{D}$, i.e., the set of states at which the agent believes E with probability at least p is an event.

Call $t(\omega, \cdot)$ the agent's *type* at ω . For each $p \in [0, 1]$, the agent's *p-belief operator* (Friedell, 1969; Monderer and Samet, 1989) $B^p : \mathcal{D} \rightarrow \mathcal{D}$ is defined as

$$B^p(E) := \{\omega \in \Omega \mid t(\omega, E) \geq p\} \text{ for each } E \in \mathcal{D}. \quad (5)$$

For each $p \in [0, 1]$, the mapping $\mathcal{I}_p : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ defined by

$$\mathcal{I}_p(\omega) := \{E \in \mathcal{D} \mid t(\omega, E) \geq p\} \text{ for each } \omega \in \Omega$$

is an information correspondence because

$$B_{\mathcal{I}_p}(E) := \{\omega \in \Omega \mid E \in \uparrow \mathcal{I}_p(\omega)\} = B^p(E) \in \mathcal{D} \text{ for each } E \in \mathcal{D}.$$

¹⁴Here, I establish the negative result even if I do not impose the assumption that each $P(\omega)$ is measurable (i.e., an event).

Since Samet (2000) shows that the p -belief operator approach is equivalent to the type space approach,¹⁵ the information correspondence approach accommodates both probabilistic and non-probabilistic beliefs in a unified manner (e.g., in the dynamic games discussed in Section 1), and enables one to study similarities and differences between them.¹⁶

3 Properties of an Information Correspondence

This section studies three kinds of properties of beliefs for an information correspondence: (i) logical properties such as consistency, necessitation, and conjunction properties; (ii) introspective properties including the truth axiom, which together with the logical properties generalize those of possibility correspondences; and (iii) “Belief-in” properties. An agent whose belief is represented by an information correspondence may violate certain properties and yet believe that her belief satisfies them.

3.1 Logical Properties

I introduce four well-known logical properties: No-Contradiction, Consistency, Necessitation, and Countable Conjunction. While No-Contradiction and Consistency are equivalent in the possibility correspondence model, Consistency is at least as strong as No-Contradiction in the information correspondence model.¹⁷

Hereafter in this section, fix a state space $(\Omega, \mathcal{D})$. I define each logical property for an information collection Γ , and say that an information correspondence $\mathcal{I}$ satisfies it if every $\mathcal{I}(\omega)$ does (e.g., $\mathcal{I}$ satisfies No-Contradiction if $\emptyset \notin \mathcal{I}(\omega)$ for all $\omega \in \Omega$).¹⁸

1. An information collection $\Gamma \in \mathcal{P}(\mathcal{D})$ satisfies *No-Contradiction* if

$$\emptyset \notin \Gamma.$$

In words, Γ does not contain a contradiction in the form of the empty set.

2. The information collection Γ satisfies *Consistency* (or it is *serial*) if

$$E \cap F \neq \emptyset \text{ for any } E, F \in \Gamma.$$

¹⁵Meier (2006) and Zhou (2010) extend the conditions to finitely-additive beliefs. Fukuda (2025) studies various properties of non-additive beliefs.

¹⁶For the literature studying how knowledge is technically and conceptually different from probability-one beliefs, see, for instance, Brandenburger and Dekel (1987), Dubra and Echenique (2004), Fukuda (2019, 2024a), Hérves-Beloso and Monteiro (2013), Lee (2018), and Tóbiás (2021).

¹⁷In the possibility correspondence model, qualitative belief is usually assumed to satisfy Consistency (equivalently, No-Contradiction). Consistency plays an important role, for instance, in formalizing weak-dominance rationality in Bonanno and Tsakas (2018).

¹⁸When it comes to probabilistic beliefs represented by a type mapping, recall that a type mapping t associates, with each state ω , a probability measure $t(\omega, \cdot)$ which satisfies, for instance, Necessitation: $t(\omega, \Omega) = 1$. That is, logical properties of beliefs (i.e., the type mapping) are encoded as logical properties of realized beliefs $t(\omega, \cdot)$.

In words, any pair of information $(E, F) \in \Gamma^2$ is not contradictory with each other. Consistency implies No-Contradiction.

3. The information collection Γ satisfies *Necessitation* if

$$\Gamma \neq \emptyset.$$

That is, Γ contains some information, and thus a tautology is inferred from it.

4. The information collection Γ satisfies *Countable Conjunction* if,

$$\text{for any } \{F_n\}_{n \in \mathbb{N}} \subseteq \Gamma, \text{ there is } F \in \Gamma \text{ with } F \subseteq \bigcap_{n \in \mathbb{N}} F_n.^{19}$$

Intuitively, for any given countable collection of information, the information collection Γ is rich enough to have another information implying the conjunction of the given family.

Since $\uparrow \Gamma$ is the collection of events that the agent believes by making inferences from Γ , the following proposition restates each property in terms of $\uparrow \Gamma$ and shows that it captures the intended property of the belief operator $B_{\mathcal{I}}$.

Proposition 2. *Let Γ be an information collection. Let $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ be an information correspondence.*

1. (a) Γ satisfies No-Contradiction iff $\uparrow \Gamma$ satisfies it.
(b) $\mathcal{I}$ satisfies No-Contradiction iff $\uparrow \mathcal{I}$ satisfies it iff $B_{\mathcal{I}}(\emptyset) = \emptyset$.
2. (a) Γ satisfies Consistency iff $\uparrow \Gamma$ satisfies it iff $E^c \notin \uparrow \Gamma$ for any $E \in \uparrow \Gamma$.
(b) $\mathcal{I}$ satisfies Consistency iff $\uparrow \mathcal{I}$ satisfies it iff

$$B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(E^c) \text{ for all } E \in \mathcal{D}.$$

3. (a) Γ satisfies Necessitation iff $\uparrow \Gamma$ satisfies it iff $\Omega \in \uparrow \Gamma$.
(b) $\mathcal{I}$ satisfies Necessitation iff $\uparrow \mathcal{I}$ satisfies it iff $B_{\mathcal{I}}(\Omega) = \Omega$.
4. (a) Γ satisfies Countable Conjunction iff $\uparrow \Gamma$ satisfies it iff $\uparrow \Gamma$ is closed under (non-empty) countable intersection:

$$\bigcap_{n \in \mathbb{N}} F_n \in \uparrow \Gamma \text{ for any } \{F_n\}_{n \in \mathbb{N}} \in \mathcal{P}(\uparrow \Gamma).$$

¹⁹One can define variants of this conjunction property. For instance, Γ satisfies *Finite Conjunction* if, for any $F_1, F_2 \in \Gamma$, there is $F \in \Gamma$ with $F \subseteq F_1 \cap F_2$. For ease of exposition, I mainly focus on Countable Conjunction.

(b) $\mathcal{I}$ satisfies Countable Conjunction iff $\uparrow \mathcal{I}$ satisfies it iff

$$\bigcap_{n \in \mathbb{N}} B_{\mathcal{I}}(F_n) \subseteq B_{\mathcal{I}} \left(\bigcap_{n \in \mathbb{N}} F_n \right) \text{ for any } \{F_n\}_{n \in \mathbb{N}} \in \mathcal{P}(\mathcal{D}).$$

Proposition 2 establishes two points, which also hold for Propositions 3 and 4 below. First, $\mathcal{I}$ satisfies a given property iff $\uparrow \mathcal{I}$ does (for the logical properties, because Γ satisfies it iff $\uparrow \Gamma$ does, by “Part (a)”). That is, the properties are preserved under the operation of taking “ $\uparrow$ ” and depend only on $\uparrow \mathcal{I}$: if $\mathcal{I}$ and $\mathcal{I}'$ induce the same beliefs in that $\uparrow \mathcal{I} = \uparrow \mathcal{I}'$, then they share the same properties. Second, each property of $\mathcal{I}$ captures the intended property of $B_{\mathcal{I}}$ (“Part (b)”). Henceforth, $B_{\mathcal{I}}$ is said to satisfy a given property (e.g., No-Contradiction) if $\mathcal{I}$ satisfies it.

By (1b), No-Contradiction means that there is no state at which the agent believes the contradiction $\emptyset$. By (2b), Consistency means that if the agent believes E , then she does not believe its negation E^c ; it implies No-Contradiction because $B_{\mathcal{I}}(\emptyset) \subseteq B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)$ by Monotonicity. By (3b), Necessitation means that the agent always believes the tautology Ω ; it holds iff $B_{\mathcal{I}}(B_{\mathcal{I}}(\Omega)) = \Omega$, i.e., iff she always believes that she believes a tautology.²⁰ By (4b), Countable Conjunction means that if the agent believes each of a countable collection of events, then she believes its conjunction. Under Countable (in fact, Finite) Conjunction, Consistency and No-Contradiction are equivalent. For example, the belief operator in Equation (4) satisfies No-Contradiction and Consistency but fails Necessitation and Countable Conjunction. Proposition 2 generalizes the characterizations of the logical properties in the possibility correspondence model.

3.2 Introspective Properties

Next, I introduce four well-known introspective properties, which generalize the reflexivity, transitivity, Euclideaness, and symmetry axioms of a possibility correspondence. In terms of the belief operator, reflexivity characterizes Truth Axiom: if the agent “knows” an event at a state then the event holds at that state. Truth Axiom distinguishes knowledge (which has to be true) from belief (which can be false). Transitivity characterizes Positive Introspection: if the agent believes an event then she believes that she believes it. Euclideaness characterizes Negative Introspection: if she does not believe an event then she believes that she does not believe it. Symmetry characterizes the following: if she considers it possible that she believes an event E at a state, then E holds at that state. It is well-known that a reflexive and transitive possibility correspondence is Euclidean iff it is symmetric. One of the technical contributions of this paper is to formulate these four properties for information correspondences, as follows.

²⁰The proof goes as follows. If $B_{\mathcal{I}}$ satisfies Necessitation, then it follows from $\Omega = B_{\mathcal{I}}(\Omega)$ that $\Omega = B_{\mathcal{I}}(\Omega) = B_{\mathcal{I}}(B_{\mathcal{I}}(\Omega))$. Conversely, if $\Omega = B_{\mathcal{I}}(B_{\mathcal{I}}(\Omega))$, then it follows from $B_{\mathcal{I}}(\Omega) \subseteq \Omega$ that $\Omega = B_{\mathcal{I}}(B_{\mathcal{I}}(\Omega)) \subseteq B_{\mathcal{I}}(\Omega) \subseteq \Omega$.

1. An information correspondence $\mathcal{I}$ is *reflexive* (or satisfies *Truth Axiom*) if, for any $(\omega, E) \in \Omega \times \mathcal{D}$,

$$E \in \mathcal{I}(\omega) \text{ implies } \omega \in E.^{21}$$

In words, the agent's information is always correct at each state.

2. The information correspondence $\mathcal{I}$ is *transitive* (or satisfies *Positive Introspection*) if, for any $(\omega, E) \in \Omega \times \mathcal{D}$ with $E \in \mathcal{I}(\omega)$, there is $F \in \mathcal{I}(\omega)$ such that:

$$\text{if } \omega' \in F \text{ then there is } E' \in \mathcal{I}(\omega') \text{ with } E' \subseteq E.$$

Put differently, for any information E at a state, there is another information F at the same state (which can possibly be E itself) such that E is always supported as long as F is true.

3. The information correspondence $\mathcal{I}$ is *Euclidean* (or satisfies *Negative Introspection*) if the following holds. If $(\omega, E) \in \Omega \times \mathcal{D}$ satisfies $E^c \cap F \neq \emptyset$ for all $F \in \mathcal{I}(\omega)$, then:

$$\text{there is } F' \in \mathcal{I}(\omega) \text{ such that if } \omega' \in F' \text{ then } E^c \cap F \neq \emptyset \text{ for any } F \in \mathcal{I}(\omega').$$

In words, at a state ω , if there is no information that supports E , then there exists a piece of information F' that supports that there is no information that supports E .

4. The information correspondence $\mathcal{I}$ is *symmetric* if the following obtains. Let $(\omega, E) \in \Omega \times \mathcal{D}$. If, for any $F \in \mathcal{I}(\omega)$,

$$\text{there are } \omega' \in F \text{ and } F' \in \mathcal{I}(\omega') \text{ with } F' \subseteq E,$$

then $\omega \in E$. This condition states that if the agent does not believe that she does not believe E then E is true.

To compare the axioms of Euclideaness and symmetry, I restate them.

3. The information correspondence $\mathcal{I}$ is Euclidean iff the following holds. Let $(\omega, E) \in \Omega \times \mathcal{D}$. If, for any $F \in \mathcal{I}(\omega)$,

$$\text{there are } \omega' \in F \text{ and } F' \in \mathcal{I}(\omega') \text{ with } F' \subseteq E,$$

then there is $F \in \mathcal{I}(\omega)$ with $F \subseteq E$. This means that if the agent does not believe that she does not believe an event E , then she believes E .

4. The information correspondence $\mathcal{I}$ is symmetric iff, for any $(\omega, E) \in \Omega \times \mathcal{D}$ with $\omega \notin E$,

$$\text{there is } F \in \mathcal{I}(\omega) \text{ such that } \omega' \in F \text{ implies } F' \cap E^c \neq \emptyset \text{ for all } F' \in \mathcal{I}(\omega').$$

²¹That is, $\mathcal{I}(\omega) \subseteq \{E \in \mathcal{D} \mid \omega \in E\}$ for any $\omega \in \Omega$.

Proposition 3. *Let $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ be an information correspondence. For each introspective property, the following are all equivalent.*

1. (a) $\mathcal{I}$ is reflexive.
(b) $\uparrow \mathcal{I}$ is reflexive.
(c) $B_{\mathcal{I}}(E) \subseteq E$ for all $E \in \mathcal{D}$.
2. (a) $\mathcal{I}$ is transitive.
(b) $\uparrow \mathcal{I}$ is transitive.
(c) For any $\omega \in \Omega$, if $E \in \uparrow \mathcal{I}(\omega)$ then $\{\omega' \in \Omega \mid E \in \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega)$.
(d) $B_{\mathcal{I}}(\cdot) \subseteq B_{\mathcal{I}}B_{\mathcal{I}}(\cdot)$.
3. (a) $\mathcal{I}$ is Euclidean.
(b) $\uparrow \mathcal{I}$ is Euclidean.
(c) If $E \notin \uparrow \mathcal{I}(\omega)$ for some $(\omega, E) \in \Omega \times \mathcal{D}$, then $\{\omega' \in \Omega \mid E \notin \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega)$.
(d) $(\neg B_{\mathcal{I}})(\cdot) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(\cdot)$.
4. (a) $\mathcal{I}$ is symmetric.
(b) $\uparrow \mathcal{I}$ is symmetric. That is, let $(\omega, E) \in \Omega \times \mathcal{D}$, and suppose that, for any $F \in \uparrow \mathcal{I}(\omega)$, there is $\omega' \in F$ with $E \in \uparrow \mathcal{I}(\omega')$. Then $\omega \in E$.
(c) $(\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E) \subseteq E$ for all $E \in \mathcal{D}$.

As noted after Proposition 2, $\mathcal{I}$ satisfies a given introspective property iff $\uparrow \mathcal{I}$ does, and each property of $\mathcal{I}$ captures the intended property of $B_{\mathcal{I}}$. Henceforth, $B_{\mathcal{I}}$ is said to satisfy it (e.g., Truth Axiom) if $\mathcal{I}$ does. For Positive and Negative Introspection, Thijsse (1993) proves the equivalence of (2c) and (2d) and that of (3c) and (3d); Parts (2) and (3) of Proposition 3 provide the conditions on the primitive $\mathcal{I}$.

If $\mathcal{I}$ is symmetric, the resulting property of $B_{\mathcal{I}}$ is often referred to as the axiom B in logic (e.g., Chellas, 1980). By the restatement above, if $\mathcal{I}$ is reflexive and transitive, then it is symmetric iff it is Euclidean,²² which generalizes the corresponding equivalence for reflexive and transitive possibility correspondences.

Moreover, if $\mathcal{I}$ is serial, symmetric, and transitive, then it satisfies all the other properties mentioned so far.²³

Corollary 1. *If $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ is a serial, symmetric, and transitive information correspondence, then it satisfies all the properties mentioned in Propositions 2 and 3.*

²²One can also recast this statement in terms of the belief operator $B_{\mathcal{I}}$ as follows: if $B_{\mathcal{I}}$ satisfies Monotonicity, Truth Axiom, and Positive Introspection, then (4c) and (3d) in Proposition 3 are equivalent. The proof goes as follows. (4c) implies $(\neg B_{\mathcal{I}})(E) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})B_{\mathcal{I}}(E)$. Since $B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}B_{\mathcal{I}}(E)$ by Positive Introspection and since $B_{\mathcal{I}}$ is monotone, it follows that $B_{\mathcal{I}}(\neg B_{\mathcal{I}})B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E)$. Thus, $(\neg B_{\mathcal{I}})(E) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E)$, as desired. Conversely, Negative Introspection and Truth Axiom yield $(\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E) \subseteq B_{\mathcal{I}}(E) \subseteq E$.

²³In fact, the information correspondence $\mathcal{I}$ satisfies all the properties in Section 3.3 as well.

In Corollary 1, seriality and transitivity (i.e., Consistency and Positive Introspection of $B_{\mathcal{I}}$) are common assumptions on qualitative beliefs in economics and game theory. The corollary states that adding symmetry yields Truth Axiom and, interestingly, Countable Conjunction.

Proposition 3 also implies that, in the information correspondence model, Negative Introspection implies Necessitation, and so does the axiom B (property (4c) in Proposition 3): if $\mathcal{I}$ is Euclidean or symmetric, then $\mathcal{I}(\omega) = \emptyset$ leads to a contradiction.²⁴

To discuss the assumptions on a “surmise function” in Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011), I introduce two additional properties of an information correspondence $\mathcal{I}$.

1. The information correspondence $\mathcal{I}$ is *strongly transitive* if, for any $(\omega, E) \in \Omega \times \mathcal{D}$ with $E \in \mathcal{I}(\omega)$,

$$\text{if } \omega' \in E \text{ then there is } E' \in \mathcal{I}(\omega') \text{ with } E' \subseteq E.$$

This is the transitivity condition in Doignon and Falmagne (1985, Definition 3.5).

2. The information correspondence $\mathcal{I}$ satisfies the *minimality* condition if

$$E = F \text{ for any } E, F \in \mathcal{I}(\omega) \text{ with } E \subseteq F.$$

Minimality removes redundant information sets: if $F \subsetneq E$ for some $E, F \in \mathcal{I}(\omega)$, then $\uparrow \mathcal{I}(\omega) = \uparrow (\mathcal{I}(\omega) \setminus \{E\})$.

A “surmise function” is a mapping $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{P}(\Omega))$ (i.e., $\mathcal{D} = \mathcal{P}(\Omega)$) satisfying (i) reflexivity, (ii) strong transitivity, (iii) Necessitation, and (iv) minimality. The agent whose knowledge is represented by a “surmise function” satisfies Truth Axiom, Positive Introspection, and Necessitation.

Three remarks on strong transitivity are in order. First, transitivity and strong transitivity are equivalent for a singleton-valued information correspondence. Second, strong transitivity implies transitivity, but the converse may not hold (Example A.1 in Appendix A.4). Third, $\uparrow \mathcal{I}$ may not be strongly transitive even if $\mathcal{I}$ is (Example A.2 in Appendix A.4).²⁵

²⁴In terms of the belief operator, suppose to the contrary that there is $\omega \in (\neg B_{\mathcal{I}})(\Omega)$. Under Negative Introspection, Monotonicity yields $\omega \in B_{\mathcal{I}}(\neg B_{\mathcal{I}})(\Omega) \subseteq B_{\mathcal{I}}(\Omega)$, a contradiction. Under the axiom B, Monotonicity yields $\omega \in (\neg B_{\mathcal{I}})(\Omega) \subseteq (\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(\emptyset) \subseteq \emptyset$, a contradiction.

²⁵Fagin and Halpern (1987) consider the following form of transitivity that implies strong transitivity: if $\omega' \in E \in \mathcal{I}(\omega)$ then $E \in \mathcal{I}(\omega')$. Thijsse (1993, Example 4) provides an example where this stronger form of transitivity is not necessary for characterizing Positive Introspection. While it can be seen that the monotone information correspondence $\mathcal{I} = \uparrow \mathcal{I}$ in his example satisfies transitivity but violates strong transitivity, for future use, I provide examples of reflexive information correspondences in Examples A.1 and A.2 in Appendix A.4. On a related point, Fagin and Halpern (1987) also consider the following stronger Euclidean property: $\omega' \in E \in \mathcal{I}(\omega)$ implies $\mathcal{I}(\omega') \subseteq \mathcal{I}(\omega)$. Thijsse (1993, Example 5) demonstrates that this stronger Euclidean property does not necessarily characterize Negative Introspection. One of the technical contributions of this paper is to provide the technical formulations of transitivity and Euclideanness as demonstrated in Proposition 3.

Finally, while a partitional (i.e., reflexive, transitive, and Euclidean) possibility correspondence in economics and game theory presupposes Negative Introspection by construction, a “surmise function” does not.²⁶ As Fagin et al. (2003, Chapter 3) argue, “there is no one ‘true’ notion of knowledge” and “the appropriate notion depends on the application,” I believe that this difference comes from the different contexts in which knowledge is analyzed in the two fields.

3.3 “Belief-in” Properties

This subsection studies four properties stating that the agent believes in a logical or introspective property of her belief, even if her belief may violate it: Belief in Correct Belief (i.e., in Truth Axiom), Belief in Consistency, Belief in Perfect Reasoning (i.e., in “modus ponens”: if she believes E and $(\neg E) \cup F$, then she believes F), and Belief in Countable Conjunction. Such properties are interesting in their own right for “boundedly rational” agents, as Fagin and Halpern (1987) extensively study some of them, and they play roles in epistemic characterizations of solution concepts of games and other applications.

1. An information correspondence $\mathcal{I}$ is *secondary reflexive* (or satisfies *Belief in Correct Belief*) if, for any $(\omega, E) \in \Omega \times \mathcal{D}$, there is $F \in \mathcal{I}(\omega)$ such that:

if $\omega' \in F$ and there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $\omega' \in E$.

Roughly, there is always information indicating that if the agent believes E then E is true.

2. The information correspondence $\mathcal{I}$ is *secondary serial* (or satisfies *Belief in Consistency*) if, for any $(\omega, E) \in \Omega \times \mathcal{D}$, there is $F \in \mathcal{I}(\omega)$ such that:

if $\omega' \in F$ and there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $H \cap E \neq \emptyset$ for all $H \in \mathcal{I}(\omega')$.

Roughly, there is always information implying that if the agent believes E then she does not believe the negation E^c .

3. The information correspondence $\mathcal{I}$ satisfies *Belief in Perfect Reasoning* if, for any $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$, there is $G \in \mathcal{I}(\omega)$ with the following property:

if $\omega' \in G$, $E' \subseteq E$ for some $E' \in \mathcal{I}(\omega')$, and if $F' \subseteq (\neg E) \cup F$ for some $F' \in \mathcal{I}(\omega')$,

then there is $G' \in \mathcal{I}(\omega')$ such that $G' \subseteq F$. Roughly, there is always information implying that if the agent believes E and $(\neg E) \cup F$ then she believes F .

²⁶Non-partitional (reflexive and transitive) possibility correspondence models have also been studied in economics and game theory; see, for example, Dekel and Gul (1997), Morris (1996), and Shin (1993). Also, an agent whose knowledge satisfies Negative Introspection cannot be unaware of any event in the sense that if she does not know an event then she knows that she does not know it. See footnote 6 for unawareness.

4. The information correspondence $\mathcal{I}$ satisfies *Belief in Countable Conjunction* if, for any $\omega \in \Omega$ and for any countable collection $\{F_n\}_{n \in \mathbb{N}}$ of events, there is $G \in \mathcal{I}(\omega)$ such that if $\omega' \in G$ and, for each $n \in \mathbb{N}$, there is $F'_n \in \mathcal{I}(\omega')$ with $F'_n \subseteq F_n$, then there is $G' \in \mathcal{I}(\omega')$ with $G' \subseteq \bigcap_{n \in \mathbb{N}} F_n$. Roughly, there is always information that implies that her belief satisfies Countable Conjunction.

Proposition 4. *Let $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ be an information correspondence. For each “Belief-in” property, the following are all equivalent.*

1. (a) $\mathcal{I}$ is secondary reflexive.
 (b) $\uparrow \mathcal{I}$ is secondary reflexive. That is, for any $(\omega, E) \in \Omega \times \mathcal{D}$, there is $F \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F$, then $E \in \uparrow \mathcal{I}(\omega')$ implies $\omega' \in E$.
 (c) $\Omega = B_{\mathcal{I}}((\neg B_{\mathcal{I}})(E) \cup E)$ for any $E \in \mathcal{D}$.
2. (a) $\mathcal{I}$ is secondary serial.
 (b) $\uparrow \mathcal{I}$ is secondary serial. That is, for any $(\omega, E) \in \Omega \times \mathcal{D}$, there is $F \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F$ and $E \in \uparrow \mathcal{I}(\omega')$, then $E^c \notin \uparrow \mathcal{I}(\omega')$.
 (c) $\Omega = B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)))$ for any $E \in \mathcal{D}$.
3. (a) $\mathcal{I}$ satisfies Belief in Perfect Reasoning.
 (b) $\uparrow \mathcal{I}$ satisfies Belief in Perfect Reasoning. That is, for any $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$, there is $G \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$, $E \in \uparrow \mathcal{I}(\omega')$, and if $(\neg E) \cup F \in \uparrow \mathcal{I}(\omega')$, then $F \in \uparrow \mathcal{I}(\omega')$.
 (c) $B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F))) \cup B_{\mathcal{I}}(F) = \Omega$ for any $E, F \in \mathcal{D}$.
4. (a) $\mathcal{I}$ satisfies Belief in Countable Conjunction.
 (b) $\uparrow \mathcal{I}$ satisfies Belief in Countable Conjunction. That is, for any $\omega \in \Omega$ and $\{F_n\}_{n \in \mathbb{N}} \in \mathcal{P}(\mathcal{D})$, there is $G \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$ and $\{F_n\}_{n \in \mathbb{N}} \subseteq \uparrow \mathcal{I}(\omega')$, then $\bigcap_{n \in \mathbb{N}} F_n \in \uparrow \mathcal{I}(\omega')$.
 (c) $B_{\mathcal{I}}(\neg(\bigcap_{n \in \mathbb{N}} B_{\mathcal{I}}(F_n))) \cup B_{\mathcal{I}}(\bigcap_{n \in \mathbb{N}} F_n) = \Omega$ for any $\{F_n\}_{n \in \mathbb{N}} \in \mathcal{P}(\mathcal{D})$.

As noted after Proposition 2, $\mathcal{I}$ satisfies a given “Belief-in” property iff $\uparrow \mathcal{I}$ does, and each property of $\mathcal{I}$ captures the intended property of $B_{\mathcal{I}}$. Henceforth, $B_{\mathcal{I}}$ is said to satisfy it (e.g., Belief in Correct Belief) if $\mathcal{I}$ does.

In terms of the belief operator, Belief in Correct Belief means that the agent always believes that either she does not believe an event E or E obtains, and Belief in Perfect Reasoning means that she always believes that if she believes E and “ E implies F ” (more precisely, $(\neg E) \cup F$) then she believes F .²⁷

²⁷Fagin and Halpern (1987) study the notion of a narrow-minded agent. Call the agent *narrow-minded* if, for any $\omega \in \Omega$, there is $E \in \mathcal{I}(\omega)$ such that $\omega' \in E$ implies $\mathcal{I}(\omega') = \{E\}$. As in Fagin and Halpern (1987), this axiom implies Belief in Consistency and Belief in Perfect Reasoning. In contrast, Parts (2) and (3) of Proposition 4 fully characterize these two properties.

Under Necessitation, Belief in Correct Belief is a weakening of Truth Axiom. The interpersonal version of Belief in Correct Belief,

$$\Omega = B_{\mathcal{I}_j}((\neg B_{\mathcal{I}_i})(E) \cup E),$$

where $\mathcal{I}_i$ and $\mathcal{I}_j$ are information correspondences of agents i and j ,²⁸ plays an important role in the existence of common prior (and consequently Aumann (1976)'s Agreement theorem) and an epistemic characterization of backward induction as an implication of common knowledge of rationality à la Aumann (1995) in a possibility correspondence model.²⁹

Negative Introspection and Monotonicity imply Belief in Correct Belief (i.e., a Euclidean $\mathcal{I}$ is secondary reflexive): for any $(\omega, E) \in \Omega \times \mathcal{D}$, Negative Introspection implies $E \in \uparrow \mathcal{I}(\omega)$ or $(\neg B_{\mathcal{I}})(E) \in \uparrow \mathcal{I}(\omega)$, and in either case $(\neg B_{\mathcal{I}})(E) \cup E \in \uparrow \mathcal{I}(\omega)$.

Given Monotonicity, each of the four “Belief-in” properties implies Necessitation. Thus, the failure of Necessitation implies the failure of all of them, as for the information correspondence of Example 1.

Belief in Countable Conjunction is at least as strong as Belief in Perfect Reasoning, because, under Monotonicity, the latter is equivalent to *Belief in Finite Conjunction*:

- For any $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$, there is $G \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$ and $E, F \in \uparrow \mathcal{I}(\omega')$ then $E \cap F \in \uparrow \mathcal{I}(\omega')$.

Under Necessitation, Consistency, Truth Axiom, and Countable Conjunction imply Belief in Consistency, Belief in Correct Belief, and Belief in Countable Conjunction, respectively. In contrast, Example A.3 in Appendix A.5 shows that the agent can believe in these properties even if $B_{\mathcal{I}}$ violates them: an information correspondence can capture an agent whose belief fails Countable Conjunction and who believes that her own belief satisfies it.

One can also consider Belief in Positive and Negative Introspection. For probabilistic beliefs given by a type mapping t (Section 2.2.2), Samet (2000) considers Belief in Positive Certainty in the context of agents' expectations:

$$B^1((\neg B^p)(E) \cup B^1 B^p(E)) = \Omega \text{ for all } E \in \mathcal{D},$$

where B^p is the p -belief operator defined by Expression (5).³⁰

²⁸One can analogously characterize this property: For any (ω, E) , there is $F \in \uparrow \mathcal{I}_j(\omega)$ such that if $\omega' \in F$ then $E \in \uparrow \mathcal{I}_i(\omega')$ implies $\omega' \in E$.

²⁹Specifically, Samet (2013) shows that, in the framework of Aumann (1995), common belief in rationality implies backward induction outcomes when agents' beliefs satisfy the interpersonal version of secondary reflexivity $\Omega = B_{\mathcal{I}_j}((\neg B_{\mathcal{I}_i})(E) \cup E)$. Bonanno and Nehring (1998a) study a similar property and characterize it as the absence of unbounded gains from betting. Bonanno and Nehring (1998b) study another similar condition in the context of correlated equilibria under incomplete information.

³⁰Assume that each $t(\omega, \cdot)$ is a countably additive probability measure. Let $\mathcal{B}(\Omega, \mathcal{D})$ be the set of bounded Borel measurable functions on $(\Omega, \mathcal{D})$, and define the expectation operator $\mathbb{E}_t : \mathcal{B}(\Omega, \mathcal{D}) \rightarrow \mathcal{B}(\Omega, \mathcal{D})$ by $\mathbb{E}_t[f](\omega) := \int_{\Omega} f(\tilde{\omega}) t(\omega, d\tilde{\omega})$. Samet (2000) shows that the *law of iterated expectations*, $\mathbb{E}_t \mathbb{E}_t[f] = \mathbb{E}_t[f]$ for all $f \in \mathcal{B}(\Omega, \mathcal{D})$, implies Belief in Positive Certainty.

Finally, consider a singleton-valued $\mathcal{I}(\cdot) = \{P(\cdot)\}$. Since the induced belief satisfies Monotonicity, Necessitation, and Countable Conjunction, $\mathcal{I}$ satisfies Belief in Perfect Reasoning and Belief in Countable Conjunction, and No-Contradiction and Consistency are equivalent. The other properties reduce to the well-known properties of P : (i) $\mathcal{I}$ is serial, equivalently satisfies No-Contradiction, iff $P(\cdot) \neq \emptyset$; (ii) reflexive iff $\omega \in P(\omega)$ for all $\omega \in \Omega$; (iii) transitive iff $P(\omega') \subseteq P(\omega)$ for any $\omega' \in P(\omega)$; (iv) Euclidean iff $\omega' \in P(\omega)$ implies $E^c \cap P(\omega') \neq \emptyset$ for any $E \in \mathcal{D}$ with $E^c \cap P(\omega) \neq \emptyset$, which in turn holds iff $\omega' \in P(\omega)$ implies $P(\omega) \subseteq P(\omega')$; (v) symmetric iff $\omega \in P(\omega')$ implies $\omega' \in P(\omega)$; (vi) secondary reflexive iff $\omega' \in P(\omega)$ implies $\omega' \in P(\omega')$; and (vii) secondary serial iff $\omega' \in P(\omega)$ implies $P(\omega') \neq \emptyset$.

4 Equivalence among Knowledge-Belief Representations

Section 3 has shown that $B_{\mathcal{I}}$ inherits the logical, introspective, and “Belief-in” properties of $\mathcal{I}$. Section 4.1 completes the equivalence between information correspondences and monotone belief operators, so that results on monotone belief operators can be replicated with information correspondences. Section 4.2 studies a “surmise function” as a reflexive and transitive information correspondence.

I begin by introducing self-evident events. Given a monotone belief operator B , an event $E \in \mathcal{D}$ is *self-evident* if $E \subseteq B(E)$, i.e., the agent believes E whenever E is true. Denote the collection of self-evident events by

$$\mathcal{J}_B := \{E \in \mathcal{D} \mid E \subseteq B(E)\}.$$

Given an information correspondence $\mathcal{I}$, an event E is self-evident (i.e., $E \subseteq B_{\mathcal{I}}(E)$) iff

$$\text{for any } \omega \in E, \text{ there is } F \in \mathcal{I}(\omega) \text{ such that } F \subseteq E.^{31}$$

In Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011), a self-evident event coincides with a “knowledge state,” i.e., a set of “questions” that an agent is capable of solving, and the collection of knowledge states is the “knowledge structure.”

³¹Two remarks are in order. First, using self-evidence, one can introduce common belief among an (at most countable and non-empty) set I of agents. For each agent i , let $\mathcal{I}_i$ be her information correspondence, and denote by $\mathcal{J}_{B_{\mathcal{I}_i}}$ the collection of self-evident events. Then, an event F is publicly evident if $F \in \bigcap_{i \in I} \mathcal{J}_{B_{\mathcal{I}_i}}$. Following Monderer and Samet (1989), an event E is *common belief* among agents I at a state ω if there is a publicly-evident event F such that $\omega \in F \subseteq \bigcap_{i \in I} B_{\mathcal{I}_i}(E)$. If E is common belief, then everybody believes E , everybody believes that everybody believes E , and so on *ad infinitum*. Heifetz (1996, 1999) and Lismont and Mongin (1994a,b) study common belief using monotone neighborhood systems. Fukuda (2020) studies common belief in a framework in which no properties of individual beliefs are assumed. Second, one can also define the dual notion of distributed knowledge (e.g., Fagin et al., 2003). In the formulation of Fukuda (2026), who uses the collections of self-evident events, an event E is *distributed knowledge* among an (at most countable and non-empty) set of agents I at a state ω if there are a non-empty sub-group $\hat{I}$ of agents and self-evident events $F_i \in \mathcal{J}_{B_{\mathcal{I}_i}}$ for each $i \in \hat{I}$ such that $\omega \in \bigcap_{i \in \hat{I}} F_i \subseteq E$. Thus, E is distributed knowledge at ω if some agents in I can come to know it at ω by pooling their individual information. Pacuit (2017), for instance, formalizes distributed belief using neighborhood systems.

4.1 Information Correspondences and Belief Operators

For a monotone belief operator $B : \mathcal{D} \rightarrow \mathcal{D}$, an information correspondence $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ is a *generator* of B (or $\mathcal{I}$ *induces* B) if $B = B_{\mathcal{I}}$.

A given B generally has multiple generators, because information correspondences $\mathcal{I}$ and $\mathcal{I}'$ with $\uparrow \mathcal{I} = \uparrow \mathcal{I}'$ induce the same belief operator. As the next proposition shows, a canonical generator of B is the information correspondence $\mathcal{I}_B : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ defined by

$$\mathcal{I}_B(\omega) := \{E \in \mathcal{D} \mid \omega \in B(E)\} \text{ for each } \omega \in \Omega. \quad (6)$$

Since B satisfies Monotonicity, $\mathcal{I}_B = \uparrow \mathcal{I}_B$ and $B = B_{\mathcal{I}_B}$. Moreover, any generator $\mathcal{I}$ of B is included in $\mathcal{I}_B$:

$$\mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}(\cdot) = \uparrow \mathcal{I}_B(\cdot) = \mathcal{I}_B(\cdot).$$

If B also satisfies Truth Axiom and Positive Introspection, one can restrict attention to self-evident events.

Proposition 5. *Let $(\Omega, \mathcal{D})$ be a measurable space.*

1. *If $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ is an information correspondence, then $B_{\mathcal{I}}$ inherits the logical, introspective, and “Belief-in” properties of Section 3 imposed on $\mathcal{I}$ and $\uparrow \mathcal{I}(\cdot) = \uparrow \mathcal{I}_{B_{\mathcal{I}}}(\cdot)$. Conversely, if B is a monotone belief operator, then $\mathcal{I}_B$ is a generator of B , i.e., $B = B_{\mathcal{I}_B}$. Any generator $\mathcal{I}$ of B satisfies the logical, introspective, and “Belief-in” properties imposed on B and $\mathcal{I}(\cdot) \subseteq \mathcal{I}_B(\cdot)$.³²*
2. *Let $B : \mathcal{D} \rightarrow \mathcal{D}$ satisfy Monotonicity, Truth Axiom, and Positive Introspection. Define $\mathcal{I}_{\mathcal{J}_B} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ by*

$$\mathcal{I}_{\mathcal{J}_B}(\omega) := \{E \in \mathcal{J}_B \mid \omega \in B(E)\} \text{ for each } \omega \in \Omega.$$

Then, $\mathcal{I}_{\mathcal{J}_B}$ is a reflexive and (strongly) transitive information correspondence that generates B .

Generators can also be used to compare agents’ beliefs. For agents i and j with generators $\mathcal{I}_i$ and $\mathcal{I}_j$ of B_i and B_j , respectively,

$$B_i(\cdot) \subseteq B_j(\cdot) \text{ iff } \uparrow \mathcal{I}_i(\cdot) \subseteq \uparrow \mathcal{I}_j(\cdot),$$

where the right-hand side is equivalent to

$$\text{for any } \omega \in \Omega \text{ and } E \in \mathcal{I}_i(\omega), \text{ there is } F \in \mathcal{I}_j(\omega) \text{ such that } F \subseteq E.$$

In mathematical psychology, Doignon and Falmagne (1999, Definition 3.16) call it an “attribution order” (see also Doignon and Falmagne, 1985, Definition 3.3). Moreover,

³²More precisely, the logical, introspective, and “Belief-in” properties of Section 3 refer to the properties of beliefs defined in Section 3 other than strong transitivity and minimality introduced in Section 3.2. These two properties are not preserved in the sense of Proposition 5. For instance, Example A.2 in Appendix A.4 provides a strongly transitive $\mathcal{I}$ such that $\uparrow \mathcal{I}$, another generator of $B_{\mathcal{I}}$, is not strongly transitive. See also the discussion in Section 4.2.

since $B_i(\cdot) \subseteq B_j(\cdot)$ implies $\uparrow \mathcal{I}_{B_i}(\cdot) \subseteq \uparrow \mathcal{I}_{B_j}(\cdot)$, and $\uparrow \mathcal{I}_i(\cdot) \subseteq \uparrow \mathcal{I}_j(\cdot)$ implies $B_{\mathcal{I}_i}(\cdot) \subseteq B_{\mathcal{I}_j}(\cdot)$ by Equation (2),

$$\uparrow \mathcal{I}_i(\cdot) \subseteq \uparrow \mathcal{I}_{B_j}(\cdot) \text{ iff } B_{\mathcal{I}_i}(\cdot) \subseteq B_j(\cdot).$$

Following Doignon and Falmagne (1985, 1999) and Falmagne and Doignon (2011), which compare knowledge-belief representations by Galois connections in order theory, Remark B.1 in Appendix B formalizes this as a Galois connection (defined there).

Proposition 5 (1) also holds for probabilistic beliefs (i.e., p -belief operators).

4.2 Reflexive and Transitive Information Correspondences

A “surmise function” is an information correspondence on $(\Omega, \mathcal{P}(\Omega))$ satisfying (i) reflexivity, (ii) strong transitivity, (iii) Necessitation, and (iv) minimality. Doignon and Falmagne (1999, Theorems 3.10 and 6.25) show a one-to-one correspondence between “surmise functions” and “granular” “knowledge structures.”³³ Dropping the minimality condition, I show a one-to-one correspondence between reflexive and (strongly) transitive information correspondences and collections of self-evident events on a state space. By Proposition 3, reflexive and transitive information correspondences in turn correspond to belief (knowledge) operators satisfying Monotonicity, Truth Axiom, and Positive Introspection.

Proposition 6. *Let $(\Omega, \mathcal{D})$ be a measurable space.*

1. (a) *Let $\mathcal{J} \in \mathcal{P}(\mathcal{D})$ satisfy*

$$\{\omega \in \Omega \mid \text{there is } F \in \mathcal{J} \text{ with } \omega \in F \subseteq E\} \in \mathcal{J} \text{ for each } E \in \mathcal{D}. \quad (7)$$

The following mapping $\mathcal{I}_{\mathcal{J}} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ is a reflexive and (strongly) transitive information correspondence:

$$\mathcal{I}_{\mathcal{J}}(\omega) := \{E \in \mathcal{J} \mid \omega \in E\} \text{ for each } \omega \in \Omega. \quad (8)$$

- (b) *If $\Omega \in \mathcal{J}$, then $\mathcal{I}_{\mathcal{J}}(\cdot) \neq \emptyset$.*

2. (a) *Conversely, if $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ is a reflexive and transitive information correspondence, then $\mathcal{J}_{\mathcal{I}} \in \mathcal{P}(\mathcal{D})$ defined below satisfies Condition (7):*

$$\mathcal{J}_{\mathcal{I}} := \{E \in \mathcal{D} \mid \text{if } \omega \in E \text{ then there is } F \in \mathcal{I}(\omega) \text{ with } F \subseteq E\}. \quad (9)$$

- (b) *If $\mathcal{I}(\cdot) \neq \emptyset$, then $\Omega \in \mathcal{J}_{\mathcal{I}}$.*

3. (a) *Moreover, starting from $\mathcal{J}$, $\mathcal{J} = \mathcal{J}_{\mathcal{I}_{\mathcal{J}}}$.*

³³A “knowledge structure” $\mathcal{J}$ is *granular* (Doignon and Falmagne, 1999, Definition 1.35) if, for any $(\omega, E) \in \Omega \times \mathcal{J}$ with $\omega \in E$, there is a minimal $F \in \mathcal{J}$ with $\omega \in F \subseteq E$. Doignon and Falmagne (1985, Theorem 3.7) establish the equivalence between a “surmise function” and a “knowledge structure” when Ω is finite.

(b) Starting from $\mathcal{I}$, $\uparrow \mathcal{I} = \uparrow \mathcal{I}_{\mathcal{I}}$.

In mathematical psychology, Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011) characterize the collection $\mathcal{J}$ of “knowledge states” as closed under arbitrary union, whereas Proposition 6 requires Condition (7). The two are equivalent if $\mathcal{D}$ is closed under arbitrary union, because then (i) the set in Condition (7) reduces to $\bigcup\{F \in \mathcal{J} \mid F \subseteq E\}$ and (ii) $E = \bigcup\{F \in \mathcal{J} \mid F \subseteq E\}$ for each $E \in \mathcal{J}$. In general, Condition (7) is equivalent to the existence of the largest event in $\mathcal{J}$ that is included in a given event $E \in \mathcal{D}$.³⁴

Since $\mathcal{I}_{\mathcal{J}}$ in Equation (8) is strongly transitive, Proposition 6 also establishes the equivalence between collections of self-evident events and reflexive and strongly transitive information correspondences. However, strong transitivity is not necessarily preserved under the operation of taking “ $\uparrow$ ”: $\uparrow \mathcal{I}_{\mathcal{J}}$ may not be strongly transitive (Example A.4 in Appendix A.7). Thus, the reflexive and transitive information correspondence

$$\mathcal{I} := \uparrow \mathcal{I}_{\mathcal{J}}, \text{ i.e., } \mathcal{I}(\omega) := \{E \in \mathcal{D} \mid \text{there is } F \in \mathcal{J} \text{ with } \omega \in F \subseteq E\},$$

can also establish the relevant part of Proposition 6 in place of Equation (8).

Doignon and Falmagne (1999, Theorem 6.25) (and Doignon and Falmagne, 1985, Theorem 3.7) establish the correspondence between a “surmise function” and a “knowledge structure” (i.e., a reflexive and transitive information correspondence and a collection of self-evident events in my context) via a Galois connection, which amounts to: for each $\mathcal{I}$ and $\mathcal{J}$,

$$\uparrow \mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}_{\mathcal{J}}(\cdot) \text{ iff } \mathcal{J}_{\mathcal{I}} \subseteq \mathcal{J}.$$

Remark B.2 in Appendix B shows that the mappings defined by Equations (8) and (9) form a Galois connection.

If a monotone belief operator B satisfies Truth Axiom and Positive Introspection, then $\mathcal{J}_B$ satisfies Condition (7), and the information correspondence $\mathcal{I}_{\mathcal{J}_B}(\omega) = \{E \in \mathcal{J}_B \mid \omega \in B(E)\}$ in Proposition 5 (2) coincides with $\mathcal{I}_{\mathcal{J}_B}(\omega) = \{E \in \mathcal{J}_B \mid \omega \in E\}$ in Proposition 6.

Proposition 6 drops the minimality condition, which can be restored exactly when the collection of self-evident events is “granular.” Call $F \in \mathcal{J}$ ω -minimal in $\mathcal{J} \in \mathcal{P}(\mathcal{D})$ if $\omega \in F$ and there is no $G \in \mathcal{J}$ with $\omega \in G \subsetneq F$. Call $\mathcal{J} \in \mathcal{P}(\mathcal{D})$ *granular* if, for any $E \in \mathcal{J}$ and $\omega \in E$, there is an ω -minimal F in $\mathcal{J}$ with $F \subseteq E$.³⁵ For such $\mathcal{J}$, define $\sigma_{\mathcal{J}} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ by

$$\sigma_{\mathcal{J}}(\omega) := \{F \in \mathcal{J} \mid F \text{ is } \omega\text{-minimal in } \mathcal{J}\} \text{ for each } \omega \in \Omega. \quad (10)$$

That is, $\sigma_{\mathcal{J}}(\omega)$ collects the smallest “knowledge states” containing ω (the “clauses” for ω).

³⁴In economics, Fukuda (2019), Salonen (2009b), and Samet (2010) study this maximality property to obtain set-algebraic representations of knowledge. In mathematical psychology, Danilov (2009) studies a collection $\mathcal{J}$ that is closed under arbitrary union as a “pre-topology” when $\mathcal{D} = \mathcal{P}(\Omega)$.

³⁵This is the notion of granularity in footnote 33, applied to an arbitrary collection $\mathcal{J} \in \mathcal{P}(\mathcal{D})$.

Corollary 2. *Let $(\Omega, \mathcal{D})$ be a measurable space.*

1. *Let $\mathcal{J} \in \mathcal{P}(\mathcal{D})$ be a granular collection that satisfies Condition (7) and $\Omega \in \mathcal{J}$. Then $\sigma_{\mathcal{J}}$ is an information correspondence satisfying reflexivity, strong transitivity, Necessitation, and minimality. Moreover, $\uparrow \sigma_{\mathcal{J}} = \uparrow \mathcal{I}_{\mathcal{J}}$ and $\mathcal{J}_{\sigma_{\mathcal{J}}} = \mathcal{J}$.*
2. *Conversely, let $\sigma : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ be an information correspondence satisfying reflexivity, strong transitivity, Necessitation, and minimality. Then $\mathcal{J}_{\sigma}$ is a granular collection that satisfies Condition (7) and $\Omega \in \mathcal{J}_{\sigma}$. Moreover, $\sigma_{\mathcal{J}_{\sigma}} = \sigma$.*

Corollary 2 establishes a one-to-one correspondence between information correspondences satisfying (i)–(iv) of a “surmise function” and granular collections $\mathcal{J}$ that satisfy Condition (7) and contain Ω . When $\mathcal{D} = \mathcal{P}(\Omega)$, Condition (7) amounts to closure under arbitrary union, and the corollary recovers, for an arbitrary (not necessarily finite) set Ω of “items,” the correspondence of Doignon and Falmagne (1999, Theorems 3.10 and 6.25). If Ω is finite, then every $\mathcal{J} \in \mathcal{P}(\mathcal{D})$ is granular.

The corollary also clarifies the role of strong transitivity: σ is strongly transitive iff every information set is self-evident, i.e., $\sigma(\omega) \subseteq \mathcal{J}_{\sigma}$ for all $\omega \in \Omega$, since, by Equation (9), $F \in \mathcal{J}_{\sigma}$ iff, for any $\omega' \in F$, there is $E' \in \sigma(\omega')$ with $E' \subseteq F$. In the language of knowledge space theory, every “clause” for a “question” is itself a “knowledge state.”

The next example shows why the minimality condition has to be dropped in general.

Example 3. Let $\Omega = [0, 1]$, let $\mathcal{D}$ be the Borel σ -algebra on $[0, 1]$, and let $\mathcal{J}$ be the collection of open subsets of $[0, 1]$ (in the relative topology). For each $E \in \mathcal{D}$, the set in Condition (7) is the interior $\text{int}(E)$ of E , which belongs to $\mathcal{J}$. Also, $\Omega \in \mathcal{J}$. By Proposition 6, $\mathcal{I}_{\mathcal{J}}$ is a reflexive and transitive information correspondence satisfying Necessitation, and $B_{\mathcal{I}_{\mathcal{J}}}(E) = \text{int}(E)$ for each $E \in \mathcal{D}$: the agent knows E at ω iff E contains every state sufficiently close to ω .

The collection $\mathcal{J}$ is not granular, because any open set F containing ω includes an open set G with $\omega \in G \subsetneq F$. Hence, no information correspondence σ satisfying reflexivity, strong transitivity, Necessitation, and minimality generates $B_{\mathcal{I}_{\mathcal{J}}}$. To see this, suppose that such σ generates $B_{\mathcal{I}_{\mathcal{J}}}$. By Proposition 5, $\uparrow \sigma = \uparrow \mathcal{I}_{\mathcal{J}}$. Since $\mathcal{J}_{\mathcal{I}}$ in Equation (9) depends on $\mathcal{I}$ only through $\uparrow \mathcal{I}$, it follows from Proposition 6 that $\mathcal{J}_{\sigma} = \mathcal{J}_{\mathcal{I}_{\mathcal{J}}} = \mathcal{J}$, which contradicts Part (2) of Corollary 2. In the language of knowledge space theory, no “question” has a smallest “clause.” $\square$

Two further connections with knowledge space theory are worth noting. First, a singleton-valued $\mathcal{I}(\cdot) = \{P(\cdot)\}$ induces a binary relation, known as an accessibility (or possibility) relation in computer science, logic, and philosophy (e.g., Chellas, 1980; Fagin et al., 2003): the agent considers ω' *possible* at ω if $\omega' \in P(\omega)$. If P is reflexive and transitive, this relation is a “surmise (or precedence) relation” (Doignon and Falmagne, 1985, 1999, 2016; Falmagne and Doignon, 2011): if $\omega' \in P(\omega)$, then a correct response to “question” ω' can be surmised from a correct response to “question” ω .

Second, if $\mathcal{I}$ is reflexive and transitive, then the possibility operator $L_{\mathcal{I}}$ in Equation (3) satisfies (i) $E \subseteq F$ implies $L_{\mathcal{I}}(E) \subseteq L_{\mathcal{I}}(F)$; (ii) $E \subseteq L_{\mathcal{I}}(E)$; and (iii) $L_{\mathcal{I}}L_{\mathcal{I}}(\cdot) \subseteq L_{\mathcal{I}}(\cdot)$.

Thus, $L_{\mathcal{I}}$ is related to a “closure operator,” and the tuple $(\Omega, \{E \in \mathcal{D} \mid L_{\mathcal{I}}(E) \subseteq E\}) = (\Omega, \{E \in \mathcal{D} \mid E^c \in \mathcal{J}_{B_{\mathcal{I}}}\})$ to a “closure space” (Doignon and Falmagne, 1985, 1999; Falmagne and Doignon, 2011).

5 Economic Applications: Deriving Qualitative Belief from Preferences

This section provides economic applications of information correspondences. As in Morris (1996), I derive an agent’s qualitative belief (or knowledge) from her preferences. While Morris (1996)’s analysis is restricted to finite state spaces, I derive qualitative belief on an arbitrary state space using an information correspondence. This is because when the state space is infinite, the belief may not satisfy the conjunction property for an arbitrary number of events.

Throughout the section, $(\Omega, \mathcal{D})$ is a measurable space and, for simplicity, the set of consequences is $\mathbb{R}$. An *act* is a bounded measurable mapping $x : \Omega \rightarrow \mathbb{R}$, where $\mathbb{R}$ is endowed with the Borel σ -algebra. Denote by $\mathcal{B}(\Omega, \mathcal{D})$ the set of acts. For $x \in \mathcal{B}(\Omega, \mathcal{D})$, $\omega \in \Omega$, and $E \in \mathcal{D}$, let $x_{\omega} := x(\omega)$ and $x_E := (x(\omega))_{\omega \in E}$, and denote by $z = (x_E, y_{E^c})$ the act with $z_{\omega} = x_{\omega}$ for all $\omega \in E$ and $z_{\omega'} = y_{\omega'}$ for all $\omega' \in E^c$.

Consider an agent with preference (i.e., complete and transitive) relations $(\succsim_{\omega})_{\omega \in \Omega}$ on acts $\mathcal{B}(\Omega, \mathcal{D})$: $x \succsim_{\omega} y$ means that, at state ω , the act x is at least as good as y . The preference relations $(\succsim_{\omega})_{\omega \in \Omega}$ induce a (qualitative) belief operator $B_{\succsim} : \mathcal{D} \rightarrow \mathcal{D}$ if, for each $E \in \mathcal{D}$,

$$B_{\succsim}(E) := \{\omega \in \Omega \mid (x_E, y_{E^c}) \succsim_{\omega} (x_E, z_{E^c}) \text{ for all } x, y, z \in \mathcal{B}(\Omega, \mathcal{D})\} \in \mathcal{D}. \quad (11)$$

In words, the agent believes E if her preferences never depend on anything that happens when E does not occur. As in Morris (1996, Theorem 2), it can be shown that $B_{\succsim}$ satisfies: (i) Monotonicity, (ii) Necessitation, and (iii) Finite Conjunction (i.e., $B_{\succsim}(E) \cap B_{\succsim}(F) \subseteq B_{\succsim}(E \cap F)$). By Monotonicity, $B_{\succsim}$ can be represented by an information correspondence.³⁶

5.1 Expected Utility Representation

When the state space is infinite, $B_{\succsim}$ may not satisfy the Kripke property, i.e., it may not be induced by a possibility correspondence. To see this, the preference relations $(\succsim_{\omega})_{\omega \in \Omega}$ have a (state-independent) *expected utility representation* if there exist a strictly increasing and continuous utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ and a type mapping $t : \Omega \times \mathcal{D} \rightarrow$

³⁶If Ω is finite (or more generally $\mathcal{D}$ is finite), then it can be shown, similarly to Morris (1996, Theorem 1), that $B_{\succsim}$ is induced by a possibility correspondence. Indeed, letting $P(\omega) := \bigcap \{E \in \mathcal{D} \mid \omega \in B_{\succsim}(E)\}$ for each $\omega \in \Omega$,

$$B_{\succsim}(E) = \{\omega \in \Omega \mid P(\omega) \subseteq E\} \text{ for each } E \in \mathcal{D}.$$

$[0, 1]$ (as in Section 2.2.2) such that each $t(\omega, \cdot)$ is a countably additive probability measure and

$$x \succsim_{\omega} y \text{ iff } \int_{\Omega} (u \circ x)(\omega') t(\omega, d\omega') \geq \int_{\Omega} (u \circ y)(\omega') t(\omega, d\omega').$$

Then, an event is believed iff it is assigned probability one, so that $B_{\succsim} = B_{\mathcal{I}}$ with $\mathcal{I}(\omega) := \{E \in \mathcal{D} \mid t(\omega, E) = 1\}$ for each $\omega \in \Omega$. Formally:

Proposition 7. *If preference relations $(\succsim_{\omega})_{\omega \in \Omega}$ have a (state-independent) expected utility representation u together with a type mapping t , then the resulting belief operator $B_{\succsim}$ satisfies*

$$B_{\succsim}(E) = \{\omega \in \Omega \mid t(\omega, E) = 1\} \text{ for all } E \in \mathcal{D}.$$

In the proposition, the resulting belief operator $B_{\succsim}$ satisfies Monotonicity, Necessitation, and Countable Conjunction. As a simple example, let $\Omega = [0, 1]$, $u(c) := c$ for all $c \in \mathbb{R}$, and $t(\omega, \cdot) := \mu$ for each $\omega \in \Omega$, where μ is the Lebesgue measure. By Example 2, $B_{\succsim}$ violates the Kripke property, so that it is induced by an information correspondence but not by a possibility correspondence.³⁷

5.2 Maxmin Expected Utility Representation

Proposition 7 derives qualitative belief from subjective expected utility preferences. Is the derivation robust to ambiguity, i.e., to preferences that are not probabilistically sophisticated? Under maxmin expected utility (Gilboa and Schmeidler, 1989), the induced belief operator coincides with the probability-one operator with respect to the agent's set of priors, and it is again generated by an information correspondence but, in general, by no possibility correspondence.

Call a mapping C that associates, with each state $\omega \in \Omega$, a non-empty set $C(\omega)$ of countably additive probability measures on $(\Omega, \mathcal{D})$ a *prior correspondence* if it satisfies the regularity condition that, for each $E \in \mathcal{D}$,

$$\{\omega \in \Omega \mid \pi(E) = 1 \text{ for all } \pi \in C(\omega)\} \in \mathcal{D}.$$

The regularity condition parallels Condition (2) on a type mapping in Section 2.2.2. The preference relations $(\succsim_{\omega})_{\omega \in \Omega}$ have a (state-independent) *maxmin expected utility representation* (u, C) if there exist a strictly increasing and continuous utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ and a prior correspondence C such that

$$x \succsim_{\omega} y \text{ iff } \inf_{\pi \in C(\omega)} \int_{\Omega} (u \circ x)(\omega') \pi(d\omega') \geq \inf_{\pi \in C(\omega)} \int_{\Omega} (u \circ y)(\omega') \pi(d\omega').$$

If $C(\cdot)$ is singleton-valued, the representation reduces to the expected utility representation.

³⁷It also violates Truth Axiom: $B_{\succsim}([0, 1] \setminus \mathbb{Q}) = [0, 1]$, where $\mathbb{Q}$ is the set of rational numbers.

Proposition 8. *If preference relations $(\succsim_\omega)_{\omega \in \Omega}$ have a maxmin expected utility representation (u, C) , then the resulting belief operator $B_{\succsim}$ defined as in Expression (11) satisfies*

$$B_{\succsim}(E) = \{\omega \in \Omega \mid \pi(E) = 1 \text{ for all } \pi \in C(\omega)\} \text{ for each } E \in \mathcal{D}.$$

Consequently, $B_{\succsim} = B_{\mathcal{I}_C}$, where the information correspondence $\mathcal{I}_C : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ is defined by

$$\mathcal{I}_C(\omega) := \{E \in \mathcal{D} \mid \pi(E) = 1 \text{ for all } \pi \in C(\omega)\} \text{ for each } \omega \in \Omega.$$

The operator $B_{\succsim}$ satisfies Monotonicity, Necessitation, and Countable Conjunction.³⁸ Also, as in Proposition 7, $B_{\succsim}$ may fail Truth Axiom and the Kripke property.³⁹

The multiplicity of priors interacts naturally with that of information sets. For a selection s of C (i.e., $s(\omega) \in C(\omega)$ for each $\omega \in \Omega$), let $B_s(E) := \{\omega \in \Omega \mid s(\omega)(E) = 1\}$ be the set of states at which a hypothetical agent whose prior at each ω is $s(\omega)$ assigns probability one to E . By Proposition 8, as sets,

$$B_{\succsim}(E) = \bigcap_s B_s(E) \text{ for each } E \in \mathcal{D},$$

where the intersection is taken over all selections s of C . An intersection of monotone operators is monotone, and hence representable within the framework of Section 2.

The following lemma provides a condition for $B_{\succsim}$ to be represented by a possibility correspondence and a sufficient condition for it not to be. To that end, for each $\omega \in \Omega$, an event $E \in \mathcal{D}$ is said to be $C(\omega)$ -full if $\pi(E) = 1$ for all $\pi \in C(\omega)$.

Lemma 1. *Let preference relations $(\succsim_\omega)_{\omega \in \Omega}$ have a maxmin expected utility representation (u, C) .*

1. *There is a possibility correspondence $P : \Omega \rightarrow \mathcal{D}$ with $B_{\succsim}(E) = \{\omega \in \Omega \mid P(\omega) \subseteq E\}$ for all $E \in \mathcal{D}$ iff, for each $\omega \in \Omega$, there is a smallest $C(\omega)$ -full event $S(\omega)$. In this case, $P(\omega) = S(\omega)$ for each $\omega \in \Omega$.*
2. *Suppose that $\{\omega\} \in \mathcal{D}$ for each $\omega \in \Omega$. Also, suppose that, at some $\omega \in \Omega$, $C(\omega)$ contains a measure π with $\pi(\{\omega'\}) = 0$ for all $\omega' \in \Omega$ and that*

$$A(\omega) := \{\omega' \in \Omega \mid \pi'(\{\omega'\}) > 0 \text{ for some } \pi' \in C(\omega)\}$$

is at most countable. Then, there is no mapping $P : \Omega \rightarrow \mathcal{P}(\Omega)$ (measurable or not) such that, for all $E \in \mathcal{D}$, $\omega \in B_{\succsim}(E)$ iff $P(\omega) \subseteq E$.

Example 4. Let $C(\cdot) = \{\mu, \delta_{\frac{1}{2}}\}$ on $([0, 1], \mathcal{B}_{[0, 1]})$, where μ is the Lebesgue measure on $([0, 1], \mathcal{B}_{[0, 1]})$ as in Example 2 and $\delta_{\frac{1}{2}}$ is the Dirac measure concentrated at $\frac{1}{2}$. Lemma 1 (2) implies that there is no possibility correspondence P which represents $B_{\succsim}$. $\square$

³⁸To see that $B_{\succsim}$ satisfies Countable Conjunction, since each $\pi \in C(\omega)$ is countably additive, it follows that if $\pi(F_n) = 1$ for all $n \in \mathbb{N}$ and all $\pi \in C(\omega)$ then $\pi(\bigcap_{n \in \mathbb{N}} F_n) = 1$ for all $\pi \in C(\omega)$.

³⁹In fact, the behavioral derivation of belief from preferences on an infinite state space may require an information correspondence whether the agent's beliefs are ambiguous or not.

6 Conclusion

This paper develops an information correspondence which associates, with each state, a collection of possibly multiple information sets, and represents an agent’s beliefs by what these information sets imply. The only assumption that an information correspondence imposes on beliefs is Monotonicity: information correspondences correspond exactly to monotone belief operators (Proposition 5). An information correspondence reduces to a possibility correspondence if and only if each state has a smallest information set (Proposition 1). Otherwise, it can capture an agent who fails to believe the conjunction of her beliefs or a tautology.

The contributions of the paper are three-fold. First, by defining information correspondences on an arbitrary measurable space, the paper represents qualitative and probabilistic beliefs in a single framework. Every p -belief operator is generated by an information correspondence (Section 2.2.2), whereas probability-one belief on an infinite state space may not be represented by a possibility correspondence (Example 2). Section 5 shows that this distinction arises from preferences. Extending the analysis of Morris (1996) beyond finite state spaces, the belief induced from expected utility preferences is probability-one belief with respect to the agent’s prior, and the belief induced from maxmin expected utility preferences (Gilboa and Schmeidler, 1989) is probability-one belief with respect to all of her priors (Propositions 7 and 8). Such beliefs are always represented by an information correspondence but, in general, not by a possibility correspondence (Lemma 1 and Example 4).

Second, the paper contributes to the logic of local reasoning (Fagin and Halpern, 1987; Thijsse, 1993). It formulates the logical, introspective, and “Belief-in” properties as conditions on the primitive (i.e., information correspondence) $\mathcal{I}$, and shows that each condition is necessary and sufficient for the corresponding property of the belief operator $B_{\mathcal{I}}$ (Propositions 2, 3, and 4). In contrast, Fagin and Halpern (1987) impose stronger conditions that do not characterize Positive and Negative Introspection, and Thijsse (1993) characterizes these two properties in terms of the monotone neighborhood system $\uparrow \mathcal{I}$ rather than the primitive $\mathcal{I}$ (footnote 25). Moreover, each of these conditions holds for $\mathcal{I}$ if and only if it holds for $\uparrow \mathcal{I}$, so that it depends only on the beliefs that $\mathcal{I}$ induces.

Third, the paper connects the knowledge representations in economics with those in knowledge space theory in mathematical psychology (Doignon and Falmagne, 1985, 1999). A “surmise function” is a particular information correspondence, and the knowledge it describes satisfies Truth Axiom, Positive Introspection, and Necessitation, but not necessarily Negative Introspection. On an arbitrary measurable space, reflexive and transitive information correspondences correspond to collections of self-evident events satisfying Condition (7) (Proposition 6). Restricting attention to granular collections recovers the correspondence between “surmise functions” and “knowledge structures” for an arbitrary, not necessarily finite, set of “items” (Corollary 2), while the minimality condition cannot be retained in general (Example 3).

Also, the comparison identifies a difference between the two literatures. The proper-

ties of beliefs studied in economics and logic are invariant under the operation $\mathcal{I} \mapsto \uparrow \mathcal{I}$, i.e., they depend only on the induced beliefs. In contrast, the two conditions specific to “surmise functions” are not. Strong transitivity is not preserved under this operation (Examples A.2 and A.4), and neither is minimality, because $\uparrow \mathcal{I}(\omega)$ contains Ω along with any information set $E \subsetneq \Omega$.

The connection suggests directions for both fields. Knowledge space theory has developed methods to construct and assess knowledge structures from data (e.g., Cosyn et al., 2021; Doignon and Falmagne, 2016). Such empirical assessments have not been developed in interactive epistemology. Conversely, the economics literature studies higher-order beliefs, probabilistic beliefs, and unawareness, which the framework of this paper makes available to knowledge space theory.

Several further directions remain. One is belief updating. In a possibility correspondence model, an agent’s type at a state is usually the posterior conditional on her information set at that state. It would be interesting to study how an agent with multiple information sets, and possibly multiple priors, at a state updates her beliefs. Another is to define information correspondences on the generalized state spaces of Heifetz, Meier, and Schipper (2006, 2013), in which a possibility correspondence represents unawareness. An information correspondence would dispense with the conjunction and necessitation properties there, while accommodating probabilistic beliefs.

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A Appendix

A.1 List of notations and terminologies

Table A.1 provides a list of notations and terminologies: for each key notation of the paper, it provides the corresponding terminologies in economics and mathematical psychology side-by-side.

Notation	Terminology in the Paper	Corresponding Terminology in the Knowledge Space Theory
Ω	State space (Set of states of the world)	Domain (of the body of knowledge)
$\omega \in \Omega$	State	Item or question
$E \in \mathcal{D}$	Event	Set of items (or questions)
$\mathcal{I}$	Collection of self-evident events	Knowledge structure
$E \in \mathcal{I}$	Self-evident event	Knowledge state
$\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$	Information correspondence	Surmise function ($\Omega, \mathcal{I}$): Surmise system
$E \in \mathcal{I}(\omega)$	Information set at ω	Clause (background or foundation) for ω
$\omega' \in P(\omega)$ ($\mathcal{I}(\cdot) = \{P(\cdot)\}$)	Possibility (or accessibility) relation	Surmise relation

Table A.1: A list of notations and terminologies. The first column lists key notations of this paper. The second does their terminologies. The third does the corresponding terminologies in Doignon and Falmagne (1985, 1999, 2016) and Falmagne and Doignon (2011).

A.2 Section 2.1

Proof of Proposition 1. Assume (1). Fix $\omega \in \Omega$, and let $P(\omega)$ be the minimum element of $\mathcal{I}(\omega)$. For each $E \in \mathcal{D}$, $\omega \in B_{\mathcal{I}}(E)$ iff $P(\omega) \subseteq E$, where the “only if” part follows because $P(\omega)$ is the minimum element of $\mathcal{I}(\omega)$. Conversely, assume (2). Fix $\omega \in \Omega$. If $E \in \mathcal{I}(\omega)$, then $\omega \in B_{\mathcal{I}}(E)$ and thus $P(\omega) \subseteq E$. Since $P(\omega) \in \mathcal{I}(\omega)$, it follows that $P(\omega)$ is the minimum element of $\mathcal{I}(\omega)$. $\square$

A.3 Section 3.1

Proof of Proposition 2. By (a) of each property, it follows that, in (b), $\mathcal{I}$ satisfies a given property if and only if $\uparrow \mathcal{I}$ satisfies the given property. Thus, for (b), it suffices to show that $\uparrow \mathcal{I}$ satisfies a given property if and only if $B_{\mathcal{I}}$ satisfies it.

1. (a) I start by showing that if Γ satisfies No-Contradiction then so does $\uparrow \Gamma$ by contraposition. If $\emptyset \in \uparrow \Gamma$, then there is $E \in \Gamma$ with $E \subseteq \emptyset$, i.e., $\emptyset \in \Gamma$. Conversely, if $\emptyset \notin \uparrow \Gamma$, then $\emptyset \notin \Gamma$ because $\Gamma \subseteq \uparrow \Gamma$.
- (b) For each $\omega \in \Omega$, $\emptyset \in \uparrow \mathcal{I}(\omega)$ iff $\omega \in B_{\mathcal{I}}(\emptyset)$. Thus, $\emptyset \notin \uparrow \mathcal{I}(\omega)$ for all $\omega \in \Omega$ iff $B_{\mathcal{I}}(\emptyset) = \emptyset$.
2. (a) First, assume that Γ satisfies Consistency. Suppose to the contrary that there are $E, F \in \uparrow \Gamma$ such that $E \cap F = \emptyset$. Then, there are $E', F' \in \Gamma$ such that $E' \subseteq E$ and $F' \subseteq F$. Thus, $E' \cap F' \subseteq E \cap F = \emptyset$, a contradiction. Second, if $\uparrow \Gamma$ satisfies Consistency then $E^c \notin \uparrow \Gamma$ for any $E \in \uparrow \Gamma$. Third, suppose that $E^c \notin \uparrow \Gamma$ for any $E \in \uparrow \Gamma$. Suppose to the contrary that there are $E, F \in \Gamma$ with $E \cap F = \emptyset$. Since $F \subseteq E^c$, it follows that $E, E^c \in \uparrow \Gamma$, a contradiction.
- (b) I start by showing that if $\mathcal{I}$ satisfies Consistency then $B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(E^c)$. If $\omega \in B_{\mathcal{I}}(E)$ then $E \in \uparrow \mathcal{I}(\omega)$. Since $E^c \notin \uparrow \mathcal{I}(\omega)$, I have $\omega \in (\neg B_{\mathcal{I}})(E^c)$. Conversely, assume $B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(E^c)$. Take any $\omega \in \Omega$ and $E \in \uparrow \mathcal{I}(\omega)$. Since $\omega \in B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(E^c)$, it follows that $E^c \notin \uparrow \mathcal{I}(\omega)$.
3. (a) First, if $\Gamma \neq \emptyset$ then $\uparrow \Gamma \neq \emptyset$. Second, if $\uparrow \Gamma \neq \emptyset$, then there is $E \in \uparrow \Gamma$. Since $E \subseteq \Omega$, it follows that $\Omega \in \uparrow \Gamma$. Third, if $\Omega \in \uparrow \Gamma$, then there is some $E \in \Gamma$ (with $E \subseteq \Omega$), and thus $\Gamma \neq \emptyset$.
- (b) The statement follows because $\Omega \in \uparrow \mathcal{I}(\omega)$ for all $\omega \in \Omega$ iff $B_{\mathcal{I}}(\Omega) = \Omega$.
4. (a) Since $\uparrow \Gamma$ is closed under set inclusion, $\uparrow \Gamma$ satisfies Countable Conjunction if and only if $\uparrow \Gamma$ is closed under (non-empty) countable intersection.
- (b) Take any (non-empty) countable collection $\{F_n\}_{n \in \mathbb{N}} \subseteq \mathcal{D}$. The statement follows because, $\bigcap_{n \in \mathbb{N}} B_{\mathcal{I}}(F_n) \subseteq B_{\mathcal{I}}(\bigcap_{n \in \mathbb{N}} F_n)$ iff, for any $\omega \in \Omega$,

$$\text{if } F_n \in \uparrow \mathcal{I}(\omega) \text{ for all } n \in \mathbb{N} \text{ then } \bigcap_{n \in \mathbb{N}} F_n \in \uparrow \mathcal{I}(\omega).$$

□

A.4 Section 3.2

Proof of Proposition 3. 1. First, I show that (1a) implies (1b). If $E \in \uparrow \mathcal{I}(\omega)$ then there is $F \in \mathcal{I}(\omega)$ with $F \subseteq E$. Since $\mathcal{I}$ is reflexive, $\omega \in F \subseteq E$.

Second, I show that (1b) implies (1c). Take any $E \in \mathcal{D}$. If $\omega \in B_{\mathcal{I}}(E)$, then $E \in \uparrow \mathcal{I}(\omega)$, and thus $\omega \in E$. Thus, $B_{\mathcal{I}}(E) \subseteq E$.

Finally, I show that (1c) implies (1a). If $E \in \mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$, then $\omega \in B_{\mathcal{I}}(E) \subseteq E$.

2. First, I show that (2a) implies (2b). If $E \in \uparrow \mathcal{I}(\omega)$, then there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq E$. Then, by (2a), there is $G \in \mathcal{I}(\omega)$ such that if $\omega' \in G$ then there is $F' \in \mathcal{I}(\omega') \subseteq \uparrow \mathcal{I}(\omega')$ such that $F' \subseteq F \subseteq E$.

Second, I show that (2b) implies (2c). Take $\omega \in \Omega$ and $E \in \uparrow \mathcal{I}(\omega)$. By (2b), there is $F \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F$ then there is $E' \in \uparrow \mathcal{I}(\omega')$ such that $E' \subseteq E$. Then, since $F \subseteq \{\omega' \in \Omega \mid E \in \uparrow \mathcal{I}(\omega')\}$, it follows that

$$\{\omega' \in \Omega \mid E \in \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega).$$

Third, I show that (2c) implies (2d). If $\omega \in B_{\mathcal{I}}(E)$, then $E \in \uparrow \mathcal{I}(\omega)$. Then, by (2c),

$$\underbrace{\{\omega' \in \Omega \mid E \in \uparrow \mathcal{I}(\omega')\}}_{=B_{\mathcal{I}}(E)} \in \uparrow \mathcal{I}(\omega), \text{ i.e., } \omega \in B_{\mathcal{I}}B_{\mathcal{I}}(E).$$

Finally, I show that (2d) implies (2a). Fix $\omega \in \Omega$ and $E \in \mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$. Then, by (2d), $\omega \in B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}B_{\mathcal{I}}(E)$. Thus,

$$\{\omega' \in \Omega \mid E \in \uparrow \mathcal{I}(\omega')\} = B_{\mathcal{I}}(E) \in \uparrow \mathcal{I}(\omega).$$

Now, there is $F \in \mathcal{I}(\omega)$ such that if $\omega' \in F$ then $E \in \uparrow \mathcal{I}(\omega')$, i.e., there is $E' \in \mathcal{I}(\omega')$ with $E' \subseteq E$.

3. First, I show that (3a) implies (3b). Let $(\omega, E) \in \Omega \times \mathcal{D}$ be such that $E^c \cap F \neq \emptyset$ for all $F \in \uparrow \mathcal{I}(\omega)$. Since $\mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$ and since $\mathcal{I}$ is Euclidean, there is $F' \in \mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F'$ then $E^c \cap F \neq \emptyset$ for all $F \in \uparrow \mathcal{I}(\omega')$.

Second, I show that (3b) implies (3c). Suppose $E \notin \uparrow \mathcal{I}(\omega)$. Then, $E^c \cap F \neq \emptyset$ for all $F \in \uparrow \mathcal{I}(\omega)$. Then, by (3b), there is $F' \in \uparrow \mathcal{I}(\omega)$ such that

$$F' \subseteq \{\omega' \in \Omega \mid E \notin \uparrow \mathcal{I}(\omega')\}.$$

Then, $\{\omega' \in \Omega \mid E \notin \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega)$.

Third, I show that (3c) implies (3d). If $\omega \in (\neg B_{\mathcal{I}})(E)$, then $E \notin \uparrow \mathcal{I}(\omega)$. Then, by (3c),

$$\underbrace{\{\omega' \in \Omega \mid E \notin \uparrow \mathcal{I}(\omega')\}}_{=(\neg B_{\mathcal{I}})(E)} \in \uparrow \mathcal{I}(\omega), \text{ i.e., } \omega \in B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E).$$

Finally, I show that (3d) implies (3a). Let $(\omega, E) \in \Omega \times \mathcal{D}$ be such that $E^c \cap F \neq \emptyset$ for all $F \in \mathcal{I}(\omega)$. Since $E \notin \uparrow \mathcal{I}(\omega)$, it follows that $\omega \in (\neg B_{\mathcal{I}})(E)$. It follows from (3d) that $\omega \in (\neg B_{\mathcal{I}})(E) \subseteq B_{\mathcal{I}}(\neg B_{\mathcal{I}})(E)$, i.e.,

$$\{\omega' \in \Omega \mid E \notin \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega).$$

Thus, there is $F' \in \mathcal{I}(\omega)$ such that if $\omega' \in F'$ then $E \notin \uparrow \mathcal{I}(\omega')$, i.e., $E^c \cap F \neq \emptyset$ for all $F \in \mathcal{I}(\omega')$.

4. First, I show that (4a) implies (4b). Fix $(\omega, E) \in \Omega \times \mathcal{D}$. Suppose that, for any $F \in \uparrow \mathcal{I}(\omega)$, there is $\omega' \in F$ with $E \in \uparrow \mathcal{I}(\omega')$. Then, for any $F \in \mathcal{I}(\omega)$, there are $\omega' \in F$ and $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$. By (4a), $\omega \in E$.

Second, I show that (4b) implies (4c). Fix $E \in \mathcal{D}$. If $\omega \in (\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E)$, then for any $F \in \uparrow \mathcal{I}(\omega)$, $F \not\subseteq (\neg B_{\mathcal{I}})(E)$, i.e., there are $\omega' \in F$ and $E \in \uparrow \mathcal{I}(\omega')$. By (4b), $\omega \in E$. Thus, $(\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E) \subseteq E$.

Finally, I show that (4c) implies (4a). Fix $(\omega, E) \in \Omega \times \mathcal{D}$. Suppose that, for any $F \in \mathcal{I}(\omega)$, there are $\omega' \in F$ and $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$. Thus, for any $F \in \mathcal{I}(\omega)$, $F \not\subseteq (\neg B_{\mathcal{I}})(E)$. By (4c), $\omega \in (\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E) \subseteq E$.

□

Proof of Corollary 1. First, I show below that $\mathcal{I}$ is reflexive. Second, then, as mentioned in the main text, since $\mathcal{I}$ is symmetric and transitive, it is Euclidean. Third, Negative Introspection (i.e., Euclideaness), Truth Axiom (i.e., reflexivity), and Monotonicity of $B_{\mathcal{I}}$ imply Necessitation and Countable Conjunction (Fukuda, 2019, Corollary 1). Then, all the other properties follow.

Hence, I show that $\mathcal{I}$ is reflexive, i.e., $B_{\mathcal{I}}$ satisfies Truth Axiom:

$$B_{\mathcal{I}}(E) \subseteq B_{\mathcal{I}}B_{\mathcal{I}}(E) \subseteq (\neg B_{\mathcal{I}})(\neg B_{\mathcal{I}})(E) \subseteq E.$$

The first set inclusion follows from Positive Introspection, the second from Consistency, and the third from Symmetry. (One can also prove the reflexivity of $\mathcal{I}$ directly from the definition in terms of $\mathcal{I}$.) □

Example A.1. I provide an example where $\mathcal{I}$ is not strongly transitive but transitive. I define a reflexive and transitive information correspondence. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Define $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ as

$$\mathcal{I}(\omega) = \begin{cases} \{\{\omega_1, \omega_3\}\} & \text{if } \omega \in \{\omega_1, \omega_3\} \\ \{\{\omega_2\}, \{\omega_1, \omega_2\}\} & \text{if } \omega = \omega_2 \end{cases}.$$

By construction, $\mathcal{I}$ is reflexive.

To see that $\mathcal{I}$ is not strongly transitive, take $E = \{\omega_1, \omega_2\} \in \mathcal{I}(\omega_2)$ and $\omega_1 \in E$. Then, $\{\omega_1, \omega_3\} \in \mathcal{I}(\omega_1)$ and $\{\omega_1, \omega_3\} \not\subseteq E = \{\omega_1, \omega_2\}$. It can be seen, however, that $\mathcal{I}$ is transitive. This can also be verified by the fact that $B_{\mathcal{I}}$ satisfies Positive Introspection:

$$B_{\mathcal{I}}(E) = \begin{cases} \emptyset & \text{if } E \in \{\emptyset, \{\omega_1\}, \{\omega_3\}\} \\ \{\omega_2\} & \text{if } E \in \{\{\omega_2\}, \{\omega_1, \omega_2\}, \{\omega_2, \omega_3\}\} \\ \{\omega_1, \omega_3\} & \text{if } E = \{\omega_1, \omega_3\} \\ \Omega & \text{if } E = \Omega \end{cases}. \quad (\text{A.1})$$

Two additional remarks are in order. First, observe that $\mathcal{I}$ satisfies the Kripke property. Thus, consider $\mathcal{I}'(\cdot) = \{P(\cdot)\}$, where

$$P(\omega_1) = \{\omega_1, \omega_3\}, P(\omega_2) = \{\omega_2\}, \text{ and } P(\omega_3) = \{\omega_1, \omega_3\}.$$

Now, $\mathcal{I}'$ is strongly transitive.

Second, $B_{\mathcal{I}}$ satisfies all the four logical properties defined in Section 3.1. Also, $B_{\mathcal{I}}$ satisfies all the four introspective properties defined in Section 3.2 (it satisfies all the “Belief-in” properties in Section 3.3 as well). $\square$

Example A.2. I provide an example where $\mathcal{I}$ is strongly transitive but $\uparrow \mathcal{I}$ is not as in Example A.1. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Define $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ as follows:

$$\mathcal{I}(\omega) = \begin{cases} \{\{\omega_1, \omega_3\}, \Omega\} & \text{if } \omega \in \{\omega_1, \omega_3\} \\ \{\{\omega_2\}, \Omega\} & \text{if } \omega = \omega_2 \end{cases}. \quad (\text{A.2})$$

First, I show that $\mathcal{I}$ is strongly transitive. Let $\omega \in \Omega$, $E \in \mathcal{I}(\omega)$, and $\omega' \in E$. There is $F = E \in \mathcal{I}(\omega')$ such that $F \subseteq E$. Second, $\uparrow \mathcal{I}$ is written as follows:

$$\uparrow \mathcal{I}(\omega) = \begin{cases} \{\{\omega_1, \omega_3\}, \Omega\} & \text{if } \omega \in \{\omega_1, \omega_3\} \\ \{\{\omega_2\}, \{\omega_1, \omega_2\}, \{\omega_2, \omega_3\}, \Omega\} & \text{if } \omega = \omega_2 \end{cases}. \quad (\text{A.3})$$

Third, I show that $\uparrow \mathcal{I}$ is not strongly transitive. Take $E = \{\omega_2, \omega_3\} \in \uparrow \mathcal{I}(\omega_2)$ and $\omega_3 \in E$. Then, $\{\omega_1, \omega_3\} \not\subseteq \{\omega_2, \omega_3\} = E$ and $\Omega \not\subseteq \{\omega_2, \omega_3\} = E$. I remark that, since $\mathcal{I}$ is transitive, it follows from Proposition 3 that $\uparrow \mathcal{I}$ is transitive. I also remark that the belief operator $B_{\mathcal{I}}$ in this example coincides with that defined by Equation (A.1) in Example A.1. $\square$

A.5 Section 3.3

Proof of Proposition 4. 1. First, I show that (1a) implies (1b). Take $(\omega, E) \in \Omega \times \mathcal{D}$. There is $F \in \mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$ with the following property: if $\omega' \in F$ and $E \in \uparrow \mathcal{I}(\omega')$ then $F' \in \mathcal{I}(\omega')$ for some $F' \subseteq E$. Then, $\omega' \in E$.

Second, I show that (1b) implies (1c). Take $(\omega, E) \in \Omega \times \mathcal{D}$. By (1b), there is $F \in \uparrow \mathcal{I}(\omega)$ such that $F \subseteq (\neg B_{\mathcal{I}})(E) \cup E$. Thus, $\omega \in B_{\mathcal{I}}((\neg B_{\mathcal{I}})(E) \cup E)$, and hence

$$\Omega = B_{\mathcal{I}}((\neg B_{\mathcal{I}})(E) \cup E).$$

Finally, I show that (1c) implies (1a). Take any $(\omega, E) \in \Omega \times \mathcal{D}$. Since

$$\omega \in \Omega = B_{\mathcal{I}}((\neg B_{\mathcal{I}})(E) \cup E),$$

there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq (\neg B_{\mathcal{I}})(E) \cup E$. If $\omega' \in F$ and if there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $\omega' \in B_{\mathcal{I}}(E)$ implies $\omega' \in E$.

2. First, I show that (2a) implies (2b). Fix $(\omega, E) \in \Omega \times \mathcal{D}$. There is $F \in \mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$ with the following property: if $\omega' \in F$ and if $E \in \uparrow \mathcal{I}(\omega')$ and thus $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $H \cap E \neq \emptyset$ for all $H \in \uparrow \mathcal{I}(\omega')$. Thus, $E^c \notin \uparrow \mathcal{I}(\omega')$.

Second, I show that (2b) implies (2c). Fix $E \in \mathcal{D}$, and take $\omega \in \Omega$. I show that

$$\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)) = (\neg B_{\mathcal{I}})(E) \cup (\neg B_{\mathcal{I}})(E^c) \in \uparrow \mathcal{I}(\omega).$$

By (2b), there is $F \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in F$ and if $\omega' \in B_{\mathcal{I}}(E)$ then $\omega' \in (\neg B_{\mathcal{I}})(E^c)$. Thus, $F \subseteq \neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c))$. Hence,

$$\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c)) \in \uparrow \mathcal{I}(\omega).$$

Finally, I show that (2c) implies (2a). Fix $(\omega, E) \in \Omega \times \mathcal{D}$. Since

$$\omega \in \Omega = B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c))),$$

there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq \neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c))$. Thus, if $\omega' \in F$ and if there is $F' \in \mathcal{I}(\omega')$ with $F' \subseteq E$, then $\omega' \in (\neg B_{\mathcal{I}})(E^c)$. Thus,

$$H \cap E \neq \emptyset \text{ for all } H \in \mathcal{I}(\omega').$$

3. First, I show that (3a) implies (3b). For any $(\omega, E, F) \in \Omega \times \mathcal{D} \times \mathcal{D}$, there is $G \in \mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$, $E \in \uparrow \mathcal{I}(\omega')$, and if $(\neg E) \cup F \in \uparrow \mathcal{I}(\omega')$, then (3a) implies that $F \in \uparrow \mathcal{I}(\omega')$.

Second, I show that (3b) implies (3c). Fix $E, F \in \mathcal{D}$ and $\omega \in \Omega$. There is $G \in \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$, $\omega' \in B_{\mathcal{I}}(E)$, and if $\omega' \in B_{\mathcal{I}}((\neg E) \cup F)$, then $\omega' \in B_{\mathcal{I}}(F)$. Thus, $G \subseteq \neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F)$. Hence,

$$\omega \in B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F)).$$

Finally, I show that (3c) implies (3a). Take $(E, F) \in \mathcal{D}^2$ and

$$\omega \in \Omega = B_{\mathcal{I}}(\neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F)).$$

There is $G \in \mathcal{I}(\omega)$ such that:

$$\text{if } \omega' \in G \text{ then } \omega' \in \neg(B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}((\neg E) \cup F)) \cup B_{\mathcal{I}}(F).$$

If $\omega' \in G$, $E' \in \mathcal{I}(\omega')$ for some $E' \subseteq E$, and if $F' \in \mathcal{I}(\omega')$ for some $F' \subseteq (\neg E) \cup F$, then $\omega' \in B_{\mathcal{I}}(F)$, i.e., there is $G' \in \mathcal{I}(\omega')$ with $G' \subseteq F$.

4. Throughout the proof, take $\{F_n\}_{n \in \mathbb{N}} \in \mathcal{P}(\mathcal{D})$. First, I show that (4a) implies (4b). Fix $\omega \in \Omega$. By (4a), there is $G \in \mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$ such that if $\omega' \in G$ and if $\{F_n\}_{n \in \mathbb{N}} \subseteq \uparrow \mathcal{I}(\omega')$ (i.e., for each $n \in \mathbb{N}$, there is $F'_n \in \mathcal{I}(\omega')$ with $F'_n \subseteq F_n$), then

$$\text{there is } G' \in \mathcal{I}(\omega') \subseteq \uparrow \mathcal{I}(\omega') \text{ such that } G' \subseteq \bigcap_{n \in \mathbb{N}} F_n.$$

Second, I show that (4b) implies (4c). For each $\omega \in \Omega$, by (4b), there is $G \in \uparrow \mathcal{I}(\omega)$ such that $G \subseteq \neg(\bigcap_{n \in \mathbb{N}} B_{\mathcal{I}}(F_n)) \cup B_{\mathcal{I}}(\bigcap_{n \in \mathbb{N}} F_n)$. Hence,

$$\omega \in B_{\mathcal{I}} \left(\neg \left(\bigcap_{n \in \mathbb{N}} B_{\mathcal{I}}(F_n) \right) \cup B_{\mathcal{I}} \left(\bigcap_{n \in \mathbb{N}} F_n \right) \right).$$

Finally, I show that (4c) implies (4a). Take

$$\omega \in \Omega = B_{\mathcal{I}} \left(\neg \left(\bigcap_{n \in \mathbb{N}} B_{\mathcal{I}}(F_n) \right) \cup B_{\mathcal{I}} \left(\bigcap_{n \in \mathbb{N}} F_n \right) \right).$$

There is $G \in \mathcal{I}(\omega)$ such that if $\omega' \in G$ and if, for each $n \in \mathbb{N}$, there is $F'_n \in \mathcal{I}(\omega')$ with $F'_n \subseteq F_n$ (i.e., $F_n \in \uparrow \mathcal{I}(\omega')$), then $\omega' \in B_{\mathcal{I}}(\bigcap_{n \in \mathbb{N}} F_n)$. Thus,

$$\text{there is } G' \in \mathcal{I}(\omega') \text{ with } G' \subseteq \bigcap_{n \in \mathbb{N}} F_n.$$

□

Example A.3. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Define $\mathcal{I} : \Omega \rightarrow \mathcal{P}(\mathcal{D})$ as follows:

$$\mathcal{I}(\omega) = \begin{cases} \{\{\omega_1, \omega_2\}, \{\omega_2, \omega_3\}\} & \text{if } \omega = \omega_1 \\ \{\{\omega_2\}\} & \text{if } \omega = \omega_2 \\ \{\emptyset\} & \text{if } \omega = \omega_3 \end{cases}.$$

The belief operator $B_{\mathcal{I}}$ is given as follows:

$$B_{\mathcal{I}}(E) = \begin{cases} \{\omega_3\} & \text{if } E \in \{\emptyset, \{\omega_1\}, \{\omega_3\}, \{\omega_1, \omega_3\}\} \\ \{\omega_2, \omega_3\} & \text{if } E = \{\omega_2\} \\ \Omega & \text{if } E \in \{\{\omega_1, \omega_2\}, \{\omega_2, \omega_3\}, \Omega\} \end{cases}.$$

The operator $B_{\mathcal{I}}$ violates No-Contradiction because $\emptyset \in \mathcal{I}(\omega_3)$. In fact, $B_{\mathcal{I}}(\emptyset) = \{\omega_3\}$. Consequently, it violates Truth Axiom and Consistency (indeed, $B_{\mathcal{I}}(E) \cap B_{\mathcal{I}}(E^c) = \{\omega_3\}$ for any $E \in \mathcal{D}$). The operator violates Countable Conjunction:

$$B_{\mathcal{I}}(\{\omega_1, \omega_2\}) \cap B_{\mathcal{I}}(\{\omega_2, \omega_3\}) = \Omega \not\subseteq \{\omega_2, \omega_3\} = B_{\mathcal{I}}(\{\omega_2\}).$$

In contrast, it can be seen that $\mathcal{I}$ is secondary reflexive and secondary serial. It also satisfies Belief in Countable Conjunction and consequently Belief in Perfect Reasoning.

□

A.6 Section 4.1

Proof of Proposition 5. 1. By Propositions 2, 3, and 4, $B_{\mathcal{I}}$ satisfies the logical, introspective, and “Belief-in” properties of beliefs imposed on $\mathcal{I}$. Next,

$$\begin{aligned}\mathcal{I}_{B_{\mathcal{I}}}(\omega) &= \{E \in \mathcal{D} \mid \omega \in B_{\mathcal{I}}(E)\} \\ &= \{E \in \mathcal{D} \mid E \in \uparrow \mathcal{I}(\omega)\} = \uparrow \mathcal{I}(\omega) \text{ for all } \omega \in \Omega.\end{aligned}$$

Since $B_{\mathcal{I}}$ is monotone, $\uparrow \mathcal{I}_{B_{\mathcal{I}}}(\cdot) = \mathcal{I}_{B_{\mathcal{I}}}(\cdot) = \uparrow \mathcal{I}(\cdot)$.

Conversely,

$$B_{\mathcal{I}_B}(E) = \{\omega \in \Omega \mid E \in \uparrow \mathcal{I}_B(\omega)\} = B(E) \text{ for each } E \in \mathcal{D}.$$

By Propositions 2, 3, and 4, any generator $\mathcal{I}$ of B satisfies the logical, introspective, and “Belief-in” properties of beliefs imposed on B .

As argued in the main text,

$$\mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}(\cdot) = \uparrow \mathcal{I}_B(\cdot) = \mathcal{I}_B(\cdot).$$

2. First, I show that $\mathcal{I}_{\mathcal{J}_B}$ is an information correspondence that generates B , i.e.,

$$B = B_{\mathcal{I}_{\mathcal{J}_B}}.$$

Take $E \in \mathcal{D}$ and $\omega \in B(E)$. Since $\omega \in B(E) \subseteq BB(E)$, it follows that $B(E) \in \mathcal{I}_{\mathcal{J}_B}(\omega)$ and $E \in \uparrow \mathcal{I}_{\mathcal{J}_B}(\omega)$. Thus, $\omega \in B_{\mathcal{I}_{\mathcal{J}_B}}(E)$. Conversely, if $\omega \in B_{\mathcal{I}_{\mathcal{J}_B}}(E)$ then there is $F \in \mathcal{I}_{\mathcal{J}_B}(\omega)$ with $\omega \in F \subseteq E$. Then, $\omega \in F \subseteq B(F) \subseteq B(E)$.

Second, $\mathcal{I}_{\mathcal{J}_B}$ is by construction reflexive.

Third, I show that $\mathcal{I}_{\mathcal{J}_B}$ is (strongly) transitive. Let $E \in \mathcal{I}_{\mathcal{J}_B}(\omega)$, i.e., $\omega \in E = B(E)$. If $\omega' \in E$ then there is $E' = E \in \mathcal{I}_{\mathcal{J}_B}(\omega')$ such that $E' \subseteq E$. □

A.7 Section 4.2

Proof of Proposition 6. 1. (a) Let $\mathcal{J}$ satisfy Condition (7). First, I show that $\mathcal{I}_{\mathcal{J}}$ is an information correspondence. For each $E \in \mathcal{D}$,

$$\begin{aligned}\{\omega \in \Omega \mid \text{there is } F \in \mathcal{I}_{\mathcal{J}}(\omega) \text{ such that } F \subseteq E\} \\ = \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J} \text{ such that } \omega \in F \subseteq E\} \in \mathcal{J} \subseteq \mathcal{D}.\end{aligned}$$

Second, $\mathcal{I}_{\mathcal{J}}$ is reflexive by construction.

Third, I show that $\mathcal{I}_{\mathcal{J}}$ is (strongly) transitive. For any $E \in \mathcal{I}_{\mathcal{J}}(\omega)$ and $\omega' \in E$, $E' = E \in \mathcal{I}_{\mathcal{J}}(\omega')$ satisfies $\omega' \in E' \subseteq E$.

(b) Suppose $\Omega \in \mathcal{J}$. For any $\omega \in \Omega$, one has $\Omega \in \mathcal{I}_{\mathcal{J}}(\omega)$ and thus $\mathcal{I}_{\mathcal{J}}(\omega) \neq \emptyset$.

2. (a) Conversely, I show that $\mathcal{J}_{\mathcal{I}}$ satisfies Condition (7). For each $E \in \mathcal{D}$, let

$$B_{\mathcal{J}_{\mathcal{I}}}(E) := \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_{\mathcal{I}} \text{ such that } \omega \in F \subseteq E\}.$$

Then, it suffices to show that $B_{\mathcal{J}_{\mathcal{I}}}(E) \in \mathcal{J}_{\mathcal{I}}$, i.e.,

$$\text{if } \omega \in B_{\mathcal{J}_{\mathcal{I}}}(E), \text{ then there is } F' \in \mathcal{I}(\omega) \text{ such that } F' \subseteq B_{\mathcal{J}_{\mathcal{I}}}(E).$$

Let $\omega \in B_{\mathcal{J}_{\mathcal{I}}}(E)$. There is $F \in \mathcal{J}_{\mathcal{I}}$ such that $\omega \in F \subseteq E$. Since $\omega \in F$, there is $E' \in \mathcal{I}(\omega)$ such that $E' \subseteq F$. Since $\mathcal{I}$ is transitive, there is $F' \in \mathcal{I}(\omega)$ such that if $\omega' \in F'$ then there is $G \in \mathcal{I}(\omega')$ such that $\omega' \in G \subseteq E' \subseteq F \subseteq E$. Thus, $\omega' \in B_{\mathcal{J}_{\mathcal{I}}}(E)$. Hence, $F' \subseteq B_{\mathcal{J}_{\mathcal{I}}}(E)$, as desired.

- (b) If $\mathcal{I}(\cdot) \neq \emptyset$, then for any $\omega \in \Omega$, there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq \Omega$. Thus, $\Omega \in \mathcal{J}_{\mathcal{I}}$.

3. (a) Next, let $E \in \mathcal{J}$. For any $\omega \in E$, I have $E \in \mathcal{I}_{\mathcal{J}}(\omega)$ and $E \subseteq E$. Thus, $E \in \mathcal{J}_{\mathcal{I}_{\mathcal{J}}}$. Conversely, if $E \in \mathcal{J}_{\mathcal{I}_{\mathcal{J}}}$, then

$$\begin{aligned} E &= \{\omega \in \Omega \mid \text{there is } F \in \mathcal{I}_{\mathcal{J}}(\omega) \text{ such that } F \subseteq E\} \\ &= \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J} \text{ such that } \omega \in F \subseteq E\} \in \mathcal{J}. \end{aligned}$$

- (b) Finally, let $\mathcal{I}$ be given. Fix $\omega \in \Omega$. First, I show

$$\uparrow \mathcal{I}_{\mathcal{J}_{\mathcal{I}}}(\omega) \subseteq \uparrow \mathcal{I}(\omega).$$

If $E \in \mathcal{I}_{\mathcal{J}_{\mathcal{I}}}(\omega)$, then $\omega \in E \in \mathcal{J}_{\mathcal{I}}$. Thus, $E \in \uparrow \mathcal{I}(\omega)$. This implies $\mathcal{I}_{\mathcal{J}_{\mathcal{I}}}(\omega) \subseteq \uparrow \mathcal{I}(\omega)$.

Conversely, I show

$$\uparrow \mathcal{I}(\omega) \subseteq \uparrow \mathcal{I}_{\mathcal{J}_{\mathcal{I}}}(\omega).$$

Suppose that $E \in \uparrow \mathcal{I}(\omega)$. Since $\mathcal{I}$ is transitive,

$$E' := \{\omega' \in \Omega \mid E \in \uparrow \mathcal{I}(\omega')\} \in \uparrow \mathcal{I}(\omega).$$

Thus, there is $F \in \mathcal{I}(\omega)$ such that $F \subseteq E'$. Since $\mathcal{I}$ is reflexive, $\omega \in F \subseteq E'$. If $\omega' \in E'$ then $\omega' \in \{\omega'' \in \Omega \mid E' \in \uparrow \mathcal{I}(\omega'')\}$. That is, if $\omega' \in E'$ then there is $G \in \mathcal{I}(\omega')$ such that $G \subseteq E'$. Thus, $E' \in \mathcal{J}_{\mathcal{I}}$. Since $\omega \in E'$, I have $E' \in \mathcal{I}_{\mathcal{J}_{\mathcal{I}}}(\omega)$. Since $\mathcal{I}$ is reflexive, $E' \subseteq E$. This is because, for any $\omega' \in E'$, it follows from $E \in \uparrow \mathcal{I}(\omega')$ and the reflexivity of $\mathcal{I}$ (and hence $\uparrow \mathcal{I}$) that $\omega' \in E$. Thus, $E \in \uparrow \mathcal{I}_{\mathcal{J}_{\mathcal{I}}}(\omega)$, as desired.

□

Proof of Corollary 2. Note that, by Expression (9),

$$\mathcal{J}_{\mathcal{I}} = \{E \in \mathcal{D} \mid \text{if } \omega \in E \text{ then } E \in \uparrow \mathcal{I}(\omega)\}$$

depends on $\mathcal{I}$ only through $\uparrow \mathcal{I}$.

1. Reflexivity of $\sigma_{\mathcal{J}}$ holds by construction. For Necessitation, fix $\omega \in \Omega$. Since $\Omega \in \mathcal{J}$ and $\mathcal{J}$ is granular, there is an ω -minimal F in $\mathcal{J}$, i.e., $F \in \sigma_{\mathcal{J}}(\omega)$. For minimality, let $E, F \in \sigma_{\mathcal{J}}(\omega)$ with $E \subseteq F$. Since $\omega \in E \in \mathcal{J}$ and F is ω -minimal, it follows that $E = F$. For strong transitivity, let $F \in \sigma_{\mathcal{J}}(\omega)$ and $\omega' \in F$. Since $F \in \mathcal{J}$, granularity yields an ω' -minimal G in $\mathcal{J}$ with $G \subseteq F$, i.e., $G \in \sigma_{\mathcal{J}}(\omega')$.

Next, I show $\uparrow \sigma_{\mathcal{J}} = \uparrow \mathcal{I}_{\mathcal{J}}$. Since $\sigma_{\mathcal{J}}(\omega) \subseteq \mathcal{I}_{\mathcal{J}}(\omega)$, one has $\uparrow \sigma_{\mathcal{J}}(\omega) \subseteq \uparrow \mathcal{I}_{\mathcal{J}}(\omega)$. Conversely, if $E \in \mathcal{I}_{\mathcal{J}}(\omega)$, then granularity yields $F \in \sigma_{\mathcal{J}}(\omega)$ with $F \subseteq E$, so that $E \in \uparrow \sigma_{\mathcal{J}}(\omega)$. Hence $\uparrow \mathcal{I}_{\mathcal{J}}(\omega) \subseteq \uparrow \sigma_{\mathcal{J}}(\omega)$. Since $\mathcal{I}_{\mathcal{J}}$ is an information correspondence by Proposition 6, $B_{\sigma_{\mathcal{J}}} = B_{\mathcal{I}_{\mathcal{J}}}$ maps $\mathcal{D}$ into itself, and thus $\sigma_{\mathcal{J}}$ is an information correspondence. Finally, $\mathcal{J}_{\sigma_{\mathcal{J}}} = \mathcal{J}_{\mathcal{I}_{\mathcal{J}}} = \mathcal{J}$ by Proposition 6.

2. Since strong transitivity implies transitivity, Proposition 6 implies that $\mathcal{J}_{\sigma}$ satisfies Condition (7) and $\Omega \in \mathcal{J}_{\sigma}$. Fix $\omega \in \Omega$ and $F \in \sigma(\omega)$. First, $F \in \mathcal{J}_{\sigma}$: for any $\omega' \in F$, strong transitivity yields $E' \in \sigma(\omega')$ with $E' \subseteq F$. Second, F is ω -minimal in $\mathcal{J}_{\sigma}$: $\omega \in F$ by reflexivity; if $G \in \mathcal{J}_{\sigma}$ satisfies $\omega \in G \subseteq F$, then there is $F' \in \sigma(\omega)$ with $F' \subseteq G \subseteq F$, and minimality yields $F' = F$ and hence $G = F$. Thus, $\sigma(\omega) \subseteq \sigma_{\mathcal{J}_{\sigma}}(\omega)$.

To see that $\mathcal{J}_{\sigma}$ is granular, let $E \in \mathcal{J}_{\sigma}$ and $\omega \in E$. There is $F \in \sigma(\omega)$ with $F \subseteq E$, and F is ω -minimal in $\mathcal{J}_{\sigma}$ by the previous paragraph.

To see $\sigma_{\mathcal{J}_{\sigma}}(\omega) \subseteq \sigma(\omega)$, let G be ω -minimal in $\mathcal{J}_{\sigma}$. Since $G \in \mathcal{J}_{\sigma}$ and $\omega \in G$, there is $F \in \sigma(\omega)$ with $F \subseteq G$. Since $F \in \mathcal{J}_{\sigma}$ and $\omega \in F$, the ω -minimality of G yields $F = G$, i.e., $G \in \sigma(\omega)$.

□

Example A.4. Consider the belief operator B defined by Equation (A.1) in Example A.1. Let

$$\mathcal{J} := \mathcal{J}_B = \{\emptyset, \{\omega_2\}, \{\omega_1, \omega_3\}, \Omega\}.$$

The information correspondence $\mathcal{I}_{\mathcal{J}}$ coincides with Equation (A.2) in Example A.2. Then, $\uparrow \mathcal{I}_{\mathcal{J}}$ coincides with Equation (A.3) in Example A.2. The arguments in Example A.2 show that, while $\mathcal{I}_{\mathcal{J}}$ is strongly transitive, $\uparrow \mathcal{I}_{\mathcal{J}}$ is not. □

A.8 Section 5

Proof of Proposition 7. First, the “ $\supseteq$ ” part follows because, if $t(\omega, E) = 1$ then

$$\begin{aligned} \int_{\Omega} (u \circ (x_E, y_{E^c}))(\omega') t(\omega, d\omega') &= \int_E (u \circ x)(\omega') t(\omega, d\omega') \\ &= \int_{\Omega} (u \circ (x_E, z_{E^c}))(\omega') t(\omega, d\omega') \text{ for any } x, y, z \in \mathcal{B}(\Omega, \mathcal{D}). \end{aligned}$$

Second, for the “ $\subseteq$ ” part, suppose to the contrary that $\omega \in B_{\succ}(E)$ and $t(\omega, E) < 1$. Choose $x, y, z \in \mathcal{B}(\Omega, \mathcal{D})$ such that $y(\omega) = a < b = z(\omega)$ for all $\omega \in \Omega$. Then,

$$\int_{\Omega} (u \circ (x_E, y_{E^c}))(\omega') t(\omega, d\omega') < \int_{\Omega} (u \circ (x_E, z_{E^c}))(\omega') t(\omega, d\omega'),$$

a contradiction. □

Proof of Proposition 8. For any act $f \in \mathcal{B}(\Omega, \mathcal{D})$ and $\pi \in C(\omega)$, denote by

$$V_\pi(f) := \int_\Omega (u \circ f)(\omega') \pi(d\omega') \text{ and } V(f) := \inf_{\pi \in C(\omega)} V_\pi(f).$$

First, I show the “ $\supseteq$ ” part. Suppose $\pi(E) = 1$ for all $\pi \in C(\omega)$. For any $x, y, z \in \mathcal{B}(\Omega, \mathcal{D})$ and any $\pi \in C(\omega)$, since $\pi(E^c) = 0$,

$$V_\pi((x_E, y_{E^c})) = \int_E (u \circ x)(\omega') \pi(d\omega') = V_\pi((x_E, z_{E^c})).$$

Taking the infimum over $\pi \in C(\omega)$ yields $(x_E, y_{E^c}) \sim_\omega (x_E, z_{E^c})$. Hence $\omega \in B_{\succsim}(E)$.

Second, I show the “ $\subseteq$ ” part by contraposition. Suppose

$$\bar{q} := \sup_{\pi \in C(\omega)} \pi(E^c) > 0.$$

Choose $\alpha < \beta < \gamma$ in $\mathbb{R}$, and let $x, y, z \in \mathcal{B}(\Omega, \mathcal{D})$ be the constant acts $x = \gamma$, $y = \alpha$, and $z = \beta$. For each $\pi \in C(\omega)$,

$$V_\pi((x_E, y_{E^c})) = u(\gamma)\pi(E) + u(\alpha)\pi(E^c) = u(\gamma) - \pi(E^c)(u(\gamma) - u(\alpha)).$$

Since u is strictly increasing, $u(\gamma) - u(\alpha) > 0$, and thus

$$V((x_E, y_{E^c})) = u(\gamma) - \bar{q}(u(\gamma) - u(\alpha)).$$

Likewise, I obtain

$$V((x_E, z_{E^c})) = u(\gamma) - \bar{q}(u(\gamma) - u(\beta)).$$

Since $\bar{q} > 0$ and $u(\alpha) < u(\beta)$, it follows that

$$V((x_E, y_{E^c})) < V((x_E, z_{E^c})), \text{ that is, } (x_E, y_{E^c}) \prec_\omega (x_E, z_{E^c}).$$

Hence $\omega \notin B_{\succsim}(E)$.

Finally, $B_{\succsim} = B_{\mathcal{I}_C}$ follows because $\mathcal{I}_C(\omega) = \{E \in \mathcal{D} \mid \omega \in B_{\succsim}(E)\}$ is closed under supersets within $\mathcal{D}$ (each π is monotone), so that $\uparrow \mathcal{I}_C(\omega) = \mathcal{I}_C(\omega)$ and $B_{\mathcal{I}_C} = B_{\succsim}$ by Equation (2). The regularity condition on C guarantees that $B_{\succsim}$ maps $\mathcal{D}$ into itself. □

Proof of Lemma 1. By Proposition 8, $\omega \in B_{\succsim}(E)$ iff E is $C(\omega)$ -full.

1. Suppose that $P : \Omega \rightarrow \mathcal{D}$ satisfies $B_{\succsim}(E) = \{\omega \in \Omega \mid P(\omega) \subseteq E\}$ for all $E \in \mathcal{D}$. Fix $\omega \in \Omega$. Since $\omega \in B_{\succsim}(P(\omega))$, the event $P(\omega)$ is $C(\omega)$ -full. If $E \in \mathcal{D}$ is $C(\omega)$ -full, then $\omega \in B_{\succsim}(E)$, and hence $P(\omega) \subseteq E$. Thus, $P(\omega)$ is the smallest $C(\omega)$ -full event. Conversely, suppose that the smallest $C(\omega)$ -full event $S(\omega)$ exists for each ω . For each $E \in \mathcal{D}$, if $S(\omega) \subseteq E$ then $\pi(E) \geq \pi(S(\omega)) = 1$ for all $\pi \in C(\omega)$. Hence, E is $C(\omega)$ -full iff $S(\omega) \subseteq E$, and $P := S$ satisfies the desired equality.

2. Suppose to the contrary that such a mapping P exists. For each $r \in \Omega \setminus A(\omega)$, the event $\Omega \setminus \{r\} \in \mathcal{D}$ is $C(\omega)$ -full because $\pi'(\{r\}) = 0$ for all $\pi' \in C(\omega)$. Hence $\omega \in B_{\sim}(\Omega \setminus \{r\})$, and thus $P(\omega) \subseteq \Omega \setminus \{r\}$. It follows that $P(\omega) \subseteq A(\omega)$. Since $A(\omega)$ is countable, $A(\omega) \in \mathcal{D}$, and $P(\omega) \subseteq A(\omega)$ implies $\omega \in B_{\sim}(A(\omega))$, i.e., $A(\omega)$ is $C(\omega)$ -full. However, the measure $\pi \in C(\omega)$ in the statement satisfies $\pi(A(\omega)) = 0$ by countable additivity, a contradiction.

□

B Additional Results

In this Appendix, Remark B.1 formalizes the equivalence between information correspondences and monotone belief operators (studied in Section 4.1) as a Galois connection. Remark B.2 formalizes the equivalence between reflexive and transitive information correspondences and self-evident collections (studied in Section 4.2) as a Galois connection.

Formally, letting $(X, \leq_X)$ and $(Y, \leq_Y)$ be pre-ordered sets (where a pre-order is a reflexive and transitive relation), a pair of order-preserving maps $\alpha : X \rightarrow Y$ and $\beta : Y \rightarrow X$ is a *Galois connection* if the following holds:

$$x \leq_X \beta(y) \text{ iff } \alpha(x) \leq_Y y \text{ for all } (x, y) \in X \times Y.$$

Remark B.1. I formulate the equivalence between information correspondences and monotone belief operators in Proposition 5 as a Galois connection. To that end, let $(\mathbb{I}, \leq_{\mathbb{I}})$ be the collection of information correspondences on a state space $(\Omega, \mathcal{D})$ with the following pre-order: $\mathcal{I} \leq_{\mathbb{I}} \mathcal{I}'$ if and only if the following holds:

$$\text{for each } \omega \in \Omega \text{ and } E \in \mathcal{I}(\omega), \text{ there is } F \in \mathcal{I}'(\omega) \text{ such that } F \subseteq E.$$

In other words, $\uparrow \mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}'(\cdot)$. Let $(\mathbb{B}, \leq_{\mathbb{B}})$ be the collection of monotone belief operators from $\mathcal{D}$ into itself, and $B \leq_{\mathbb{B}} B'$ if and only if $B(\cdot) \subseteq B'(\cdot)$.

Let $\alpha : (\mathbb{I}, \leq_{\mathbb{I}}) \rightarrow (\mathbb{B}, \leq_{\mathbb{B}})$ be such that

$$\alpha(\mathcal{I}) = B_{\mathcal{I}}$$

defined as in Equation (2). I show that α is order-preserving. If $\mathcal{I} \leq_{\mathbb{I}} \mathcal{I}'$ then $\uparrow \mathcal{I} \leq_{\mathbb{I}} \uparrow \mathcal{I}'$ and thus $B_{\mathcal{I}} \leq_{\mathbb{B}} B_{\mathcal{I}'}$ by Equation (2).

Next, let $\beta : (\mathbb{B}, \leq_{\mathbb{B}}) \rightarrow (\mathbb{I}, \leq_{\mathbb{I}})$ be

$$\beta(B) = \mathcal{I}_B$$

as in Equation (6). By construction, β is order-preserving, i.e., if $B \leq_{\mathbb{B}} B'$ then $\mathcal{I}_B \leq_{\mathbb{I}} \mathcal{I}_{B'}$.

Now, I show that (α, β) is a Galois connection, that is, the order-preserving maps α and β on the pre-ordered spaces satisfy

$$\mathcal{I} \leq_{\mathbb{I}} \beta(B) \text{ if and only if } \alpha(\mathcal{I}) \leq_{\mathbb{B}} B \text{ for any } (\mathcal{I}, B) \in \mathbb{I} \times \mathbb{B}.$$

Indeed, if $\alpha(\mathcal{I}) \leq_{\mathbb{B}} B$ then $\mathcal{I} \leq_{\mathbb{I}} \uparrow \mathcal{I} = \mathcal{I}_{B_{\mathcal{I}}} \leq_{\mathbb{I}} \mathcal{I}_B = \beta(B)$. Conversely, if $\mathcal{I} \leq_{\mathbb{I}} \beta(B)$ then $\alpha(\mathcal{I}) = B_{\mathcal{I}} \leq_{\mathbb{B}} B_{\mathcal{I}_B} = B$. □

Remark B.2. Let $(\mathbb{I}^{rt}, \leq^{rt})$ be the collection of reflexive and transitive information correspondences with the following pre-order as in Remark B.1: $\mathcal{I} \leq^{rt} \mathcal{I}'$ if and only if the following holds:

for each $\omega \in \Omega$ and $E \in \mathcal{I}(\omega)$, there is $F \in \mathcal{I}'(\omega)$ such that $F \subseteq E$.

In other words, $\uparrow \mathcal{I}(\cdot) \subseteq \uparrow \mathcal{I}'(\cdot)$. Let $(\mathbb{J}, \subseteq)$ be the space consisting of collections of events $\mathcal{J}(\in \mathcal{P}(\mathcal{D}))$ satisfying Condition (7).

Define $\alpha : (\mathbb{I}^{rt}, \leq^{rt}) \rightarrow (\mathbb{J}, \subseteq)$ by

$$\alpha(\mathcal{I}) = \mathcal{J}_{\mathcal{I}}$$

as in Equation (9). I show that the mapping α is order-preserving. Let $\mathcal{I} \leq^{rt} \mathcal{I}'$, and take $E \in \mathcal{J}_{\mathcal{I}}$. If $\omega \in E$ then there is $F' \in \mathcal{I}'(\omega)$ such that $F' \subseteq E$. Thus, $E \in \mathcal{J}_{\mathcal{I}'}$.

Next, define $\beta : (\mathbb{J}, \subseteq) \rightarrow (\mathbb{I}^{rt}, \leq^{rt})$ by

$$\beta(\mathcal{J}) = \mathcal{I}_{\mathcal{J}}$$

as in Equation (8). I show that β is order-preserving. If $\mathcal{J} \subseteq \mathcal{J}'$, then $\mathcal{I}_{\mathcal{J}}(\cdot) \subseteq \mathcal{I}_{\mathcal{J}'}(\cdot)$. This implies $\mathcal{I}_{\mathcal{J}} \leq^{rt} \mathcal{I}_{\mathcal{J}'}$.

Now, I establish that (α, β) is a Galois connection, that is,

$$\mathcal{I} \leq^{rt} \beta(\mathcal{J}) \text{ if and only if } \alpha(\mathcal{I}) \subseteq \mathcal{J} \text{ for any } (\mathcal{I}, \mathcal{J}) \in \mathbb{I}^{rt} \times \mathbb{J}.$$

If $\mathcal{I} \leq^{rt} \beta(\mathcal{J}) = \mathcal{I}_{\mathcal{J}}$ then $\alpha(\mathcal{I}) \subseteq \alpha \circ \beta(\mathcal{J}) = \mathcal{J}_{\mathcal{I}_{\mathcal{J}}} = \mathcal{J}$. Conversely, if $\alpha(\mathcal{I}) \subseteq \mathcal{J}$ then $\mathcal{I} \leq^{rt} \beta(\mathcal{J})$ because $\uparrow \mathcal{I} = \uparrow \mathcal{I}_{\mathcal{J}} \leq^{rt} \uparrow \mathcal{I}_{\mathcal{J}}$. $\square$